

Invariant Metric f -Structures on Specific Homogeneous Reductive Spaces

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Abstract: For homogeneous reductive spaces G/H with reductive complements decomposable into an orthogonal sum $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ of three $Ad(H)$ -invariant irreducible mutually inequivalent submodules we establish simple conditions under which an invariant metric f -structure (f, g) belongs to the classes $\mathbf{G}_1\mathbf{f}$, \mathbf{NKf} , and \mathbf{Killf} of generalized Hermitian geometry. The statements obtained are then illustrated with four examples. Namely we consider invariant metric f -structures on the manifolds of oriented flags $SO(n)/SO(2) \times SO(n-3)$ ($n \geq 4$), the Stiefel manifold $SO(4)/SO(2)$, the complex flag manifold $SU(3)/T_{max}$, and the quaternionic flag manifold $Sp(3)/SU(2) \times SU(2) \times SU(2)$.

Key words: Homogeneous reductive space, f -structure, invariant structure, nearly Kähler structure, flag manifold.

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INTRODUCTION

The concept of generalized Hermitian geometry (see, for example, [16]) was created in the 1980s as a natural consequence of the development of Hermitian geometry and the theory of almost contact structures. One of the central objects in this concept is the metric f -structure (f, g) , that is, an f -structure [21] f compatible with an invariant Riemannian metric g .

An interesting problem that arises in this context is to determine whether a given metric f -structure belongs to the main classes of generalized Hermitian geometry, for example, to the classes $\mathbf{G}_1\mathbf{f}$ (see [16]), \mathbf{NKf} (see [6] and [7]), and \mathbf{Killf} (see [14] and [15]). It should be emphasized that in the case of naturally reductive manifolds [18] there exist a number of results that transform this problem into an easy computational task ([6], [8], [4], [5]). However, in the case of an arbitrary Riemannian metric this problem is not an easy one, at least because it involves the calculation of the implicitly defined Levi-Civita connection.

In this paper we consider invariant metric f -structures (f, g) on specific

homogeneous reductive spaces G/H , namely on homogeneous reductive spaces that satisfy the following set of conditions:

- 1) G is a compact semisimple Lie group (hence the Killing form B of G is negative definite).
- 2) The reductive complement \mathfrak{m} admits the decomposition

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$$

into the direct sum of $Ad(H)$ -invariant irreducible mutually non-equivalent submodules and this decomposition is B -orthogonal.

3)

$$0 \neq [\mathfrak{m}_i, \mathfrak{m}_{i+1}] \subset \mathfrak{m}_{i+2} \pmod{3}, \quad i = 1, 2, 3.$$

4)

$$[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h}, \quad i = 1, 2, 3,$$

where \mathfrak{h} is the Lie algebra of H .

In this case it is not difficult to obtain an explicit formula for the Levi-Civita connection of a Riemannian manifold $(G/H, g)$. At the same time, for any nontrivial invariant f -structure which is not an almost complex structure [18] there exists such $i \in \{1, 2, 3\}$ that either $\text{Im } f = \mathfrak{m}_i$ or $\text{Ker } f = \mathfrak{m}_i$. This, in its turn, has enabled us to obtain easy-to-check characteristic conditions (Theorem 2 and Theorem 3) for metric f -structures (f, g) under which they belong to the aforementioned classes of generalized Hermitian geometry.

Note that this paper was initiated by the study of the manifolds of oriented flags $SO(n)/SO(2) \times SO(n-3)$ ($n \geq 4$). In [10] it was shown that these homogeneous spaces satisfy the conditions 1) – 4). In the last section of this paper we provide other examples of such spaces. Namely, by making use of Theorem 2 and Theorem 3, we consider invariant metric f -structures on the Stiefel manifold $SO(4)/SO(2)$, the complex flag manifold $SU(3)/T_{max}$, and the quaternionic flag manifold $Sp(3)/SU(2) \times SU(2) \times SU(2)$.

1. PRELIMINARIES

1.1. INVARIANT f -STRUCTURES ON HOMOGENEOUS REDUCTIVE SPACES

Homogeneous reductive spaces make up the main subject of our further considerations. Therefore we begin with recollecting some basic facts related to them.

DEFINITION 1. [19] Let G be a connected Lie group, H its closed subgroup, \mathfrak{g} and \mathfrak{h} the corresponding Lie algebras. G/H is called a *homogeneous reductive space* if there exists $\mathfrak{m} \subset \mathfrak{g}$ such that

- 1) $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.
- 2) For any $h \in H$ $Ad(h)\mathfrak{m} \subset \mathfrak{m}$.

$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is the *reductive decomposition* corresponding to G/H and \mathfrak{m} is the *reductive complement*.

For any homogeneous reductive space G/H its reductive complement \mathfrak{m} can be identified with the tangent space to G/H at the point $o = H$ in the following sense:

for any $h \in H$, $d\tau(h)_o = Ad(h)$, where $\tau(g) : G/H \rightarrow G/H$, $xH \rightarrow (gx)H$.

Since all homogeneous spaces to be discussed in this paper are reductive, we agree to identify their reductive complements and their tangent spaces at the point o .

An *affinor structure* on a smooth manifold is known to be a tensor field of type (1,1) realized as a field of endomorphisms acting on its tangent bundle. In this paper we will be primarily interested in the *almost complex structure* [18] (such an affinor structure J that $J^2 = -\text{id}$) and the *f -structure* [21] (an affinor structure f satisfying $f^3 + f = 0$).

