EXTRACTA MATHEMATICAE Vol. **23**, Núm. 1, 43 – 48 (2008)

A Note on the Stability of Linear Combinations of Algebraic Operators

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Presented by Mostafa Mbekhta Received April 9, 2008

Abstract: The main purpose of this note is to characterize all the algebraic operators S and T having the same minimal polynomial and for which certain spectral properties of linear combinations of S and T do not depend on their coefficients.

Key words: Stability, invertibility, algebraic operator.

AMS *Subject Class.* (2000): 47A05.

1. Introduction

Let X be a Banach space, and T, S two idempotent operators on X. Several papers (see [2, 5] and the references within) have addressed stability properties of the linear combination $c_1T + c_2S$; it has been proved that a large number of properties (e.g., injectivity, invertibility, Fredholmness) are shared by all such linear combinations, provided $c_1, c_2 \neq 0$ and $c_1 + c_2 \neq 0$.

An idempotent T is defined by the relation $T^2 = T$; in other words, it is an algebraic operator, and its minimal polynomial (except in trivial cases) is $p(z) = z^2 - z$. A natural question is whether the stability results above can be extended to more general situations. Thus, we may consider two algebraic operators T, S with the same minimal polynomial p, and look for similar stability results. We will show below that essentially there is no such extension; in other words, these properties of idempotents are rather special. The situation is the same even if we restrict ourselves to matrices instead of operators.

On the positive side, if we assume that the two operators T, S commute, then we can easily obtain stability results of the type discussed, even if their

minimal polynomials are different. This is a consequence of (multidimensional) spectral theory.

2. Main result

As in [2], instead of $c_1T + c_2S$ we will rather consider the operator $T - zS$, and thus work with a single parameter z.

THEOREM 2.1. *Let p be a unital polynomial of degree $d \geq 1$. The following assertions are equivalent:*

a) *$p(z) = z - a$ or $p(z) = z^2 - bz$ where $b \neq 0$;*

b) *there exists a finite set F such that for all matrices S, T whose minimal polynomial is p, $z \mapsto \dim\ker(T - zS)$ is constant on $\mathbb{C} \setminus F$.*

Proof. *a) $\Rightarrow$ b)*: If $p(z) = z - a$, then $T - zS = (1 - z)aI$ and the result is obvious with $F = \{1\}$. If $p(z) = (z^2 - bz)$ with $b \neq 0$, then S/b and T/b are idempotents. Since $\dim\ker(T - zS) = \dim\ker(T/b - zS/b)$, using the main result of [2] or [5], we obtain the statement with $F = \{0, 1\}$.

b) $\Rightarrow$ a): Suppose p is not of the required form. We will discuss the several possible cases.

I. Degree of $p = 2$.

Ia. If $p(z) = (z-a)(z-b)$ with $a, b \in \mathbb{C} \setminus \{0\}$ and $a \neq b$, take $S_{a,b} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $T_\theta = R_\theta S_{a,b} R_\theta^{-1}$ where $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. The minimal polynomial of $S_{a,b}$ and T_θ is $p(z) = (z - a)(z - b)$ since $S_{a,b}$ and T_θ are unitarily equivalent. The determinant of $T_\theta - zS_{a,b}$ is equal to

$$d(z) = abz^2 - z(2ab + (a - b)^2 \sin^2\theta) + ab.$$

We have $\dim\ker(T_\theta - zS_{a,b}) = 0$ if z is not a root of $d(z)$; since the set of values of these roots, when $\theta \in [0, 2\pi)$, is infinite, there is no set F as required.

Ib. If $p(z) = (z - a)^2$ with $a \neq 0$, consider $S_a = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ and $T_\theta = R_\theta S R_\theta^{-1}$. The minimal polynomial of S_a and T_θ is $p(z) = (z - a)^2$. The determinant of $T_\theta - zS_a$ is equal to

$$d(z) = a^2z^2 - z(\sin\theta + 2a^2) + a^2.$$

As above, the set of its roots is infinite when $\theta \in [0, 2\pi)$.

