EXTRACTA MATHEMATICAE Vol. **22**, Núm. 2, 115 – 126 (2007)
Satellite conference on Banach spaces · ICM2006

Gaussian Behaviour and Average of Marginals for Convex Bodies *

Jesús Bastero, Julio Bernués

Departamento de Matemáticas, Universidad de Zaragoza, 50009 *Zaragoza, Spain*
bastero@unizar.es, bernues@unizar.es

Received March 29, 2007

Abstract: We recall several instances of the so called Central Limit Theorem for convex bodies under different metrics. We prove a Central Limit type theorem which holds for random k-dimensional subspace $E \subset \mathbb{R}^n$ and for a certain class of isotropic bodies.

Key words: isotropic convex bodies, central limit, concentration phenomena.

AMS *Subject Class.* (2000): 46B20, 52A22, 60F05.

1. Introduction and notation

Dvoretzky's theorem [5] in its infinitely dimensional setting says that ℓ_2 is finitely representable in any infinite dimensional normed space X. More precisely, in Milman's approach [6], given $\varepsilon > 0$ and any n-dimensional normed space $(X, \|\cdot\|)$, for every $1 \le k \le C\varepsilon^2 \log n$, there exists a k-dimensional subspace $E \subseteq X$ such that

$$B_X \cap E \subset T(B_2^k) \subset (1+\varepsilon) B_X \cap E \tag{1}$$

where $C > 0$ is an absolute constant and T some isomorphism $T : \ell_2^k \to E$ $(B_X = \{x \in X \,:\, \|x\| \le 1\})$.

How many k-dimensional subspaces E verify (1)? In order to answer this question we introduce some euclidean structure. Assume $(X, \|\cdot\|) = (\mathbb{R}^n, \|\cdot\|_K)$ so that the Euclidean ball is the ellipsoid of maximal volume contained K, then, there exists $r > 0$ such that for every $0 < \varepsilon < 1$

$$rB_2^n \cap E \subset K \cap E \subset (1+\varepsilon) rB_2^n \cap E \tag{2}$$

and

$$\nu_{n,k} \{E \in G_{n,k} \,:\, E \text{ verifies } (2)\} \ge 1 - \exp(-ck)$$

* Authors partially supported by DGA (Spain) E64, MEyC (Spain) MTM2004-03036 and Marie Curie RTN CT-2004-511953.

for some absolute constant $c > 0$ ($\nu_{n,k}$ denotes the Haar measure in the grasmannian of the k dimensional subspaces in $\mathbb{R}^n$).

Let K be a centrally symmetric convex body in $\mathbb{R}^n$ (K is compact convex with the origen in its interior and $K = -K$). We can state the theorem in a more geometrical way as follows:

Theorem 1.1. *Every centrally symmetric convex body in $\mathbb{R}^n$ has a position TK so that given any $\varepsilon > 0$, random k-dimensional central sections of K are almost a multiple of the Euclidean ball, whenever $1 \le k \le C\varepsilon^2 \log n$. The same is true for random orthogonal projections of K.*

Recall that in the finite dimensional context, the dual of a subspace is a quotient space or equivalently an orthogonal projection. One of the main tools in the proof of Dvoretzky's theorem is the *concentration of measure phenomenon*, which can be expressed in the following way:

Theorem 1.2. *There exist absolute constants $c_1, c_2 > 0$ such that for any continuous function $f : S^{n-1} \to \mathbb{R}$ we have*

$$\mu_n \left\{ \theta \,:\, \left| f(\theta) - \int_{S^{n-1}} f \, \mathrm{d}\mu_n \right| > \omega_f(a) \right\} \le c_1 \exp\left(-c_2 n a^2\right),$$

where

$$\omega_f(a) = \sup_{d(\theta_1,\theta_2) \le a} |f(\theta_1) - f(\theta_2)|$$

is the modulus of continuity of f and μ_n is the normalized measure on S^{n-1}.

The same phenomenon happens in the Grasmaniann $G_{n,k}$ equipped with the measure $\nu_{n,k}$ and the Haussdorf distance between the unit spheres of k-dimensional subspaces.

