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On the Functoriality of Stratified Desingularizations

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Abstract: This article is devoted to the study of smooth desingularizations, a geometric tool usually employed in the definition of the De Rham Intersection Cohomology with differential forms [12]. In this paper we work with the category of Thom-Mather simple spaces [10], [14]. We construct a functor which sends each Thom-Mather simple space into a smooth manifold called its primary unfolding, and prove that this construction is functorially preserved under Thom-Mather morphisms.

Key words: Intersection cohomology, stratified pseudomanifolds.

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To our families.

Introduction

Stratified spaces were initially defined by Thom [14], a more recent exposition can be found in [10]. These spaces are related to the Intersection Homology defined by Goresky and MacPherson, a loose homology theory extending the Poincaré Duality to spaces with singular points [6]. On the other hand, we can define the De Rham Intersection Cohomology with differential forms by means of an unfolding; it is a suitable smooth desingularization that can be made up by removing the singular part and gluing together two diffeomorphic copies of the regular part along a smooth hypersurface, for more details see Davis [4], Ferrarotti [5] and Saralegi [12].

Simple pseudomanifolds are almost everywhere smooth spaces, they can be separated in two disjoint pieces: A closed subset $\Sigma \subset X$ called the singular part, and its complement $R = X - \Sigma$ which turns to be a dense open subset called the regular part. The local model of this situation is the product $U \times c(L)$ where U and L are manifolds and L is compact; the name pseudomanifold arises from this kind of charts. A familiarized reader will notice that these are just stratified pseudomanifolds whose depth is $d \leq 1$.

This article is structured as follows: first we introduce the notion of simple pseudomanifolds and their unfoldings. Next we study the Thom-Mather simple spaces and prove that each simple pseudomanifold has an unfolding if and only if it satisfies the Thom-Mather condition. Although the unfoldings are in general neither unique nor funtorial objects [1], [3]; in the last section we prove that every radium preserving morphism between Thom-Mather simple spaces induces a smooth map between their respective unfoldings; hence the unfolding of a Thom-Mather simple space is essentially unique. We construct a functor from the category of Thom-Mather simple spaces to the category of smooth manifolds. This is done with the tool of primary unfoldings, which allows us to work in a subcategory of spaces.

For higher depth stratified pseudomanifolds ($d \geq 2$), any attempt to extend these results should deal with the problem of mutual incidence between the intermediate (non maximal and non minimal) strata. The authors are working on this purpose and hope to provide a more general discussion in the future.

All along this article, each time we use the word *manifold* we mean a smooth differentiable manifold of class C^∞ without boundary.

1. Simple pseudomanifolds

In this section we introduce the family of simple pseudomanifolds. Recall that, in general, a stratified space is a pair $(X, \mathcal{S})$ where $\mathcal{S}$ is a partition of X in disjoint smooth pieces called strata, satisfying a border incidence condition. Some geometric properties (such as dimension) turn to be semilocal, i. e. they remain constant along the strata. A characteristic semilocal property of stratified spaces is the depth; a non negative integer d that measures the number of strata along which one can approach to a singular point. As said before, a familiarized reader will notice that simple spaces are just stratified spaces whose depth is ≤ 1. We will avoid all these definitions in regard of a clear exposition; for more details see [6], [10].

Definition 1.1. A *simple space* is a 2nd countable metric topological space X which can be written as the disjoint union of two manifolds $X = R \sqcup \Sigma$, such that R is an open dense set (and therefore Σ is closed). We refer to R (resp. Σ) as the *regular* (resp. *singular*) part of X. A regular (resp. singular) *stratum* S of X is a connected component of R (resp. Σ). The pair (R, Σ) is a *decomposition* of X.

A *simple subspace* of X is a subset $Y \subset X$ such that $(R \cap Y, \Sigma \cap Y)$ is a decomposition of Y with the induced topology.

If X' is another simple space, then a *morphism* (resp. *isomorphism*) is a continuous map $X \xrightarrow{f} X'$ which preserves the decomposition in a smooth (resp. diffeomorphic) way. An *embedding* is a morphism $X \xrightarrow{f} X'$ such that $f(X)$ is a simple subspace of X' and $X \xrightarrow{f} f(X)$ is an isomorphism.

Examples 1.2.

(1) Each manifold M is a simple space whose singular part $\Sigma = \emptyset$ is the empty set.

