

The Set of Exponents for Isomorphic Extensions of Hölder Maps is not Always Closed

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Presented by Jesús M. F. Castillo

Received March 3, 2008

Abstract: We give examples of Banach spaces X and Y such that the sets of $\alpha \in (0, 1]$, which satisfy that every α -Hölder maps from a subset of X into Y can be isomorphically extended to all X , are not closed. This answers a question of Lancien and Randrianantoanina.

Key words: Extension, Hölder map.

AMS *Subject Class.* (2000): 46B20, 46B42, 54C20.

1. INTRODUCTION

We refer to [7] and bibliography therein for all background on the subject of this note. Let us just recall the following definitions and notations. If (X, d_X) and (Y, d_Y) are metric spaces, $\alpha \in (0, 1]$ and $K > 0$, a map $f : X \rightarrow Y$ is said (K, α) -Hölder if for all $x, y \in X$

$$d_Y(f(x), f(y)) \leq K d_X(x, y)^\alpha.$$

When $\alpha = 1$, f is said K -Lipschitz. For $C \geq 1$, $\mathcal{B}_C(X, Y)$ denotes the set of exponents $\alpha \in (0, 1]$ such that every (K, α) -Hölder map from a subset of X into Y can be extended to a (CK, α) -Hölder map defined on all X . Such an extension is called an isomorphic extension (resp. an isometric extension) if $C > 1$ (resp. $C = 1$). We study the following set:

$$\mathcal{B}(X, Y) = \bigcup_{C \geq 1} \mathcal{B}_C(X, Y).$$

In this note we present Banach spaces X and Y , such that the set $\mathcal{B}(X, Y)$ is of the form $(0, \alpha)$ for some α depending on X and Y . This answers a part of Problem 5.2, raised by Lancien and Randrianantoanina in [7], showing that $\mathcal{B}(X, Y)$ is not always closed.

2. PRELIMINARIES

2.1. THE MAZUR MAP. Let us consider $1 \leq p < q < \infty$. The Mazur map $\phi_{p,q} : \ell_p \rightarrow \ell_q$ is defined for $x = (x_i)_{i \in \mathbb{N}}$ by the formula

$$\phi_{p,q}(x) = \left(|x_i|^{\frac{p}{q}} \text{sign}(x_i) \right)_{i \in \mathbb{N}}.$$

As for all real numbers a and b and for all $0 < \alpha \leq 1$,

$$||a|^\alpha \text{sign}(a) - |b|^\alpha \text{sign}(b)| \leq 2|a - b|^\alpha,$$

the Mazur map $\phi_{p,q}$ is $(2, \frac{p}{q})$ -Hölder. For $n \in \mathbb{N}$, we denote by $\phi_{p,q}^n$ the finite-dimensional version of the Mazur map, between ℓ_p^n and ℓ_q^n . Fix $0 < \varepsilon \leq 1$ and $\frac{p}{q} < \alpha \leq 1$. As explained in [8], for all ε -net $\mathcal{N}(\varepsilon)$ in $B(\ell_p^n)$, the closed unit ball of ℓ_p^n , and for all $x \neq y \in \mathcal{N}(\varepsilon)$, we have

$$\begin{aligned} \|\phi_{p,q}^n(x) - \phi_{p,q}^n(y)\| &\leq 2\|x - y\|^{\frac{p}{q}} \\ &\leq \frac{2}{\varepsilon^{\alpha - \frac{p}{q}}} \|x - y\|^\alpha \end{aligned}$$

(because $\|x - y\| \geq \varepsilon$), so that $\phi_{p,q}^n$ is (a_n, α) -Hölder on $\mathcal{N}(\varepsilon)$ with constant $a_n = 2\varepsilon^{\frac{p}{q} - \alpha}$. An important result, needed in our study, appears in the proof of Proposition 4. in [8]. Namely:

FACT 1. If $g : B(\ell_p^{2n}) \rightarrow \ell_q^{2n}$ is (K, α) -Hölder and coincide with $\phi_{p,q}^{2n}$ on $\mathcal{N}(\varepsilon)$, then

$$2K\varepsilon^\alpha + 4\varepsilon^{\frac{p}{q}} \geq 1 - \frac{K}{n^{\frac{\alpha}{p} - \frac{1}{q}}}.$$

