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Representations of Codimension One Non-Abelian Nilradical Lie Algebras

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Abstract: A Theorem is proved that shows that for a solvable Lie algebra $\mathfrak{h}$ of dimension $n+2$ whose nilradical is codimension one and for which the nilradical has a one-dimensional derived algebra there is a subgroup of $\mathrm{GL}(n+2,\mathbb{R})$ whose Lie algebra is isomorphic to $\mathfrak{h}$. The Theorem helps to give a more conceptual understanding of the classification of the algebras in dimensions four, five and six. Finally the main Theorem is applied to a particularly interesting class of algebras for which the nilradical is isomorphic to the five-dimensional Heisenberg algebra.

Key words: Low-dimensional Lie algebras, right-invariant vector field, Lie algebra extension, codimension one nilradical.

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1. Introduction

A well known theorem in the theory of Lie algebras due to Ado asserts that every real Lie algebra $\mathfrak{g}$ of dimension n has a faithful representation as a subalgebra of $\mathrm{GL}(p,\mathbb{R})$ for some p. The theorem does not give much information about the value of p but leads one to believe that p may be very large in relation to the size of n and consequently it seems to be of limited practical value. If $\mathfrak{g}$ has a trivial center then the adjoint representation furnishes a faithful representation of $\mathfrak{g}$ and in the notation used above $p=n$.

The classification of low-dimensional Lie algebras is a complicated and laborious task that enjoys none of the elegance of the theory of semi-simple Lie algebras. Nonetheless Lie algebras are of fundamental importance and we consider it to be desirable to obtain matrix representations for them. In Section 2 we develop some Theorems that explain how to obtain representations in the case where the algebra has a non-trivial center. In particular we shall show that if $\mathfrak{g}$ has a codimension one abelian nilradical then it has a faithful representation as a subalgebra of $\mathrm{GL}(n,\mathbb{R})$. The main result of the paper shows that the results just mentioned can be generalized to the case where the nilradical of $\mathfrak{g}$ is non-abelian but is isomorphic to the direct sum of Heisenberg and abelian Lie algebras.

We apply our results to algebras of dimensions four, five and six in Section 3 and Section 4. We refer to the 1976 list given by Patera et al. [8] for a comprehensive list of the *indecomposable* algebras of dimension five and less, which in turn was based on [6]. We have followed the list given in [8] and made allowance for slight typographical changes. In dimension four there are four algebras that have a non-trivial center, none of which involves a parameter. As the dimension of $\mathfrak{g}$ increases the algebras form moduli; that is to say, there are families of inequivalent algebras that depend on several parameters. The six-dimensional indecomposable algebras have been classified by Mubarakzyanov [7] and we discuss these algebras in detail in Section 3. Our results help to give a slightly more conceptual understanding of the classification of the algebras in dimensions four, five and six although in the latter two cases our results are far from exhaustive. However, we study in detail a class of Mubarakzyanov algebras for which the nilradical is isomorphic to the five-dimensional Heisenberg algebra. The classification of this class of algebras is particularly interesting because it depends on classifying the orbits of the adjoint representation of the ten-dimensional symplectic group. Most of the calculations were done with the MAPLE symbolic manipulation program.

2. Representations of codimension one-nilradical algebras

We consider a Lie algebra g of dimension $n+1$. Suppose that the only non-zero brackets are given by

$$[e_i, e_{n+1}] = A_i^j e_j\,, \tag{1}$$

where A_i^j is an $n \times n$ matrix and the summation over j ranges from 1 to n.

Proposition 2.1. *Suppose that $\mathfrak{g}$ is an $n+1$ dimensional vector space with a skew-symmetric product $[\,,]$ for which the only non-zero brackets are given by equation* (1). *Then $\mathfrak{g}$ is automatically a Lie algebra for all choices of the $n \times n$ matrix A.*

Theorem 2.1. *Suppose that the n-dimensional Lie algebra $\mathfrak{g}$ has a codimension one abelian ideal. Then $\mathfrak{g}$ has a faithful representation as a subalgebra of* $\mathrm{GL}(n,\mathbb{R})$.

Proof. If $\mathfrak{g}$ is not itself nilpotent the abelian ideal in question will necessarily be the nilradical of $\mathfrak{g}$. On the other hand $\mathfrak{g}$ itself may be nilpotent. If $\{e_1, e_2, \ldots, e_{n-1}\}$ is a basis for the codimension one abelian ideal we can extend it to a basis for $\mathfrak{g}$ by means of the vector e_n. Define the endomorphism A to be $\mathrm{ad}(e_n)$ and let its matrix be a^i_j. The non-zero brackets of g are given by $[e_n, e_i] = \sum_{k=1}^{n-1} a^k_i e_k$. To obtain the representation, map e_n to A; for each vector e_i $(i \leq n-1)$ map it to the $n \times n$ matrix E_i whose only non-zero entry is a 1 in the $(i, n)^{th}$ position. Clearly the $E'_i s$ commute. Then note that the matrix product $E_i A$ is zero and so

$$[A, E_i] = \sum_{k=1}^{n-1} a^k_i E_k \tag{2}$$

and we have the required representation. ∎

Corollary 2.2. *An n-dimensional Lie algebra $\mathfrak{g}$ that has a codimension one abelian ideal is isomorphic to the Lie algebra of a subgroup of* $\mathrm{GL}(n, \mathbb{R})$, *that can be described explicitly.*

Proof. We resume from the previous Corollary. The subgroup of $\mathrm{GL}(n, \mathbb{R})$ that we seek is given by

$$S = \begin{bmatrix} e^{(x_n A)} & x \\ 0 & 1 \end{bmatrix}$$

where x denotes the column $n-1$-vector, with entries x_1,$x_2, \ldots, x_{n-1}$. Clearly it is a group since the first $n-1$ entries in the last column are arbitrary and its Lie algebra is isomorphic to $\mathfrak{g}$ as can be seen by differentiating with respect to each of the parameters and setting them equal to zero. ∎

Referring to [11] there are eleven classes of algebra in Tables 2 and 5 for which faithful representations cannot be found by using the adjoint representation. Several of the algebras in Tables 2 contain a parameter ϵ whose value is 0 or 1. The significance of ϵ is that when it is zero the nilradical has an *abelian* complement in the algebra $\mathfrak{g}$. For such algebras we can find faithful representations by appealing to the following Theorem [9].

