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On the Wild Rank Conjecture of Han

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Abstract: The concept of rank of a wild algebra, introduced by Y. Han in [8], is discussed and, after a slight modification, investigated by means of Tarski's quantifier elimination theorem. This method allows, in particular, to prove that if there is a regular one-parameter family of d-dimensional algebras, uncountably many of them wild, then the whole family consists of wild algebras. A possible approach to the general question if *tame is open* is discussed.

Key words: wild algebra, variety of algebras, wild rank conjecture, quantifier elimination

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1. Introduction

It is still not known if *tame is open*, that is, if tame algebras define Zariski-open sets in varieties of algebras. The analogous fact about representation-finite algebras was proved long ago by Gabriel [5]. By the result of Geiss [7] a degeneration of a wild algebra is wild. Clearly, the Geiss's result would follow from openness of tame, in fact the latter property would be much stronger. It is proved in [11] that *tame is open* is equivalent to finite axiomatizability of the class of tame algebras (of any fixed dimension, over algebraically closed fields of fixed characteristic).

Beside of potential applications, a positive solution of the problem would enhance our understanding of tame-wild dichotomy. There are also some rather unexpected connections with other problems in representation theory, see [12].

In [8] Y. Han formulates wild rank conjecture and proves that it implies that tame is open. He also proves the conjecture for several classes of algebras (see Corollary 3.2 below). The conjecture is equivalent to finite axiomatizability of tameness (Remark 3.11).

We modify slightly the definition of the rank of a wild algebra introduced by Han. The modification conserves the main feature of the rank: wild algebra has finite wild rank, and the (suitably modified) wild rank conjecture implies that tame is open. Using Crawley-Boevey's proof of tame-wild dichotomy [2] we give in Theorem 2.2 and Corollary 2.4 some relations between the ranks. Calculating our rank is easier by elementary means, in particular we prove that d-dimensional algebras with ranks bounded by a fixed number form constructible sets in varieties of algebras (Proposition 3.4). We collect some consequences of this fact, in particular we prove in Corollary 3.7 that if there is a regular one-parameter family of d-dimensional algebras, uncountably many of them wild, then the whole family consists of wild algebras.

Finally we look at the problem from another point of view. Having observed that wildness of an algebra can be expressed in terms of certain first order properties of matrix rings, we look for some "compactness theorems" (see Proposition 4.1). It has not helped with the question if tame is open so far. But we believe it is a reasonable approach to the problem.

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2. On the definition of wildness

Throughout this paper K denotes an algebraically closed field. We denote by $K\langle X, Y\rangle$ the free associative K-algebra with two free generators X, Y. Given a ring R we denote by $\mathbb{M}_m(R)$ the ring of all $m \times m$-matrices with coefficients in R.

Recall the classical definition of wild algebra due to Drozd [4]:

Definition 2.1. A finite dimensional K-algebra A is *wild* if there exists an A-$K\langle X, Y\rangle$-bimodule M, free of finite rank over $K\langle X, Y\rangle$, such that the functor

$$M \otimes_{K\langle X,Y\rangle} (-) : \mathrm{fin}(K\langle X, Y\rangle) \to \mathrm{mod}(A)$$

preserves indecomposability and sends nonisomorphic modules to nonisomorphic ones.

Here $\mathrm{fin}(K\langle X, Y\rangle)$ is the category of the left finite dimensional $K\langle X, Y\rangle$-modules, whereas $\mathrm{mod}(A)$ is the category of the left finitely generated A-modules. The complementary concept is tame algebra. The reader is referred to [4], see also [3], [16] for the original definition of tameness. See also the monograph [18, Chapter XIX] for a discussion of conditions equivalent to wildness.

Let us call a bimodule M satisfying the condition in Definition 2.1 a *wild parametrization* for A. We call the $K\langle X,Y\rangle$-rank of M the *rank* of the parametrization. If A is wild then Y. Han [8] calls the minimal rank of a wild parametrization for A the *rank* of A. Let us call it *wild rank* of A and denote by $wrk_1(A)$. Set $wrk_1(A) = \infty$ when A is tame.

The *Wild Rank Conjecture* of Han asserts that there is a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that $wrk_1(A) \leq f(\dim_K A)$ whenever A is wild. It is not known if this is true in general. The conjecture is confirmed for some classes of algebras. For instance, if A is a d-dimensional wild algebra which is either local or a two-point algebra or a radical square zero algebra, then $wrk_1(A) \leq 10b$, where b is the fixed number defined in [8, Lemma 7].

