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# Lifting Infinitesimal Automorphisms to Higher Order Adapted Frame Bundles

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*Abstract*: We describe all $\mathcal{F}ol_{m,n}$-natural operators $\mathcal{A}$ lifting infinitesimal automorphisms $X$ on foliated $(m+n)$-dimensional manifolds $(M,\mathcal{F})$ with $n$-dimensional foliations $\mathcal{F}$ into vector fields $\mathcal{A}(X)$ on the $r$-th order adapted frame bundle $P^r(M,\mathcal{F})$. Next, we describe all $\mathcal{F}ol_{m,n}$-natural affinors on $P^r(M,\mathcal{F})$.

*Key words*: foliated manifold, infinitesimal automorphism, natural operator, natural affinor, higher order adapted frame bundle.

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## 0. Introduction

The present paper is devoted to extend results from our previous papers [4] and [3] to similar results for foliated manifolds instead of manifolds. We modify and joint in respective way the texts of papers [4] and [3]. All manifolds and maps are assumed to be of class $\mathcal{C}^\infty$.

The notion on foliated manifolds can be found in many papers, e.g. [5]. Let $\mathcal{F}ol_{m,n}$ denote the category of all $(m+n)$-dimensional foliated manifolds with $n$-dimensional foliations and their foliation respecting local diffeomorphisms. Let $(M,\mathcal{F})$ be a $\mathcal{F}ol_{m,n}$-object. We have the $r$-th order adapted frame bundle

$$P^r(M,\mathcal{F}) = \{j^r_0\varphi \mid \varphi : (\mathbf{R}^{m+n},\mathcal{F}^{m,n}) \to (M,\mathcal{F}) \text{ is a } \mathcal{F}ol_{m,n}\text{-map}\}$$

over $M$ of $(M,\mathcal{F})$ with the target projection, where $\mathcal{F}^{m,n} = \{\{a\}\times\mathbf{R}^n\}_{a\in\mathbf{R}^m}$ is the standard $n$-dimensional foliation on $\mathbf{R}^{m+n}$. Clearly, $P^r(M,\mathcal{F})$ is a principal bundle with the group $G^r_{m,n} = P^r(\mathbf{R}^{m+n},\, \mathcal{F}^{m,n})_0$ (with the multiplication given by the composition of jets) acting on the right on $P^r(M,\mathcal{F})$ by the composition of jets. Every $\mathcal{F}ol_{m,n}$-map $\psi : (M_1,\mathcal{F}_1) \to (M_2,\mathcal{F}_2)$ can be extended (via composition of jets) into principal bundle (local) isomorphism

$P^r\psi : P^r(M_1, \mathcal{F}_1) \to P^r(M_2, \mathcal{F}_2)$ covering $\psi$ given by $P^r\psi(j_0^r\varphi) = j_0^r(\psi \circ \varphi)$ . Thus we have the bundle functor $P^r : \mathcal{F}ol_{m,n} \to \mathcal{PB}_m(G^r_{m,n})$ in the sense of [1].

Let $(M, \mathcal{F})$ be a $\mathcal{F}ol_{m,n}$-object. A vector field $X$ on $M$ is called an *infinitesimal automorphism* of $(M, \mathcal{F})$ if its flow is formed by local $\mathcal{F}ol_{m,n}$-maps $(M, \mathcal{F}) \to (M, \mathcal{F})$ or (equivalently) if $[X, Y]$ is tangent to $\mathcal{F}$ for any vector field $Y$ tangent to $\mathcal{F}$. The space $\mathcal{X}(M, \mathcal{F})$ of all infinitesimal automorphisms of $(M, \mathcal{F})$ is a Lie subalgebra in $\mathcal{X}(M)$.

The general concept of natural operators can be found in [1]. In this paper we need the following partial definition.

Definition 1. A $\mathcal{F}ol_{m,n}$-natural operator $\mathcal{A} : T_{Inf-Aut} \rightsquigarrow TP^r$ is a family of $\mathcal{F}ol_{m,n}$-invariant regular operators (functions)

$$\mathcal{A} = \mathcal{A}_{(M,\mathcal{F})} : \mathcal{X}(M, \mathcal{F}) \to \mathcal{X}(P^r(M, \mathcal{F}))$$

for any $\mathcal{F}ol_{m,n}$-object $(M, \mathcal{F})$. (Of course, for some $(M, \mathcal{F})$ one can have $\mathcal{X}(M, \mathcal{F}) = \emptyset$; then $\mathcal{A}_{(M,\mathcal{F})} = \emptyset$.) The invariance means that if $X_1 \in \mathcal{X}(M_1, \mathcal{F}_1)$ and $X_2 \in \mathcal{X}(M_2, \mathcal{F}_2)$ are two related infinitesimal automorphisms of $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ (respectively) by a $\mathcal{F}ol_{m,n}$-map $\psi : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ then $\mathcal{A}_{(M_1,\mathcal{F}_1)}(X_1)$ and $\mathcal{A}_{(M_2,\mathcal{F}_2)}(X_2)$ are related by $P^r\psi$. The regularity means that $\mathcal{A}$ transforms smoothly parametrized families of infinitesimal automorphisms into smoothly parametrized families of vector fields.