Next let us consider a probabilistic framework. For K a convex body of volume 1, the restriction of the Lebesgue measure on K is a probability, $\mathbb{P} = |\cdot|$, on $\mathbb{R}^n$. Project this probability onto a k dimensional subspace E to produce the marginal probability $\mathbb{P}_E$ defined by

$$\mathbb{P}_E(B) = \mathbb{P}(B + E^\perp) = |\{x \in \mathbb{R}^n \,:\, P_E(x) \in B\}|$$

for every $B \subset E$ (P_E is the orthogonal projection onto E). According to Dvoretzky's theorem the support of $\mathbb{P}_E$ is almost a multiple of an Euclidean ball for random k-dimensional subspaces E (and $1 \le k \le C\varepsilon^2 \log n$), whenever K is in John's position.

The question now is, how is $\mathbb{P}_E$ distributed? This question actually comes from Sudakov [15] in the seventies. Is $\mathbb{P}_E$ also like a probability on a k-dimensional euclidean ball?, i.e., is it gaussian? This is the content of the so called *Central Limit Problem for convex bodies*.

Let us recall the classical Central Limit Theorem in a particular case:

THEOREM 1.3. *Let X be a random variable uniformly distributed on the real interval $[-1/2, 1/2]$, and let $\{X_i\}_1^n$ be a sequence of n i.i.d. copies of X, then*

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} \underset{n\to\infty}{\longrightarrow} N\left(0, \frac{1}{\sqrt{12}}\right)$$

in the sense that in the total variation metric

$$\left\| \mu_n - N\left(0, \frac{1}{\sqrt{12}}\right) \right\|_{TV} = \int_{-\infty}^{\infty} \left| f_n(t) - \sqrt{\frac{6}{\pi}} \exp\left(\frac{-t^2}{24}\right) \right| \mathrm{d}t \underset{n\to\infty}{\longrightarrow} 0$$

f_n is the density of the probability μ_n given by the r.v. $X_1 + \cdots + X_n/\sqrt{n}$.

The geometric interpretation of this theorem says that given

$$\theta = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right) \in S^{n-1},$$

the 1-dimensional marginal probability $\mathbb{P}_\theta$ of the uniform distribution on the cube $[-1/2, 1/2]^n$ is close to a normal distribution, when the dimension is large. The same thing happens for random $\theta \in S^{n-1}$. It was conjectured among the specialists that a version of the central limit theorem should be true for a general class of convex bodies, those which are in isotropic position.

In this note we want to survey several recent results on the central limit problem. In order to answer it, first we need to put the convex body in a special position in which all the 1-dimensional marginals have the same variance. This is the isotropic position.

A convex body K (no necessarilly centrally symmetric) of volume one is in isotropic position if

- the mean is 0:
$$\int_K x \,\mathrm{d}x = 0,$$
- the variance of all the 1-dimensional marginals is constant:
$$\int_K \langle\theta, x\rangle^2 \,\mathrm{d}x = C \qquad \text{for all } \theta \in S^{n-1};$$

or equivalently $\int_K x_i x_j \mathrm{d}x = C\delta_{i,j}$. The constant $C = L_K^2$ is the isotropy constant. Every convex body of volume one has a unique, up to orthogonal transformations, isotropic position.

A key ingredient in the solution of the problem is the concentration of the euclidean norm on K. This fact is already considered in works by [4], [16].

