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Cartan Domains and Indefinite Euclidean Spaces

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Abstract: We prove that the complex hyperbolic space is the only Cartan domain which admits a Kähler immersion into the indefinite complex Euclidean space of finite index.

Key words: Kähler metrics; bounded domains; diastasis function; symmetric space; complex space form.

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1. Introduction and statement of the main result

Let (Ω, g_B) be a bounded symmetric domain of $\mathbb{C}^n$ equipped with its Bergman metric g_B. In [3] it is proven that (Ω, g_B) does not admit a holomorphic and isometric (from now on Kähler) immersion into the flat Euclidean space $\mathbb{C}^N$ for any $N \in \mathbb{N}$ and it admits a Kähler immersion into $\ell^2(\mathbb{C})$ (the infinite dimensional complex Euclidean space with the flat metric) if and only if the rank of Ω equals 1, namely Ω is the complex hyperbolic space $\mathbb{C}\mathrm{H}^n$ and $g_B = (n+1)g_{hyp}$. Here g_{hyp} denotes the hyperbolic metric on $\mathbb{C}\mathrm{H}^n$, namely the metric whose associated Kähler form is given by

$$\omega_{hyp} = -\frac{i}{2}\,\partial\bar\partial \log\left(1 - |z|^2\right), \qquad |z|^2 = |z_1|^2 + \cdots + |z_n|^2.$$

Moreover, one can verify (see also Calabi [2, Theorem 13]) that the map

$$\begin{array}{rcl} f : \mathbb{C}\mathrm{H}^n & \longrightarrow & \ell^2(\mathbb{C}) \\ (z_1, \dots, z_n) & \longmapsto & (n+1)\left(\dots, \sqrt{\frac{(|m_j|-1)!}{m_j!}}\, z^{m_j}, \dots\right) \end{array} \tag{1}$$

is a Kähler immersion of $(\mathbb{C}\mathrm{H}^n, (n+1)g_{hyp})$ into $\ell^2(\mathbb{C})$ (we are using the multi-index notation of Section 2). In this paper we address the problem of extending this result when the ambient space is the indefinite complex Euclidean space

$$\mathbb{C}^{r,s} = \left(\mathbb{C}^{r+s}, g_{r,s}\right), \qquad r, s \in \mathbb{N} \cup \{\infty\},\ (r,s) \neq (\infty,\infty).$$

Here $g_{r,s}$ is the indefinite Kähler metric on $\mathbb{C}^{r+s}$ whose associated (indefinite) Kähler form is given by

$$\omega_{r,s} = \frac{i}{2}\,\partial\bar{\partial}\left(\sum_{j=1}^{r}|z_j|^2 - \sum_{k=r+1}^{r+s}|z_k|^2\right) \tag{2}$$

when $r \in \mathbb{N}$ and $s \in \mathbb{N} \cup \{\infty\}$, and

$$\omega_{\infty,s} = \frac{i}{2}\,\partial\bar{\partial}\left(-\sum_{j=1}^{s}|z_j|^2 + \sum_{k=s+1}^{+\infty}|z_k|^2\right) \tag{3}$$

when $s \in \mathbb{N}$ and $r = \infty$. One calls s the *index* of $g_{r,s}$. Notice that we are excluding the case when both $s = \infty$ and $r = \infty$, since by Theorem 1 in Calabi [2] every real analytic Kähler manifold admits a local Kähler immersion into $\mathbb{C}^{\infty,\infty}$. Observe also that $(\mathbb{C}\mathrm{H}^n, (n+1)g_{hyp})$ can be Kähler immersed into $\mathbb{C}^{\infty,s}$ via the map $i \circ f \colon \mathbb{C}\mathrm{H}^n \to \mathbb{C}^{\infty,s}$, where $i \colon \ell^2(\mathbb{C}) \to \mathbb{C}^{\infty,s}$ denotes the natural inclusion and f is the map in (1). It is worth pointing out that one can construct infinitely many non congruent Kähler immersion of $(\mathbb{C}\mathrm{H}^n, (n+1)g_{hyp})$ into $\mathbb{C}^{\infty,s}$. For example for any holomorphic function ψ on $\mathbb{C}\mathrm{H}^n$ the map $z \mapsto (\psi(z), \psi(z), f(z))$ is a Kähler immersion of $\mathbb{C}\mathrm{H}^n$ into $\mathbb{C}^{\infty,1}$.

Behind the pure mathematical interest, indefinite Kähler geometry can be viewed in the case when $s = 1$, as a combination of the Lorentzian geometry of space-time and the symplectic geometry of phase space. Among the authors that have been studying the geometry of Kähler submanifolds of *finite* indefinite space forms we cite M. Barros, A. Romero, Y.J. Suh and T. Umehara (see [1], [5], [7]).