(2) The canonical decomposition of any manifold with (nonempty) boundary M is $(M - \partial M, \partial M)$.

(3) Any open subspace of a simple space is itself a simple space.

(4) If M is a manifold and X is a simple space, then the product space $M \times X$ is a simple space, its decomposition is $(M \times R, M \times \Sigma)$.

(5) Let L be a compact manifold, the *open cone* of L is the quotient space

$$c(L) = \frac{L \times [0, \infty)}{\sim},$$

where $(l, 0) \sim (l', 0)$ for any $l, l' \in L$. The equivalence class of a point (l, r) will be written as $[l, r]$. The *vertex* is the class of any point $(l, 0)$; it will be denoted by v. For convenience, we agree that $c(\emptyset) = \{v\}$ is a singleton. The space $c(L)$ is simple, its decomposition is $(L \times \mathbb{R}^+, \{v\})$.

(6) A *pseudo-Euclidean model* (or *pem* for short), is a product $U \times c(L)$ with the decomposition given by the above examples, i.e., $(V \times L \times \mathbb{R}^+, U \times \{v\})$; where U, L are manifolds and L is compact; this L is said to be the *link* of U on $U \times c(L)$. Our convention for $L = \emptyset$ implies that any euclidean nbhd is a pem.

(7) Since any pem is the quotient of a product manifold; each morphism $U \times c(L) \xrightarrow{f} V \times c(N)$ can be written as

$$f(u, [l, r]) = \big(a_1(u, l, r), [a_2(u, l, r), a_3(u, l, r)]\big),$$

where a_1, a_2, a_3 are maps defined on $U \times L \times [0, \infty)$, and they are *smooth by pieces*. Notice that $a_3(u, l, 0) = 0$ for any u, l.

Definition 1.3. A *simple pseudomanifold* or *spm* for short, is a simple space X such that each singular point $x \in \Sigma$ has an open nbhd $x \in V \subset X$ which is the image of an embedding

$$U \times c(L) \xrightarrow{\alpha} X .$$

We call $V = \mathrm{Im}(\alpha)$ a *pem-nbhd* of x, while the pair (U, α) is a *chart*. Since any euclidean nbhd is a pem-nbhd (see example §1.2-(6)), the above condition is non-trivial just for singular points.

Up to some minor details, we assume that $\alpha(u, v) = u$ for all $u \in U$, so $U = V \cap \Sigma$ is an open nbhd of x on the corresponding stratum $S \subset \Sigma$ containing x. We usually ask the points of S to have the same link L, so it does not depend on the choice of x. We call L the *link* of S. ∎

Examples 1.4.

(a) The examples §1.2-(1), (2), (5) and (6) are spm's.

(b) If M is a manifold and X is a spm then $M \times X$ is a spm.

(c) Any open set of a spm is also a spm.

2. Desingularizations

It is well known how some usual (co)homological properties of smooth manifolds are lost when we add singularities. This is the case of the Poincaré Duality, for instance. In order to recover these properties on a larger family of spaces, the original works of Goresky and MacPherson defined the Intersection Homology with singular chains [6]. Later on [7], [12], Hector and Saralegi provided a smooth approach to the Intersection Cohomology. For more details see also [1], [5]. Their viewpoint strongly depends on two geometric objects associated to any spm X, which are built in order to study the way we reach the singular part and how we can recover the usual cohomological data; these are the Thom-Mather tubular neighborhoods [14] and the smooth unfoldings [4].

2.1. Unfoldings Recall the definition of smooth unfoldings [12]. Such an object is obtained from an spm X with decomposition (R, Σ) by gluing a finite number of copies of R and replacing Σ with a suitable smooth hypersurface.

DEFINITION 2.1. An *unfolding* of a spm X is a manifold $\widetilde{X}$ together with a continuous proper map $\widetilde{X} \xrightarrow{\mathcal{L}} X$ such that:

(1) The restriction $\mathcal{L}^{-1}(R) \xrightarrow{\mathcal{L}} R$ is a smooth trivial covering.