The natural idea of a potential proof of Wild Rank Conjecture is to investigate the proof of tame-wild dichotomy and bocs reduction algorithm, see [2], [4]. The obstruction is the following: in order to check if an algebra A is wild one applies the algorithm not just to A but, roughly speaking, to the class of A-modules of dimension bounded by m, separately for each $m \in \mathbb{N}$. A is wild if we find a "wild configuration" for some m. The problem is to approximate *a priori* this m by a function of the dimension of A.

The condition that a bimodule M is a wild parametrization for A is rather complicated. But there are several well known reformulations of the definition allowing a translation into more accessible language. We present one of them in the theorem below.

First recall the concept of module varieties. Given a number m let $\mathbf{mod}_A(m)$ be the variety of all algebra homomorphisms $A \to \mathrm{End}_K(K^m)$. We identify the points of $\mathbf{mod}_A(m)$ with the corresponding modules. By a plane in this variety we mean a two dimensional affine subvariety of the affine space $\mathrm{Hom}_K(A, \mathrm{End}_K(K^m))$ contained in $\mathbf{mod}_A(m)$.

THEOREM 2.2. *The following conditions are equivalent for an algebra A:*

(1) *A is wild,*

(2) *There is a number m and a plane in the variety of m-dimensional A-modules, consisting of pairwise nonisomorphic indecomposable modules.*

Moreover, the number m in (2) *can be chosen such that $m \leq 4 \cdot wrk_1(A)$.*

Proof. For the proof of the implication (1)⇒(2) we can either refer to the third main theorem of [6], see also [17], or make the following consideration.

Assume that M is a wild parametrization for A. Consider the following 4×4-matrices with coefficients in $K\langle X, Y\rangle$.

$$\mathcal{X} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & X & Y & 0 \end{bmatrix}; \quad \mathcal{Y} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & X & 0 \end{bmatrix}$$

Let N be the $K\langle X, Y\rangle$-$K\langle X, Y\rangle$-bimodule isomorphic to $K\langle X, Y\rangle^4$ as a right $K\langle X, Y\rangle$-module and equipped by the left action of X and Y defined by $\mathcal{X}$ and $\mathcal{Y}$ respectively (in the standard basis). One can check directly that the functor

$$(-) \otimes_{K\langle X,Y\rangle} N : \mathrm{fin}(K\langle X, Y\rangle) \ \rightarrow \ \mathrm{fin}(K\langle X, Y\rangle)$$

preserves indecomposability and sends nonisomorphic modules to nonisomorphic ones (but it is not full!), see [2, p. 479].

Moreover, the matrices $\mathcal{X}^2$, $\mathcal{Y}^2$, $\mathcal{X}\mathcal{Y} = \mathcal{Y}\mathcal{X}$ have coefficients of degree at most one and every (noncommutative) monomial in two variables of degree at least 3 applied to $\mathcal{X}$ and $\mathcal{Y}$ vanishes. We conclude that the A-modules

$$M \otimes_{K\langle X,Y\rangle} N \otimes_{K\langle X,Y\rangle} (K\langle X, Y\rangle/(X - \lambda, Y - \mu)), \ \ \lambda, \mu \in K$$

form a plane of indecomposable pairwise nonisomorphic modules of dimension $4\cdot$ (the rank of M), as required.

(2)$\Rightarrow$(1) Under the assumption (2) the well known arguments, see [4], [14], prove that A is not tame. Then apply Tame-Wild Dichotomy [4], [2]. ∎

Recall that $wrk_1(A)$ denotes the minimal rank of a wild parametrization for A. Denote by $wrk_2(A)$ the minimal number m in the condition (2) in the above theorem. We agree that $wrk_2(A) = \infty$ if A is tame.

We have proved that $wrk_1(A)$ is finite if and only if $wrk_2(A)$ is finite and $wrk_2(A) \leq 4 \cdot wrk_1(A)$. There is also an upper bound for $wrk_1(A)$ in terms of $wrk_2(A)$.