Our main result is the following theorem, which shows that $(\mathbb{C}\mathrm{H}^n, (n+1)g_{hyp})$ can be characterized among irreducible bounded symmetric domains as the only one which admits a Kähler immersion into $\mathbb{C}^{\infty,s}$, $s < \infty$.

Theorem 1. *Let (Ω, g_B) be a Cartan domain. Assume that there exists a local Kähler immersion (Ω, g_B) into $\mathbb{C}^{r,s}$, then $r = \infty$, $s \in \mathbb{N}$ and $(\Omega, g_B) = (\mathbb{C}\mathrm{H}^n, (n+1)g_{hyp})$.*

The paper is organized as follows. In the next section we collect the basic results about bounded symmetric domains and their Calabi's diastasis functions needed in the proof of Theorem 1, to whom Section 3 is dedicated.

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2. Preliminaries

Let (M, g) be a Kähler manifold. A Kähler potential for g is a real valued function Φ such that in a neighbourhood of a point $p \in M$ endowed with complex coordinates $\{z\} = \{z_1, \dots, z_n\}$, one has

$$g_{\alpha\bar\beta} = 2g\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar z_\beta}\right) = \frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar z_\beta}\,.$$

If g (and hence Φ) is assumed to be real analytic, by duplicating the variables z and $\bar z$, Φ can be complex analytically continued to a function $\tilde\Phi$ defined in a neighbourhood U of the diagonal containing $(p, \bar p) \in M \times \bar M$ (here $\bar M$ denotes the manifold conjugated to M). Thus one can consider the power expansion around the origin of Φ with respect to z and $\bar z$ and write it as

$$\Phi(z, \bar z) = \sum_{j,k=0}^{\infty} a_{jk} z^{m_j} \bar z^{m_k}, \tag{4}$$

where we arrange every n-tuple of nonnegative integers as a sequence $m_j = (m_{j,1}, \dots, m_{j,n})$ and order them as follows: $m_0 = (0, \dots, 0)$ and if $|m_j| = \sum_{\alpha=1}^n m_{j,\alpha}$, $|m_j| \leq |m_{j+1}|$ for all positive integer j. Further z^{m_j} denotes the monomial in n variables $\prod_{\alpha=1}^n z_\alpha^{m_{j,\alpha}}$, $z = (z_1, \dots, z_n)$.

A Kähler potential is not unique, it is defined up to an addition with the real part of a holomorphic function. The *diastasis function* D^g for g is the Kähler potential around p characterized by the fact that in every coordinate system $\{z\}$ centered at p, the matrix (a_{jk}) in (4) with $\Phi = \mathrm{D}^g$ satisfies $a_{j0} = a_{0j} = 0$ for every nonnegative integer j.

Example 1. Let $\mathbb{C}\mathrm{H}^1 = \{z \in \mathbb{C} : |z|^2 < 1\}$ be the unitary disk of $\mathbb{C}$, equipped with the hyperbolic metric g_{hyp} whose associated Kähler form is given by

$$\omega_{hyp} = -\frac{i}{2}\partial\bar\partial \log\left(1 - |z|^2\right).$$

Then the globally defined diastasis around the origin is given by

$$\mathrm{D}^{g_{hyp}}(z, \bar z) = -\log\left(1 - |z|^2\right) = \sum_{j=1}^{+\infty} \frac{|z|^{2j}}{j}\,, \qquad a_{jk} = \frac{\delta_{jk}}{j}\,.$$

The conditions for the existence of a Kähler immersion into the finite or infinite dimensional complex Euclidean space $\mathbb{C}^{N,0} = \mathbb{C}^N$ endowed with the flat metric $g_0 = g_{N,0}$ have been studied by Calabi in [2], and can be summarized as follows:

Theorem 2. (Calabi's criterion) *Let (M, g) be a Kähler manifold and $p \in M$. There exists a neighbourhood $U \subset M$ of p which admits a Kähler immersion $f\colon (U, g) \to \mathbb{C}^{N \leq \infty}$ if and only if the matrix of coefficients (a_{jk}) in (4) with $\Phi = \mathrm{D}^g$ is positive semidefinite of rank at most N. Moreover, if f is full then the rank of (a_{jk}) is exactly N.*

Recall that a holomorphic map $f : (M, \mathrm{g}) \to \mathbb{C}^{N \leq \infty}$ is said to be *full* if $f(M)$ is not contained in any complex totally geodesic submanifold of $\mathbb{C}^N$.