A $\mathcal{F}ol_{m,n}$-natural operator $\mathcal{A} : T_{Inf-Aut} \rightsquigarrow TP^r$ is said to be of vertical type if $\mathcal{A}_{(M,\mathcal{F})}(X)$ is a vertical vector field on $P^r(M, \mathcal{F}) \to M$ for any infinitesimal automorphism $X$ of an arbitrary $\mathcal{F}ol_{m,n}$-object $(M, \mathcal{F})$.

Let $k$ be a non-negative integer. A $\mathcal{F}ol_{m,n}$-natural operator $\mathcal{A} : T_{Inf-Aut} \rightsquigarrow TP^r$ is said to be of order $\leq k$ if for any infinitesimal automorphisms $X_1$ and $X_2$ of $(M, \mathcal{F})$ and $x \in M$ the equality of $k$-jets $j^k_x(X_1) = j^k_x(X_2)$ implies $\mathcal{A}_{(M,\mathcal{F})}(X_1) = \mathcal{A}_{(M,\mathcal{F})}(X_2)$ on the fiber $(P^r(M, \mathcal{F}))_x$ of $P^r(M, \mathcal{F})$ over $x$.

Example 1. An example of a $\mathcal{F}ol_{m,n}$-natural operator $\mathcal{A} : T_{Inf-Aut} \rightsquigarrow TP^r$ of order $\leq r$ is the flow operator $\mathcal{P}^r$ sending an infinitesimal automorphism $X$ of a $\mathcal{F}ol_{m,n}$-object $(M, \mathcal{F})$ into the complete lift $\mathcal{P}^rX$ of $X$ to $P^r(M, \mathcal{F})$. We recall that $\mathcal{P}^rX$ is the vector field on $P^r(M, \mathcal{F})$ such that if $\{\Phi_t\}$ is the flow of $X$ then $\{P^r(\Phi_t)\}$ is the flow of $\mathcal{P}^rX$. (We observe that to the flow of $X$ we can apply functor $P^r$ because the flow is formed by $\mathcal{F}ol_{m,n}$-maps.)

EXAMPLE 2. Let $E \in \mathcal{L}(G^r_{m,n})$. Let $E^*$ denote the fundamental vector field on $P^r(M,\mathcal{F})$ corresponding to $E$ for any $\mathcal{F}ol_{m,n}$-object $(M,\mathcal{F})$. We have the (constant) $\mathcal{F}ol_{m,n}$-natural operator $E^* : T_{Inf-Aut} \rightsquigarrow TP^r$ defined by $(E^*)_{(M,\mathcal{F})}(X) = E^*$ for any infinitesimal automorphism $X$ of $(M,\mathcal{F})$. Clearly, the $\mathcal{F}ol_{m,n}$-natural operator $E^*$ is of vertical type.

DEFINITION 2. A $\mathcal{F}ol_{m,n}$-natural affinor on $P^r$ is a $\mathcal{F}ol_{m,n}$-invariant family of tensor fields of type $(1,1)$ (affinors)

$$B = B_{(M,\mathcal{F})} : TP^r(M,\mathcal{F}) \to TP^r(M,\mathcal{F})$$

on $P^r(M,\mathcal{F})$ for any $\mathcal{F}ol_{m,n}$-object $(M,\mathcal{F})$. The invariance means that affinors $B_{(M_1,\mathcal{F}_1)}$ and $B_{(M_2,\mathcal{F}_2)}$ are $P^r\psi$-related for any $\mathcal{F}ol_{m,n}$-map $\psi : (M_1,\mathcal{F}_1) \to (M_2,\mathcal{F}_2)$.

A $\mathcal{F}ol_{m,n}$-natural affinor $B$ on $P^r$ is said to be of vertical type if $B : TP^r(M,\mathcal{F}) \to VP^r(M,\mathcal{F})$ for any $\mathcal{F}ol_{m,n}$-object $(M,\mathcal{F})$, where $VP^r(M,\mathcal{F})$ is the vertical bundle of $P^r(M,\mathcal{F}) \to M$.