Theorem 2.2. *Suppose that the n-dimensional Lie algebra $\mathfrak{g}$ has a basis $\{e_1, e_2, \ldots, e_n\}$ and only the following non-zero brackets: $[e_a, e_i] = C^j_{ai} e_j$, where $1 \leq i, j \leq r$, $r+1 \leq a, b, c \leq n$. Suppose that $\mathfrak{g}$ has an abelian nilradical for which a basis is $\{e_1, e_2, \ldots, e_r\}$ and $\{e_{r+1}, \ldots, e_n\}$ is a basis for* *an abelian subalgebra complementary to the nilradical. Then $\mathfrak{g}$ has a faithful representation as a subalgebra of* $\mathrm{GL}(r+1,\mathbb{R})$.

We want to consider now the following problem: starting from an $(n+1)$-dimensional Lie algebra $\mathfrak{g}$ that has a codimension one abelian ideal construct a one-dimensional non-split extension $\mathfrak{h}$ of $\mathfrak{g}$ by the center of the nilradical of $\mathfrak{h}$, which is assumed to be one-dimensional. Thus starting from equation (1) we want to construct a Lie algebra $\mathfrak{h}$ of dimension $n+2$ with the following non-z ero brackets:

$$\begin{aligned}
[e_0, e_{n+1}] &= a e_0\,,\\
[e_i, e_{n+1}] &= A_i^j e_j + a_i e_0\,,\\
[e_i, e_j] &= B_{ij} e_0\,,
\end{aligned}\tag{3}$$

where a is a scalar, a_i is an n-vector and B_{ij} is a skew-symmetric $n\times n$ matrix.

Proposition 2.3. *With the brackets above $\mathfrak{h}$ is a Lie algebra if and only if*

$$aB_{ij} + A_i^k B_{jk} + A_j^k B_{ki} = 0\,.\tag{4}$$

Proof. In principle there are four types of expressions in the Jacobi identity corresponding to the indices $0,i,j,\ 0,i,n+1,\ i,j,k$ and $i,j,n+1$. However, it is clear that only the fourth of these choices is not identically satisfied. A straightforward calculation shows that (4) is the required condition. ∎

We note that equation (4) can be written in matrix form as

$$BA + A^t B = aB\,.\tag{5}$$

The following Lemma sometimes can help to simplify (3):

Lemma 2.4. *For the Lie algebra $\mathfrak{h}$ if B_{ij} is non-singular we can assume, without loss of generality, that the vector a_i is zero.*

Proof. Make a change of basis in which only the vector e_{n+1} is changed according to $\overline{e_{n+1}} = e_{n+1} + \lambda^i e_i$. To eliminate the $a_i e_0$ we need to be able to solve $\lambda^k B_{ik} + a_i = 0$ and we can do so provided B_{ij} is non-singular. ∎

Lemma 2.4 can be modified in the case where B_{ij} is singular so that only some components of a_i can be removed.

Instead of the the extension embodied in (3) let us consider an $n+2$ dimensional codimension one nilradical Lie algebra $\mathfrak{h}$ whose nilradical is isomorphic to a direct sum of a Heisenberg Lie algebra and an abelian algebra: equivalently, we are looking at a solvable Lie algebra whose nilradical is codimension one and for which the nilradical has a one-dimensional derived algebra. Then the non-zero brackets are necessarily of the form:

$$\begin{aligned} [e_0, e_{n+1}] &= ae_0 + b_i e_i\,, \\ [e_i, e_{n+1}] &= A_i^j e_j + a_i e_0\,, \\ [e_i, e_j] &= B_{ij} e_0\,. \end{aligned} \tag{6}$$

However, the Jacobi identity coming from $\{e_i, e_j, e_{n+1}\}$ implies that the b_i are all zero so that e_0 spans a one-dimensional ideal. Again (4) is the remaining condition needed to ensure that $\mathfrak{h}$ is a Lie algebra and we are essentially back to the situation comprised by (3) except that we do not need to assume that center of the nilradical of $\mathfrak{h}$ is one-dimensional.

THEOREM 2.3. *For a solvable Lie algebra $\mathfrak{h}$ of dimension $n+2$ whose nilradical is codimension one and for which the nilradical has a one-dimensional derived algebra there is a subgroup of* $\mathrm{GL}(n+2,\mathbb{R})$ *whose Lie algebra is isomorphic to $\mathfrak{h}$.*

Proof. We consider a matrix group of the following form where a typical element is denoted by S:

$$S = \begin{bmatrix} e^{aw} & b^t & z \\ 0 & e^{w\sigma} & x \\ 0 & 0 & 1 \end{bmatrix}$$

where b and x are column n-vectors. The idea is that the lower right hand $(n+1)\times(n+1)$ block corresponds to the group for the quotient abelian nilradical Lie algebra. Note that S^{-1} is given by

$$S^{-1} = \begin{bmatrix} e^{-aw} & b_1^t & z_1 \\ 0 & e^{-w\sigma} & x_1 \\ 0 & 0 & 1 \end{bmatrix},$$

where b_1, z_1, and x_1 are given by $b_1 = -e^{-aw}e^{-w\sigma^t}b$, $z_1 = e^{-aw}(b^t e^{-w\sigma}x - z)$, and $x_1 = -e^{-w\sigma}x$, respectively. The right-invariant Maurer-Cartan form

dSS^{-1} is given by:

$$dSS^{-1} = \begin{bmatrix} adw & (db^t - ab^t dw)e^{-w\sigma} & dz - db^t e^{-w\sigma}x + a(b^t e^{-w\sigma}x - z)dw \\ 0 & \sigma dw & dx - \sigma x dw \\ 0 & 0 & 0 \end{bmatrix}.$$

Suppose that there exists an $n \times n$ constant matrix C and column n-vector λ such that

$$e^{-w\sigma^t}(db - abdw) + \lambda dw = C(dx - \sigma x dw)\,. \tag{7}$$

Then computing the exterior derivative we find that

$$e^{-w\sigma^t}(-adbdw) + \sigma^t(db - abdw)dw = C\sigma dwdx\,.$$

Now use (7) to eliminate b and equate the coefficients of $dxdw$ to obtain

$$\sigma^t C + C\sigma = aC\,. \tag{8}$$

Now (8) is the necessary and sufficient condition for there to exist vectors b and λ such that (7) is satisfied.

