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# When is a Group Homomorphism a Covering Homomorphism?

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*Abstract*: Let $G$ be a topological group which acts in a continuous and transitive way on a topological space $M$. Sufficient conditions are given that assure that, for every $m \in M$, the map from $G$ onto $M$ defined by $g \mapsto g \cdot m$ is an open map. Some consequences of the existence of these conditions, concerning spinor groups and covering homomorphisms between Lie groups, are obtained.

*Key words*: covering, group homomorphism, Lie group, open map, spinor.

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## Introduction

A standard reference concerning Clifford algebras and spinor groups is [2, Part I]. In that article, the authors define, for each $k \in \mathbb{N}$, the spinor group $\mathrm{Spin}(k)$ as a group of invertible elements of the real Clifford algebra $C_k$. There is a natural continuous homomorphism $\rho$ from $\mathrm{Spin}(k)$ to $SO(k, \mathbb{R})$ and the authors state that $\rho$ is a covering homomorphism (see [2, Proposition 3.13]). However, what is actually proved is just that $\rho$ is surjective and that the kernel has two elements and this is not enough to prove the statement. The same problem arises in [3, §I.6], in [5, §20.2] and in [6, §4.7]. The goal of this article is to state and prove a theorem concerning topological groups that assures that $\rho$ is really a covering homomorphism. Another way of doing this, using Lie theory, can be found in [4, §II.XI]. We also give a new proof of a theorem concerning covering homomorphisms between Lie groups.

## 1. The main theorem

In what follows, every topological space (and, in particular, every topological group) is Hausdorff. The unit element of a group $G$ will be denoted by $e_G$ or simply by $e$, when there is only one group involved. The concepts and basic facts concerning topological groups which will be needed here can be found at [4, Chapter II].

If $\varphi$ is a continuous homomorphism from a topological group $G$ to a topological group $H$, in order that $\varphi$ is a covering homomorphism it is necessary that $\varphi$ is surjective and that the kernel of $\varphi$ is discrete. In general, these conditions are not sufficient to assure that $\varphi$ is a covering homomorphism. As an example, let $\alpha$ be a real irrational number and let $G$ be the subgroup of the torus $S^1 \times S^1$ whose elements are those of the form $(\exp(\mathrm{i}t), \exp(\mathrm{i}\alpha t))$, for some $t \in \mathbb{R}$. Consider the homomorphism of the group $(\mathbb{R}, +)$ onto $G$ that maps each $t \in \mathbb{R}$ into $(\exp(\mathrm{i}t), \exp(\mathrm{i}\alpha t))$. If you consider in $\mathbb{R}$ and in $G$ the usual topologies, then this map is a continuous and bijective homomorphism, but it is not a homeomorphism since $G$ is not locally compact. An even simpler example is given by the identity map from $(\mathbb{R}, +)$ (with the discrete topology) onto $(\mathbb{R}, +)$ (with the usual topology).

In order to give general conditions concerning two topological groups $G$ and $H$ that assure that each continuous and surjective homomorphism from $G$ onto $H$ with discrete kernel is a covering homomorphism, we shall have to prove a theorem concerning group actions on topological spaces.

THEOREM 1. *Let $G$ be a Lindelöf and locally compact topological group which acts in a continuous and transitive way on a Baire space $M$. If $m \in M$, then the map*

$$\begin{array}{ccc} G & \longrightarrow & M \\ g & \longmapsto & g \cdot m \end{array}$$

*is an open map.*

*Proof.* It will be enough to prove that if $V$ is a neighborhood of $e$, then $V \cdot m$ is a neighborhood of $m$. Let $W$ be a neighborhood of $e$ such that $W^{-1} \cdot W \subset V$ and suppose that $W \cdot m$ is a neighborhood of some of its points; in other words, suppose that, for some $w_0 \in W$, $W \cdot m$ is a neighborhood of $w_0 \cdot m$. Then ${w_0}^{-1} \cdot (W \cdot m)$ is a neighborhood of $m$ and therefore

$$\bigcup_{w \in W} w^{-1} \cdot (W \cdot m) \quad (= V \cdot m)$$

is a neighborhood of $m$.