(2) For each $z \in \mathcal{L}^{-1}(\Sigma)$ there is a commutative square diagram:

$$\begin{array}{ccc} U \times L \times \mathbb{R} & \xrightarrow{\widetilde{\alpha}} & \widetilde{X} \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \mathcal{L}} \\ U \times c(L) & \xrightarrow[\alpha]{} & X \end{array}$$

such that

(a) (U, α) is a chart (see §1.3),
(b) $c(u, l, t) = (u, [l, |t|])$,
(c) $\widetilde{\alpha}$ is a diffeomorphism on $\mathcal{L}^{-1}(\mathrm{Im}(\alpha))$.

We will refer to the above diagram as an *unfolded chart* at $x = \mathcal{L}(z)$.

A spm X is *unfoldable* if there is (at least) a (smooth) unfolding as above. An *unfoldable morphism* is a commutative square diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{X}' \\ {\scriptstyle \mathcal{L}}\downarrow & & \downarrow{\scriptstyle \mathcal{L}'} \\ X & \xrightarrow[f]{} & X' \end{array} \tag{1}$$

such that f is a morphism, $\widetilde{f}$ is smooth and the vertical arrows are unfoldings.

EXAMPLES 2.2.

(1) For any manifold M the identity map $M \xrightarrow{id} M$ is an unfolding.

(2) The map c given in §2.1-(2).(b) is an unfolding of the pem $U \times c(L)$.

(3) For any manifold M with non-empty border $\partial M \neq \emptyset$; the link of ∂M is a point. Define $\widetilde{M} = M \times \{\pm 1\}/\sim$ as the quotient set obtained by gluing two copies of M along ∂M, i.e., $(m, 1) \sim (m, -1)$ for all $m \in \partial M$. Write $[m, j]$ for the class of (m, j). Consider the map

$$\widetilde{M} \xrightarrow{\mathcal{L}} M, \quad \mathcal{L}([m, j]) = m.$$

Since the border submanifold ∂M has always a smooth collar then, locally, $\mathcal{L}$ behaves like c at §2.1-(2).(b). So $\widetilde{M}$ has a unique smooth structure such that $\mathcal{L}$ is an unfolding.

(4) The restriction of an unfolding to any open subset is also and unfolding.

(5) If $\widetilde{X} \xrightarrow{\mathcal{L}} X$ is an unfolding, then for any singular stratum $S \subset \Sigma$ the restriction $\mathcal{L}^{-1}(S) \xrightarrow{\mathcal{L}} S$ is a smooth bundle with typical fiber $F = L$ and $\mathcal{L}^{-1}(S)$ is a hypersurface of $\widetilde{X}$.

(6) The 2-torus $\mathbb{T}^2$ provides an unfolding for the real projective plane $\mathbb{T}^2 \xrightarrow{\mathcal{L}} \mathbb{R}P^2$. We replace the singular point $\Sigma = \{\infty\}$ with two disjoint circles $\mathbb{S}^1 \times \{\pm 1\}$ and glue them with two copies of the cylinder $\mathbb{S}^1 \times \mathbb{R}$, obtaining so $\mathbb{T}^2$. The map $\mathcal{L}$ is given by $\mathcal{L}(e^{i\theta}, e^{i\varphi}) = [e^{i2\theta}, e^{i\varphi}]$.

2.2. Tubular neighborhoods The construction of tubular neighborhoods is due, among others, to Gleason and Palais. It was developed in the context of compact transformation groups. In the smooth context, a tubular neighborhood can be constructed trough an invariant riemannian metric and they are related to the existence of equivariant slices [2]. All these results have been extended to the singular context [7], [10]. When we move on a spm X, as we approach to the singular part we must preserve the geometric notion of conical radium. The unfoldings and the tubular neighborhoods can be related through another geometric construction, the normalizations; so although we we do not explicitly mention it, we always can assume that the links are connected [9].

Definition 2.3. Given an spm X let us consider a singular stratum $S \subset \Sigma$. A *tubular neighborhood* of S (in the sequel a *tubular nbhd*, for short) is a fiber bundle $\xi = (T, \tau, S, c(L))$ satisfying

(1) T is an open nbhd of S in X.

(2) The abstract fiber of ξ is $F = c(L)$, the open cone of the link L of S.

(3) $\tau(x) = x$ for any $x \in S$. In other words, the inclusion $S \subset T$ is a section of the fiber bundle.