Turning next to $dx - \sigma x dw$ we find that $d(dx - \sigma x dw) = \sigma dwdx$. Finally considering the form θ defined by $dz - azdw + (ab^t dw - db^t)e^{-w\sigma}x$ we find that

$$d\theta = -a\theta dw - \lambda^t dw(dx - \sigma x dw) + (dx - \sigma x dw)^t C^t (dx - \sigma x dw)\,.$$

Now the structure equations of the $n+2$ dimensional Lie algebra $\mathfrak{h}$ are given by

$$\begin{aligned} de^0 &= -ae^0 e^{n+1} + a_i e^i e^{n+1} - B_{ij} e^i e^j\,, \\ de^i &= -\sigma^i_j e^j e^{n+1}\,, \\ de^{n+1} &= 0\,. \end{aligned}$$

Thus we can choose $\lambda = a$ and $C = B$. The second of these conditions is consistent with (8) and is valid because we are assuming that (4) is satisfied. With these choices the Maurer-Cartan form dSS^{-1} engenders the required structure equations and we obtain a group whose Lie algebra is isomorphic to $\mathfrak{h}$. ∎

3. Applications

3.1. Four-dimensional algebras. We refer to [8] for a list of the four-dimensional indecomposable Lie algebras. The first six classes of algebra have a codimension-one abelian ideal. As such and referring to equation (1) the classes are obtained by putting the matrix A into Jordan normal form. Algebras 7-11 have a non-abelian codimension-one nilradical: as such it can only be isomorphic to the three-dimensional Heisenberg algebra. Using the theory of the previous Section we can write the structure equations of the algebra $\mathfrak{g}$ in the form:

$$\begin{aligned}
[e_2, e_3] &= e_1 \,, \\
[e_1, e_4] &= (a+d)e_1 \,, \\
[e_2, e_4] &= ae_2 + be_3 \,, \\
[e_3, e_4] &= ce_2 + de_3 \,.
\end{aligned}$$

Notice that e_1 spans the center of the nilradical and that these algebras are all split extensions in view of 2.4. If we quotient by the ideal generated by e_1 we obtain a solvable three-dimensional algebra with a codimension-one nilradical. Conversely, starting from *any* non-abelian, possibly decomposable three-dimensional algebra , we obtain uniquely a non-abelian four-dimensional indecomposable codimension-one nilradical algebra.

Theorem 3.1. *There is a one to one correspondence between four-dimensional indecomposable Lie algebras that have a non-abelian codimension-one nilradical and three-dimensional algebras that have an abelian codimension-one nilradical.*

To complete the classification of the four-dimensional algebras we remark that in an indecomposable solvable algebra $\mathfrak{g}$ the codimension of the nilradical is always less than or equal to $\frac{n}{2}$ where n is the dimension of $\mathfrak{g}$. Thus for $n = 4$ it remains only to discuss the case where the nilradical is two-dimensional. However, in that case it can be shown that the nilradical is abelian and that the algebra is isomorphic to algebra 12 in [8].

3.2. Five-dimensional algebras. Now let $\mathfrak{g}$ be a five-dimensional indecomposable algebra. The situation now is more complicated than in dimension four for several reasons: firstly the extension may not split; secondly, in the codimension-one nilradical case, the nilradical may be isomorphic to

$\mathbb{R}^4, \mathbb{R} \bigoplus H$ or the four-dimensional indecomposable nilpotent algebra, 4.1 in [8]; thirdly in the codimension-one nilradical case, the nilradical itself may not be abelian.

Of the 40 five-dimensional algebras listed in [8] the following have a non-trivial center and therefore the adjoint representation is not faithful:

$$1\,,\ 2\,,\ 3\,,\ 4\,,\ 5\,,\ 6\,,\ 8c\,,\ 9bc\ (b=0)\,,\ 10\,,\ 14pq\,,\ 15\ (a=0)\,,$$

$$19b\ (a=0)\,, 20\ (a=0)\,,\ 22\,,\ 25b\ (p=0)\,,\ 26\ (\epsilon=\pm1, p=0)\,,$$

$$28\ (a=0)\,,\ 29\,,\ 30\ (a=-1)\,,\ 38\,,\ 39\,.$$

Thus we have 22 cases to consider of which four depend on one parameter and one of which depends on two. Of these algebras the first six are nilpotent of which the first two have codimension-one abelian ldeals. Algebras 7-18 have a codimension-one abelian nilradical. Algebras 19-29 have a codimension-one nilradical that is isomorphic to $\mathbb{R} \bigoplus H$ where H denotes the three-dimensional Heisenberg algebra. As such it follows that the non-zero brackets may be assumed to be of the of the form:

$$\begin{aligned}
[e_2, e_3] &= e_1\,,\\
[e_1, e_5] &= ve_1 + ce_4\,,\\
[e_2, e_5] &= ae_2 + *e_3 + *e_4\,,\\
[e_3, e_5] &= be_2 + *e_3 + *e_4\,,\\
[e_4, e_5] &= *e_1 + *e_2 + *e_3 + de_4\,.
\end{aligned}$$

Here we have used Lemma 2.4 and the fact that the center of the nilradical is an ideal. If we now compute the Jacobi identity for e_2, e_3, e_5 we deduce that $[e_1, e_5] =$ is a multiple of e_1 so that the coefficient c above must be zero and e_1 generates a one-dimensional ideal. It is interesting to observe that the Jacobi identity is now satisfied provided only that we add the condition $v = a + b$.