DEFINITION 2. [1] Let G/H be a homogeneous manifold, F an affinor structure. F is called *invariant* with respect to G if for any $g \in G$

$$d\tau(g) \circ F = F \circ d\tau(g).$$

It is known that any invariant affinor structure F on a reductive homogeneous space G/H is completely determined by its value F_o at the point $o = H$, where F_o is a linear operator on the reductive complement \mathfrak{m} such that

$$F_o \circ Ad(h) = Ad(h) \circ F_o \text{ for any } h \in H.$$

For this reason, further we will not distinguish an invariant structure F on G/H and its value F_o at the point $o = H$.

1.2. SOME IMPORTANT CLASSES IN GENERALIZED HERMITIAN GEOMETRY The concept of generalized Hermitian geometry appeared in the 1980s and is mostly associated with the works of V. F. Kiritchenko (see, for example, [16] and [17]). It should be mentioned that this theory is a natural consequence of the development of Hermitian geometry and the theory of almost contact structures with many applications.

In the sequel by $\mathfrak{X}(M)$ we will denote the set of all smooth vector fields on a manifold M .

One of the central objects in generalized Hermitian geometry is a *metric f -structure* [16] (f, g) , where f is an f -structure compatible with a (pseudo) Riemannian metric $g = \langle \cdot, \cdot \rangle$ in the following sense:

$$\langle fX, Y \rangle + \langle X, fY \rangle = 0 \text{ for any } X, Y \in \mathfrak{X}(M).$$

Evidently, this definition generalizes the notion of an almost Hermitian structure J in Hermitian geometry. A manifold M equipped with a metric f -structure is called a *metric f -manifold*.

It is worth noticing that the main classes of generalized Hermitian geometry (see [16], [7], [8], [14], and [15]) in the special case $f = J$, where J is an almost complex structure, coincide with those of Hermitian geometry (see [13]). In this section we will mainly concentrate on the classes **Kill** f , **NK** f , and **G** $_1$ f of metric f -structures.

A fundamental role in generalized Hermitian geometry is played by the tensor T of type $(2, 1)$ which is called a *composition tensor* [16]. In [16] it was shown that such a tensor exists on any metric f -manifold and it is possible to evaluate it explicitly:

$$T(X, Y) = \frac{1}{4}f(\nabla_{fX}(f)fY - \nabla_{f^2X}(f)f^2Y),$$

where ∇ is the Levi-Civita connection of a (pseudo) Riemannian manifold (M, g) , $X, Y \in \mathfrak{X}(M)$.

With the help of this tensor one can define the structure of a so-called *adjoint Q -algebra* (see [16]) on $\mathfrak{X}(M)$ by the formula $X * Y = T(X, Y)$. It gives the opportunity to introduce some classes of metric f -structures in terms of natural properties of the adjoint Q -algebra.

For example, if

$$T(X, X) = 0 \text{ for any } X \in \mathfrak{X}(M) \tag{1}$$

(that is, if $\mathfrak{X}(M)$ is an anticommutative Q -algebra) then f is referred to as a G_1f -structure. $\mathbf{G}_1\mathbf{f}$ denotes the class of G_1f -structures, which was first introduced (in a more general situation) in [16].

A metric f -structure on (M, g) is said to be a *Killing f -structure* [14, 15] if

$$\nabla_X(f)X = 0 \text{ for any } X \in \mathfrak{X}(M) \quad (2)$$

(that is, if f is a Killing tensor). The class of Killing f -structures is denoted by $\mathbf{Kill}\mathbf{f}$.

The defining property of *nearly Kähler f -structures* (or *NK f -structures*) is

$$\nabla_{fX}(f)fX = 0 \text{ for any } X \in \mathfrak{X}(M). \quad (3)$$

This class of metric f -structures, which is denoted by \mathbf{NKf} , was first determined in [3] (see also [7, 6]). It is not difficult to see that for $f = J$ the classes $\mathbf{Kill}\mathbf{f}$ and \mathbf{NKf} coincide with the well-known class \mathbf{NK} of *nearly Kähler structures* [12].

The following relations between the classes mentioned are evident:

$$\mathbf{Kill}\mathbf{f} \subset \mathbf{NKf} \subset \mathbf{G}_1\mathbf{f}. \quad (4)$$

The classical result bellow will be used to rewrite formulas (1), (2) and (3) in a form more suitable for further considerations.

THEOREM 1. ([18]) *Let (M, g) be a Riemannian manifold, $M = G/H$ a homogeneous reductive space with the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then the Levi-Civita connection with respect to g can be expressed in the form*

$$\nabla_X Y = \frac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y), \quad (5)$$

where U is the symmetric bilinear mapping $\mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$ defined by the formula

$$2g(U(X, Y), Z) = g(X, [Z, Y]_{\mathfrak{m}}) + g([Z, X]_{\mathfrak{m}}, Y) \text{ for any } X, Y, Z \in \mathfrak{m}. \quad (6)$$

It can be shown in the standard way that the application of (5) to (1), (2) and (3) produces the following result.

LEMMA 1. *Let (M, g) be a Riemannian manifold, $M = G/H$ a reductive homogeneous space with the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then for an invariant metric f -structure (f, g) on M the following holds.*

1) $f \in \mathbf{G_1f}$ if and only if

$$f(2U(fX, f^2X) - f(U(fX, fX)) + f(U(f^2X, f^2X))) = 0 \quad (7)$$

for any $X \in \mathfrak{m}$;

2) $f \in \mathbf{NKf}$ if and only if

$$\frac{1}{2}[fX, f^2X]_{\mathfrak{m}} + U(fX, f^2X) - f(U(fX, fX)) = 0 \quad \text{for any } X \in \mathfrak{m}; \quad (8)$$

3) $f \in \mathbf{Killf}$ if and only if

$$\frac{1}{2}[X, fX]_{\mathfrak{m}} + U(X, fX) - f(U(X, X)) = 0 \quad \text{for any } X \in \mathfrak{m}. \quad (9)$$

2. MAIN RESULTS

ASSUMPTION 1. Suppose that for a homogeneous reductive space G/H with the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ the following is true.