We consider the following theorem by Antilla, Ball and Perisinaki [2]:

Theorem 1.4. *There exists an absolute constant $c > 0$ such that, given $0 < \varepsilon < 1/2$ and any isotropic symmetric convex body K in $\mathbb{R}^n$ verifying the ε-concentration hypothesis*

$$\left|\left\{x \in K \,:\, \big||x| - \sqrt{n}L_K\big| \geq \varepsilon\sqrt{n}L_K\right\}\right| \leq \varepsilon\,,$$

then

$$\nu\left\{\theta \in S^{n-1} \,:\, \|F_\theta(t) - \Gamma_\theta(t)\|_\infty < \varepsilon + \frac{c}{n}\right\} \geq 1 - n\exp\left(-cn\varepsilon^2\right),$$

where $F_\theta(t) = |\{x \in K \,:\, \langle x, \theta\rangle \leq t\}|$ and

$$\Gamma_\theta(t) = \frac{1}{\sqrt{2\pi}L_K}\int_{-\infty}^{t} \exp\left(-|x|^2/2L_K^2\right) \mathrm{d}x\,.$$

That is, under the extra ε-concentration hypothesis, the authors proved that the 1-dimensional marginals of convex bodies are close to gaussian for random directions.

The main references for the solution of the problem are those by Klartag [8], [9] where the author proves that *every* isotropic convex body (symmetric or not) verifies a concentration hypothesis even stronger than the ε-concentration. Moreover, he solves the central limit problem for k-dimensional marginals when k increases up to $1 \leq k \leq cn^\kappa$. His result is also stronger than the conclusion one would expected according to Dvoretzky's theorem, where we can only get $1 \leq k \leq c\log n$, although it is worth to mention here that it is not kwnon whether or not John's and isotropic positions coincides. Also it is not known if the bound given by Klartag is sharp.

Theorem 1.5. (B. Klartag, 2006-07) *There exist universal constants $C, c > 0, \kappa > 0$ for which the following holds: Let $0 < \varepsilon < 1$ and $1 \leq k \leq cn^\kappa$ an integer. If K is an isotropic convex body in $\mathbb{R}^n$ there is a k-dimensional subspace E such that*

$$\|\mathbb{P}_E - \Gamma_E\|_{TV} < \varepsilon\,, \tag{3}$$

where Γ_E is the standard gaussian measure on E with variance L_K. Moreover the measure of the set $\mathcal{E}$ of k-dimensional subspaces E verifying (3) *is*

$$\nu_{n,k}(\mathcal{E}) \geq 1 - \exp\left(-Cn^{0.99}\right).$$

Three are the main ingredients in Klartag's proof:

(i) the use of the methods of Dvoretzky's theorem and the concentration of measure phenomenon on S^{n-1} and directly on $G_{n,k}$;

(ii) establish a new concentration inequality for the euclidean norm in all the isotropic convex bodies

$$\left|\left\{x \in K : \left|\frac{|x|}{\sqrt{n}L_K} - 1\right| > t\right\}\right| \leq A \exp\left(-Bn^{0.33}t^2\right)$$

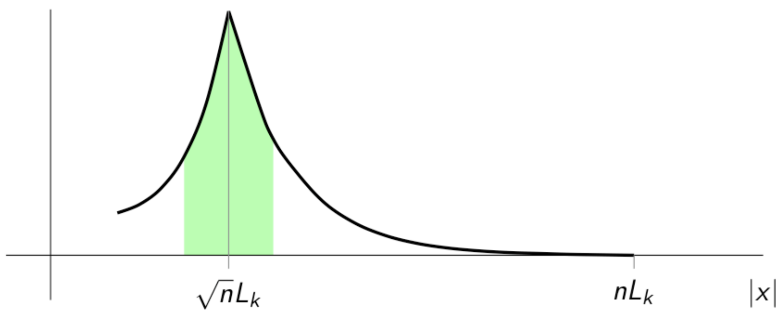
for all $0 \leq t \leq 1$ and for some universal constants $A, B > 0$;

(iii) compare the uniform distribution on K with some other measure invariant under the action of the orthogonal group and having a better concentration inequality.

It is worth to mention here that (ii) together with a recent result by Paouris [13]

$$\left|\left\{x \in K : \frac{|x|}{\sqrt{n}L_K} > Ct\right\}\right| \leq \exp\left(-t\sqrt{n}\right) \qquad \text{for all } t > 1$$

show that every isotropic convex body has the mass concentrated in a thin shell with radius $L_K\sqrt{n}$.