We continue by dividing out by the ideal generated by e_1. As such using the obvious abuse of notation we change basis in the subspace spanned by e_2, e_3, e_4 but we only allow e_4 to be replaced by a non-zero multiple of itself. We have a four-dimensional algebra with a codimension-one abelian nilradical. In the case of split extensions, which means that the coefficient d above is zero, algebras 5.19, 5.21 and 5.25, correspond to the four-dimensional algebras, 4.5, 4.4 and 4.6, respectively. In the case of algebras 4.2 and 4.3 each of them yield two inequivalent forms given that e_4 can only be changed by a non-zero multiple of itself, namely, 5.23 and 5.28 and 5.22 and 5.29, respectively. The algebras 5.20, 5.24 and 5.26 correspond to 4.5, 4.2 and 4.6 whereas 5.27 corresponds to a decomposable four-dimensional algebra. In the case of the non-split extensions it is possible to reduce the number of parameters by one.

3.3. Six-dimensional algebras with codimension one nilradical. Let us explain how the indecomposable, solvable Lie algebras of dimension six are classified. There is a general inequality that bounds the dimension of the nilradical of a solvable Lie algebra $\mathfrak{g}$ by $\frac{1}{2}\dim(\mathfrak{g})$. According to Turkowski if the dimension of the nilradical of a solvable Lie algebra of dimension six is three, then the algebra is decomposable and hence there are just three subclasses to consider [11]: the nilpotents and the algebras that have a codimension one nilradical or codimension two nilradical, respectively. The latter class of algebras have been obtained in [11] and in [9] we gave matrix representations for them. Morozov and Winternitz et al considered the six-dimensional nilpotent Lie algebras [5, 8] and matrix representations for them were given in [2].

The algebras that have a codimension one nilradical were classified by Mubarakzyanov who found 99 classes of algebras [7]. Mubarakzyanov's paper is divided into nine different cases depending on the possible forms for the nilradical that we denote by $\mathfrak{n}$. Cases 1 to 12 have $\mathfrak{n}$ abelian (Section 1); cases 13 to 38 have $\mathfrak{n}$ isomorphic to the 3 dimensional Heisenberg algebra (Section 2); cases 39 to 53 have $\mathfrak{n}$ isomorphic to the unique 4-dimensional indecomposable nilpotent Lie algebra (Section 3); cases 54 to 70 have $\mathfrak{n}$ isomorphic to the nilpotent Lie algebra 5.1 in (Section 4); cases 71 to 75 have $\mathfrak{n}$ isomorphic to the nilpotent Lie algebra 5.2 in (Section 5); cases 76 to 81 have $\mathfrak{n}$ isomorphic to the nilpotent Lie algebra 5.3 in (Section 6);cases 82 to 93 have $\mathfrak{n}$ isomorphic to the nilpotent Lie algebra 5.4 in (Section 7); cases 94 to 98 have $\mathfrak{n}$ isomorphic to the nilpotent Lie algebra 5.5 in (Section 8); finally case 99 has $\mathfrak{n}$ isomorphic to the nilpotent Lie algebra 5.6 in (Section 9). In his list [7] Mubarakzyanov always uses a basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ for the Lie algebra $\mathfrak{g}$ in which e_6 is *not* in $\mathfrak{n}$.

Representations for algebras 1 to 12 in Mubarakzyanov's list can be obtained from Theorem 2.1. We consider the case where the nilradical is isomorphic to $H \bigoplus \mathbb{R}^2$. The reductions that can made analogously to the five-dimensional case can be summarized by means of the matrix $-\operatorname{ad}(e_5)$ which can be assumed to have the following form:

$$S = \begin{bmatrix} a+b & 0 & 0 & * & * & 0 \\ 0 & a & * & 0 & 0 & 0 \\ 0 & * & b & 0 & 0 & 0 \\ 0 & * & * & * & * & 0 \\ 0 & * & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is not our purpose in this venue to give a comprehensive appraisal of [7] and so we shall not discuss further algebras 13-38. It is apparent that Mubarakzyanov's list is defective. In this article we have chosen to focus on a particularly interesting class of algebras for which the nilradical is isomorphic to the five-dimensional Heisenberg algebra and which correspond to algebras 82-93 in [7]. These algebras provide a particularly nice application of Theorem 2.3.

4. Six-dimensional algebras with Heisenberg nilradical

Following Mubarakzyanov for algebras 82-93 we choose a basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ so that $[e_2, e_4] = e_1, [e_3, e_5] = e_1$ and e_6 is not in the nilradical. We can make two initial simplifications. First of all, e_1 spans the center of the nilradical and as such is an ideal in $\mathfrak{g}$. It follows that $[e_1, e_6] = ae_1$ for some $a \in \mathbb{R}$. Furthermore Lemma 2.4 allows us to assume that $[e_i, e_6]$, $2 \leq i \leq 5$, is a linear combination of $\{e_2, e_3, e_4, e_5\}$.

To further normalize $\mathfrak{g}$ we consider $E_6 := -\mathrm{ad}(e_6)$: its $(1,1)$-entry is a but otherwise, the top and bottom rows and first and last columns consist of zeroes. Mubarakzyanov denotes the central 4×4 block of E_6 by E_6'. Mubarakzyanov modifies E_6 by replacing it by $\tilde{E_6} := E_6 - \frac{a}{2}I$: we could reverse this step, if necessary, after normalizing E_6'. If we now impose the Jacobi identity we find that it is precisely equivalent to the statement that $\tilde{E_6'}$ has the following block form $\left[\begin{smallmatrix} K & L \\ M & -K^t \end{smallmatrix}\right]$ where K is an arbitrary 2×2 matrix, and L and M are arbitrary symmetric 2×2 matrices. Thus far we have reached line 15 of Mubarakzyanov's paper on page 112. Notice that the algebra $\mathfrak{g}$ has been reduced to a dependence on just 11 parameters. Now the theory is closed very nicely by the observation that the matrix $\left[\begin{smallmatrix} K & L \\ M & -K^t \end{smallmatrix}\right]$ is precisely an element of the Lie algebra of the symplectic group. We now make a change of basis in which we leave e_1 and e_6 fixed and we can use the symplectic group in the subspace spanned by $\{e_2, e_3, e_4, e_5\}$ which will preserve the structure of the nilradical and it is the largest group that will do so. Thus the classification of algebras 82-93 is equivalent to classifying the orbits of the adjoint representation of the symplectic group Sp(2).