$A_1)$ G is a compact semisimple Lie group (hence the Killing form B on \mathfrak{g} is negative definite).

$A_2)$ The reductive complement \mathfrak{m} admits the decomposition

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \quad (10)$$

into the direct sum of $Ad(H)$ -invariant irreducible mutually non-equivalent submodules and this decomposition is B -orthogonal.

$A_3)$

$$0 \neq [\mathfrak{m}_i, \mathfrak{m}_{i+1}] \subset \mathfrak{m}_{i+2} \pmod{3}, \quad i = 1, 2, 3. \quad (11)$$

$A_4)$

$$[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h}, \quad i = 1, 2, 3. \quad (12)$$

In the view of $A_1)$ and $A_2)$ any invariant Riemannian metric g on G/H is uniquely determined by the triple of positive real numbers (a_1, a_2, a_3) which implies that

$$g = a_1 g_0 |_{\mathfrak{m}_1 \times \mathfrak{m}_1} + a_2 g_0 |_{\mathfrak{m}_2 \times \mathfrak{m}_2} + a_3 g_0 |_{\mathfrak{m}_3 \times \mathfrak{m}_3}, \quad (13)$$

where g_0 is an invariant inner product generated by the negative of the Killing form B . Further we will refer to (a_1, a_2, a_3) as to the *characteristic numbers* of g . We will also denote the projection of X onto \mathfrak{m}_i by X_i for any $X \in \mathfrak{m}$.

Assumption 1 makes it possible to calculate the symmetric bilinear mapping $U(X, Y)$ defined in the previous section. The proof of the following result, which is nothing but the simplification of (6) in the view of Assumption 1, can be found in [20].

LEMMA 2. *Suppose that G/H satisfies Assumption 1. Then the symmetric bilinear mapping U is defined by the formula*

$$\begin{aligned} U(X, Y) = & \frac{a_3 - a_2}{2a_1}([X_2, Y_3] + [Y_2, X_3]) + \frac{a_3 - a_1}{2a_2}([X_1, Y_3] + [Y_1, X_3]) \\ & + \frac{a_2 - a_1}{2a_3}([X_1, Y_2] + [Y_1, X_2]). \end{aligned} \quad (14)$$

Here and below we assume that G/H satisfies Assumption 1.

LEMMA 3. *For any invariant affinor structure f on G/H $f(\mathfrak{m}_i)$ ($i = 1, 2, 3$) is $Ad(H)$ -invariant.*

Proof. A_2 yields that $Ad(h)\mathfrak{m}_i \subset \mathfrak{m}_i$ for any $h \in H$. Hence

$$f(Ad(H)\mathfrak{m}_i) \subset f(\mathfrak{m}_i).$$

f is an invariant affinor structure, therefore

$$Ad(H)(f(\mathfrak{m}_i)) \subset f(\mathfrak{m}_i). \quad \blacksquare$$

PROPOSITION 1. *Let f be an invariant affinor f -structure on G/H with $\text{Ker } f \neq \{0\}$ and $\text{Im } f \neq \{0\}$, and assume that G/H satisfies Assumption 1. Then there exists $i \in \{1, 2, 3\}$ such that either $\text{Im } f = \mathfrak{m}_i$ or $\text{Ker } f = \mathfrak{m}_i$.*

Proof. As $\mathfrak{m} = \text{Ker } f \oplus \text{Im } f$, for any $i \in \{1, 2, 3\}$ we have

$$\mathfrak{m}_i = (\text{Ker } f)_i \oplus (\text{Im } f)_i,$$

where

$$(\text{Ker } f)_i = \mathfrak{m}_i \cap \text{Ker } f, \quad (\text{Im } f)_i = \mathfrak{m}_i \cap \text{Im } f.$$

Suppose that there exists $k \in \{1, 2, 3\}$ such that $(\text{Ker } f)_k \neq 0$ and $(\text{Im } f)_k \neq 0$. Obviously, for any $X \in (\text{Ker } f)_k$ and $h \in H$

$$f(\text{Ad}(h)X) = \text{Ad}(h)(f(X)) = 0,$$

which implies that $(\text{Ker } f)_k$ is $\text{Ad}(H)$ -invariant.

The same is true for $(\text{Im } f)_k$. Indeed, for any $X \in (\text{Im } f)_k$ we have $\text{Ad}(h)X \in \text{Im } f$ (by Lemma 3) and $\text{Ad}(h)X \in \mathfrak{m}_k$ (by A_2).

In this way we have obtained that \mathfrak{m}_k is decomposed into the sum of the two non-trivial $\text{Ad}(H)$ -invariant subspaces, which contradicts Assumption 1. ■

Proposition 1 yields that for any non-trivial invariant affiner f -structure f which is not an almost complex structure the following is true:

$$1) \text{ either } f|_{\mathfrak{m}_i} = J, f|_{\mathfrak{m}_j \oplus \mathfrak{m}_k} = 0,$$

$$2) \text{ or } f|_{\mathfrak{m}_i} = 0, f|_{\mathfrak{m}_j \oplus \mathfrak{m}_k} = J,$$

where $\{i, j, k\} = \{1, 2, 3\}$, and J is an almost complex structure.