Klartag's result nicely extends to a larger class of isotropic probabilities on $\mathbb{R}^n$, the log-concave (a) isotropic (b) probabilities, that is, those for which:

(a) $\mathbb{P}(\lambda A + (1-\lambda)B) \geq \mathbb{P}(A)^\lambda \mathbb{P}(B)^{1-\lambda}$ for all $\lambda \in [0,1]$ and $A, B \subset \mathbb{R}^n$ borelian subsets, and

(b) all 1-dimensional marginals have mean 0 and the same variance σ^2.

A full account on the history around this problem can be found in [8]. But we want to point out one recent result due to Naor and Romik [12] concerning the central limit problem which uses a weaker metric but the closeness to a gaussian distribution remains valid for other classes of probabilities on $\mathbb{R}^n$ (not necessarily log-concave).

Theorem 1.6. *Let $\mathbb{P}$ be a non-atomic, compactly supported, isotropic Borel probability on $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} x_i^2 x_j^2 \, \mathrm{d}\mathbb{P}(x) \leq \int_{\mathbb{R}^n} x_i^2 \, \mathrm{d}\mathbb{P}(x) \int_{\mathbb{R}^n} x_j^2 \, \mathrm{d}\mathbb{P}(x) , \qquad 1 \leq i, j \leq n .$$

Set $B^4 := \int_{\mathbb{R}^n} \sum_{i=1}^n x_i^4 \, \mathrm{d}\mathbb{P}(x)$. Then for every $0 < \varepsilon < 1$ and $1 \leq k \leq \frac{c\varepsilon^4 n^2}{B^4}$ we have

$$\nu_{n,k} \{ E \in G_{n,k} \, : \, T(\mathbb{P}_E, \Gamma_E) \leq \varepsilon \} \geq 1 - \frac{c_1}{\varepsilon} \exp\left(- \frac{c_2 \varepsilon^4 n^2}{B^4} \right),$$

where

$$T(\mathbb{P}_E, \Gamma_E) = \sup \{ |\mathbb{P}_E(H) - \Gamma_E(H)| \, : \, H \text{ affine half-space} \}$$

is the Tsirelson distance.

For instance, normalized unit balls of ℓ_p^n spaces, $0 < p$, verify their hypothesis.

2. The results

For $E \in G_{n,k}$ define the distribution function

$$\mathbb{P}_E(tB_2^n) = \mathbb{P}\{ x \in \mathbb{R}^n \, : \, |P_E(x)| \leq t \} , \qquad t \geq 0$$

(that is, the marginal measure of $\mathbb{P}$ on E of a t-dilate of the euclidean ball on $\mathbb{R}^n$). Klartag's result implies that for isotropic log-concave probabilities with variance σ we have

$$\sup_t |\mathbb{P}_E(tB_2^n) - \Gamma_E(tB_2^n)| \leq \varepsilon ,$$

where B_2^n is the standard euclidean ball on $\mathbb{R}^n$ and

$$\Gamma_E(tB_2^n) \;=\; \frac{1}{(\sqrt{2\pi}\sigma)^k}\int_{\{x\in E:|x|\le t\}} \exp\left(-\frac{|x|^2}{2\sigma^2}\right)\mathrm{d}x := \int_{\{|x|\le t\}} \gamma_k(x)\,\mathrm{d}x\,.$$

We are interested in the behavior of

$$\sup_{t\ge 0}\left|\frac{\mathbb{P}_E(tB_2^n)}{\Gamma_E(tB_2^n)}-1\right|.$$

This was done in the 1-dimensional case by A. Sodin [14] (1-dimensional marginal), by introducing a strong concentration hypothesis. Our purpose in this work is to extend his result to other dimensions. We will see that some of the results involved actually hold for a quite large class of probabilities.

Let us introduce some notation. Let $\mathbb{P}$ be a Borel probability on $\mathbb{R}^n$,

$$M_2^2(\mathbb{P}) = M_2^2 := \frac{1}{n}\int_{\mathbb{R}^n}|x|^2\mathrm{d}\mathbb{P}(x)$$

which also is the average of the variances of all 1-dimensional marginal,

$$M_{\mathbb{P}} := \sup_{t>0}\frac{\mathbb{P}\{tD_n\}}{|tD_n|}$$

the Hardy-Littlewood maximal function of the probability $\mathbb{P}$ in the origin.