2.2. A FIRST INCLUSION. We refer to [6] for notions of Banach lattice, type and cotype. If B denotes a Banach space, we set $p_B = \sup\{1 \leq p \leq 2 : B \text{ has type } p\}$ and $q_B = \inf\{q \geq 2 : B \text{ has cotype } q\}$. The following lemma is needed in the sequel:

LEMMA 1. Let X and Y be Banach lattices such that $1 < p_X \leq q_X < \infty$ and $1 < p_Y \leq q_Y < \infty$. Then:

$$\left(0, \frac{p_X}{q_Y}\right] \subseteq \mathcal{B}(X, Y).$$

Proof. According to [6, p.100-101], for $\varepsilon > 0$ small enough, X (resp. Y) admits an equivalent renorming with modulus of smoothness (resp. convexity) of power type $1 < p_X - \varepsilon \leq 2$ (resp. $q_Y + \varepsilon$). We equip X and Y with these equivalent norms (note that $\mathcal{B}(X, Y)$ is invariant under equivalent renormings). Using results of [1] and [9], we have that X has Markov-type $p_X - \varepsilon$ and Y has Markov-cotype $q_Y + \varepsilon$. Let us equip X with the metric $d(x, y) = \|x - y\|^m$ with $m = \frac{p_X - \varepsilon}{q_Y + \varepsilon}$. The metric space $X_d = (X, d)$ has a Markov-type which is the same than Markov-cotype of Y . Using Theorem 1.7 in [1] with obvious modifications (see Theorem 5 in [8]), we obtain $1 \in \mathcal{B}(X_d, Y)$. This gives $\frac{p_X - \varepsilon}{q_Y + \varepsilon} \in \mathcal{B}(X, Y)$. By [2], if $\alpha \in \mathcal{B}(X, Y)$ and $0 < \beta \leq \alpha$, then $\beta \in \mathcal{B}(X, Y)$ so we obtain:

$$\left(0, \frac{p_X - \varepsilon}{q_Y + \varepsilon}\right] \subseteq \mathcal{B}(X, Y).$$

Let $\varepsilon \rightarrow 0$ to conclude. ■

3. A FIRST EXAMPLE

Let us consider $X = \ell_2$ and $Y = \left(\sum \oplus \ell_{q_n}^{2n}\right)_2$ for $q_n \downarrow 2$ such that $\lim_{n \rightarrow \infty} n^{\frac{1}{q_n} - \frac{1}{2}} = 0$. Take, for example, for all $n \in \mathbb{N}$, $q_n = 2 + \frac{1}{1 + \sqrt{\ln(n)}}$. Then

$$\mathcal{B}(X, Y) = (0, 1).$$

We have $p_X = q_X = 2$ and $p_Y = q_Y = 2$ so Lemma 1 gives $(0, 1) \subseteq \mathcal{B}(X, Y)$. The inclusion $\mathcal{B}(X, Y) \subseteq (0, 1]$ being obvious, let us show that $1 \notin \mathcal{B}(X, Y)$. Suppose that the contrary holds and consider $C \geq 1$ such that $1 \in \mathcal{B}_C(X, Y)$. For $n \in \mathbb{N}$, we identify ℓ_2^{2n} as a subspace of X and $\ell_{q_n}^{2n}$ as a complemented subspace of Y , with a norm-one projection $P_n : Y \rightarrow \ell_{q_n}^{2n}$. We denote by B_X the closed unit ball of X . Fix, in $4B_X$, a 3-separated sequence (x_n) , that is for all $n \neq m$, $\|x_n - x_m\| \geq 3$. For $n \in \mathbb{N}$ we consider $0 < \varepsilon_n \leq 1$, determined later, and $\mathcal{N}(\varepsilon_n)$ a ε_n -net in $B(\ell_2^{2n})$. Moreover, we set $D_n = x_n + \mathcal{N}(\varepsilon_n)$ and the disjoint union $D = \bigcup_{n \in \mathbb{N}} D_n \subset 4B_X$. With notations introduced in preliminaries, we consider

$$\begin{aligned} f : D &\longrightarrow Y \\ D_n \ni x &\longmapsto f(x) = \frac{1}{a_n} \phi_{2, q_n}^{2n}(x - x_n). \end{aligned}$$