Therefore, all that remains to be proved is that among all neighborhoods $W$ of $e$ such that $W^{-1} \cdot W \subset V$ there is at least one such that $W \cdot m$ is a neighborhood of some of its points, and this is equivalent to saying that the interior of $W \cdot m$ is not empty. Let $W$ be a compact neighborhood of $e$ such that $W^{-1} \cdot W \subset V$; such a neighborhood exists since we are supposing that $G$ is locally compact. It is clear that the interior of $W \cdot m$ is not empty if and only if, for some $g \in G$, the interior of $g \cdot (W \cdot m)$ is not empty. It follows from the fact that $G$ is a Lindelöf space and from the fact that $\bigcup_{g \in G} g \cdot \mathring{W} = G$ that there is a sequence $(g_n)_{n \in \mathbb{N}}$ of elements of $G$ such that $\bigcup_{n \in \mathbb{N}} g_n \cdot W = G$ and, therefore, such that $\bigcup_{n \in \mathbb{N}} g_n \cdot (W \cdot m) = M$, since the action of $G$ on $M$ is transitive. For each $n \in \mathbb{N}$, $g_n \cdot (W \cdot m)$ is a compact set, since $W$ is compact and the action is continuous, and, in particular, each set $g_n \cdot (W \cdot m)$ is a closed set. Since $M$ is a Baire space, there is at least one $n \in \mathbb{N}$ such that the interior of $g_n \cdot (W \cdot m)$ is not empty and, as it has already been observed, this is equivalent to the assertion that the interior of $W \cdot m$ is not empty. ∎

This proof is adapted from the proof of the corollary in [1, §9] (see Corollary 2 below).

It should be observed that if $G$ is a connected and locally compact topological group, then $G$ is also a Lindelöf space. In fact, since $G$ is connected, it is generated by any neighborhood of $e$ (see [4, §II.IV, Theorem 1]) and therefore if $V$ is a compact neighborhood of $e$ then $G = \bigcup_{n \in \mathbb{N}} V^n$. This proves that $G$ is $\sigma$-compact and therefore Lindelöf. Of course, it follows from this observation and from the fact that any connected component of a topological group is homeomorphic to the connected component of the unit element that, more generally, if a locally compact group $G$ has only a finite or countable set of connected components, then $G$ is Lindelöf.

Before we proceed, let us see an interesting consequence of the previous theorem. This corollary is the corollary of [1, §9] that was mentioned above; we prove it for completeness and because the proof is very short.

Corollary 2. *Let $G$ be a Lindelöf and locally compact group which acts in a continuous and transitive way on a Baire space $M$. Given $m \in M$, if $H$ is the stabilizer of $m$ in $G$ and if in $G/H$ one considers the final topology with respect to the natural projection from $G$ onto $G/H$, then the map*

$$\begin{array}{ccc} G/H & \longrightarrow & M \\ gH & \longmapsto & g \cdot m \end{array}$$

*is a homeomorphism.*

*Proof.* The map is clearly a continuous bijection and all that remains to be proved is that it is an open map. If $A$ is an open set of $G/H$ and $\pi : G \to G/H$

denotes the natural projection, then $A$ is mapped onto $\pi^{-1}(A) \cdot m$ and this set is an open set, by the previous theorem. ∎

THEOREM 3. *Let $G$ and $H$ be topological groups and suppose that, as topological spaces, $G$ is Lindelöf and locally compact and $H$ is a Baire space. If $\varphi$ is a continuous homomorphism from $G$ onto $H$, then $\varphi$ is a covering homomorphism if and only if its kernel is discrete.*

*Proof.* The homomorphism $\varphi$ induces the action from $G$ on $H$ defined by

$$\begin{array}{ccc} G & \longrightarrow & \operatorname{Aut}(H) \\ g & \longmapsto & \left( \begin{array}{ccc} H & \to & H \\ h & \mapsto & \varphi(g) \cdot h \end{array} \right). \end{array}$$