PROPOSITION 2.3. *For any basic K-algebra Λ:*

$$wrk_1(\Lambda) \leq 2064 \cdot \dim_K(\Lambda)(\dim_K(\Lambda) + 1)wrk_2(\Lambda).$$

Proof. Assume that Λ is a wild K-algebra. Let n be the number of isomorphism classes of simple Λ-modules and $d = \dim_K(\Lambda)$. We follow the steps of the proof of Tame-Wild Dichotomy in [2]. We use the terminology and notation from there. Denote by $P_1(\Lambda)$ the category of Λ-homomorphisms $\alpha : P \to Q$ between projective Λ-modules, such that $Im(\alpha) \subseteq \mathrm{rad}(Q)$, the Jacobson radical of Q. We shall write an object of $P_1(\Lambda)$ as a triple (P, Q, α) [2, Section 6]. The dimension vector of such an object (P, Q, α) is the pair $(\underline{\dim}\,\mathrm{top}(P), \underline{\dim}\,\mathrm{top}(Q)) \in K_0(\Lambda) \times K_0(\Lambda)$ of dimension vectors of $\mathrm{top}(P)$ and $\mathrm{top}(Q)$.

Given an element $z \in K_0(\Lambda)$ (identified with $\mathbb{Z}^n$ via $\underline{dim}$) let $|z|$ be the sum of the coordinates of z.

There is an additive Roiter bocs $\mathcal{A} = (A, V)$, a dimension vector preserving equivalence

$$\Xi : rep\,(\mathcal{A}) \longrightarrow P_1(\Lambda)$$

($rep\,(\mathcal{A})$ denotes the category of representations of the bocs $\mathcal{A}$) and a finitely generated Λ-A-bimodule T such that if $\Xi(M) = (P, Q, \alpha)$ then $Coker(\alpha) \cong T \otimes_A M$, [2, Proposition 6.1].

If (P, Q, α) is indecomposable in $P_1(\Lambda)$ and $\dim(Coker(\alpha)) = m$ then $|\underline{\dim}\,\mathrm{top}(P)| + |\underline{\dim}\,\mathrm{top}(Q)| \leq m + md$.

If $\mathbf{mod}_\Lambda(m)$ contains a plane of pairwise nonisomorphic indecomposable modules it follows that the bocs $\mathcal{A}$ is wild and one of the wild configurations showed in [2, Proposition 3.10] occur when the reduction algorithm is applied to the representations of $\mathcal{A}$ of some dimension $(\underline{d}, \underline{d}')$ such that $|(\underline{d}, \underline{d}')| := |\underline{d}| + |\underline{d}'| \leq m$.

The proof of Proposition 3.10 in [2] shows that there exists a functor

$$\mathcal{F} : \mathrm{fin}(K\langle X, Y\rangle) \longrightarrow rep(\mathcal{A})$$

defining wildness of $\mathcal{A}$ and such that

$$|\underline{dim}(\mathcal{F}(U))| \leq 172 \cdot m(d+1) \dim_K(U)$$

for any U in $\mathrm{fin}(K\langle X, Y\rangle)$. We also use the fact that the functors Θ_I^* (see Section 3 in [2]) induce an injection of the corresponding Grothendieck groups of the categories of representations of bocses.

Next, there is a finitely generated Λ-$K\langle X, Y\rangle$-bimodule S such that the functor

$$S \otimes_{K\langle X,Y\rangle} (-) : \mathrm{fin}(K\langle X, Y\rangle) \longrightarrow \mathrm{mod}(\Lambda)$$

preserves indecomposability and heteromorphisms. This functor is equivalent to the composition of $T \otimes_A (-)$, $\mathcal{F}$ and a full and faithful endofunctor Φ of $\mathrm{fin}(K\langle X, Y\rangle)$ such that $\dim_K \Phi(U) = 3 \cdot \dim_K U$ for any U in $\mathrm{fin}(K\langle X, Y\rangle)$. Remember that $T \otimes_A (-)$ is equivalent to the composition of the dimension vector preserving equivalence Ξ with taking the cokernel. The cokernel of a homomorphism between projective Λ-modules with the tops at most m'-dimensional, has the dimension less than or equal $m'd$. Thus

$$\dim_K(S \otimes_{K\langle X,Y\rangle} U) \leq 3 \cdot 172 \cdot md(d+1) \cdot \dim_K U.$$

Now, following the proof of Theorem B in 6.4 of [2] we conclude that there is a nonzero element $p \in K[X,Y]$ and an R-Λ-bimodule S'', where $R = K[X,Y,p^{-1}]$, free of rank less than or equal $516 \cdot d(d+1)$ over R and such that the functor

$$(-) \otimes_R S'' : \mathrm{fin}(R) \longrightarrow \mathrm{mod}(\Lambda)$$

preserves indecomposability and heteromorphisms.