(4) The structure group of ξ is a subgroup of Difeo(L); i.e., if (U, α), (V, β)

are two bundle charts and $U \cap V \neq \emptyset$ then the change of charts is

$$\beta^{-1}\alpha : U \cap V \times c(L) \longrightarrow U \cap V \times c(L) \ ,$$

$$\beta^{-1}(\alpha(u, [l, r])) = (u, [g_{\alpha\beta}(u)(l), r]),$$

where $g_{\alpha\beta}(u)$ is a diffeomorphism of L for all $u \in U \cap V$.

The spm X is a *Thom-Mather* if and only if every singular stratum has a tubular nbhd. If X, X' are Thom-Mather spms a *Thom-Mather morphism* is a morphism f in the sense of §1.1 which preserves the tubular nbhd.

Remark 2.4. Given a tubular nbhd $T \xrightarrow{\tau} S$, by §2.3-(4), there is global sense of radium $T \xrightarrow{\rho} [0, \infty)$ in the whole tube T given by $\rho(u, [l, r]) = r$, this function is the *radium* of T. Notice that $\rho^{-1}(\{0\}) = S$ and $\rho^{-1}(\mathbb{R}^+) = (T - S)$. There is also an action $\mathbb{R}^+ \times T \longrightarrow T$ by *radium stretching*, given by $\lambda \cdot \alpha(u, [l, r]) = \alpha(u, [l, \lambda r])$.

Remark 2.5. Given a spm X, the singular strata are disjoint connected manifolds. Since X is normal, they can be separated trough a disjoint family of open subsets. Therefore, and up to some minor details, if X is Thom-Mather then we can find a disjoint family of tubular nhbds. This allows us to simplify some things. In the rest of this section we fix a Thom-Mather spm X and we will assume, without loss of generality, that X has a unique singular stratum $S = \Sigma$, with a tubular nbhd as above.

The main goal of this section is to study how the unfoldings and the tubes relate to each other. We state this relationship in the following theorem.

Theorem 2.6. *A spm is Thom-Mather if and only if it is unfoldable.*

We will prove separately each implication, so next we show how to obtain an unfolding from a tubular nbhd. For an equivalent way of constructing unfoldings, the reader can see [4] [7].

Proposition 2.7. *Every Thom-Mather spm is unfoldable.*

Proof. For the singular stratum $S = \Sigma$ and a the tubular nbhd $T \xrightarrow{\tau} S$, let's fix a bundle atlas $\mathcal{U} = \{(U_\alpha, \alpha)\}_{\alpha \in \mathfrak{J}}$. We proceed in three steps.

Unfolding a chart: For any chart $(U,\alpha)\in\mathcal{U}$, the unfolding of $\tau^{-1}(U)$ is the composition

$$U\times L\times\mathbb{R}\xrightarrow{c}U\times c(L)\xrightarrow{\alpha}\tau^{-1}(U)\ , \tag{2}$$

where c is the map given at §2.1-(2).(b).

Unfolding the tube: Define

$$\widetilde{T}=\frac{\bigsqcup_\alpha U_\alpha\times L\times\mathbb{R}}{\sim},\quad (u,l,t)\sim(u,g_{\alpha\beta}(u)(l),t)\ \forall\alpha,\beta,\ \forall u\in U_\alpha\cap U_\beta \tag{3}$$

as the quotient of the disjoint union, with the above equivalence relation. Write $[u,l,t]$ for the equivalence class of a triple (u,l,t). According to [13, p. 14], the above operation defines a fiber bundle

$$\widetilde{T}\xrightarrow{\widetilde{\tau}}S\ ,\quad \widetilde{\tau}([u,l,t])=u \tag{4}$$

with abstract fiber $F=L\times\mathbb{R}$ and the same structure group of T. Since the cocycles are smooth, notice that $\widetilde{T}$ is a manifold. Let's now define

$$\widetilde{T}\xrightarrow{\mathcal{L}}T\ ,\quad \mathcal{L}([u,l,r])=\alpha(u,[l,|t|]),\ \forall u\in U_\alpha,\ \forall\alpha. \tag{5}$$

In order to show that the above arrow is an unfolding of T, the reader only needs to check, for any chart $(U_\alpha,\alpha)\in\mathcal{U}$, the smoothness of $\widetilde{\alpha}$ in the next commutative square diagram:

$$\begin{array}{ccc} U_\alpha\times L\times\mathbb{R} & \xrightarrow{\widetilde{\alpha}} & \widetilde{T}\\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle\mathcal{L}}\\ U_\alpha\times c(L) & \xrightarrow[\alpha]{} & T\end{array}$$

where $\widetilde{\alpha}(u,l,r)=[u,l,r]$ is just to pick the respective equivalence class.