EXAMPLE 3. We have the identity $\mathcal{F}ol_{m,n}$-natural affinor $Id$ on $P^r$ such that $Id : TP^r(M,\mathcal{F}) \to TP^r(M,\mathcal{F})$ is the identity map for any $\mathcal{F}ol_{m,n}$-object $(M,\mathcal{F})$.

In the present article we solve the following two problems.

PROBLEM 1. To classify all $\mathcal{F}ol_{m,n}$-natural operators $\mathcal{A} : T_{Inf-Aut} \rightsquigarrow TP^r$.

PROBLEM 2. To classify all $\mathcal{F}ol_{m,n}$-natural affinors on $P^r$.

The solution of Problem 1 is given in Theorem 1. We prove that the set of all $\mathcal{F}ol_{m,n}$-natural operators $\mathcal{A} : T_{Inf-Aut} \rightsquigarrow TP^r$ is a free finite-dimensional module over some algebra. We will introduce the module structure and construct explicitly a basis of this module. The solution of Problem 2 is given in Theorem 2.

For $n = 0$, $\mathcal{F}ol_{m,0}$ is the category $\mathcal{M}f_m$ of $m$-dimensional manifolds and their local diffeomorphisms. Thus we reobtain the respective results from [4] and [3]. The part of the present paper concerning Problem 1 (resp. Problem 2) is a respective modification (adaptation) of the paper [4] (resp. [3]).

Natural affinors play a very important role in the differential geometry. They can be applied to study torsions of connections [2]. In our situation given a $\mathcal{F}ol_{m,n}$-natural affinor $B : TP^r(M,\mathcal{F}) \to TP^r(M,\mathcal{F})$ gives a torsion $\tau_B(\Gamma) = [B,\Gamma]$ of a principal connection $\Gamma : TP^r(M,\mathcal{F}) \to VP^r(M,\mathcal{F})$ on $P^r(M,\mathcal{F})$, where the bracket is the Frolicher-Nijenhuis one. That is why, natural affinors have been studied in many papers.

## 1. Preliminaries

Lemma 1. *Let $X, Y \in \mathcal{X}(M,\mathcal{F})$ be infinitesimal automorphisms of $(M,\mathcal{F})$ and $x \in M$ be a point. Suppose that $j^r_x X = j^r_x Y$ and $X_x$ is not-tangent to $\mathcal{F}$. Then there exists a (locally defined) $\mathcal{F}ol_{m,n}$-map $\psi : (M,\mathcal{F}) \to (M,\mathcal{F})$ such that $j^{r+1}_x(\psi) = j^{r+1}_x(id_M)$ and $\psi_* X = Y$ near $x$.*

*Proof.* A direct modification of the proof of Lemma 42.4 in [1]. ∎

Proposition 1. *Any $\mathcal{F}ol_{m,n}$-natural operator $\mathcal{A} : T_{Inf-Aut} \rightsquigarrow TP^r$ is of order $\le r$.*

*Proof.* A replica of the proof of Proposition 42.5 in [1]. We use Lemma 1 instead of Lemma 42.4 in [1]. ∎

The following lemma can be found in some previous our paper (in printing). For the reader convenience we cite its proof.

Lemma 2. *Any vector $v \in T_w P^r(M,\mathcal{F})$, $w \in (P^r(M,\mathcal{F}))_x$, $x \in M$ is of the form $\mathcal{P}^r X_w$ for some $X \in \mathcal{X}(M,\mathcal{F})$. Moreover $j^r_x X$ is uniquely determined.*

*Proof.* We can assume that $(M,\mathcal{F}) = (\mathbf{R}^{m+n},\mathcal{F}^{m,n})$ and $w$ is over 0. Since $P^r(\mathbf{R}^{m+n},\mathcal{F}^{m,n})$ is in usual way a sub-principal bundle of $P^r\mathbf{R}^{m+n}$, then by well-known manifold version of the lemma, we find $X \in \mathcal{X}(\mathbf{R}^{m+n})$ such that $v = \mathcal{P}^r X_w$ and $j^r_0 X$ is determined uniquely. Any infinitesimal automorphism $Y$ of $(\mathbf{R}^{m+n},\mathcal{F}^{m,n})$ gives $\mathcal{P}^r Y_w$ which is tangent to $P^r(\mathbf{R}^{m+n},\mathcal{F}^{m,n})$. On the other hand the dimension of $P^r(\mathbf{R}^{m+n},\mathcal{F}^{m,n})$ and the dimension of the space of $r$-jets $j^r_0 Y$ of infinitesimal automorphisms $Y$ of $(\mathbf{R}^{m+n},\mathcal{F}^{m,n})$ are equal. Then the lemma follows from the dimension argument because the flow operator is linear. ∎