Then the final correction is made: we construct a wild parametrization of the form $L \otimes_R S''$ for a suitable $K\langle X,Y\rangle$-R-bimodule L, free of rank 4 as a left $K\langle X,Y\rangle$-module, see the proof of Theorem B in [2]. Let us remark that the left multiplication by X and Y in L is defined by the matrices $\mathcal{X}$ and $\mathcal{Y}$ described in the proof of Theorem 2.2. ∎

Corollary 2.4. *Let Λ be a (not necessarily basic) K-algebra. Assume that μ is the maximum of the dimensions of simple Λ-modules (clearly $\mu \leq \dim \Lambda$). Then*

$$\tfrac{1}{4}\, wrk_2(\Lambda) \leq wrk_1(\Lambda) \leq 2064 \cdot \mu \dim_K(\Lambda)(\dim_K(\Lambda)+1)\, wrk_2(\Lambda).$$

Proof. The left hand side inequality follows from Theorem 2.2. Let Λ' be basic algebra Morita equivalent to Λ. The right hand side inequality follows since there is an exact equivalence of categories

$$F : \mathrm{mod}(\Lambda') \to \mathrm{mod}(\Lambda)$$

such that $\dim_K F(X) \leq \mu \dim_K(X)$ for any Λ'-module X. ∎

3. Application of quantifier elimination

Throughout this section we fix a natural number d. Let $\mathrm{Alg}_K(d)$ be the variety of the d-dimensional K-algebras (associative, with unit), see [13]. That is, $\mathrm{Alg}_K(d)$ is a subset of K^{d^3} consisting of the tuples $\gamma = (\gamma_{ijk})_{i,j,k=1,\dots,d}$ such that the multiplication

$$\cdot : K^d \times K^d \to K^d$$

defined by

$$e_i \cdot e_j = \sum_{k=1}^{d} \gamma_{ijk} e_k, \quad i, j = 1, ..., d,$$

is associative and admits a unit. Here we denote by e_i the ith standard basis element of K^d. We consider $\mathrm{Alg}_K(d)$ as a topological space with the Zariski topology.

Given $\gamma \in \mathrm{Alg}_K(d)$ let $A(\gamma)$ be the K-algebra isomorphic to K^d as vector space and equipped with the multiplication defined by γ with respect to the standard basis $e_1, ..., e_d$ of K^d. Let $\mathcal{W}$ be the set of the points γ of $\mathrm{Alg}_K(d)$ corresponding to wild algebras. Given $i = 1, 2$ and a number r set

$$\mathcal{W}_i^{\leq r} = \{\gamma \in \mathrm{Alg}_K(d) : wrk_i(A(\gamma)) \leq r\}$$

Lemma 3.1. *$\overline{\mathcal{W}_i^{\leq r}} \subset \mathcal{W}$ for any $i = 1, 2$ and natural r.*

Proof. For $i = 1$ the proof is well-known [8], [7], [11]. The idea is the following. Applying algebraic geometry arguments one shows that an algebra A is wild if and only if, for some natural m, the following condition ϕ_m holds:

there exists an A-$K\langle X, Y\rangle$-bimodule M, free of rank m over $K\langle X, Y\rangle$, such that the A-modules $M \otimes_{K\langle X,Y\rangle} U$ and $M \otimes_{K\langle X,Y\rangle} V$ are isomorphic if and only if $U \cong V$, for any m-dimensional left $K\langle X, Y\rangle$-modules U, V.

Moreover, if $wrk_1(A) = m$, then ϕ_m holds. The condition ϕ_m is chosen in such way that, applying upper semi-continuity arguments and variety dimension characterizations of wildness, see e.g. [15], one shows that the Zariski closure of the set of points γ such that $A(\gamma)$ has the property ϕ_m, is contained in $\mathcal{W}$.