Unfolding the whole spm X: Remark that

$$\mathcal{L}^{-1}(T-S)=T^+\sqcup T^-$$

has two connected components, each of them being a smooth bundle over S with abstract fibers $F^+=L\times\mathbb{R}^+$ and $F^-=L\times\mathbb{R}^-$ respectively. These two components are disjoint because the cocycles are radium-independent. The unfolding of the whole space X can be made by taking two copies of the regular part $R=X-\Sigma$; say R^+,R^-, and gluing them together and suitably with $\widetilde{T}$ along $\mathcal{L}^{-1}(T-S)$. ∎

In order to prove the converse statement, let's recall that any manifold M with nonempty boundary $\partial M \neq \emptyset$ has a *collar*, i.e., a transverse nbhd given by a smooth embedding $\Gamma : \partial M \times [0,\infty) \longrightarrow M$ such that $\mathrm{Im}(\Gamma)$ is open in M and $\Gamma(m,0) = m$ for all $m \in \partial M$.

Proposition 2.8. *Every unfoldable pseudomanifold X is Thom-Mather.*

Proof. Let $\widetilde{X} \xrightarrow{\mathcal{L}} X$ be an unfolding of X. Then $\mathcal{L}^{-1}(X-\Sigma)$ is a finite trivial smooth covering of $R = (X - \Sigma)$, i.e., a disjoint union of finitely many diffeomorphic copies of R. Pick one, say $R_0 \cong R$, such that $\overline{R_0}$ is a manifold with boundary $\partial(\overline{R_0}) = \mathcal{L}^{-1}(S)$, and take a collar $\mathcal{L}^{-1}(S) \times \mathbb{R} \xrightarrow{\Gamma} \overline{R_0}$ of $\mathcal{L}^{-1}(S)$ in $\overline{R_0}$. Define $T = \mathcal{L}(\mathrm{Im}(\Gamma))$ and

$$T \xrightarrow{\tau} S\ , \quad \tau(\mathcal{L}(\Gamma(z,r))) = \mathcal{L}(z).$$

Following [7], the above map provides a tubular nbhd of S and each unfolded chart as in §2.1-(2) induces a bundle chart

$$U \times c(L) \xrightarrow{\widehat{\alpha}} \tau^{-1}(U)\ , \quad \widehat{\alpha}(u,[l,r]) = \mathcal{L}(\Gamma(\widetilde{\alpha}(u,l,0),r)).$$

We leave the details to the reader. ∎

3. Functorial constructions

In the previous sections we dealed with the existence of unfoldings in terms of the tubular nbhds. Now we will see in more detail their categorical properties. It can be easily deduced from §2.1 that the unfoldings are neither unique nor functorial objects. We will restrict ourselves to a narrower family of spaces in order to develope some ideas concerning the smooth desingularization of pseudomanifolds and their functoriality, when considered as a topological process.

3.1. Primary unfoldings

Primary unfoldings are the smallest unfoldings one can find for a given pseudomanifold. They where originally presented by Brasselet, Hector and Saralegi [1] and later redefined by Dalmagro [3], whose points of view constitute the aim of this section.

Definition 3.1. A *primary unfolding* is an unfolding $\widetilde{X} \xrightarrow{\mathcal{L}} X$ in our previous sense, such that the preimage of the regular part $\mathcal{L}^{-1}(R) = R_0 \sqcup R_1$ is a double (smooth, trivial) covering, i.e., the union of exactly two diffeomorphic copies of R.

Remark 3.2. The family of primary unfoldings is representative in the category of unfoldable spms. Starting from any unfolding $\widetilde{X} \xrightarrow{\mathcal{L}} X$, we can construct a primary unfolding $\widetilde{X}' \xrightarrow{\mathcal{L}'} X$ by taking a manifold with border $M = \overline{R_0}$ as in the proof of §2.8 and then proceeding as in example §2.2-(3) in order to get $\widetilde{M} \xrightarrow{\mathcal{L}''} M$. Then $\widetilde{X}' = \widetilde{M}$ and $\mathcal{L}' = \mathcal{L}\mathcal{L}''$ is the composition.