Concentration hypothesis. A probability $\mathbb{P}$ on $\mathbb{R}^n$ verifies (CH) if:

$$\mathbb{P}\left\{x\in\mathbb{R}^n\,:\,\left|\frac{|x|}{\sqrt{n}M_2}-1\right|>t\right\}\;\le\; A\,\exp\left(-Bn^\alpha t^\beta\right)$$

for all $0\le t\le 1$ and for some constants $\alpha,\beta,A,B>0$.

For instance, normalized ℓ_p^n, $p>0$, balls verify (CH) for $2\alpha=\beta=\min\{p,2\}$. Also uniformly convex balls contained in small Euclidean balls verify (CH) (see [14]). After Klartag's result [8] every isotropic convex body satisfies (CH) with $\alpha=0.33$ and $\beta=2$.

The result we obtain for isotropic convex bodies K (i.e. $\mathbb{P}$ is the uniform measure on K and $\sigma=L_K$) is the following:

Theorem 2.1. ([3]) *Let $\mathcal{M}$ be the class of isotropic convex bodies in $\mathbb{R}^n$ that satisfy $L_K\le c_1$. Given $K\in\mathcal{M}$, $0<\varepsilon<1$, we have*

$$\sup_{t\ge 0}\left|\frac{\mathbb{P}_E(tB_2^n)}{\Gamma_E(tB_2^n)}-1\right|\le\varepsilon \tag{*}$$

for random k-dimensional subspace E, whenever

$$1 \le k \le \frac{C\varepsilon \log n}{(\log\log n)^2}\,,$$

where "E random subspace" means that

$$\nu_{n,k}\left\{E \in G_{n,k} \,:\, (*) \text{ occurs}\right\} \ge 1 - \exp\left(-cn^{0.9}\right),$$

where C, c only depend on c_1.

In the proof below we will point out the steps that hold with more generality and those that need extra hypothesis.

Sketch of the proof. The proof will be done in three steps. We compare the individual distributions with the average and the average with 1 via the triangle inequality

$$\begin{aligned}\left|\frac{\mathbb{P}_E(tB_2^n)}{\Gamma_E(tB_2^n)} - 1\right| &\le \left|\frac{\mathbb{P}_E(tB_2^n)}{\Gamma_E(tB_2^n)} - \int_{G_{n,k}} \frac{\mathbb{P}_E(tB_2^n)}{\Gamma_E(tB_2^n)}\,\mathrm{d}\nu_{n,k}\right| \\ &\quad + \left|\int_{G_{n,k}} \frac{\mathbb{P}_E(tB_2^n)}{\Gamma_E(tB_2^n)}\,\mathrm{d}\nu_{n,k} - 1\right|.\end{aligned}$$

Step 1. We first study the average distribution

$$\frac{F(t)}{\Gamma(t)} := \int_{G_{n,k}} \frac{\mathbb{P}_E(tB_2^n)}{\Gamma_E(tB_2^n)}\,\mathrm{d}\nu_{n,k}\,,$$

where

$$\begin{aligned}F(t) &= \int_{G_{n,k}} \mathbb{P}_E(tB_2^n)\,\mathrm{d}\nu_{n,k}\,,\\ \Gamma_E(tB_2^n) = \Gamma(t) &= \int_{\{x\in\mathbb{R}^k : |x|\le t\}} \gamma_k(x)\,\mathrm{d}x\,.\end{aligned}$$

The following result expresses the average distribution in a suitable way. A more general formula is actually valid for any probability $\mathbb{P}$ (see [3]).