The argument used above for the Mubarakzyanov algebras 82-93 can be extended easily to algebras of dimension $2n+2$ that have a $2n+1$-dimensional Heisenberg Lie algebra. However, it is convenient to choose as a basis $\{e_0, e_1, \dots, e_{2n+1}\}$ so that e_0 spans the center of the nilradical and $[e_i, e_{n+j}] = \delta_{ij} e_0$ where δ_{ij} denotes the Kronecker delta.

Starting from equation (4) we rewrite the condition as

$$B\left(A - \frac{a}{2}I\right) + \left(A - \frac{a}{2}I\right)^t B = 0\,. \tag{9}$$

In this class of algebras the matrix B is skew-symmetric and non-singular and therefore (9) says that the matrix $A - \frac{a}{2}I$ belongs to the Lie algebra of the symplectic group.

We can make a change of basis of the following form:

$$\overline{e_0} = \alpha e_0\,, \quad \overline{e_i} = p_i^j e_j + *e_0\,, \quad \overline{e_{n+1}} = \beta e_{n+1} + *e_i + *e_0\,,$$

where the asterisks denote unspecified coefficients. As such the transformed non-zero brackets are

$$\begin{aligned} [\overline{e_0}, \overline{e_{n+1}}] &= a\beta \overline{e_0}\,, \\ [\overline{e_i}, \overline{e_j}] &= \frac{1}{\alpha} p_i^k p_j^l B_{ij} \overline{e_0}\,, \\ [\overline{e_i}, \overline{e_{n+1}}] &= \beta p_i^k (p^{-1})_j^l A_k^j \overline{e_l} + *e_0\,. \end{aligned}$$

Of these non-zero brackets we can choose β so that $a = 2$ unless $a = 0$; in the latter case the coefficient β remains at our disposal. In the third class of brackets the matrix is conjugated, scaled by β giving again a symplectic matrix and again the terms involving e_0 may be eliminated using Lemma 2.4. If we choose $\alpha = 1$ we can reduce to the following two cases of non-zero brackets:

$$\begin{aligned} [e_i, e_j] &= B_{ij} e_0\,, \\ [e_i, e_{n+1}] &= \beta A_i^j e_j\,, \\ [e_0, e_{n+1}] &= a e_0 \quad (a = 0, 2)\,, \\ [e_i, e_j] &= B_{ij} e_0\,, \\ [e_i, e_{n+1}] &= A_i^j e_j\,, \end{aligned}$$

where $A = \frac{a}{2}I + \sigma$ $(a = 0, 2)$ and σ is an element of the Lie algebra of the symplectic group.

5. Representations of the Mubarakzyanov algebras

We now give the list of normalized Lie algebras corresponding to the Mubarakzyanov algebras 82-93: our numbering is chosen so that it corresponds as far as is practicable to the list in [7]. We have made corrections and simplified the notation in several cases. We give a 6×6 matrix which parametrizes a Lie group whose matrix Lie algebra coincides with each of of the algebras. Some cases can be subdivided so that they correspond to special values of the non-group parameters, denoted variously by α,a and b below. For each of of the algebras we give also a basis for the right-invariant vector fields. The representations provide an illustration of Theorem 2.3 in practice.

$$\begin{aligned}[6.82]:\quad & [e_2,e_4]=e_1\,, \qquad [e_3,e_5]=e_1\,, \qquad [e_1,e_6]=2e_1\,,\\ & [e_2,e_6]=(a+1)e_2\,, \qquad [e_3,e_6]=(b+1)e_3\,,\\ & [e_4,e_6]=(1-a)e_4\,, \qquad [e_5,e_6]=(1-b)e_5 \quad (\alpha=1)\,.\end{aligned}$$

$$\begin{aligned}[6.82]':\quad & [e_2,e_4]=e_1\,, \quad [e_3,e_5]=e_1\,, \quad [e_2,e_6]=e_2\,, \quad [e_3,e_6]=ae_3\,,\\ & [e_4,e_6]=-e_4\,, \qquad [e_5,e_6]=-ae_5 \quad (\alpha=0\,,\ a=1\,,\ b=a)\,.\end{aligned}$$

$$\begin{bmatrix} e^{2\alpha w} & e^{(\alpha+a)w}z & e^{(\alpha+b)w}q & 0 & 0 & p\\ 0 & e^{(\alpha+a)w} & 0 & 0 & 0 & x\\ 0 & 0 & e^{(\alpha+b)w} & 0 & 0 & y\\ 0 & 0 & 0 & e^{(\alpha-a)w} & 0 & z\\ 0 & 0 & 0 & 0 & e^{(\alpha-b)w} & q\\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Right-invariant vector fields:

$$\begin{aligned}&\big(D_p\,, D_x\,, D_y\,, D_z+xD_p\,, D_q+yD_p\,,\\ &\quad D_w+(\alpha-b)qD_q+(\alpha+a)xD_x+(\alpha+b)yD_y+(\alpha-a)zD_z2\alpha pD_p\big)\,.\end{aligned}$$

$$
\begin{aligned}
[6.83]: \quad & [e_2,e_4]=e_1\,, \qquad [e_3,e_5]=e_1\,, \qquad [e_1,e_6]=2e_1\,,\\
& [e_2,e_6]=(a+1)e_2+e_3\,, \qquad [e_3,e_6]=(a+1)e_3\,,\\
& [e_4,e_6]=(1-a)e_4\,, \qquad [e_5,e_6]=(1-a)e_5-e_4 \quad (\alpha=1)\,.
\end{aligned}
$$

$$
\begin{aligned}
[6.83]': \quad & [e_2,e_4]=e_1\,, \quad [e_3,e_5]=e_1\,, \quad [e_2,e_6]=e_2+e_3\,, \quad [e_3,e_6]=e_3\,,\\
& [e_4,e_6]=-e_4\,, \qquad [e_5,e_6]=-e_4-e_5 \quad (\alpha=0\,,\ a=1)\,.
\end{aligned}
$$

$$
\begin{bmatrix}
e^{2\alpha w} & e^{(\alpha+a)w}z & e^{(\alpha+a)w}(y-zw) & 0 & 0 & p\\
0 & e^{(\alpha+a)w} & -we^{(\alpha+a)w} & 0 & 0 & x\\
0 & 0 & e^{(\alpha+a)w} & 0 & 0 & q\\
0 & 0 & 0 & e^{(\alpha-a)w} & we^{(\alpha-a)w} & y\\
0 & 0 & 0 & 0 & e^{(\alpha-a)w} & z\\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