A bounded symmetric domain Ω of complex dimension n is an open connected bounded subset of $\mathbb{C}^n$, such that for every $x \in \Omega$ there is a biholomorphism σ_x of Ω for which x is an isolated fixed point. The reproducing kernel K for the Hilbert space of holomorphic L^2-functions on Ω defines the Kähler form $\omega_B = \frac{i}{2}\partial\bar\partial \log \mathrm{K}$ on Ω, whose associated metric is called the Bergman metric. Every bounded symmetric domain is the product of irreducible bounded symmetric domains, called Cartan domains.

From E. Cartan classification, Cartan domains can be divided into two categories, classical and exceptional ones. Classical domains can be described in terms of complex matrices as follows (m and n are nonnegative integers, $n \geq m$):

$$\Omega_1[m,n] = \{Z \in M_{m,n}(\mathbb{C}) \,:\, I_m - ZZ^* > 0\} \qquad (\dim(\Omega_1) = nm)\,,$$

$$\Omega_2[n] = \left\{Z \in M_n(\mathbb{C}) \,:\, Z = Z^T,\ I_n - ZZ^* > 0\right\} \qquad \left(\dim(\Omega_2) = \tfrac{n(n+1)}{2}\right),$$

$$\Omega_3[n] = \left\{Z \in M_n(\mathbb{C}) \,:\, Z = -Z^T,\ I_n - ZZ^* > 0\right\} \qquad \left(\dim(\Omega_3) = \tfrac{n(n-1)}{2}\right),$$

$$\Omega_4[n] = \left\{ Z = (z_1, \dots, z_n) \in \mathbb{C}^n \,:\, n \neq 2\,,\ \sum_{j=1}^n |z_j|^2 < 1, \right.$$
$$\left. 1 + \left|\sum_{j=1}^n z_j^2\right|^2 - 2\sum_{j=1}^n |z_j|^2 > 0 \right\} \qquad (\dim(\Omega_4) = n)\,,$$

where I_m (resp. I_n) denotes the $m \times m$ (resp. $n \times n$) identity matrix, and $A > 0$ means that A is positive definite. In the last domain we are assuming $n \neq 2$ since $\Omega_4[2]$ is not irreducible (and hence it is not a Cartan domain). Indeed, (see [3] for details) the biholomorphism

$$\begin{aligned} f : \Omega_4[2] &\longrightarrow \mathbb{C}\mathrm{H}^1 \times \mathbb{C}\mathrm{H}^1 \\ (z_1, z_2) &\longmapsto (z_1 + iz_2, z_1 - iz_2) \end{aligned}$$

satisfies

$$f^*(2(g_{hyp} \oplus g_{hyp})) = g_B.$$

Notice also that we have the following equalities

$$\begin{aligned}
(\Omega_1[m,1], g_B) &= (\mathbb{C}\mathrm{H}^m, (m+1)g_{hyp}),\\
(\Omega_1[1,n], g_B) &= (\mathbb{C}\mathrm{H}^n, (n+1)g_{hyp}),\\
(\Omega_2[1], g_B) &= (\Omega_3[2], g_B) = (\Omega_4[1], g_B) = (\mathbb{C}\mathrm{H}^1, 2g_{hyp}),\\
(\Omega_3[3], g_B) &= (\mathbb{C}\mathrm{H}^3, 4g_{hyp}).
\end{aligned}$$

The reproducing kernels of classical Cartan domains are given by

$$\begin{aligned}
\mathrm{K}_{\Omega_1}(z,z) &= \frac{1}{V(\Omega_1)}\left[\det(I_m - ZZ^*)\right]^{-(n+m)},\\
\mathrm{K}_{\Omega_2}(z,z) &= \frac{1}{V(\Omega_2)}\left[\det(I_n - ZZ^*)\right]^{-(n+1)},\\
\mathrm{K}_{\Omega_3}(z,z) &= \frac{1}{V(\Omega_3)}\left[\det(I_n - ZZ^*)\right]^{-(n-1)},\\
\mathrm{K}_{\Omega_4}(z,z) &= \frac{1}{V(\Omega_4)}\left(1 + \left|\sum_{j=1}^n z_j^2\right|^2 - 2\sum_{j=1}^n |z_j|^2\right)^{-n}, \qquad (5)
\end{aligned}$$

where $V(\Omega_j)$ is the total volume of Ω_j with respect to the Euclidean measure of the ambient complex Euclidean space (see [3] for details).