Let us consider the first of these two cases. The following statement is valid.

THEOREM 2. *Suppose that G/H satisfies Assumption 1, and g is an arbitrary invariant Riemannian metric on G/H . Let (f, g) be an invariant metric f -structure such that $f|_{\mathfrak{m}_i} = J, f|_{\mathfrak{m}_j \oplus \mathfrak{m}_k} = 0$, where $\{i, j, k\} = \{1, 2, 3\}$, and J is an almost complex structure. Then*

1) (f, g) is not a Killing f -structure;

2) (f, g) belongs to the class **NKf** (and, consequently, to the class **G₁f**).

Proof. We assume that

$$f|_{\mathfrak{m}_1} = J, f|_{\mathfrak{m}_2 \oplus \mathfrak{m}_3} = 0 \tag{15}$$

(the results for the other cases are obtained via cyclic rearrangement of indices).

1) **Kill f** is defined by the formula (9). Taking (14), (15) and Assumption 1 into account we obtain

$$U(X, fX) = \frac{a_3 - a_1}{2a_2} [(fX)_1, X_3] + \frac{a_2 - a_1}{2a_3} [(fX)_1, X_2],$$

$$f(U(X, X)) = \frac{a_3 - a_2}{a_1} f([X_2, X_3]).$$

Besides,

$$\frac{1}{2}[X, fX]_{\mathfrak{m}} = \frac{1}{2}[X_2, (fX)_1] + \frac{1}{2}[X_3, (fX)_1].$$

Hence, (9) is equivalent to the following relation:

$$\begin{aligned} \frac{a_3 - a_2 - a_1}{2a_2} [(fX)_1, X_3] + \frac{a_2 - a_1 - a_3}{2a_3} [(fX)_1, X_2] \\ - \frac{a_3 - a_2}{a_1} f([X_2, X_3]) = 0 \end{aligned}$$

for any $X \in \mathfrak{m}$.

By A_3), $[\mathfrak{m}_i, \mathfrak{m}_j] \neq 0$ ($i, j \in \{1, 2, 3\}$, $i \neq j$). Therefore (f, g) belongs to **Kill f** if and only if the characteristic numbers of g satisfy the following set of conditions:

$$\begin{cases} \frac{a_3 - a_2 - a_1}{2a_2} = 0, \\ \frac{a_2 - a_1 - a_3}{2a_3} = 0, \\ \frac{a_3 - a_2}{a_1} = 0. \end{cases}$$

Evidently, this system is inconsistent.

2) The defining property of **NKf** is (8). As (15) holds, (14) yields that

$$U(fX, fX) = U(fX, f^2X) = 0.$$

Moreover, by Assumption 1,

$$\frac{1}{2}[fX, f^2X]_{\mathfrak{m}} = \frac{1}{2}[(fX)_1, (f^2X)_1]_{\mathfrak{m}} = 0.$$

Thus (8) holds for any Riemannian metric. As a particular case, any f satisfying (15) is a G_1f -structure. ■

Now let us consider the second group of f -structures.

THEOREM 3. *Suppose that G/H satisfies Assumption 1, and g is an arbitrary invariant Riemannian metric on G/H with the characteristic numbers (a_1, a_2, a_3) . Let (f, g) be an invariant metric f -structure such that $f|_{\mathfrak{m}_i} = 0$, $f|_{\mathfrak{m}_j \oplus \mathfrak{m}_k} = J$, where $\{i, j, k\} = \{1, 2, 3\}$, and J is an almost complex structure. Then*

- 1) (f, g) is a G_1f -structure;
 2) (f, g) is a nearly Kähler f -structure if and only if $a_j = a_k$ and

$$[fX, f^2X]_{\mathfrak{m}} = 0 \text{ for any } X \in \mathfrak{m}; \quad (16)$$

- 3) (f, g) is a Killing f -structure if and only if $a_j = a_k = \frac{4}{3}a_i$ and

$$\begin{cases} [Z, fZ]_{\mathfrak{m}} = 0, \\ [Y, fZ] + f([Y, Z]) = 0 \end{cases} \quad \text{for any } Y \in \mathfrak{m}_i, Z \in \mathfrak{m}_j \oplus \mathfrak{m}_k.$$

Proof. Without loss of generality it can be assumed that

$$f|_{\mathfrak{m}_1} = 0, \quad f|_{\mathfrak{m}_2 \oplus \mathfrak{m}_3} = J. \quad (17)$$

- 1) It is evident that both

$$U(fX, f^2X) = \frac{a_3 - a_2}{2a_1}([(fX)_2, (f^2X)_3] + [(f^2X)_2, (fX)_3]) \quad (18)$$

and

$$U(fX, fX) = \frac{a_3 - a_2}{a_1}[(fX)_2, (fX)_3] \quad (19)$$

belong to $\text{Ker } f$ for any $X \in \mathfrak{m}$. Therefore (7) holds regardless of the choice of (a_1, a_2, a_3) .

- 2) Clearly,

$$\frac{1}{2}[fX, f^2X]_{\mathfrak{m}} = \frac{1}{2}[(fX)_2, (f^2X)_3] + \frac{1}{2}[(fX)_3, (f^2X)_2]. \quad (20)$$

Using (18), (19) and (20) we can rewrite (8) as follows:

$$\frac{a_3 - a_2 + a_1}{2a_1}[(fX)_2, (f^2X)_3] + \frac{a_3 - a_2 - a_1}{2a_1}[(f^2X)_2, (fX)_3] = 0$$

for any $X \in \mathfrak{m}$.