Lemma 3.3. *Let X, X' be two unfoldable pseudomanifolds. Then, for any pair of unfoldings $\widetilde{X} \xrightarrow{\mathcal{L}} X$, $\widetilde{X}' \xrightarrow{\mathcal{L}'} X$ and any morphism $X \xrightarrow{f} X'$; there is a unique continuous and almost everywhere smooth map $\widetilde{X} \xrightarrow{\widetilde{f}} \widetilde{X}'$ such that the square diagram §(1) p.4, is commutative; we call $\widetilde{f}$ the lifting of f.*

Proof. In order to simplify the exposition, by the above remark §3.2, we assume that $\widetilde{X} \xrightarrow{\mathcal{L}} X$ and $\widetilde{X}' \xrightarrow{\mathcal{L}'} X$ are primary unfoldings.

(a) *Lifting f on the regular part*: If $\Sigma = \emptyset$ then there is no singular part, and $X = R$ is a manifold. It follows that $\widetilde{X} = R_0 \sqcup R_1$. Take $\widetilde{f} = f \times \mathrm{id}$, then §3.3 trivially holds. So let's suppose that $\Sigma \neq \emptyset$ and moreover, by §2.5, we will assume that $\Sigma = S$ is a single stratum. By these arguments, we have already defined a continuous function $\widetilde{f}$ satisfying §3.3 on $\mathcal{L}^{-1}(R)$. Therefore, we only must find a continuous extension of $\widetilde{f}$ to the entire $\widetilde{X}$, i.e., to $\mathcal{L}^{-1}(S)$.

(b) *Extension of the lifting*: Pick some $\widetilde{z} \in \mathcal{L}^{-1}(S)$. We must show a way to choose $\widetilde{f}(\widetilde{z})$. For this sake, let $\{\widetilde{z}_n\}_n \subset \mathcal{L}^{-1}(R)$ be a sequence converging to $\widetilde{z}$. Since $\mathcal{L}, f$ are continuous maps and $\mathcal{L}'$ is a continous and proper map; by an argument of compactness, and up to some little adjusts, we may assume that the sequence $\{\widetilde{f}(\widetilde{z}_n)\}_n$ converges in $\widetilde{X}'$. We define

$$\widetilde{f}(\widetilde{z}) = \lim_{n\to\infty} \widetilde{f}(\widetilde{z}_n).$$

If our limit-definition makes sense then it is also continuous; so next we will show the non ambiguity of $\widetilde{f}$. Since the former is a local definition, we first study the

(c) *Local writing of the lifting*: By §2.1-(2) we can restrict to unfoldables charts; so we will assume that X= $U \times c(L)$ and $Y = V \times c(N)$ are trivial pem nbhds and their respective unfoldings are the canonical ones - see §2.2-(2). Then f can be written as in §1.2-(7). The point

$$\widetilde{z} = (u, l, 0) \in U \times L \times \{0\}$$

is the limit of a sequence

$$\{\widetilde{z}_n = (u_n, l_n, t_n)\}_n \subset U \times L \times (\mathbb{R} - \{0\}).$$

So the sequences $\{u_n\}_n$, $\{l_n\}_n$ and $\{t_n\}_n$ respectively converge to u, l and 0. Since $\mathcal{L}, f$ are continuous maps, the sequence

$$w_n = f(\mathcal{L}(\widetilde{z}_n)) = \big(a_1(u_n, l_n, |t_n|), [a_2(u_n, l_n, |t_n|), a_3(u_n, l_n, |t_n|)]\big)$$

converges to $w = f(\mathcal{L}(\widetilde{z})) = (a_1(u, l, 0), v)$. By the continuity of the functions a_j for $j = 1, 2, 3$ and up to some little adjust on a_2 concerning the compactness arguments; we get that

$$\widetilde{w}_n = \big(a_1(u_n, l_n, |t_n|), a_2(u_n, l_n, |t_n|), \pm a_3(u_n, l_n, |t_n|)\big)$$

converges to $\widetilde{\omega} = (a_1(u, l, 0), a_2(u, l, 0), 0)$.

(d) *The lifting is well defined*: From the continuity of the functions a_i, the element $\widetilde{w}$ does not depend on the choice of a particular sequence $\{\widetilde{z}_n\}_n$.