THEOREM 2.2. ([3]) *Let $\mathbb{P}$ be a Borel probability on $\mathbb{R}^n$ such that $\mathbb{P}\{0\} = 0$. Then, for all $1 \le k < n$ and $t \ge 0$ we have*

$$F(t) = \int_{\{|s|\le t\}} \varphi_k(s)\,\mathrm{d}s\,,$$

where

$$\varphi_k(s) = \frac{|S^{n-k-1}|}{|S^{n-1}|} \int_{\{|x|\geq|s|\}} \left(1 - \frac{|s|^2}{|x|^2}\right)^{\frac{n-k-2}{2}} \frac{\mathrm{d}\mathbb{P}(x)}{|x|^k}\,.$$

Step 2. We compute the distance between the average distribution and 1. For *any* probability satisfying hypothesis (CH) we have a result that measures proximity of the density of the average distribution $\varphi^k(s)$ and the gaussian density $\gamma^k(s)$, this estimate passes to the distribution functions for log-concave probabilities. The theorem in Step 1 is crucial in order to go on with computations.

THEOREM 2.3. ([3]) *Let* $\mathbb{P}$ *be a probability on* $\mathbb{R}^n$ *satisfying (CH) and let*

$$\gamma = \min\left\{\frac{\alpha}{\max\{2, 2\beta\}}, \frac{1}{4}\right\}.$$

If $\log(M_{\mathbb{P}}^{1/n} M_2) < c_1 n^\gamma$ *and* $k < c_2 n^\gamma$ *(for suitable* c_1, c_2 *depending only on the constants in (CH)), then*

$$\left|\frac{\varphi_k(s)}{\gamma_k(s)} - 1\right| \leq \frac{c}{n^\gamma} \qquad \text{whenever } |s| \leq c M_2 n^{\gamma/2}$$

for some constant $c = c(\alpha, A, \beta, B) > 0$.

Moreover, if $\mathbb{P}$ *is log-concave then*

$$\sup_{t>0}\left|\frac{F(t)}{\Gamma(t)} - 1\right| \leq \frac{c}{n^\gamma}\,.$$

In particular, when $K \subset \mathbb{R}^n$ is an isotropic convex body then Klartag's result says it satisfies (CH) with $\alpha = 0.33$ and $\beta = 2$. Also $M_{\mathbb{P}}^{1/n} M_2 = L_K$ and $\log(M_{\mathbb{P}}^{1/n} M_2) < c_1 n^\gamma$ holds (since $L_K \leq c_1 n^{1/4}$, [7]) and so for $k \leq c_2 n^\gamma$ we have

$$\sup_{|s|\in[0, cn^{\gamma/2}]}\left|\frac{\varphi_k(s)}{\gamma_k(s)} - 1\right| \leq \frac{c}{n^\gamma}$$

and

$$\sup_{t>0}\left|\frac{F(t)}{\Gamma(t)} - 1\right| \leq \frac{c}{n^\gamma}\,.$$

Step 3. We estimate the distance between an individual $\frac{\mathbb{P}_E(tB_2^n)}{\Gamma(t)}$, $E \in G_{n,k}$, and the average $F(t)/\Gamma(t)$.

For that matter we take into account the concentration of measure phenomenon on $G_{n,k}$ stated in the introduction. In this final step we must restrict ourselves to probabilities given by isotropic convex bodies. Recall:

THEOREM 2.4. (CONCENTRATION OF MEASURE [11]) *Let $f\colon G_{n,k} \to \mathbb{R}$ continuous. There exist absolute constants $c_1, c_2 > 0$ such that for every $a > 0$ we have*

$$\nu_{n,k}\left\{E : \left|f(E) - \int_{G_{n,k}} f \,\mathrm{d}\nu_{n,k}\right| > \omega_f(a)\right\} \leq c_1 \exp\left(-c_2 n a^2\right),$$

where

$$\omega_f(a) = \sup_{d(E_1,E_2)\leq a} |f(E_1) - f(E_2)|$$

is the modulus of continuity of f.

We want to apply it to the function $f(E) = \mathbb{P}_E(tB_2^n)$ and so first compute the modulus of continuity ω_f.