We claim that f is 1-Lipschitz on D . This is clear on each D_n . For $x \in D_n$ and $y \in D_m$ with $n \neq m$, we have:

$$\begin{aligned} \|f(x) - f(y)\| &\leq \frac{1}{a_n} \|\phi_{2,q_n}^{2n}(x - x_n)\| + \frac{1}{a_m} \|\phi_{2,q_m}^{2m}(y - x_m)\| \\ &\leq \frac{1}{a_n} + \frac{1}{a_m} \quad (\text{because for } n \text{ (or } m), \phi_{2,q_n}^{2n}(B(\ell_2^{2n})) \subseteq B(\ell_{q_n}^{2n})) \\ &\leq 1 \quad (\text{because for } n \text{ (or } m), a_n = 2\varepsilon_n^{\frac{2}{q_n}-1} \geq 2) \\ &\leq \|x - y\|. \end{aligned}$$

By assumption, $1 \in \mathcal{B}_C(X, Y)$, so there exists $g : X \rightarrow Y$, C -Lipschitz, such that for all $x \in D$, $g(x) = f(x)$. For all $n \in \mathbb{N}$, the map

$$\begin{aligned} g_n : \ell_2^{2n} &\longrightarrow \ell_{q_n}^{2n} \\ x &\longmapsto g_n(x) = a_n P_n \circ g(x + x_n) \end{aligned}$$

is $a_n C$ -Lipschitz and coincide with ϕ_{2,q_n}^{2n} on $\mathcal{N}(\varepsilon_n)$. Apply Fact 1 with $\varepsilon = \varepsilon_n$, $p = 2$, $q = q_n$ and $K = a_n C = 2C\varepsilon_n^{\frac{2}{q_n}-1}$, to obtain successively

$$\begin{aligned} 2K\varepsilon_n + 4\varepsilon_n^{2/q_n} &\geq 1 - Kn^{1/q_n-1/2}, \\ 4C\varepsilon_n^{2/q_n} + 4\varepsilon_n^{2/q_n} &\geq 1 - 2C\varepsilon_n^{2/q_n-1}n^{1/q_n-1/2}, \\ 8C\varepsilon_n^{2/q_n} &\geq 1 - 2C\varepsilon_n^{2/q_n-1}n^{1/q_n-1/2} \quad (\text{because } C \geq 1), \\ \varepsilon_n^{2/q_n} \left(8 + 2\frac{n^{1/q_n-1/2}}{\varepsilon_n} \right) &\geq \frac{1}{C} > 0. \end{aligned}$$

For all $n \in \mathbb{N}$, we take $\varepsilon_n = n^{1/q_n-1/2}$. By the choice of (q_n) , letting $n \rightarrow \infty$, we obtain a contradiction in the last inequality above.

4. COMPLEMENTARY RESULTS

We improve Lemma 1 and provide other examples of Banach spaces X and Y such that $\mathcal{B}(X, Y)$ is not closed.

4.1. A SECOND INCLUSION. If B denotes a Banach space, with notations p_B and q_B above, it is well known that B contains $\ell_{p_B}^n$'s and $\ell_{q_B}^n$'s uniformly. That is, for $r = p_B$ or $r = q_B$, there exists $L > 0$ such that for all $n \in \mathbb{N}$,

there exists an isomorphism $T_n : \ell_r^n \rightarrow T_n(\ell_r^n) \subset B$, such that for all $x \in \ell_r^n$, $\|x\| \leq \|T_n(x)\| \leq L\|x\|$. If, in addition, for all $n \in \mathbb{N}$, there exist projections $P_n : B \rightarrow T_n(\ell_r^n)$ such that $\sup_{n \in \mathbb{N}} \|P_n\| < \infty$, then we say that B contains uniformly well-complemented copies of ℓ_r^n 's.

THEOREM 1. *Let X and Y be Banach lattices such that $1 < p_X \leq q_X < \infty$ and $1 < p_Y \leq q_Y < \infty$. Suppose that Y contains uniformly well-complemented copies of $\ell_{q_Y}^n$'s. Then*

$$\left(0, \frac{p_X}{q_Y} \left[\subseteq \mathcal{B}(X, Y) \subseteq \left(0, \frac{p_X}{q_Y} \right] \right).$$