Right-invariant vector fields:

$$
\begin{aligned}
\big(D_p\,, -D_q\,, D_x\,, -(D_y+qD_p)\,, D_z+xD_p\,,&\\
D_w+(a+l)qD_q+((a+l)x-q)D_x&\\
+(z+(a-l)y)D_y+(a-l)zD_z+2apD_p\big)\,.&
\end{aligned}
$$

$$
\begin{aligned}
[6.84]: \quad & [e_2,e_4]=e_1\,, \quad [e_3,e_5]=e_1\,, \quad [e_2,e_6]=e_2\,,\\
& [e_4,e_6]=-e_4\,, \qquad [e_5,e_6]=e_3\,.
\end{aligned}
$$

$$
S=\begin{bmatrix}
1 & w & \frac{w^2}{2} & e^z q & x\\
0 & 1 & w & 0 & y\\
0 & 0 & 1 & 0 & z\\
0 & 0 & 0 & e^z & p\\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

Right-invariant vector fields:

$$
\big(-D_x\,, -D_p\,, D_y\,, D_q+pD_x\,, -(D_w+yD_x+zD_y)\,, D_z+pD_p-qD_q\big)\,.
$$

$$[6.85]: \quad [e_2,e_4]=e_1\,,\quad [e_3,e_5]=e_1\,,\quad [e_1,e_6]=2e_1\,,\quad [e_2,e_6]=(a+1)e_2\,,$$
$$[e_3,e_6]=e_3\,,\quad [e_4,e_6]=(1-a)e_4\,,\quad [e_5,e_6]=\pm e_3+e_5\,.$$

$$\begin{bmatrix} e^{2w} & ye^{(a+1)w} & -ze^{w} & qe^{(1-a)w} & (x\pm zw)e^{w} & p \\ 0 & e^{(a+1)w} & 0 & 0 & 0 & q \\ 0 & 0 & e^{w} & 0 & \mp we^{w} & x \\ 0 & 0 & 0 & e^{(1-a)w} & 0 & y \\ 0 & 0 & 0 & 0 & e^{w} & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Right-invariant vector fields:

$$\big(2D_p\,, D_q+yD_p\,, -(D_x+zD_p)\,, D_y+qD_p\,, D_z-xD_p\,,$$
$$D_w+2pD_p+(a+1)qD_q+(x\mp z)D_x+zD_z+(1-a)yD_y\big)\,.$$

$$[6.86]: \quad [e_2,e_4]=e_1\,,\quad [e_3,e_5]=e_1\,,\quad [e_1,e_6]=2e_1\,,\quad [e_2,e_6]=e_2\,,$$
$$[e_3,e_6]=e_2+e_3\,,\quad [e_4,e_6]=e_4-e_5\,,\quad [e_5,e_6]=e_5\,.$$

$$\begin{bmatrix} e^{2w} & -ze^{w} & -(y+z+zw)e^{w} & xe^{w} & (q-x-xw)e^{w} & p \\ 0 & e^{w} & we^{w} & 0 & 0 & q \\ 0 & 0 & e^{w} & 0 & 0 & x \\ 0 & 0 & 0 & e^{w} & -we^{w} & y \\ 0 & 0 & 0 & 0 & e^{w} & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Right-invariant vector fields:

$$\big(2D_p\,, -D_x+(z-y)D_p\,, -(zD_p+D_q)\,, D_y-xD_p\,, D_z-(q+x)D_p\,,$$
$$D_w+2pD_p+(q+x)D_q+xD_x+zD_z+(y-z)D_y\big)\,.$$

$[6.86']: \quad [e_2,e_4]=e_1\,,\quad [e_3,e_5]=e_1\,,\quad [e_1,e_6]=2e_1\,,\quad [e_2,e_6]=e_2\,,\quad [e_3,e_6]=e_3\mp e_5\,,\quad [e_4,e_6]=e_4\pm e_2\,,\quad [e_5,e_6]=e_5\,.$

$$\begin{bmatrix} e^{2\,w} & -ye^w & ze^w & (q\mp y(w+1))e^w & -(x\pm z(w+1))e^w & p \\ 0 & e^w & 0 & \pm we^w & 0 & q \\ 0 & 0 & e^w & 0 & \mp we^w & x \\ 0 & 0 & 0 & e^w & 0 & y \\ 0 & 0 & 0 & 0 & e^w & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Right-invariant vector fields:

$$\big(-2D_p\,,\ D_q+yD_p\,,\ D_z+(x\mp z)D_p\,,\ D_y-(q\pm y)D_p\,,\ D_x-zD_p, D_w+2pD_p+(q\pm y)D_q+(x\mp z)D_x+yD_y+zD_z\big)\,.$$