This action is continuous (since $\varphi$ is continuous) and transitive (since $\varphi$ is surjective). Therefore, it follows from the theorem 1 (with $m = e_H$) that $\varphi$ is an open map. Let $V$ be a neighborhood of $e_G$ such that $V \cap \ker \varphi = \{e_G\}$, let $W$ be an open neighborhood of $e_G$ such that $W \cdot W^{-1} \subset V$ and define $W' = \varphi(W)$. Since $\varphi$ is an open map, $W'$ is a neighborhood of $e_H$. Then

$$\varphi^{-1}(W') = \bigcup_{g \in \ker \varphi} g \cdot W$$

and, furthermore, this is a disjoint union, because if $g, h \in \ker \varphi$ and $v, w \in W$ are such that $g \cdot v = h \cdot w$, then $v \cdot w^{-1} = g^{-1} \cdot h \in \ker \varphi$; since $v \cdot w^{-1} \in V$, it follows that $g = h$. Therefore $\varphi^{-1}(W')$ is homeomorphic to $\ker(\varphi) \times W'$ when we consider in $\ker \varphi$ the discrete topology. This proves that $\varphi$ is a covering homomorphism. ∎

In order to apply this theorem to the spinor groups, it will be enough to prove that these groups are Lindelöf and locally compact. But it is a consequence of the definition of $\operatorname{Spin}(k)$ (see [2, pp. 6–8]) that this group can be seen as a closed subset of a finite-dimensional real vector space (with the usual topology); therefore, it is both a Lindelöf space and a locally compact space. Since $SO(k, \mathbb{R})$ is compact (and therefore a Baire space) the natural homomorphism from $Spin(k)$ onto $SO(k, \mathbb{R})$ is a covering homomorphism. As it was observed before (see [2, Part I] and [3, §I.6]), this fact can be used to prove that $\operatorname{Spin}(k)$ has a Lie group structure.

## 2. Lie group homomorphisms

Let us extract another consequence of Theorem 3. If $\varphi$ is an analytic homomorphism from a Lie group $G$ to a Lie group $H$, let $\varphi^*$ denote the differential of $\varphi$ at $e_G$. Note that, since every connected Lie group is locally compact, Lindelöf and a Baire space, theorem 3 implies that an analytic homomorphism $\varphi$ from a connected Lie group $G$ to a Lie group $H$ is a covering homomorphism if and only if $\varphi$ is surjective and $\ker\varphi$ is discrete.

Theorem 4. *If $G$ and $H$ are connected Lie groups and $\varphi$ is an analytic homomorphism from $G$ onto $H$, then $\varphi$ is a covering homomorphism if and only if $\varphi^*$ is an isomorphism.*

*Proof.* Using the exponential map it is easy to prove that if $\varphi^*$ is surjective then $\varphi$ is also surjective. In fact, these statements are equivalent. If $\varphi$ is surjective, then it induces a bijective analytic homomorphism $\psi : G/\ker(\varphi) \to H$. It is in fact a homeomorphism; this can be seen as a consequence of Corollary 2 or as an application of the theorem of invariance of domain. Since every continuous homomorphism between Lie groups is analytic (see [4, §IV.XIII] or [7, Theorem 3.39]), it follows that $\psi^{-1}$ is also analytic. Therefore, $\psi^*$ is an isomorphism and this implies that $\varphi^*$ is surjective; in fact, if $\pi$ denotes the natural projection from $G$ onto $G/\ker(\varphi)$, then $\pi^*$ is surjective and

$$\varphi = \psi \circ \pi \qquad \Longrightarrow \qquad \varphi^* = \psi^* \circ \pi^*.$$

Finally, observe that $\varphi^*$ is injective if and only if the kernel of $\varphi$ is discrete. Indeed, if $\varphi^*$ is not injective, then there is some $X$ in the Lie algebra $\mathfrak{g}$ of $G$ such that $X \neq 0$ and that $\varphi^*(X) = 0$, and this would imply that

$$\varphi(\exp(tX)) = \exp(t\varphi^*(X)) = e_H \qquad \text{for all } t \in \mathbb{R}.$$

On the other hand, if $\varphi^*$ is injective and if $U$ is neighborhood of 0 in $\mathfrak{g}$ such that $\exp|_U$ and $\exp|_{\varphi^*(U)}$ are injective and that $\exp(U)$ is a neighborhood $V$ of $e_G$, then every $g \in V \setminus \{e_G\}$ has the form $\exp(X)$ for some $X \in U \setminus \{0\}$ and therefore

$$\varphi(g) = \varphi(\exp(X)) = \exp(\varphi^*(X))\,;$$

since $\varphi^*(X) \neq 0$ and $\exp|_{\varphi^*(U)}$ is injective, this proves that $\varphi(g) \neq e_H$. ∎

Cf. [7, p. 100] for another proof of Theorem 4.

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