## 2. The $\mathcal{F}ol_{m,n}$-natural operators $\mathcal{B} : T_{Inf-Aut} \rightsquigarrow T^{(0,0)}P^r$

If (in the definition of $\mathcal{F}ol_{m,n}$-natural operators $\mathcal{A} : T_{Inf-Aut} \rightsquigarrow TP^r$) we replace the space $\mathcal{X}(P^r(M,\mathcal{F}))$ by the space $C^\infty(P^r(M,\mathcal{F}))$ of mappings $P^r(M,\mathcal{F}) \to \mathbf{R}$, we obtain the concept of $\mathcal{F}ol_{m,n}$-natural operators $\mathcal{B} : T_{Inf-Aut} \rightsquigarrow T^{(0,0)}P^r$ lifting infinitesimal automorphisms of $(M,\mathcal{F})$ into maps $P^r(M,\mathcal{F}) \to \mathbf{R}$.

Example 4. We have the following general example of $\mathcal{F}ol_{m,n}$-natural operators $T_{Inf-Aut} \rightsquigarrow T^{(0,0)}P^r$. Let

$$\lambda : J_0^{r-1}(T_{Inf-Aut}(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})) \to \mathbf{R}$$

be a map, where $J_0^{r-1}(T_{Inf-Aut}(\mathbf{R}^{m+n}, \mathcal{F}^{m,n}))$ is the vector space of all $(r-1)$-jets $j_0^{r-1}X$ at $0 \in \mathbf{R}^{m+n}$ of infinitesimal automorphism $X \in \mathcal{X}(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})$. Then given an infinitesimal automorphisms $X$ on a $\mathcal{F}ol_{m,n}$-object $(M,\mathcal{F})$ we have $\mathcal{B}^{<\lambda>}(X) : P^r(M,\mathcal{F}) \to \mathbf{R}$ given by

$$\mathcal{B}^{<\lambda>}(X)(j_0^r\psi) = \lambda(j_0^{r-1}(\psi_*^{-1}X))$$

for all $j_0^r\psi \in (P^r(M,\mathcal{F}))_x$, $x \in M$, where $\psi : (\mathbf{R}^{m+n}, \mathcal{F}^{m,n}) \to (M,\mathcal{F})$ is a $\mathcal{F}ol_{m,n}$-map with $\psi(0) = x$. The correspondence $\mathcal{B}^{<\lambda>} : T_{Inf-Aut} \rightsquigarrow T^{(0,0)}P^r$ is a $\mathcal{F}ol_{m,n}$-natural operator of order $\leq r-1$ transforming infinitesimal automorphisms of $(M,\mathcal{F})$ into maps $P^r(M,\mathcal{F}) \to \mathbf{R}$.

The set of $\mathcal{F}ol_{m,n}$-natural operators $\mathcal{B} : T_{Inf-Aut} \rightsquigarrow T^{(0,0)}P^r$ is (in obvious way) an algebra. Actually, given $\mathcal{F}ol_{m,n}$-natural operators $\mathcal{B}_1, \mathcal{B}_2 : T_{Inf-Aut} \rightsquigarrow T^{(0,0)}P^r$ we have $\mathcal{F}ol_{m,n}$-natural operator $\mathcal{B}_1\mathcal{B}_2 : T_{Inf-Aut} \rightsquigarrow T^{(0,0)}P^r$ given by

$$(\mathcal{B}_1\mathcal{B}_2)_{(M,\mathcal{F})}(X) = (\mathcal{B}_1)_{(M,\mathcal{F})}(X)(\mathcal{B}_2)_{(M,\mathcal{F})}(X)$$

for any infinitesimal automorphism $X$ of a $\mathcal{F}ol_{m,n}$-object $(M,\mathcal{F})$, where in the right of the above formula we have the multiplication of real valued functions. Similarly we define the sum $\mathcal{B}_1 + \mathcal{B}_2 : T_{Inf-Aut} \rightsquigarrow T^{(0,0)}P^r$.

Proposition 2. *The map $\lambda \to \mathcal{B}^{<\lambda>}$ is an algebra isomorphism from the algebra of smooth maps $J_0^{r-1}(T_{Inf-Aut}(\mathbf{R}^{m+m}, \mathcal{F}^{m,n})) \to \mathbf{R}$ onto the algebra of all $\mathcal{F}ol_{m,n}$-natural operators $T_{Inf-Aut} \rightsquigarrow T^{(0,0)}P^r$.*

*Proof.* Clearly, the map $\lambda \to \mathcal{B}^{<\lambda>}$ is an algebra monomorphism. Any $\mathcal{F}ol_{m,n}$-natural operator $\mathcal{B} : T_{Inf-Aut} \rightsquigarrow T^{(0,0)}P^r$ of order $\leq r-1$ defines $\lambda : J_0^{r-1}(T_{Inf-Aut}(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})) \to \mathbf{R}$ by

$$\lambda(j_0^{r-1}X) = \mathcal{B}(X)_{j_0^r(id_{\mathbf{R}^{m+n}})} .$$

By the order argument $\lambda$ is well-defined. It is smooth because of the regularity of $\mathcal{B}$ (standard argument using the Boman theorem, [1]). Then by the invariance with respect to local trivialization one can easily see that $\mathcal{B} = \mathcal{B}^{<\lambda>}$.