There are two kinds of exceptional domains $\Omega_5[16]$ of dimension 16 and $\Omega_6[27]$ of dimension 27, that can be described in terms of 3×3 matrices with entries in the 8-dimensional algebra of complex octonions $\mathbb{O}_{\mathbb{C}}$. We refer to [6] for a more complete description of these domains.

3. Proof of the main result

The proof of the Theorem 1 is based on the following two lemmas.

Lemma 3. *Let (M, g) be a Kähler manifold and let $A = (a_{jk})$ be the $\infty \times \infty$ Hermitian matrix given by*

$$\mathrm{D}^g(z,\bar z) = \sum_{j,k} a_{jk} z^{m_j} \bar z^{m_k}, \qquad (6)$$

where $\{z_1, \dots, z_n\}$ *are complex coordinates centered at the origin of a neighbourhood* U *of* $p \in M$ *and* D^g *is the diastasis function of* g*. If* $(U, g_{|U})$ *admits a Kähler immersion into* $\mathbb{C}^{\infty,s}$ *(resp.* $\mathbb{C}^{r,\infty}$*) then the number of negative eigenvalues is less or equal than* s *(resp.* r*).*

Proof. In order to prove the lemma we introduce the following notations. Denote by $\{v_i\}_{i\in\mathbb{N}}$ a sequence of complex numbers and consider the complex vector space

$$V = \left\{ \{v_i\}_{i\in\mathbb{N}} \,:\, \sum_{i=0}^{+\infty} v_i z^{m_i} < \infty \right\}.$$

Every holomorphic function $\psi = \sum_{i=0}^{+\infty} a_i z^{m_i}$ induces a linear functional $\widehat{\psi} \in V^* = \mathrm{Hom}(V, \mathbb{C})$ by

$$\widehat{\psi}(v) = \sum_{i=0}^{+\infty} a_i v_i \,, \qquad v = \{v_i\}_{i\in\mathbb{N}} \in V\,.$$

Define a sesquilinear form on V by

$$\begin{array}{rcl} \widehat{D}^g : V \times V & \longrightarrow & \mathbb{R} \\ (u, v) & \longmapsto & uAv^*\,. \end{array} \tag{7}$$

Assume now that $f\colon U \to \mathbb{C}^{\infty,s}$ is a Kähler immersion of a neighbourhood of a point $p \in M$ into $\mathbb{C}^{\infty,s}$ (the case $\mathbb{C}^{r,\infty}$ is treated similarly). We can assume that $f(p) = 0$. In local coordinates f is given by

$$f(z) = (f_1(z), \dots, f_s(z), f_{s+1}(z), \dots) \in \mathbb{C}^{\infty,s},$$

for suitable holomorphic functions f_j. Since f is a Kähler immersion it follows by (3) and by the very definition of the diastasis function that

$$\mathrm{D}^g(z, \bar{z}) = -|f_1(z)|^2 - \cdots - |f_s(z)|^2 + \sum_{k=s+1}^{+\infty} |f_k(z)|^2.$$

Hence in our notation

$$\widehat{\mathrm{D}}^g(u, v) = -\widehat{f}_1(u)\overline{\widehat{f}_1}(v) - \cdots - \widehat{f}_s(u)\overline{\widehat{f}_s}(v) + \sum_{k=s+1}^{+\infty} \widehat{f}_k(u)\overline{\widehat{f}_k}(v)\,. \tag{8}$$

Let $W \subset V$ be the complex subspace of V consisting of those $w \in V$ such that $\widehat{\mathrm{D}}^g(w, w) < 0$. By (7) each eigenvector of a negative eigenvalue of the matrix

A belongs to W. Hence, in order to prove the lemma we need to show that $\dim W \le s$. Assume, by a contradiction, that $\dim W > s$ and let $\widehat{\xi}_1, \dots, \widehat{\xi}_m$, $m \le s$ be a basis for the subspace of V^* spanned by the linear functionals $\widehat{f}_1, \dots, \widehat{f}_s$. If $m = 0$ then $f(U) \subset \ell^2(\mathbb{C})$ and by Calabi's criterion the matrix A does not have negative eigenvalues. On the other hand, if $m \ge 1$, the $\mathbb{C}$-linear map $L\colon W \to \mathbb{C}^m$ defined by

$$L(w) = \left(\widehat{\xi}_1(w), \dots, \widehat{\xi}_m(w)\right)$$

is surjective. Thus there exists $0 \ne w_0 \in W$ such that $\widehat{\xi}_1(w_0) = \dots = \widehat{\xi}_m(w_0) = 0$ and hence $\widehat{f}_1(w_0) = \dots = \widehat{f}_s(w_0) = 0$. By (8) we have $\widehat{\mathrm{D}}^g(w_0, w_0) \ge 0$, which contradicts the fact that $w_0 \in W$. ∎

LEMMA 4. *The diastasis function around the origin for the Bergman metric g_B of a Cartan domain Ω is globally defined and given by*

$$\mathrm{D}^{\Omega}(z) = \log\left(V(\Omega)\mathrm{K}_{\Omega}(z, z)\right). \tag{9}$$

Furthermore, if (a_{jk}) is the matrix in (6), we have $a_{jk} = 0$ whenever $|m_j| \ne |m_k|$.