The proof for $i = 2$ follows now, since $\mathcal{W}_2^{\leq r} \subseteq \mathcal{W}_1^{\leq 2064d^2(d+1)r}$ by Proposition 2.3, Corollary 2.4. ∎

This way we get the following corollary.

Corollary 3.2. [8] *The wild rank conjecture implies that tame is open.*

Let $\mathbb{L} = (x_1, x_2, ..., +, \cdot, 0, 1)$ be the first order language of the theory of fields. Let $\phi = \phi(x_1, ..., x_m)$ be a formula in this language in d variables. Given a field L and elements $a_1, ..., a_m \in L$ we define the satisfiability of ϕ by the sequence $a_1, ..., a_m$ in L in the usual way, under the natural interpretation of the symbols $\cdot, +, 0, 1$. The fact that $a_1, ..., a_m$ satisfy the formula ϕ in L is denoted by $L \models \phi(a_1, ..., a_m)$. The reader is referred to [1] and [10] for basic concepts of model theory.

The following assertion is a direct consequence of well-known Tarski's quantifier elimination theorem for algebraically closed fields [10, Theorem 12.4].

THEOREM 3.3. *For any first order formula $\phi = \phi(x_1, ..., x_m)$ in the language $\mathbb{L}$ with free variables $x_1, ..., x_m$ there exists a quantifier-free formula $\psi(x_1, ..., x_m)$ of the form*

$$\bigvee_{i=1,...,s} [F_{i1}(x_1, ..., x_m) = ... = F_{in_i}(x_1, ..., x_m) = 0 \wedge G_i(x_1, ..., x_m) \neq 0]$$

where $F_{ij}, G_i \in \mathbb{Z}[x_1, ..., x_m]$, $i = 1, ..., s$, $j = 1, ..., n_i$, such that

$$K \models \phi(a_1, ..., a_m) \Leftrightarrow K \models \psi(a_1, ..., a_m)$$

for any algebraically closed field K and any tuple $a = (a_1, ..., a_m) \in K^m$. The numbers s, n_i, for $i = 1, ..., s$ and the degrees of the polynomials F_{ij}, G_i are bounded by a number depending only on the complexity of ϕ.

That is, any first-order formula can be replaced by a constructible condition, if we are restricted to algebraically closed fields. We do not give here a precise definition of the complexity of a formula, it is an intuitive "by-product" of the usual recursive definition of the set of first-order formulas.

PROPOSITION 3.4. *Let r be a natural number. There exist numbers s, $t_1, ..., t_s$ and polynomials $F_{ab}, G_a \in \mathbb{Z}[X_{ijk} : 1 \leq i, j, k \leq d]$, $a = 1, ..., s$, $b = 1, ..., t_a$, such that*

$$\mathcal{W}_2^{\leq r} = \bigcup_{a=1}^{s} (\bigcap_{b=1}^{t_a} \{\gamma \in \mathrm{Alg}_K(d) : F_{ab}(\gamma) = 0\} \cap \{\gamma \in \mathrm{Alg}_K(d) : G_a(\gamma) \neq 0\})$$

The polynomials F_{ab}, G_a are chosen independently on the algebraically closed field K. The numbers $s, t_1, ..., t_s$ and the degrees of the polynomials are bounded by a number depending only on d and r.

Proof. The property $wrk_2(A(\gamma)) \leq r$ can be written as first order formula with variables γ. Indeed, the inequality $wrk_2(A(\gamma)) \leq r$ means that, for some $m \leq r$, there are three d-tuples of $m \times m$-matrices

$$(M_s)_{1 \leq s \leq d},\ (M_s^x)_{1 \leq s \leq d},\ (M_s^y)_{1 \leq s \leq d}$$

such that

1) for any $\lambda, \mu \in K$ the matrices $M_s^{\lambda,\mu} = M_s + \lambda M_s^x + \mu M_s^y$ define a $A(\gamma)$-module structure on K^m, that is:

$$M_s^{\lambda,\mu} M_t^{\lambda,\mu} = \sum_{u=1}^{d} \gamma_{stu} M_u^{\lambda,\mu},$$

and $\sum_{u=1}^{d} \iota_u M_u^{\lambda,\mu}$ equals the identity matrix if $(\iota_1, ..., \iota_d)$ is the unit of $A(\gamma)$.