$[6.87]: \quad [e_2,e_4]=e_1\,,\ [e_3,e_5]=e_1\,,\ [e_1,e_6]=2e_1\,,\ [e_2,e_6]=e_2\pm e_5\,,\ [e_3,e_6]=e_3\pm e_4\,,\ [e_4,e_6]=e_4\,,\ [e_5,e_6]=\pm e_3+e_5\,.$

$$\begin{bmatrix} e^{2\,w} & qe^w & (\pm q(w+1)-z)e^w & \Big(x+(w+1)(q\mp z)+\frac{qw^2}{2}\Big)e^w & \Big((w+1)(\pm(x+q)-z)-y+\frac{(\pm q-z)w^2}{2}+\frac{\pm qw^3}{6}\Big)e^w & p \\ 0 & e^w & \pm we^w & \frac{w^2e^w}{2} & frac\pm w^3e^w6 & y \\ 0 & 0 & e^w & \pm we^w & \frac{w^2e^w}{2} & x \\ 0 & 0 & 0 & e^w & \pm we^w & z \\ 0 & 0 & 0 & 0 & e^w & q \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Right-invariant vector fields:

$$\big(-2D_p\,,\ yD_p\pm D_z+D_q\,,\ D_x+zD_p\pm D_y\,, D_y-qD_p\,,\ D_z-xD_p\pm D_x\,,$$
$$D_w+2pD_p+qD_q+(x\pm z)D_x+(y\pm x)D_y+(z\pm q)D_z\big)\,.$$

$$
\begin{aligned}
[6.88]:\quad & [e_2,e_4]=e_1\,,\qquad [e_3,e_5]=e_1\,,\qquad [e_1,e_6]=2e_1\,,\\
& [e_2,e_6]=(a+1)e_2-be_3\,,\quad [e_3,e_6]=be_2+(a+1)e_3\,,\\
& [e_4,e_6]=(1-a)e_4-be_5\,,\quad [e_5,e_6]=be_4+(1-a)e_5\quad(\alpha=1)\,.
\end{aligned}
$$

$$
\begin{aligned}
[6.88]':\quad & [e_2,e_4]=e_1\,,\ [e_3,e_5]=e_1\,,\ [e_2,e_6]=ae_2-e_3\,,\ [e_3,e_6]=e_2+ae_3\,,\\
& [e_4,e_6]=-ae_4-e_5\,,\qquad [e_5,e_6]=e_4-ae_5\quad(\alpha=0\,,\ b=1)\,.
\end{aligned}
$$

$$
\begin{bmatrix}
e^{2\alpha w} & q & z & 0 & 0 & p\\
0 & e^{(\alpha-a)w}\cos(bw) & -e^{(\alpha-a)w}\sin(bw) & 0 & 0 & x\\
0 & e^{(\alpha-a)w}\sin(bw) & e^{(\alpha-a)w}\cos(bw) & 0 & 0 & y\\
0 & 0 & 0 & e^{(\alpha-a)w}\cos(bw) & -e^{(\alpha-a)w}\sin(bw) & 0\\
0 & 0 & 0 & e^{(\alpha-a)w}\sin(bw) & e^{(\alpha-a)w}\cos(bw) & 0\\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

Right-invariant vector fields:

$$
\begin{aligned}
\big(&-D_p\,, D_p-\sin(bw)D_z+\cos(bw)D_q\,,\ yD_p+\sin(bw)D_q+\cos(bw)D_z\,,\ D_x\,,\ D_y\,,\\
&D_w+(\alpha+a)(qD_q+zD_z)+((\alpha-a)x-by)D_x+((\alpha-a)y+bx)D_y+2\alpha pD_p\big)\,.
\end{aligned}
$$

$$
\begin{aligned}
[6.89]:\quad & [e_2,e_4]=e_1\,,\ [e_3,e_5]=e_1\,,\ [e_1,e_6]=2e_1\,,\ [e_2,e_6]=(a+1)e_2\,,\\
& [e_3,e_6]=e_3+be_5,\ [e_4,e_6]=(1-a)e_4,\ [e_5,e_6]=-be_3+e_5\ \ (\alpha=1)\,.
\end{aligned}
$$

$$
\begin{aligned}
[6.89]':\quad & [e_2,e_4]=e_1\,,\quad [e_3,e_5]=e_1\,,\quad [e_2,e_6]=e_2\,,\quad [e_3,e_6]=ae_5\,,\\
& [e_4,e_6]=-e_4\,,\quad [e_5,e_6]=-ae_3\ \ (\alpha=0\,,\ a=1\,,\ b=a)\,.
\end{aligned}
$$

$$
\begin{bmatrix}
e^{2\alpha w} & 0 & xe^{(\alpha-a)w} & \Gamma_1 & \Lambda_1 & z\\
0 & e^{(\alpha+a)w} & 0 & 0 & 0 & x\\
0 & 0 & e^{(\alpha-a)w} & 0 & 0 & y\\
0 & 0 & 0 & e^{\alpha w}\cos(bw) & e^{\alpha w}\sin(bw) & p\\
0 & 0 & 0 & -e^{\alpha w}\sin(bw) & e^{\alpha w}\cos(bw) & q\\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

where $\Gamma_1=-e^{\alpha w}\big(q\cos(bw)+p\sin(bw)\big)$, $\Lambda_1=e^{\alpha w}\big(-q\sin(bw)+p\cos(bw)\big)$.

Right-invariant vector fields:

$$\big(D_z\,,\ -2(D_x+yD_z)\,,\ D_q-pD_z\,,\ \tfrac{1}{2}D_y\,,\ D_p+qD_z\,,\\ D_w+(\alpha p+bq)D_p+(\alpha q-bp)D_q+x(\alpha+a)D_x+y(\alpha-a)D_y+2\alpha zD_z\big)\,.$$

$$[6.90]:\quad [e_2,e_4]=e_1\,,\quad [e_3,e_5]=e_1\,,\quad [e_1,e_6]=2e_1\,,\quad [e_2,e_6]=e_2\,,\\ [e_3,e_6]=e_3+ae_5\,,\quad [e_4,e_6]=e_4\pm e_2\,,\quad [e_5,e_6]=-ae_3+e_5\,.$$