Ic. If $p(z) = z^2$, take

$$S_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $T_\theta = U_\theta S_0 U_\theta^{-1}$ where U_θ is the unitary matrix defined by

$$U_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$

Since we have

$$T_\theta - zS_0 = \begin{pmatrix} 0 & \cos\theta - z & 0 & -\sin\theta \\ 0 & 0 & 0 & 0 \\ 0 & \sin\theta & 0 & \cos\theta - z \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

the dimension of $\ker(T_\theta - zS_0)$ is 2 for all $z \in \mathbb{C} \setminus \{e^{i\theta}, e^{-i\theta}\}$ and is 3 or 4 (for $\theta = k\pi$, $k \in \mathbb{Z}$) otherwise. Therefore, in each of the above cases, there is no finite set F such that for all $z \in \mathbb{C} \setminus F$ the dimension of $\ker(T - zS)$ is constant independently of the choice of S and T.

II. Degree of $p \geq 3$.

IIa. Suppose that the roots of p are all distinct. Then p has at least two nonzero distinct roots a, b. Consider $S = S_{a,b} \oplus A$ and $T_\theta = R_\theta S_{a,b} R_\theta^{-1} \oplus A$, where A is a matrix whose minimal polynomial is p. As for the case $p(z) = (z-a)(z-b)$, considering $T_\theta - zS$, there is no finite set F on which $z \mapsto \dim\ker(T_\theta - zS)$ is constant on $\mathbb{C} \setminus F$.

IIb. If p has a root a of multiplicity at least 2, take A an arbitrary matrix whose minimal polynomial is p. Consider $S = S_a \oplus A$, $T_\theta = R_\theta S_a R_\theta^{-1} \oplus A$ if $a \neq 0$, and $S = S_0 \oplus A$, $T_\theta = U_\theta S_0 U_\theta^{-1}$ if $a = 0$. As above, we obtain that there exists no finite set F, independent of the choice of S and T, on which $z \mapsto \dim\ker(T_\theta - zS)$ is constant on $\mathbb{C} \setminus F$. ∎

Remark 2.2. The remarkable property of a pair of idempotents cannot be extended to more than two. One might hope for instance that if P, Q, R are three idempotents, then $\dim\ker(P + zQ + wR)$ is constant outside a fixed algebraic variety (not depending on the idempotents). But this is easily seen not to be true. Indeed, denote $P_t = \begin{pmatrix} \cos^2 t & \cos t \sin t \\ \cos t \sin t & \sin^2 t \end{pmatrix}$. Then the determinant of $P_0 + zP_t + wP_\theta$ is $z\sin^2 t + w\sin^2\theta + zw\sin^2(t-\theta)$, whose zero set is not independent of t and θ.

3. Commuting operators

As opposed to the general case, it is rather simple to obtain stability if the two operators $T, S \in \mathcal{L}(X)$ commute.

Remember that the left spectrum $\sigma^l(T,S)$ is defined as the set of $(z,w) \in \mathbb{C}^2$ for which $T - zI$ and $S - wI$ generate a proper left ideal of $\mathcal{L}(X)$. A similar definition gives the right spectrum $\sigma^r(T,S)$, while the Harte spectrum is $\sigma^H(T,S) = \sigma^l(T,S) \cup \sigma^r(T,S)$. We have then the spectral mapping theorem [3]:

Lemma 3.1. *If $f : U \to \mathbb{C}$ is holomorphic on an open set $U \subset \mathbb{C}^2$ containing $\sigma^H(T,S)$, then $\sigma^l(f(T,S)) = f(\sigma^l(T,S))$, $\sigma^r(f(T,S)) = f(\sigma^r(T,S))$, and $\sigma^H(f(T,S)) = f(\sigma^H(T,S))$.*

Theorem 3.2. *Suppose $T, S \in \mathcal{L}(X)$ are two commuting algebraic operators, with corresponding minimal polynomials p, q. Suppose that the roots of p are λ_i, $i = 1, \dots, m$ and those of q are μ_j, $j = 1, \dots, n$. Define the set $F = \{\frac{\lambda_i}{\mu_j} : i = 1, \dots, m, j = 1, \dots, n, \mu_j \neq 0\}$. Then, for all $z \notin F$, $T - zS$ is simultaneously left invertible or not.*

Proof. Applying Lemma 3.1 to the function $f(\lambda, \mu) = \lambda - z\mu$, it follows that $T - zS$ is left invertible if and only if $\lambda - z\mu \neq 0$ for all $(\lambda, \mu) \in \sigma^l(T,S)$. If $(0,0) \in \sigma^l(T,S)$, then this last condition is not satisfied for any z, and thus $T - zS$ is not invertible for all $z \in \mathbb{C}$.