Quite similarly as Proposition 1, one can show that any $\mathcal{B}$ in question is of order $\leq r-1$. Then the map $\lambda \to \mathcal{B}^{<\lambda>}$ is an isomorphism. ∎

## 3. The $\mathcal{F}ol_{m,n}$-natural operators $\mathcal{A} : T_{Inf-Aut} \rightsquigarrow TP^r$ of vertical type

Let $E_\nu \in \mathcal{L}(G^r_{m,n})$ ($\nu = 1, \dots, \dim(G^r_{m,n})$) be a basis of $\mathcal{L}(G^r_{m,n})$. Then the fundamental vector fields $(E_\nu)^*$ for $\nu = 1, \dots, \dim(G^r_{m,n})$ form a basis over $C^\infty(P^r(M,\mathcal{F}))$ of the module of vertical vector fields on $P^r(M,\mathcal{F})$ for any $\mathcal{F}ol_{m,n}$-object $(M,\mathcal{F})$.

The space of all $\mathcal{F}ol_{m,n}$-natural operators $T_{Inf-Aut} \rightsquigarrow TP^r$ transforming infinitesimal automorphisms of $\mathcal{F}ol_{m,n}$-objects $(M,\mathcal{F})$ into vector fields on $P^r(M,\mathcal{F})$ is (in obvious way) a module over the algebra of $\mathcal{F}ol_{m,n}$-natural operators $T_{Inf-Aut} \rightsquigarrow T^{(0,0)}P^r$. (Actually, given $\mathcal{F}ol_{m,n}$-natural operators $\mathcal{A} : T_{Inf-Aut} \rightsquigarrow TP^r$ and $\mathcal{B} : T_{Inf-Aut} \rightsquigarrow T^{(0,0)}P^r$ we have $\mathcal{F}ol_{m,n}$-natural operator $\mathcal{B}\mathcal{A} : T_{Inf-Aut} \rightsquigarrow TP^r$ given by

$$(\mathcal{B}\mathcal{A})_{(M,\mathcal{F})}(X) = \mathcal{B}_{(M,\mathcal{F})}(X)\mathcal{A}_{(M,\mathcal{F})}(X)$$

for any infinitesimal automorphism $X$ on a $\mathcal{F}ol_{m,n}$-object $(M,\mathcal{F})$, where in right of the above formula is the multiplication of vector fields by real valued functions.) Then by Proposition 2 it is the module over the algebra of all maps $J^{r-1}_0(T_{Inf-Aut}(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})) \to \mathbf{R}$.

Proposition 3. *The (sub)module of all vertical type $\mathcal{F}ol_{m,n}$-natural operators $\mathcal{A} : T_{Inf-Aut} \rightsquigarrow TP^r$ is free. The $\mathcal{F}ol_{m,n}$-natural operators $(E_\nu)^*$ in question form a basis over $\mathcal{C}^\infty(J^{r-1}_0(T_{Inf-Aut}(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})))$ of this module.*

*Proof.* Since the fundamental vector fields $(E_\nu)^*$ on $P^r(M,\mathcal{F})$ form the basis of the module of vertical vector fields on $P^r(M,\mathcal{F})$, then any $\mathcal{F}ol_{m,n}$-natural operator $\mathcal{A}$ (of vertical type) in question is of the form

$$\mathcal{A}(X) = \sum \lambda_\nu(X)(E_\nu)^*$$

for some uniquely determined maps $\lambda_\nu(X) : P^r(M,\mathcal{F}) \to \mathbf{R}$, where $X$ is an infinitesimal automorphism of a $\mathcal{F}ol_{m,n}$-object $(M,\mathcal{F})$. Because of the invariance of $\mathcal{A}$ with respect to $\mathcal{F}ol_{m,n}$-maps, $\lambda_\nu : T_{Inf-Aut} \rightsquigarrow T^{(0,0)}P^r$ are $\mathcal{F}ol_{m,n}$-natural operators. ∎