$a_3 - a_2 + a_1 \neq 0$ (otherwise $a_3 - a_2 - a_1 = 0$ and hence $a_1 = 0$). Thus $f \in \mathbf{NKf}$ with respect to (a_1, a_2, a_3) if and only if

$$\begin{cases} [(fX)_2, (f^2X)_3] = \frac{a_3 - a_2 - a_1}{a_3 - a_2 + a_1}[(fX)_3, (f^2X)_2], \\ [(fX)_3, (f^2X)_2] = \frac{a_3 - a_2 - a_1}{a_3 - a_2 + a_1}[(fX)_2, (f^2X)_3] \end{cases} \quad \text{for any } X \in \mathfrak{m}$$

(to obtain the second equation we substitute X for fX in the first one). It follows in the standard way that

$$\left\{ \begin{array}{l} \left(1 - \left(\frac{a_3 - a_2 - a_1}{a_3 - a_2 + a_1} \right)^2 \right) [(fX)_2, (f^2X)_3] = 0, \\ [(fX)_2, (f^2X)_3] = \frac{a_3 - a_2 - a_1}{a_3 - a_2 + a_1} [(fX)_3, (f^2X)_2] \end{array} \right. \quad \text{for any } X \in \mathfrak{m}.$$

The first equation yields that $\frac{a_3 - a_2 - a_1}{a_3 - a_2 + a_1} = \pm 1$. As a_1, a_2 and a_3 are positive numbers, we have $a_2 = a_3$. Then

$$[(fX)_2, (f^2X)_3] + [(fX)_3, (f^2X)_2] = 0.$$

In the view of (17) and Assumption 1 this means that $[fX, f^2X]_{\mathfrak{m}} = 0$ for any X in \mathfrak{m} . Thus 2) is proved.

3) As (4) holds, here we consider f -structures satisfying (16) and invariant Riemannian metrics with characteristic numbers (a_1, a_2, a_2) ($a_1, a_2 > 0$) only.

As above, we check that

$$U(X, X) = \frac{a_2 - a_1}{a_2} [X_1, X_2 + X_3],$$

$$U(X, fX) = \frac{a_2 - a_1}{2a_2} [X_1, (fX)_2 + (fX)_3].$$

Since (16) holds,

$$\begin{aligned} \frac{1}{2} [X, fX]_{\mathfrak{m}} &= \frac{1}{2} [X_1, (fX)_2 + (fX)_3] + \frac{1}{2} [X_2 + X_3, (fX)_2 + (fX)_3]_{\mathfrak{m}} \\ &= \frac{1}{2} [X_1, (fX)_2 + (fX)_3]. \end{aligned}$$

Thus (9) can be represented as follows:

$$\frac{2a_2 - a_1}{2a_2} [X_1, (fX)_2 + (fX)_3] - \frac{a_2 - a_1}{a_2} f([X_1, X_2 + X_3]) = 0$$

for any $X \in \mathfrak{m}$.

For convenience we shall rewrite it in this way:

$$\frac{2a_2 - a_1}{2a_2} [Y, fZ] - \frac{a_2 - a_1}{a_2} f([Y, Z]) = 0 \quad \text{for any } Y \in \mathfrak{m}_1, Z \in \mathfrak{m}_2 \oplus \mathfrak{m}_3.$$

Then it follows that

$$[Y, fZ] = \frac{2(a_2 - a_1)}{2a_2 - a_1} f([Y, Z]) \quad \text{for any } Y \in \mathfrak{m}_1, Z \in \mathfrak{m}_2 \oplus \mathfrak{m}_3 \quad (21)$$

($2a_2 - a_1 \neq 0$, because otherwise $a_1 = a_2 = 0$). If we replace Z by fZ in (21) and then apply f to its both sides, we obtain

$$f([Y, Z]) = \frac{2(a_2 - a_1)}{2a_2 - a_1} [Y, fZ] \quad \text{for any } Y \in \mathfrak{m}_1, Z \in \mathfrak{m}_2 \oplus \mathfrak{m}_3. \quad (22)$$

(21) and (22) produce the following system of equations

$$\begin{cases} \frac{4(a_2 - a_1)^2}{2a_2 - a_1} = 1, \\ [Y, fZ] = \frac{2(a_2 - a_1)}{2a_2 - a_1} f([Y, Z]) \end{cases} \quad \text{for any } Y \in \mathfrak{m}_1, Z \in \mathfrak{m}_2 \oplus \mathfrak{m}_3.$$

To conclude the proof, it remains to note that this system is equivalent to

$$\begin{cases} a_2 = \frac{4}{3}a_1, \\ [Y, fZ] + f([Y, Z]) = 0 \end{cases} \quad \text{for any } Y \in \mathfrak{m}_1, Z \in \mathfrak{m}_2 \oplus \mathfrak{m}_3. \quad \blacksquare$$

3. EXAMPLES

3.1. THE MANIFOLDS OF ORIENTED FLAGS In [10] we considered manifolds of oriented flags of the form

$$SO(n)/SO(2) \times SO(n-3) \quad (n \geq 4) \quad (23)$$

as homogeneous Φ -spaces [11] of order 6. We proved that for any $n \geq 4$ the reductive complement \mathfrak{m} of any such space is decomposed into the direct sum $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ of irreducible $Ad(H)$ -invariant summands. For the canonical f -structures on this homogeneous Φ -space of order 6 we obtained the following result (in the notations of [10]).