Suppose now $(0,0) \notin \sigma^l(T,S)$. Take $(\lambda, \mu) \in \sigma^l(T,S)$. If $\mu = 0$, then $\lambda \neq 0$, and thus $\lambda - z\mu \neq 0$; therefore $T - zS$ is left invertible. If $\mu \neq 0$, but $\lambda - z\mu = 0$, then $z = \frac{\lambda}{\mu}$. Since $\sigma^l(T,S) \subset \sigma^l(T) \times \sigma^l(S)$, it follows that $z \in F$. Therefore $T - zS$ is left invertible for any $z \notin F$. ∎

Remark 3.3. Note that if T, S are commuting algebraic operators, then $T - zS$ is also algebraic, since the algebras generated by T and S are finite

dimensional, while the algebra generated by $T - zS$ is contained in their product. As the spectrum of an algebraic operator is equal to its point spectrum, injectivity is equivalent to either left, right or simple invertibility, or boundedness below (they are all equivalent to the fact that $0 \notin \sigma(T)$). One can therefore reformulate Theorem 3.2 in each of these terms.

An operator $T \in \mathcal{L}(X)$ is called semi-Fredholm if its range $R(T)$ is closed and either $X/R(T)$ or $\ker T$ have finite dimension, and Fredholm if both have finite dimension. More precisely, it is upper semi-Fredholm if $\dim \ker T < \infty$ and lower semi-Fredholm if $\dim X/R(T) < \infty$. Also, T upper semi-Fredholm implies T left essentially invertible, T lower semi-Fredholm implies T right essentially invertible, and T Fredholm implies T essentially invertible ("essentially" meaning modulo compact operators). A procedure introduced in [6, 1, 4] allows us to extend the results above to these classes. Namely, if X is a Banach space, one can define the spaces

$$\begin{aligned} \ell^\infty(X) &= \{\mathbf{x} = (x_n) : x_n \in X,\ \sup \|x_n\| < \infty\}, \\ \tau(X) &= \{\mathbf{x} \in \ell^\infty(X) : \{x_n : n \in \mathbb{N}\} \text{ is totally bounded in } X\}, \\ \tilde{X} &= \ell^\infty(X)/\tau(X), \end{aligned}$$

and one has the following result [6, 1, 4]:

PROPOSITION 3.4. *If $T \in \mathcal{L}(X)$, then T is upper semi-Fredholm if and only if $\tilde{T}$ is injective.*

If T is algebraic then $\tilde{T}$ is also algebraic (with the same minimal polynomial), and Remark 3.3 applies to $\tilde{T}, \tilde{S}$. We obtain thus the following corollary.

COROLLARY 3.5. *With the above notation, for all $z \notin F$ the operator $T - zS$ is simultaneously lower semi-Fredholm, upper semi-Fredholm, Fredholm, left essentially invertible, right essentially invertible, essentially invertible.*

We may compare Theorem 3.2, Remark 3.3 and Corollary 3.5 with the Main Theorem in [2], or with [5, Theorem 3.1].

ACKNOWLEDGEMENTS

The authors wish to thank Mostafa Mbekhta for drawing their attention to this problem.

References

[1] J. J. Buoni, R. Harte, T. Wickstead, Upper and lower Fredholm spectra. I., *Proc. Amer. Math. Soc.* **66** (2) (1977), 309 – 314.

[2] H.-K. Du, C.-Y. Deng, M. Mbekhta, V. Müller, On spectral properties of linear combinations of idempotents, *Studia Math.* **180** (3) (2007), 211 – 217.

[3] R. Harte, A spectral mapping theorem for holomorphic functions. *Math. Z.* **154** (1) (1977), 67 – 69.

[4] R. Harte, "Invertibility and Singularity for Bounded Linear Operators", Monographs and Textbooks in Pure and Applied Mathematics, 109, Marcel Dekker Inc., New York, 1988.

[5] J. J. Koliha, V. Rakočević, Stability theorems for linear combinations of idempotents, *Integral Equations Operator Theory* **58** (4) (2007), 597 – 601.

[6] B. N. Sadovskiĭ, Limit-compact and condensing operators, *Uspehi Mat. Nauk* **27** (1(163)) (1972), 81 – 146.