## 4. A decomposition

Proposition 4. *Let $\mathcal{A} : T_{Inf-Aut} \rightsquigarrow TP^r$ be a $\mathcal{F}ol_{m,n}$-natural operator of order $\leq r$. There is a unique smooth map $\lambda : J^{r-1}_0(T_{Inf-Aut}(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})) \to \mathbf{R}$ such that $\mathcal{A} - \mathcal{B}^{<\lambda>}\mathcal{P}^r$ is of vertical type, where $\mathcal{P}^r : T_{Inf-Aut} \rightsquigarrow TP^r$ is the flow operator.*

*Proof.* Let $X$ be an infinitesimal automorphism of $(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})$. We can write $\mathcal{A}(X)_{j^r_0(id_{\mathbf{R}^{m+n}})} = \mathcal{P}^r\tilde{X}_{j^r_0(id_{\mathbf{R}^{m+n}})}$ for some infinitesimal automorphism $\tilde{X}$ (see Lemma 2). Suppose that $\tilde{X}_0 \neq 0$ and $X_0 \neq \mu\tilde{X}_0$ for all $\mu \in \mathbf{R}$. Then there is an $\mathcal{F}ol_{m,n}$-map $\psi : (\mathbf{R}^{m+n}, \mathcal{F}^{m,n}) \to (\mathbf{R}^{m+n}, \mathcal{F}^{m,n})$ preserving $j^r_0(id_{\mathbf{R}^{m+n}})$ such that

$$J^rT\psi(j^r_0X) = j^r_0X \text{ and } J^rT\psi(j^r_0\tilde{X}) \neq j^r_0\tilde{X}\,.$$

Then

$$\mathcal{A}(X)_{j^r_0(id_{\mathbf{R}^{m+n}})} = \mathcal{P}^r(\psi_*\tilde{X})_{j^r_0(id_{\mathbf{R}^{m+n}})} \neq \mathcal{P}^r(\tilde{X})_{j^r_0(id_{\mathbf{R}^{m+n}})} = \mathcal{A}(X)_{j^r_0(id_{\mathbf{R}^{m+n}})}\,.$$

This is a contradiction. Consequently, we have

$$(*) \qquad T\pi^r \circ \mathcal{A}(X)_{j^r_0(id_{\mathbf{R}^{m+n}})} = \lambda(j^{r-1}_0X)X_0$$

for some (not necessarily unique and not necessarily smooth) map

$$\lambda : J^{r-1}_0(T_{Inf-Aut}(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})) \to \mathbf{R}$$

and all infinitesimal automorphisms of $(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})$ with coefficients (with respect to the basis of canonical vector fields on $\mathbf{R}^{m+n}$) being polynomials of degree $\leq r-1$, where $\pi^r : P^r(\mathbf{R}^{m+m}, \mathcal{F}^{m,n}) \to \mathbf{R}^{m+m}$ is the usual projection $j^r_0\psi \to \psi(0)$.

We are going to show that $\lambda$ can be chosen smooth. Of course (since the left hand side of $(*)$ depends smoothly on $j^r_0X$), the map

$$\Phi : J^{r-1}_0(T_{Inf-Aut}(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})) \to \mathbf{R}$$

given by

$$\Phi(j^{r-1}_0X) = \lambda(j^{r-1}_0X)X^1(0)$$

is smooth and $\Phi(j^{r-1}_0X) = 0$ if $X^1(0) = 0$, where $X_0 = \sum_i X^i(0)\frac{\partial}{\partial x^i}_0$. Then (this is a known fact from the mathematical analysis) there is a smooth map

$$\Psi : J^{r-1}_0(T_{Inf-Aut}(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})) \to \mathbf{R}$$

such that $\Phi(j_0^{r-1}X) = \Psi(j_0^{r-1}X)X^1(0)$. Then we can define new $\lambda = \Psi$. This new $\lambda$ is equal to the old one for $X^1(0) \neq 0$. Then for the new $\lambda$ we have $(*)$ if additionally $X^1(0) \neq 0$. Then we have $(*)$ for all $X$ in question because of the smooth and density arguments.

Then $(\mathcal{A}(X) - \mathcal{B}^{<\lambda>}(X)\mathcal{P}^r X)_{j_0^r(id_{\mathbf{R}^{m+n}})}$ is vertical for all infinitesimal automorphisms $X$ of $(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})$ with coefficients (with respect to the basis of vector fields) being polynomials of degree $\leq r-1$.