1) For $f_1(\theta) = \frac{1}{\sqrt{3}}(\theta - \theta^5)$

$$\text{Im } f_1 = \mathfrak{m}_1 \oplus \mathfrak{m}_2, \quad \text{Ker } f_1 = \mathfrak{m}_3.$$

$$2) \text{ For } f_2(\theta) = \frac{1}{2\sqrt{3}}(\theta - \theta^2 + \theta^4 - \theta^5)$$

$$\text{Im } f_2 = \mathfrak{m}_2, \text{ Ker } f_2 = \mathfrak{m}_1 \oplus \mathfrak{m}_3.$$

$$3) \text{ For } f_3(\theta) = \frac{1}{2\sqrt{3}}(\theta + \theta^2 - \theta^4 - \theta^5)$$

$$\text{Im } f_3 = \mathfrak{m}_1, \text{ Ker } f_3 = \mathfrak{m}_2 \oplus \mathfrak{m}_3.$$

$$4) \text{ For } f_4(\theta) = \frac{1}{\sqrt{3}}(\theta^2 - \theta^4)$$

$$\text{Im } f_4 = \mathfrak{m}_1 \oplus \mathfrak{m}_2, \text{ Ker } f_4 = \mathfrak{m}_3.$$

In [10] it was checked that for any $i \in \{1, 2, 3, 4\}$ f_i is compatible with any invariant Riemannian metric (13), where $g_0 = -B(X, Y) = -(n-2) \text{Tr}(X \cdot Y)$.

The application of Theorem 2 immediately gives us that (f_2, g) and (f_3, g) are not Killing f -structures for any invariant Riemannian metric g . Nevertheless, (f_2, g) and (f_3, g) are nearly Kähler f -structures (and, hence, G_1f -structures) with respect to any invariant Riemannian metric g .

Taking account of the facts that $[f_1X, f_1^2X] = 0$, $[f_4X, f_4^2X] \neq 0$, and $[Y, f_1Z] + f_1([Y, Z]) = 0$ for any $X \in \mathfrak{m}$, $Y \in \mathfrak{m}_3$, and $Z \in \mathfrak{m}_1 \oplus \mathfrak{m}_2$, by Theorem 3, we obtain

- 1) (f_1, g) and (f_4, g) are G_1f -structures for any invariant Riemannian metric g ;
- 2) (f_1, g) belongs to **NKf** if and only if the characteristic numbers of g are (s, s, t) ($s, t > 0$); (f_4, g) is not a nearly Kähler f -structure for any invariant Riemannian metric g ;
- 3) (f_1, g) belongs to **Killf** if and only if the characteristic numbers of g are $(3s, 3s, 4s)$, where $s > 0$. (f_4, g) is not a Killing f -structure for any invariant Riemannian metric g .

The same results were obtained in [10] by means of direct calculations.

3.2. THE COMPLEX FLAG MANIFOLD All invariant metric f -structures on the complex flag manifold $SU(3)/T_{max}$ (T_{max} is a maximal torus of $SU(3)$) were considered in the view of generalized Hermitian geometry in [9]. Therefore, here we restrict ourselves to mentioning that $SU(3)/T_{max}$ satisfies the conditions of Assumption 1. Hence Theorems 2 and 3 are applicable in this case.

3.3. THE STIEFEL MANIFOLD Let us consider $G/H = SO(4)/SO(2)$ (a Stiefel manifold). Then

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & a & b_1 & b_2 \\ -a & 0 & c_1 & c_2 \\ -b_1 & -c_1 & 0 & 0 \\ -b_2 & -c_2 & 0 & 0 \end{pmatrix} : a, b_1, b_2, c_1, c_2 \in \mathbb{R} \right\}.$$

It is not difficult to see that the manifold in question satisfies Assumption 1. Indeed, there is a decomposition of \mathfrak{m} into the sum of three $Ad(H)$ -invariant mutually inequivalent irreducible submodules $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ (see [2]), where

$$\begin{aligned} \mathfrak{m}_1 &= \left\{ \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : a \in \mathbb{R} \right\}, \\ \mathfrak{m}_2 &= \left\{ \begin{pmatrix} 0 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & 0 \\ -b_1 & 0 & 0 & 0 \\ -b_2 & 0 & 0 & 0 \end{pmatrix} : b_1, b_2 \in \mathbb{R} \right\}, \\ \mathfrak{m}_3 &= \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 & c_2 \\ 0 & -c_1 & 0 & 0 \\ 0 & -c_2 & 0 & 0 \end{pmatrix} : c_1, c_2 \in \mathbb{R} \right\}. \end{aligned}$$

The conditions A_3) and A_4) are easily checked by straightforward calculations.

Let us consider the following f -structures on this manifold:

$$\begin{aligned} f_1 : \begin{pmatrix} 0 & a & b_1 & b_2 \\ -a & 0 & c_1 & c_2 \\ -b_1 & -c_1 & 0 & 0 \\ -b_2 & -c_2 & 0 & 0 \end{pmatrix} &\longrightarrow \begin{pmatrix} 0 & 0 & b_2 & -b_1 \\ 0 & 0 & 0 & 0 \\ -b_2 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 \end{pmatrix}, \\ f_2 : \begin{pmatrix} 0 & a & b_1 & b_2 \\ -a & 0 & c_1 & c_2 \\ -b_1 & -c_1 & 0 & 0 \\ -b_2 & -c_2 & 0 & 0 \end{pmatrix} &\longrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & c_2 & -c_1 \\ 0 & -c_2 & 0 & 0 \\ 0 & c_1 & 0 & 0 \end{pmatrix}, \end{aligned}$$

$$f_3 : \begin{pmatrix} 0 & a & b_1 & b_2 \\ -a & 0 & c_1 & c_2 \\ -b_1 & -c_1 & 0 & 0 \\ -b_2 & -c_2 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & b_2 & -b_1 \\ 0 & 0 & c_2 & -c_1 \\ -b_2 & -c_2 & 0 & 0 \\ b_1 & c_1 & 0 & 0 \end{pmatrix},$$

$$f_4 : \begin{pmatrix} 0 & a & b_1 & b_2 \\ -a & 0 & c_1 & c_2 \\ -b_1 & -c_1 & 0 & 0 \\ -b_2 & -c_2 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & b_2 & -b_1 \\ 0 & 0 & -c_2 & c_1 \\ -b_2 & c_2 & 0 & 0 \\ b_1 & -c_1 & 0 & 0 \end{pmatrix}.$$