Proof. The left-hand side inclusion is given by Lemma 1. For readability we write $p = p_X$ and $q = q_Y$. Fix $\frac{p}{q} < \alpha \leq 1$. To prove the right-hand side inclusion, we proceed like in the first example. We use notations given there, in order to apply Fact 1, to obtain that $\alpha \notin \mathcal{B}(X, Y)$. There exists $L > 0$ (resp. $L' > 0$) such that for all $n \in \mathbb{N}$, there exists an isomorphism $T_n : \ell_p^{2n} \rightarrow T_n(\ell_p^{2n}) \subset X$, such that for all $x \in \ell_p^{2n}$, $\|x\| \leq \|T_n(x)\| \leq L\|x\|$ (resp. $R_n : \ell_q^{2n} \rightarrow R_n(\ell_q^{2n}) \subset Y$, such that for all $x \in \ell_q^{2n}$, $\|x\| \leq \|R_n(x)\| \leq L'\|x\|$). Let (x_n) be a $3L$ -separated sequence in $4LB_X$. For $n \in \mathbb{N}$, set $D_n = x_n + T_n(\mathcal{N}(\varepsilon_n))$, with $\varepsilon_n = n^{\frac{1}{q} - \frac{1}{p}}$ and $\mathcal{N}(\varepsilon_n)$ a ε_n -net in $B(\ell_p^{2n})$. Consider the disjoint union $D = \bigcup_{n \in \mathbb{N}} D_n \subset 4LB_X$. The function that cannot be extended is

$$\begin{aligned} f : D &\longrightarrow Y \\ D_n \ni x &\longmapsto f(x) = \frac{1}{a_n} R_n \circ \phi_{p,q}^{2n} \circ T_n^{-1}(x - x_n). \end{aligned}$$

Arguing as before, f is (L'', α) -Hölder on D , with $L'' = \max(L', L'/L^\alpha)$. Suppose that $g : X \rightarrow Y$ is Hölder with exponent α and coincides with f on D . Then the map $x \mapsto a_n P_n \circ g(x_n + T_n(x))$ (where $P_n : Y \rightarrow R_n(\ell_q^{2n})$ are projections such that $\sup_{n \in \mathbb{N}} \|P_n\| < \infty$) is still Hölder with exponent α and coincide with $\phi_{p,q}^{2n}$ on $\mathcal{N}(\varepsilon_n)$. Apply Fact 1 and let $n \rightarrow \infty$ to obtain a contradiction. ■

Remarks. (1) Suppose that the Banach lattices X and Y satisfy the assumptions of Theorem 1. If, in addition, X does have type p_X and Y does have cotype q_Y , then they admit equivalent renormings of appropriate power type p_X and q_Y . Thus the proof shows that

$$\mathcal{B}(X, Y) = \left(0, \frac{p_X}{q_Y} \right].$$

(2) For $Y = \ell_q$, $1 \leq q < \infty$, the right-hand side inclusion is proved in [3, Proposition 5] using the connection between approximation and extension by Hölder mappings.

4.2. EXAMPLES IN THE SETTING OF ORLICZ SPACES. We give now examples of both occurrences in Theorem 1. They complete Examples 3.11 and 4.4 in [4]. We refer to [5] for background about Orlicz functions and Orlicz spaces. Consider the Orlicz function M defined for all $u > 0$ by $M(u) = u^p(1 + |\ln(u)|)$, $1 < p \leq 2$. We claim that, for all $2 \leq q < \infty$,

$$\mathcal{B}(L_M[0, 1], \ell_q) = \left(0, \frac{p}{q}\right] \text{ and } \mathcal{B}(\ell_M, \ell_q) = \left(0, \frac{p}{q}\right).$$

Indeed, for $X = L_M[0, 1]$ or $X = \ell_M$, we have $p_X = p$ because $\frac{uM'(u)}{M(u)} \rightarrow p$ whenever $u \rightarrow +\infty$ or $u \rightarrow 0$. Recall that for $Y = \ell_q$, with $2 \leq q < \infty$, we have $q_Y = q$. If $X = L_M[0, 1]$, X does have type p . Thus the remark after Theorem 1 gives the first part of the previous claim. If $X = \ell_M$, X does not have type p . By arguing like in Example 3.11 in [4], together with Lemma 3.6 and Proposition 4.1 therein, we have $\frac{p}{q} \notin \mathcal{B}(\ell_M, \ell_q)$, because $\lim_{n \rightarrow \infty} n^{1/q} M^{-1}(1/n)^{p/q} = 0$.

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