Moreover, $M_s^x M_t^x = M_s^x M_t^y = M_s^y M_t^x = M_s^y M_t^y = 0$ for any s, t. This means that the points $M^{\lambda,\mu}$ of $\mathbf{mod}_m(A(\gamma))$ defined by $M^{\lambda,\mu}(e_i) \mapsto M_i^{\lambda,\mu}$, $i = 1, ..., d$, $\lambda, \mu \in K$, define a plane in $\mathbf{mod}_m(A(\gamma))$.

2) for any $\lambda, \mu \in K$ any idempotent matrix commuting with all $M_i^{\lambda,\mu}$ is either 0 or the identity (the plane consists of indecomposable modules).

3) The modules are pairwise nonisomorphic: if there is an invertible matrix F such that $M_i^{\lambda,\mu} F = F M_i^{\lambda',\mu'}$ for every $i = 1, ..., d$ then $\lambda = \lambda'$ and $\mu = \mu'$.

The assertion follows now from Theorem 3.3. ∎

Remark 3.5. The formula considered in the proof can be regarded as a first order formula in the two-sorted first order language of the theory of algebras over fields, see Section 4 below and [10]: we substitute the variables of the second sort by matrices. The shape of the formula does not depend on the size of the matrices. Moreover, we can restrict to the constructible subset of $\mathrm{Alg}_K(d)$ containing the points corresponding to basic algebras with fixed Gabriel quiver. Then rearrange the formula and require that the matrices whose existence is postulated satisfy the nilpotency condition Nil_d, see Section 4, of degree d.

Consider a regular map $\ell : K \to \mathrm{Alg}_K(d)$. By the degree of ℓ we mean the maximum of the degrees of the coordinate maps of ℓ. Set $A_\lambda := A(\ell(\lambda))$. Under the hypothesis that *tame is open* the following holds: if A_λ is wild for infinitely many $\lambda \in K$, then A_λ is wild for every λ. We prove some weaker facts.

THEOREM 3.6. *Fix d and let $i = 1$ or $i = 2$. There exists a function $\beta^{(i)} : \mathbb{N}^2 \to \mathbb{N}$, such that if*

$$|\{\lambda \in K : wrk_i(A_\lambda) \le r\}| \ge \beta^{(i)}(r, \deg(\ell)),$$

for some r, then A_λ is wild for any $\lambda \in K$.

Proof. Consider the case $i = 2$ first. Let $C := \ell^{-1}(\mathcal{W}_2^{\leq r})$. By Proposition 3.3

$$C = \bigcup_{a=1}^{s}(\bigcap_{b=1}^{t_a}\{\lambda \in K : F'_{ab}(\gamma) = 0\} \cap \{\gamma \in \mathrm{Alg}_K(d) : G'_a(\gamma) \neq 0\})$$

for some polynomials F_{ab}, G_a in one variable with coefficients in K. By Proposition 3.4 the number s and the degrees of the polynomials can be bounded by a number T depending only on d, r and $\deg(\ell)$. Set $\beta^{(2)}(r, \deg(\ell)) = T^2$.

If $|C| > T^2$ then C contains a subset of the form $\{G_a \neq 0\}$, that is, C is cofinite in K, hence dense. By Lemma 3.1 the proof is finished in case $i = 2$.

In the case $i = 1$ we use Corollary 2.4: it is enough to set

$$\beta^{(1)}(r, \deg(\ell)) = \beta^{(2)}(4r, \deg(\ell)).$$

∎

The following is a direct consequence of the above result.

Corollary 3.7. *If A_λ is wild for uncountably many values of λ, then A_λ is wild for every $\lambda \in K$.*

Let L be the subfield of K generated by the coefficients of ℓ.

Theorem 3.8. *Fix d and $i = 1$ or $i = 2$. There exists a function $\eta^{(i)} : \mathbb{N} \to \mathbb{N}$ such that if $wrk_i(A_\lambda) \leq r$ for some $\lambda \in K$ such that the transcendence degree $trdeg_L L(\lambda)$ of λ over L is at least $\eta^{(i)}(r)$, then A_λ is wild for any $\lambda \in K$.*

Proof. Let $i = 2$. We keep the notation from the proof of Theorem 3.6. The polynomials F'_{ab}, G'_a have coefficients in L. Let $\eta^{(2)}(r)$ be the maximum of the degrees of the polynomials F'_{ab}. If $trdeg_L L(\lambda) \geq \eta^{(2)}(r)$ and $\lambda \in C$ then C contains a subset of the form $\{G_a \neq 0\}$ and we finish the proof as in 3.6. ∎