Notice that, the lifting $\widetilde{f}$ is always smooth on $\mathcal{L}^{-1}(X - \Sigma)$, the preimage of the regular part. ∎

Remark 3.4. Given an unfolding $\widetilde{X} \xrightarrow{\mathcal{L}} X$; a *bubble* of $\widetilde{X}$ is a connected component of $\mathcal{L}^{-1}(X - \Sigma)$.The above proof is still valid if we take any other permutation of the bubbles. Along the rest of this paper, we assume that we are working with the identity permutation, unless we state the opposite.

Definition 3.5. A morphism f between pems nhbds is *liftable* if its lifting $\widetilde{f}$ is globally smooth on every $\widetilde{X}$. This is equivalent to ask $\widetilde{f}$ to be smooth on a nbhd of $\mathcal{L}^{-1}(\Sigma)$.

Proposition 3.6. *A morphism between pem nhbds*

$$U\times c(L)\xrightarrow{f}U'\times c(L')\ ,\quad f(u,[l,r])=\big(a_1(u,l,r),[a_2(u,l,r),a_3(u,l,r)]\big),$$

is liftable into

$$U\times L\times\mathbb{R}\xrightarrow{\widetilde{f}}U'\times L'\times\mathbb{R}\ ,\quad \widetilde{f}(u,l,t)=\big(\widetilde{a}_1(u,l,t),\widetilde{a}_2(u,l,t),\widetilde{a}_3(u,l,t)\big),$$

if and only if

(*a*) $\widetilde{a}_1$, $\widetilde{a}_2$ *are smooth even extensions of, respectively,* a_1, a_2.

(*b*) $\widetilde{a}_3$ *is either an odd (and therefore smoooth) extension of* a_3 *or it is a smooth even extension and* $\widetilde{a}_3(u,l,0)=a_3(u,l,0)=0$ *for all* u,l.

Proof. If $\widetilde{f}$ is a lifting of f, then $fc=c'\widetilde{f}$, where c and c' are canonical unfoldings as in §2.1-(2). Checking both sides of this equality we get

$$f(c(u,l,t))=f(u,[l,|t|])=\big(a_1(u,l,|t|),[a_2(u,l,|t|),a_3(u,l,|t|)]\big)$$

and

$$\begin{aligned}c'(\widetilde{f}(u,l,t))&=c'\big(\widetilde{a}_1(u,l,t),\widetilde{a}_2(u,l,t),\widetilde{a}_3(u,l,t)\big)\\&=\big(\widetilde{a}_1(u,l,t),[\widetilde{a}_2(u,l,t),|\widetilde{a}_3(u,l,t)|]\big).\end{aligned}$$

We conclude that

$$\big(a_1(u,l,|t|),[a_2(u,l,|t|),a_3(u,l,|t|)]\big)=\big(\widetilde{a}_1(u,l,t),[\widetilde{a}_2(u,l,t),|\widetilde{a}_3(u,l,t)|]\big).$$

There are two cases; $t=0$ and $t\neq 0$, from which we get §3.6. ∎

Lemma 3.7. *Let* X *be a Thom-Mather spm, then the cocycles of a tubular neighborhoood are liftable.*

Proof. It is enough to take $\varphi=\beta^{-1}\alpha$ as in §2.3-(4) and check that $a_1(u,l,r)=u$, $a_2(u,l,r)=g(u)(l)$, $a_3(u,l,r)=r$ satisfy the hypothesis of §3.6. ∎

Lemma 3.8. *Let* $f,\, f':U\times c(L)\longrightarrow U'\times c(L')$ *be morphisms and* $\varphi:U\times c(L)\longrightarrow U\times c(L)$, $\varphi':U'\times c(L')\longrightarrow U'\times c(L')$ *be isomophisms as in §2.3-(4). Then* $f'\varphi=\varphi' f$ *if and only if* a_1,a_3 *are invarant with respect* *to the group action on the coordinate l and a_2 conmutes with the cocycles, i.e, a_1, a_2 and a_3 satisfy:*

$$\begin{aligned} a_1(u,l,r) &= a_1'\big(u, g(u)(l), r\big), \\ g'\big(a_1(u,l,r)\big)a_2(u,l,r) &= a_2'\big(u, g(u)(l), r\big), \\ a_3(u,l,r) &= a_3'\big(u, g(u)(l), r\big). \end{aligned} \tag{6}$$