Since the union of orbits with respect to the $\mathcal{F}ol_{m,n}$-maps preserving $j_0^r(id_{\mathbf{R}^{m+n}})$ of all $j_0^r X$ for infinitesimal automorphisms $X$ with coefficients (with respect to the basis as above) being polynomials of degree $\leq r-1$ is dense in $J_0^r(T_{Inf-Aut}(\mathbf{R}^{m+n}, \mathcal{F}^{m,n}))$ (see Lemma 1), the vector $(\mathcal{A}(X) - \mathcal{B}^{<\lambda>}(X)\mathcal{P}^r X)_{j_0^r(id_{\mathbf{R}^{m+n}})}$ is vertical for all infinitesimal automorphisms $X$ of $(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})$ with coefficients (with respect to the basis) being polynomials of degree $\leq r$. Then $(\mathcal{A}(X) - \mathcal{B}^{<\lambda>}(X)\mathcal{P}^r X)_{j_0^r(id_{\mathbf{R}^{m+n}})}$ is vertical for all infinitesimal automorphisms $X$ of $(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})$ because of the order argument. Then $\mathcal{A} - \mathcal{B}^{<\lambda>}\mathcal{P}^r$ is of vertical type because of the $\mathcal{F}ol_{m,n}$-invariance and the fact that $P^r$ is a transitive bundle functor (i.e. $P^r(M, \mathcal{F})$ is the $\mathcal{F}ol_{m,n}$-orbit of $j_0^r(id_{\mathbf{R}^{m+m}})$). ∎

## 5. Solution of Problem 1

We know that any $\mathcal{F}ol_{m,n}$-natural operator $\mathcal{A} : T_{Inf-Aut} \rightsquigarrow TP^r$ is of order $\leq r$ (see Proposition 1). Then summing up Propositions 3 and 4 we get.

THEOREM 1. *All $\mathcal{F}ol_{m,n}$-natural operators $T_{Inf-Aut} \rightsquigarrow TP^r$ form a free finite-dimensional module over the algebra of all smooth functions*

$$J_0^{r-1}(T_{Inf-Aut}(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})) \to \mathbf{R}.$$

*The operators $\mathcal{P}^r$ and $(E_\nu)^*$ for $\nu = 1, \ldots, \dim(G^r_{m,n})$ form a basis in this module, where $(E_\nu)$ is a basis of $\mathcal{L}(G^r_{m,n})$ and given $E \in \mathcal{L}(G^r_{m,n})$ the fundamental vector field on $P^r(M, \mathcal{F})$ is denoted by $E^*$.*

## 6. A decomposition for $\mathcal{F}ol_{m,n}$-natural affinors

PROPOSITION 5. *Let $B$ be a $\mathcal{F}ol_{m,n}$-natural affinor on $P^r$. There is a unique real number $\lambda$ such that $B - \lambda Id$ is of vertical type.*

*Proof.* Using $B$ we define a linear $\mathcal{F}ol_{m,n}$-natural operator $\mathcal{A} : T_{Inf-Aut} \rightsquigarrow TP^r$ by $\mathcal{A}(X) = B(\mathcal{P}^r X)$ for any $X \in \mathcal{X}(M, \mathcal{F})$ (the linearity means that

$\mathcal{A}(X)$ is linear in $X$). By Proposition 4 and the homogeneous function theorem [1], since $\mathcal{A}$ is linear, there exists a unique real number $\lambda$ such that $\mathcal{A}-\lambda\mathcal{P}^r$ is vertical. Then $(B-\lambda Id)(\mathcal{P}^r X_\sigma)$ is vertical for any infinitesimal automorphism $X \in \mathcal{X}(M,\mathcal{F})$ and $\sigma \in P^r(M,\mathcal{F})$. Then $(B-\lambda Id)(v)$ is vertical for any $v \in TP^r(M,\mathcal{F})$ because of Lemma 2. Then $B-\lambda Id$ is vertical. ∎

## 7. An example of $\mathcal{F}ol_{m,n}$-natural affinors of vertical type

We have the following example.

Example 5. Let

$$C : J_0^{r-1}(T_{Inf-Aut}(\mathbf{R}^{m+n},\mathcal{F}^{m,n})) \to (J_0^r(T_{Inf-Aut}(\mathbf{R}^{m+n},\mathcal{F}^{m,n})))_0$$

be a linear map, where

$$J_0^{r-1}(T_{Inf-Aut}(\mathbf{R}^{m+n},\mathcal{F}^{m,n})) = \{j_0^{r-1}X \mid X \in \mathcal{X}(\mathbf{R}^{m+n},\mathcal{F}^{m,n})\}$$

and $(J_0^r(T_{Inf-Aut}(\mathbf{R}^{m+n},\mathcal{F}^{m,n})))_0 = \{j_0^r X \mid X \in \mathcal{X}(\mathbf{R}^{m+n},\mathcal{F}^{m,n})\ , X_0 = 0\}$. Define a vertical $\mathcal{F}ol_{m,n}$-natural affinor $B^C : TP^r(M,\mathcal{F}) \to VP^r(M,\mathcal{F})$ on $P^r$ by