Proof. We refer the reader to [4] for a proof. ∎

In order to simplify the proof of our theorem we introduce the following definition. We say that a square submatrix C of a square matrix M is *central* if its diagonal lies on the diagonal of M. Furthermore we say that M is a *block* matrix if it is of the form

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & M_1 & 0 & 0 & \dots \\ \vdots & 0 & M_2 & 0 & \dots \\ & \vdots & 0 & \ddots & \end{pmatrix},$$

where each block M_i is a central submatrix of M.

Notice that Lemma 4 says that the matrix (a_{jk}) of the diastasis D^{Ω} of a Cartan domain Ω is a block matrix, where each block M_i contains the elements a_{jk} with $|m_j| = |m_k| = i$.

Proof of Theorem 1. Let (Ω, g_B) be a Cartan domain. Then it is easily seen that $(\mathbb{C}\mathrm{H}^1, \gamma g_{hyp})$ admits a Kähler immersion in (Ω, g_B) where γ is the genus of Ω. By Example 1 the matrix (a_{jk}) for γg_{hyp} is a diagonal matrix with infinite positive eigenvalues (given by $\gamma/j,\ j=1,2,\dots$). By Lemma 3 it follows that $\mathbb{C}\mathrm{H}^1$, and hence (Ω, g_B), can not be Kähler immersed into $\mathbb{C}^{r,s}$ with $r \in \mathbb{N}$, $s \leq \infty$.

Thus it remains to prove that a Cartan domain of rank greater than 1 can not admit a Kähler immersion into $\mathbb{C}^{\infty,s}$ for $s \in \mathbb{N}$.

Assume, by contradiction, that there exists $f : \Omega \to \mathbb{C}^{\infty,s}$. Without loss of generality we can assume $f(0) = 0$. We are going to prove that the matrix (a_{jk}) in (6) has infinite negative eigenvalues. By Lemma 3 this will be the desired contradiction. Since it is known that any irreducible bounded symmetric domain of rank at least two can be exhausted by totally geodesic submanifolds isomorphic to $\Omega_4[3]$, we need only to prove the assertion for the case $\Omega_4[3]$.

By Lemma 4 and equation (5) the diastasis of $(\Omega_4[3], g_B)$ is given by

$$\begin{aligned}\mathrm{D}^{\Omega_4}(z,\bar z) = -3\log\Big(1 - 2\big(|z_1|^2 + |z_2|^2 + |z_3|^2\big) \\ + \left(z_1^2 + z_2^2 + z_3^2\right)\left(\bar z_1^2 + \bar z_2^2 + \bar z_3^2\right)\Big)\,.\end{aligned}$$

We will show that every block (except the first one) has at least one negative eigenvalue. Consider the 3×3 matrix

$$B = \frac{12\left(2|z_3|^2 + 1\right)}{\left(1 - 2|z_3|^2 + |z_3|^4\right)^2}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Let B_0 be the matrix obtained by evaluating B at $z_3 = \bar z_3 = 0$. Then B_0 is the submatrix of (a_{jk}) with j,k corresponding to the triples $(2,0,0)$, $(0,2,0)$ and $(0,0,2)$. Further let B_n be the submatrix of (a_{jk}) with j,k corresponding to the triples $(2,0,n)$, $(0,2,n)$, $(0,0,n+2)$. The matrix B_n can be obtained from B by deriving each of its entries n times with respect to z_3, n times with respect to $\bar z_3$ and evaluating at $z_3 = \bar z_3 = 0$. From

$$\frac{\partial^{2n}}{\partial z_3^n \partial \bar z_3^n}\, \frac{12\big(2|z_3|^2 + 1\big)}{\big(1 - 2|z_3|^2 + |z_3|^4\big)^2}\Big|_{z_3 = \bar z_3 = 0} > 0\,,$$

we have $\det(B_n) < 0$ for all $n \in \mathbb{N}$. Thus every B_n must have a negative eigenvalue. This implies that the $(n+2)$th block of (a_{jk}) which contains B_n as a central submatrix, has at least one negative eigenvalue and hence (a_{jk}) has infinite negative eigenvalues. ∎

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