Again we get

Corollary 3.9. *If A_λ is wild for some $\lambda \in K$ transcendent over L, then A_λ is wild for every $\lambda \in K$.*

Remark 3.10. The proof of Proposition 2.3 is in fact a comment to a proof due to Crawley-Boevey in [2]. In order to prove Corollaries 3.7 and 3.9 one does not have to trace this complicated consideration: it is possible to define another "wild rank" based on an idea from the proof of Lemma 3.1. Namely, let the new wild rank be the least m such that the condition ϕ_m holds. Then quantifier elimination proves an analogue of Theorems 3.6 and 3.8. But such a rank is not a natural concept, contrary to wrk_1 and wrk_2 - the first coming from the standard definition of wildness, the latter related with a method of proving wildness, frequently used in practice.

Remark 3.11. The Wild Rank Conjecture is equivalent to finite axiomatizability of the classes of d-dimensional tame algebras over algebraically closed fields, for every $d \in \mathbb{N}$. Indeed, assume first that Wild Rank Conjecture is true. The left hand side inequality of Corollary 2.4 shows that, given a natural number d, there is a number r such that $wrk_2(A) \leq r$, for any d-dimensional wild algebra A. It follows from (the proof of) Proposition 3.4 that $wrk_2(A) \leq r$ is a first order property of the algebra A and therefore it is an axiom for the class of wild d-dimensional algebras. The finite axiomatizability of the class of tame d-dimensional algebras follows now by standard model theory arguments, see e.g. [1, Theorem 4.12] and [10, Theorem 2.13].

Conversely, if the class of d-dimensional tame algebras is finitely axiomatizable, then again by basic facts of model theory we conclude that there is a number r such that $wrk_2(A) \leq r$, for any d-dimensional wild algebra A. Now the assertion of Wild Rank Conjecture follows thanks to the right hand side inequality of Corollary 2.4. See also Corollary 3.7 in [11] and the comments there.

4. Final remarks

In this section we discuss some "compactness theorem" for properties of matrix algebras. We have noticed in 3.5 that wildness of a K-algebra $A(\gamma)$ determined by structure constants γ is equivalent to the fact that some matrix algebra $\mathbb{M}_m(K)$ has a property $\psi(\gamma)$ depending on A. If we could find a bound for the number m, depending only on complexity of ψ we would prove that *tame is open*. Unfortunately we are not able to do it - our compactness theorem works only for some class of properties.

Let L be a field, not necessarily algebraically closed. Now we are going to define a class of properties, or rather formulas in a suitable language of algebras over fields.

Denote by $\mathbb{A}$ the *two-sorted first order language of algebras over fields*, see [10], that is, the disjoint union $\mathbb{L}_1 \coprod \mathbb{L}_2$ of two copies of $\mathbb{L}$ (see Section 3 above) equipped with another function symbol $\cdot$. The terms from $\mathbb{L}_1$ (resp.

$\mathbb{L}_2$) are called terms of the first (resp. second) sort. The new function symbol associates to a pair of elements of the first and the second sort an element of the second sort. The language $\mathbb{A}$ has the usual logical connectives: $\wedge$, $\vee$, $\neg$, $\rightarrow$ and allows quantifiers on both sorts of variables. By a model for this language we mean a pair (L, R), where L and R are models for $\mathbb{L}_1$ and $\mathbb{L}_2$ respectively and the new function symbol is interpreted as a function

$$\cdot : L \times R \rightarrow R.$$

It is clear that if L is a field and R is an L-algebra with identity then the obvious interpretation of the symbols of the language $\mathbb{A}$ allows us to treat the pair (L, R) as a model for $\mathbb{A}$.