There is no difficulty in checking that these f -structures are invariant and compatible with any invariant Riemannian metric (13), where $g_0 = -B(X, Y) = -2 \operatorname{Tr}(X \cdot Y)$.

By Theorem 2, we obtain that both (f_1, g) and (f_2, g) , where g is an arbitrary invariant Riemannian metric, belong to the classes \mathbf{NKf} and $\mathbf{G_1f}$, but they are not Killing f -structures.

By Theorem 3, we immediately see that (f_3, g) and (f_4, g) are G_1f -structures for any invariant Riemannian metric.

As f_3 does not satisfy (16), (f_3, g) is not an NKf -structure, and, consequently, not a Killing f -structure with respect to any invariant Riemannian metric.

The verification of the respective conditions of Theorem 3 yields that (f_4, g) is an NKf -structure if and only if the characteristic numbers of g are (s, t, t) , where $s, t > 0$. (f_4, g) belongs to \mathbf{Killf} if and only if the characteristic numbers of g are $(4s, 3s, 3s)$, where $s > 0$.

3.4. THE QUATERNIONIC FLAG MANIFOLD To conclude this paper, we consider the example of the quaternionic flag manifold $G/H = Sp(3)/SU(2) \times SU(2) \times SU(2)$, which also satisfies Assumption 1 [20]. In this case

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & x & y \\ -\bar{x} & 0 & z \\ -\bar{y} & -\bar{z} & 0 \end{pmatrix} : x, y, z \in \mathbb{H} \right\}, \quad \mathfrak{m}_1 = \left\{ \begin{pmatrix} 0 & x & 0 \\ -\bar{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : x \in \mathbb{H} \right\},$$

$$\mathfrak{m}_2 = \left\{ \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ -\bar{y} & 0 & 0 \end{pmatrix} : y \in \mathbb{H} \right\}, \quad \mathfrak{m}_3 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & -\bar{z} & 0 \end{pmatrix} : z \in \mathbb{H} \right\}.$$

Let $\{p, q, r\} = \{1, 2, 3\}$. Suppose that $a_1, a_2, a_3 \in \mathbb{R}$ are such that $a_1^2 + a_2^2 + a_3^2 = 1$. It is straightforward to check that

$$f(X) = \begin{cases} (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})X, & X \in \mathfrak{m}_p, \\ 0, & X \in \mathfrak{m}_q \oplus \mathfrak{m}_r \end{cases}$$

is both invariant and compatible with any invariant Riemannian metric (13), where $g_0 = -\operatorname{Re}(B(X, Y)) = -8 \operatorname{Re} \operatorname{Tr}(X \cdot Y)$. By Theorem 2, f is both NKf - and G_1f -structure. At the same time, it is not a Killing f -structure with respect to any invariant Riemannian metric.

Also invariant and compatible with any invariant Riemannian metric (13) are f -structures of the form

$$f_1 : \begin{pmatrix} 0 & x & y \\ -\bar{x} & 0 & z \\ -\bar{y} & -\bar{z} & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & h_1x & h_2y \\ -\bar{h}_1x & 0 & 0 \\ -\bar{h}_2y & 0 & 0 \end{pmatrix}, \quad (24)$$

where $h_1, h_2 \in \mathbb{H}$ are such that $\operatorname{Re} h_1 = \operatorname{Re} h_2 = 0$, $|h_1| = |h_2| = 1$.

In this case we have

$$[f_1X, f_1^2X]_{\mathfrak{m}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{h}_1xy - \bar{x}h_2y \\ 0 & \bar{h}_2yx - \bar{y}h_1x & 0 \end{pmatrix},$$

where

$$X = \begin{pmatrix} 0 & x & y \\ -\bar{x} & 0 & z \\ -\bar{y} & -\bar{z} & 0 \end{pmatrix} \in \mathfrak{m}.$$

For this reason, $[f_1X, f_1^2X]_{\mathfrak{m}} = 0$ for any $X \in \mathfrak{m}$ if and only if $h_1 = \bar{h}_2 = -h_2$. At the same time, there exist such $Y \in \mathfrak{m}_3$, $Z \in \mathfrak{m}_1 \oplus \mathfrak{m}_2$ that, regardless of the choice of h_1 and h_2 , $[Y, fZ] + f([Y, Z]) \neq 0$.

Thus, an invariant metric f -structure (f_1, g) , where f_1 is of the form (24), g is an arbitrary Riemannian metric, belongs to the class $\mathbf{G}_1\mathbf{f}$ and does not belong to the class $\mathbf{Kill}\mathbf{f}$. In this case (f_1, g) is an NKf -structure if and only if $h_1 = -h_2$ and the characteristic numbers of g are (λ, λ, μ) , where $\lambda, \mu > 0$.