$$\begin{bmatrix} e^{2w} & \Gamma_2 & \Lambda_2 & -qe^w & (z+qw)e^w & p\\ 0 & \cos(aw)e^w & \sin(aw)e^w & 0 & 0 & x\\ 0 & -\sin(aw)e^w & \cos(aw)e^w & 0 & 0 & y\\ 0 & 0 & 0 & e^w & -we^w & z\\ 0 & 0 & 0 & 0 & e^w & q\\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\Gamma_2=e^w(\cos(aw)(y-x)+\sin(aw)(x+y))$, $\Lambda_2=e^w(\sin(aw)(y-x)-\cos(aw)(x+y))$. Right-invariant vector fields:

$$\big(-2D_p\,,D_z+qD_p\,,\ D_y+(x-y)D_p\,,\ D_q-zD_p\,,\ D_x-(x+y)D_p\,,\\ D_w+2pD_p+qD_q+(x+ay)D_x+(y-ax)D_y+(z-q)D_z\big)\,.$$

$$[6.90]':\quad [e_2,e_4]=e_1\,,\quad [e_3,e_5]=e_1\,,\quad [e_3,e_6]=e_5\,,\\ [e_4,e_6]=\pm e_2\,,\quad [e_5,e_6]=-e_3\,.$$

$$\begin{bmatrix} 1 & -y\cos(w)+x\sin(w) & y\sin(w)+x\cos(w) & z & \frac{z^2}{2} & p\\ 0 & \cos(w) & -\sin(w) & 0 & 0 & x\\ 0 & \sin(w) & \cos(w) & 0 & 0 & y\\ 0 & 0 & 0 & 1 & z & \pm q\\ 0 & 0 & 0 & 0 & 1 & w\\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Right-invariant vector fields:

$$\Big(-D_p\,,\ -D_q\,,\ \tfrac{1}{\sqrt{2}}(D_y-xD_p)\,,D_z+qD_p\pm wD_q\,,\\ -\tfrac{1}{\sqrt{2}}(yD_p+D_x)\,,\ D_w-yD_x+xD_y\Big)\,.$$

$[6.92]: \quad [e_2, e_4] = e_1\,, \quad [e_3, e_5] = e_1\,, \quad [e_1, e_6] = 2e_1\,, \quad [e_2, e_6] = e_2 - ae_4\,, \quad [e_3, e_6] = e_3 - be_5\,,$
$[e_4, e_6] = e_4 + ae_2\,, \quad [e_5, e_6] = e_5 + be_3 \;\; (\alpha = 1)\,.$

$[6.92]': \quad [e_2, e_4] = e_1\,, \quad [e_3, e_5] = e_1\,, \quad [e_2, e_6] = -e_4\,, \quad [e_3, e_6] = -ae_5\,, \quad [e_4, e_6] = e_2\,, \quad [e_5, e_6] = ae_3 \;\; (\alpha = 0\,,\ b = a)\,.$

$$\left[\begin{array}{cccccc} e^{2\,\alpha w} & \big(z\cos(bw) + q\sin(bw)\big)e^{\alpha w} & \big(z\sin(bw) - q\cos(bw)\big)e^{\alpha w} & \big(y\cos(aw) + x\sin(aw)\big)e^{\alpha w} & \big(y\sin(aw) - x\cos(aw)\big)e^{\alpha w} & p \\ 0 & e^{\alpha w}\cos(bw) & e^{\alpha w}\sin(bw) & 0 & 0 & q \\ 0 & -e^{\alpha w}\sin(bw) & e^{\alpha w}\cos(bw) & 0 & 0 & z \\ 0 & 0 & 0 & e^{\alpha w}\cos(aw) & e^{\alpha w}\sin(aw) & x \\ 0 & 0 & 0 & -e^{\alpha w}\sin(aw) & e^{\alpha w}\cos(aw) & y \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Right-invariant vector fields:

$$\big(2D_p\,,\ D_z - xD_p\,,\ D_q - yD_p\,,\ D_x + zD_p\,,\ D_y + qD_p\,,\ D_w + 2pD_p + (q + by)D_q + (az + x)D_x + (y - bq)D_y + (z - ax)D_z\big)\,.$$

$[6.93]: \quad [e_2, e_4] = e_1\,, \quad [e_3, e_5] = e_1\,, \quad [e_1, e_6] = 2\alpha e_1\,, \quad [e_2, e_6] = \alpha e_2 + ae_3\,, \quad [e_3, e_6] = \alpha e_3 - ae_2\,,$
$[e_4, e_6] = \alpha e_4 \pm e_2 + ae_5\,, \quad [e_5, e_6] = \alpha e_5 \pm e_3 - ae_4\,.$

$$\left[\begin{array}{cccccc} e^{2\,\alpha w} & -\big(q\sin(aw) + z\cos(aw)\big)e^{\alpha w} & \Gamma_3 & \big((y \mp qw)\sin(aw) + (x \mp zw)\cos(aw)\big)e^{\alpha w} & \Lambda_3 & p \\ 0 & \cos(aw)e^{\alpha w} & -\sin(aw)e^{\alpha w} & \pm w\cos(aw)e^{\alpha w} & -w\sin(aw)e^{\alpha w} & x \\ 0 & \sin(aw)e^{\alpha w} & \cos(aw)e^{\alpha w} & w\sin(aw)e^{\alpha w} & \pm w\cos(aw)e^{\alpha w} & y \\ 0 & 0 & 0 & \cos(aw)e^{\alpha w} & -\sin(aw)e^{\alpha w} & z \\ 0 & 0 & 0 & \sin(aw)e^{\alpha w} & \cos(aw)e^{\alpha w} & q \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

where $\Gamma_3 = \big(z\sin(aw) - q\cos(aw)\big)e^{\alpha w}$, $\Lambda_3 = \big((y \mp qw)\cos(aw) + (\pm zw - x)\sin(aw)\big)e^{\alpha w}$. Right-invariant vector fields:
$\big(-2D_p, D_x + zD_p, D_y + qD_p, D_z - xD_p, D_q - yD_p, D_w + 2\alpha pD_p + (\alpha q + az)D_q + (\alpha x - ay \pm z)D_x + (\alpha y + ax \pm q)D_y + (\alpha z - aq)D_z\big)\,.$

Remark. It is to be noted that the last Lie algebra has been corrected from the version that appears in [7] where there is an obvious error.

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