$$B^C(v) = VP^r\psi((\mathcal{P}^r\tilde{v})_\theta)\ ,\ v \in T_{j_0^r\psi}P^r(M,\mathcal{F})\ ,\ j_0^r\psi \in P^r(M,\mathcal{F})\,,$$

where $\theta = j_0^r(id_{\mathbf{R}^{m+m}}) \in P^r(\mathbf{R}^{m+n},\mathcal{F}^{m,n})$ is the element and $\tilde{v} \in \mathcal{X}$ $(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})$ is an infinitesimal automorphism of $(\mathbf{R}^{m+n},\mathcal{F}^{m,n})$ such that $j_0^r\tilde{v} = C(j_0^{r-1}((\psi^{-1})_*\overline{v}))$ and $v = (\mathcal{P}^r\overline{v})_{j_0^r\psi}$. One can standardly show that $B^C(v)$ is well-defined. More precisely (by Lemma 2), $j^r_{\psi(0)}\overline{v}$ is uniquely determined by $v$. Then $j_0^{r-1}((\psi^{-1})_*\overline{v}) \in J_0^{r-1}(T_{Inf-Aut}(\mathbf{R}^{m+n}, \mathcal{F}^{m,n}))$ is determined by $v$. Then $j_0^r(\tilde{v}) \in (J_0^r(T_{Inf-Aut}(\mathbf{R}^{m+n},\mathcal{F}^{m,n})))_0$ is determined by $v$. Then $(\mathcal{P}^r\tilde{v})_\theta$ is determined by $v$ and vertical. Then $B^C(v)$ is determined by $v$ and vertical.

Using the linearity of the flow operator, we deduce that $B^C : TP^r(M,\mathcal{F}) \to VP^r(M,\mathcal{F})$ is a vertical affinor on $P^r(M,\mathcal{F})$. Clearly the family $B^C$ is a $\mathcal{F}ol_{m,n}$-natural affinor on $P^r$.

## 8. Solution of Problem 2

Theorem 2. *Any $\mathcal{F}ol_{m,n}$-natural affinor on $P^r$ is of the form*

$$B = \lambda Id + B^C$$

*for a unique real number $\lambda$ and a unique linear map $C : J_0^{r-1}(T_{Inf-Aut}(\mathbf{R}^{m+n},\mathcal{F}^{m,n})) \to (J_0^r(T_{Inf-Aut}(\mathbf{R}^{m+n},\mathcal{F}^{m,n})))_0$.*

*Proof.* Because of Proposition 5, we can assume that $B$ is vertical. Define a linear map

$$C : J_0^{r-1}(T_{Inf-Aut}(\mathbf{R}^{m+m}, \mathcal{F}^{m,n})) \to (J_0^r(T_{Inf-Aut}(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})))_0$$

by $C(j_0^{r-1}X) = j_0^r\tilde{X}$ , where $\tilde{X}$ is an infinitesimal automorphism of $(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})$ such that $(\mathcal{P}^r\tilde{X})_\theta = B((\mathcal{P}^r\overline{X})_\theta)$ and $\overline{X}$ is an unique infinitesimal automorphism of $(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})$ such that $j_0^{r-1}X = j_0^{r-1}\overline{X}$ and $\overline{X}$ has coefficients with respect to the basis of canonical vector fields $\frac{\partial}{\partial x^i}$ on $\mathbf{R}^{m+n}$ being polynomials of degree $\le r-1$.

Then $B((\mathcal{P}^rX)_\theta) = B^C((\mathcal{P}^rX)_\theta)$ for all infinitesimal automorphisms of $(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})$ such that $X$ has coefficients (with respect to the basis as above) being polynomials of degree $r-1$. Since the union of all orbits with respect to the $\mathcal{F}ol_{m,n}$-maps preserving $\theta$ of jets $j_0^rX$ of infinitesimal automorphisms $X$ of $(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})$ with coefficients (with respect to the basis as above) being polynomials of degree $\le r-1$ is dense in $J_0^r(T_{Inf-Aut}(\mathbf{R}^{m+n}, \mathcal{F}^{m,n}))$ (see Lemma 1), $B((\mathcal{P}^rX)_\theta) = B^C((\mathcal{P}^rX)_\theta)$ for all infinitesimal automorphisms $X$ of $(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})$. Then $B(v) = B^C(v)$ for all $v \in T_\theta P^r(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})$ because of Lemma 2. Then $B = B^C$ because of the $\mathcal{F}ol_{m,n}$-invariance and the fact that $P^r$ is a transitive bundle functor (i.e., $P^r(M, \mathcal{F})$ is the $\mathcal{F}ol_{m,n}$-orbit of $\theta$). ∎

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