Arguing as above, we obtain that for any invariant Riemannian metric g (f_2, g) and (f_3, g) are G_1f structures and are not NKf -structures (and, consequently, not Killing f -structures). Here

$$f_2 : \begin{pmatrix} 0 & x & y \\ -\bar{x} & 0 & z \\ -\bar{y} & -\bar{z} & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & h_1 x & 0 \\ -\overline{h_1 x} & 0 & h_2 z \\ 0 & -\overline{h_2 z} & 0 \end{pmatrix},$$

$$f_3 : \begin{pmatrix} 0 & x & y \\ -\bar{x} & 0 & z \\ -\bar{y} & -\bar{z} & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & h_1 y \\ 0 & 0 & h_2 z \\ -\overline{h_1 y} & -\overline{h_2 z} & 0 \end{pmatrix},$$

where $h_1, h_2 \in \mathbb{H}$ are such that $\operatorname{Re} h_1 = \operatorname{Re} h_2 = 0$, $|h_1| = |h_2| = 1$.

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REFERENCES

- [1] D. V. ALEKSEEVSKY, V. V. Lychagin, A. M. Vinogradov, Main concepts and notions of differential geometry, *Itoji Nauki i Tekhniki: Fundamental Research VINITI* **28** (1988), 5–289 (in Russian).
- [2] A. Arvanitoyeorgos, An introduction to Lie groups and the geometry of homogeneous spaces, Student Mathematical Library, vol. 22, American Mathematical Society, Providence, RI, 2003.
- [3] V. V. Balashchenko, Riemannian geometry of canonical structures on regular Φ -spaces, Preprint No. 174/1994, Fakultät für Mathematik der Ruhr-Universität Bochum, 1994, 1–19.
- [4] V. V. Balashchenko, Naturally reductive Killing f -manifolds, *Uspekhi Mat. Nauk* **54** (3) (1999), 151–152; (English translation: *Russian Math. Surveys* **54** (3) (1999), 623–625).
- [5] V. V. Balashchenko, Homogeneous Hermitian f -manifolds, *Uspekhi Mat. Nauk* **56** (3) (2001), 159–160; (English translation: *Russian Math. Surveys* **56** (3) (2001), 575–577).
- [6] V. V. Balashchenko, Homogeneous nearly Kähler f -manifolds, *Dokl. Akad. Nauk* **376** (4) (2001), 439–441; (English translation: *Dokl. Math.* **63** (1) (2001), 56–58).
- [7] V. V. Balashchenko, Invariant nearly Kähler f -structures on homogeneous spaces, *Contemp. Math.* **288** (2001), 263–267.
- [8] V. V. Balashchenko, Invariant structures generated by Lie group automorphisms on homogeneous spaces, Proceedings of the Workshop “Contemporary Geometry and Related Topics” (Belgrade, Yugoslavia, 15–21 May, 2002) (N. Bokan, M. Djoric, A. T. Fomenko, Z. Rakic, J. Wess., eds.), World Sci. Publ., River Edge, NJ, 2004, 1–32.
- [9] V. V. Balashchenko, Invariant f -structures in the generalized Hermitian geometry, <http://arxiv.org/abs/math.DG/0503533>.

- [10] V. V. BALASHCHENKO, A. SAKOVICH, Invariant f -structures on the flag manifolds $SO(n)/SO(2) \times SO(n-3)$, *Int. J. Math. Math. Sci.* **2006** (2006), 1–15.
- [11] V. V. BALASHCHENKO, N. A. STEPANOV, Canonical affinor structures of classical type on regular Φ -spaces, *Mat. Sb.* **186** (11) (1995), 3–34, ; (English translation: *Sb. Math.* **186** (11) (1995), 1551–1580).
- [12] A. GRAY, Nearly Kähler manifolds, *J. Differential Geom.* **4** (3) (1970), 283–309.
- [13] A. GRAY, L. M. HERVELLA, The sixteen classes of almost Hermitian manifolds and their linear invariants, *Ann. Mat. Pura Appl.* **123** (4) (1980), 35–58.
- [14] A. S. GRITSANS, Geometry of Killing f -manifolds, *Uspekhi Mat. Nauk* **45** (4) (1990), 149–150; (English translation: *Russian Math. Surveys* **45** (4) (1990), 168–169).
- [15] A. S. GRITSANS, On construction of Killing f -manifolds, *Izv. Vyssh. Uchebn. Zaved. Mat.* **6** (1992), 49-57; (English translation: *Soviet Math. (Iz. VUZ)* **36** (6) (1992)).
- [16] V. F. KIRICHENKO, Methods of generalized Hermitian geometry in the theory of almost contact manifolds, *Itogi Nauki i Tekhniki, Problems of Geometry, VINITI* **18** (1986), 25–71; (English translation: *J. Soviet Math.* **42** (5) (1988), 1885–1919).
- [17] V. F. KIRICHENKO, Generalized quasi-Kaehlerian manifolds and axioms of CR-submanifolds in generalized Hermitian geometry, I, *Geom. Dedicata* **51** (1) (1994), 75–104.
- [18] S. KOBAYASHI, K. NOMIZU, “Foundations of Differential Geometry, Vol. II”, Interscience Publishers, John Willey & Sons, Inc., New York-London-Sydney, 1969.
- [19] K. NOMIZU, Invariant affine connections on homogeneous spaces, *Amer. J. Math.* **76** (1) (1954), 33–65.
- [20] N. WALLACH, Compact homogeneous Riemannian manifolds with strictly positive curvature, *Ann. Math.* **96** (2) (1972), 277–295.
- [21] K. YANO, On a structure defined by a tensor field f of type $(1, 1)$ satisfying $f^3 + f = 0$, *Tensor (N.S.)* **14** (1963), 99–109.