Proof. Let f as above, and φ, φ' be cocycles as in §2.3-(4), then

$$\begin{aligned} f'\big(\varphi(u,[l,r])\big) &= f'\big(u,[g(u)(l),r]\big) \\ &= \big(a_1'(u,g(u)(l),r),[a_2'(u,g(u)(l),r),a_3'(u,g(u)(l),r)]\big). \end{aligned}$$

On the other hand

$$\begin{aligned} \varphi'\big(f(u,[l,r])\big) &= \varphi'\big(a_1(u,l,r),[a_2(u,l,r),a_3(u,l,r)]\big) \\ &= \big(a_1(u,l,r),[g'(a_1(u,l,r)(a_2(u,l,r)),a_3(u,l,r))]\big). \end{aligned}$$

If $f'\varphi = \varphi' f$ then, after checking the cases $r = 0$ and $r \neq 0$, we get §3.8. ∎

Lemma 3.9. *The restriction of a Thom-Mather morphism (see* §2.3*) to the local trivializations satisfy* §3.8.

Proof. Let $X \xrightarrow{\Psi} X'$ be a Thom-Mather morphism. Without any loss of generality, we can assume that $T = X$ and $T' = X'$ (see §2.5). Let α, β and α', β' be two bundle charts of X and X' respectively defined on $U \times c(L)$ and $U' \times c(L')$.

If $\varphi = \beta^{-1}\alpha$ and $\varphi' = (\beta')^{-1}\alpha'$ are their respective cocycles (c.f., §2.3-(4)), then $f = (\alpha')^{-1}\Psi\alpha$ and $f' = (\beta')^{-1}\Psi\beta$ satisfy §3.8. ∎

These results imply that there is a functor from the category of Thom-Mather spaces to the category of the smooth manifolds. Formally we state the next theorem.

Theorem 3.10. *Every Thom-Mather morphism has a smooth Thom-Mather lifting.*

Proof. Let $X \xrightarrow{f} X'$ be a Thom-Mather morphism. According to §3.3 there is a unique continuous lifting $\widetilde{X} \xrightarrow{\widetilde{f}} \widetilde{X}'$; moreover, $\widetilde{f}$ is an almost everywhere smooth map. Recall that by §3.7, the cocycles of the tubular nbhds at X and X' are liftable. Since f preserves the Thom-Mather structure, by 3.9, the composition of f with any pair of trivilizing charts at X and X' turns to be liftable. By a uniqueness argument, we deduce that $\widetilde{f}$ is locally smooth on a nbhd of $\mathcal{L}^{-1}(\Sigma)$ and, therefore, $\widetilde{f}$ is smooth. ∎

3.2. Uniqueness of the primary unfoldings Theorem §3.10 shows that the primary unfoldings have a quite nice, functorial behaviour.

Theorem 3.11. *Let X and X' be two Thom-Mather spaces and let $X \xrightarrow{f} X'$ be a Thom-Mather isomorphism. Then, the smooth lifting of f, $\widetilde{f}$, is a diffeomorphism between the manifolds $\widetilde{X}$ and $\widetilde{X}'$.*

Proof. We know that by §3.3 the lifting of f, $\widetilde{f}$ is unique up to permutation of the bubbles. Let us fix a bubble permutation σ(as in §3.4) on $\mathcal{L}^{-1}(X-\Sigma)$. As f is a Thom-Mather morphism its lifting $\widetilde{f}$ is a smooth Thom-Mather map between $\widetilde{X}$ and $\widetilde{X}'$ (see, §3.10). On the another hand, the inverse morphism of f, g is also a Thom-Mather map which lifts into a smooth manifold map $\widetilde{g}$ between $\widetilde{X}'$ and $\widetilde{X}$. In order to satisfy the equations $\widetilde{g}\widetilde{f} = id_{\widetilde{X}}$ and $\widetilde{f}\widetilde{g} = id_{\widetilde{X}'}$, we take the lifting $\widetilde{g}$ of g induced with the inverse permutation σ^{-1} of the bubbles. So, therefore $\widetilde{f}$ is a diffeomorphism. ∎

Corollary 3.12. *The primary unfolding of a Thom-Mather space is unique.*

Proof. Take $X = X'$ as in §3.11 and apply the same argumentation to the identity $X \xrightarrow{id} X$. ∎

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