Given a sequence of variables $\underline{x} = (x_1, ..., x_u)$ let $Nil_m(\underline{x})$ denote the conjunction of all equations $F = 0$, where F runs through the set of all monomials of degree m in the free associative $\mathbb{Z}$-algebra $\mathbb{Z}\langle \underline{x} \rangle$ with free generators $\underline{x}$. For example

$$Nil_2(x, y) =' x^2 = xy = yx = y^2 = 0'.$$

Given two numbers m, b let $\mathcal{C}_{m,b}$ be the class of formulas ψ in the language $\mathbb{A}$ such that

1. The free variables in ψ are of the first kind only. Let $\underline{z} = (z_1, ..., z_a)$ be a sequence of the free variables of ψ.
2. $\psi = \psi(\underline{z})$ has the form

$$\exists_{\underline{x}} (\phi(\underline{z}, \underline{x}) \wedge \forall_{\underline{y}} \eta(\underline{z}, \underline{x}, \underline{y}))$$

   where $\phi(\underline{z}, \underline{x})$ is quantifier-free, has free variables $\underline{z}$ of the first kind and b free variables $\underline{x} = (x_1, ..., x_b)$ of the second kind, and $\eta(\underline{z}, \underline{x}, \underline{y})$ is positive (=built without negation) and quantifier-free, $\underline{y}$ is a sequence of variables of (possibly) both kinds.
3. $(\phi(\underline{z}, \underline{x}) \wedge \forall_{\underline{y}} \eta(\underline{z}, \underline{x}, \underline{y})) \rightarrow Nil_m(\underline{x})$.

Set $\mathcal{C} = \bigcup_{m,b=1}^{\infty} \mathcal{C}_{m,b}$. Let $\mu_{m,b}$ denote the number of (noncommutative) monomials in b variables of degree less than m. Given a formula $\psi \in \mathcal{C}_{m,b} \setminus \mathcal{C}_{m-1,b}$ let $g(\psi)$ equal the number of inequalities appearing in ϕ multiplied by $\mu_{m,b}$.

Note that the formulas described in 3.5 do not belong to $\mathcal{C}$. In order to include them we would have to resign on the positivity of η.

PROPOSITION 4.1. *Let L be a field and $\psi(\underline{z}) \in \mathcal{C}$, $\underline{z} = (z_1, ..., z_a)$. Assume that there is a natural number n such that*

$$\mathbb{M}_n(L) \models \psi(\underline{\lambda})$$

for some $\underline{\lambda} \in L^a$. Then there is such n less than or equal $g(\psi)$.

Proof. We identify $\mathbb{M}_n(L)$ with $\mathrm{End}_L(L^n)$. Let $X_1, ..., X_b \in \mathbb{M}_n(L)$ be elements, whose existence is postulated by the outer existential quantifiers. Since $Nil_m(X_1, ..., X_b)$ is satisfied, the subalgebra Λ of $\mathbb{M}_n(L)$ generated by $X_1, ..., X_b$ has dimension less than or equal $\mu_{m,b}$.

Let $F_1(X_1, ..., X_b) \neq 0$,..., $F_t(X_1, ..., X_b) \neq 0$ be all inequalities appearing in $\phi(\underline{\lambda})$ and satisfied by $X_1, ..., X_b$. Choose $v_i \in L^n$ such that for $i = 1, ..., t$: $F_1(X_1, ..., X_b)(v_i) \neq 0$. Let W be the Λ-submodule of L^n generated by $v_1, ..., v_t$, clearly $\dim_L W \leq g(\psi)$. Then $F(X_1, ..., X_b)_{|W} \neq 0$. Every L-endomorphism of W extends to an endomorphism of L^n and since η is positive quantifier-free we see that the restrictions of $X_1, ..., X_b$ to W satisfy $(\phi(\underline{\lambda}, \underline{x}) \wedge \forall_{\underline{y}} \eta(\underline{\lambda}, \underline{x}, \underline{y}))$ and therefore

$$\mathrm{End}_L(W) \models \psi(\underline{\lambda}).$$

∎

Remark 4.2. The well-known fact that the matrix algebra of size $m \times m$ does not satisfy any polynomial identity of degree less than or equal $2m$, [9], can be considered from such a point of view. Namely, take an element $F \in \mathbb{Z}\langle X_1, ..., X_b\rangle$ and consider the formula (a sentence, actually)

$$\psi =' \exists_{M_1, ..., M_b} F(M_1, ..., M_b) \neq 0',$$

where $M_1, ..., M_s$ are matrix variables - variables of the second sort. Set $g(\psi) = [\frac{\deg F + 1}{2}]$. If $\mathbb{M}_n(L) \models \psi$ for some m then, clearly, $F \neq 0$ and $\mathbb{M}_{g(\psi)}(L) \models \psi$, by the fact mentioned above.

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