LEMMA 2.5. ([3]) *Let $0 < \varepsilon < 1$, $t > 0$ and $K \subset \mathbb{R}^n$ isotropic. Then for every $E_1, E_2 \in G_{n,k}$ we have*

$$|f(E_1) - f(E_2)| \leq \epsilon$$

provided that $d(E_1, E_2) \leq a$, where

$$a = \begin{cases} \dfrac{c^k \varepsilon^2 t^2}{\mathcal{L}_k^k}, & \text{if } t \leq c\sqrt{k}, \\ \dfrac{c\varepsilon^2}{\sqrt{k} t^{k-1}}, & \text{otherwise;} \end{cases}$$

c is an absolute constant and $\mathcal{L}_k = \sup\{L_M : M \subset \mathbb{R}^k \text{ isotropic}\}$.

Therefore, by Lemma 2.5 and Step 2 we have

$$\nu\left\{E \in G_{n,k} : \left|\frac{\mathbb{P}_E(tB_2^n)}{\Gamma(t)} - 1\right| > \varepsilon + \frac{1}{n^\gamma}\right\} \leq c_1 \exp\left(-c_2\, a^2 n\right)$$

with a as before.

The previous inequality holds for *each* $t > 0$ and $K \in \mathcal{M}$ (a also depending on t). A somewhat technical but standard approximation argument yields the desired inequality for *all* $t \geq \varepsilon$

$$\nu_{n,k}\left\{E \in G_{n,k} \,:\, \sup_{t\geq\varepsilon}\left|\frac{\mathbb{P}_E(tB_2^n)}{\Gamma(t)} - 1\right| \leq \varepsilon\right\} \geq 1 - \exp\left(-Cn^{0.9}\right).$$

The remaining estimate for $\sup_{t\leq\varepsilon}$ relies on different arguments concerning the behavior at $t = 0$. For that matter it is necessary to use concentration of measure phenomena for the function $E \to |K \cap E^{\perp}|$ and to compute its lipschitz constant. Details will appear in a forthcoming paper [1].

References

[1] Alonso, D., Bastero, J., Bernués, J., Paouris, G., High dimensional random sections of isotropic convex bodies, *in preparation.*

[2] Anttila, M., Ball, K., Perissinaki, I., The central limit problem for convex bodies, *Trans. Amer. Math. Soc.* **355** (2003), 4723 – 4735.

[3] Bastero, J., Bernués, J., Asymptotic behaviour of averages of k-dimensional marginals of measures on $\mathbb{R}^n$, *preprint.*

[4] Brehm, U., Hinow, P., Vogt, H., Voigt, J., Moment inequalities and central limit properties of isotropic convex bodies, *Math. Z.* **240** (2002), 37 – 51.

[5] Dvoretzky, A., Some results on convex bodies and Banach spaces, in "Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1961)", Jerusalem Academic Press, Jerusalem, 1961, 123 – 160.

[6] Figiel, T., Lindenstrauss, J., Milman, V.D., The dimension of almost spherical sections of convex bodies, *Acta Math.* **139** (1977), 53 – 94.

[7] Klartag, B., On convex perturbations with a bounded isotropic constant, *Geom. Funct. Anal.* **16** (2006), 1274 – 1290.

[8] Klartag, B., A central limit theorem for convex sets, *Invent. Math.* **168** (2007), 91 – 131.

[9] Klartag, B., Power-law estimates for the central limit theorem for convex sets, *J. Funct. Anal.* **245** (2007), 284 – 310.

[10] Milman, V.D., Pajor, A., Isotropic positions and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space, in "Geometric Aspects of Functional Analysis (1987–88)", Lecture Notes in Math., 1376, Springer, Berlin, 1989, 64 – 104.

[11] Milman, V.D., Schechtman, G., "Asymptotic Theory of Finite Dimensional Normed Spaces", Lecture Notes in Math., 1200, Springer-Verlag, Berlin, 1986.

[12] Naor, A., Romik, D., Projecting the surface measure of the sphere of ℓ_p^n, *Ann. Inst. H. Poincaré Probab. Statist.* **39** (2) (2003), 241 – 261.

[13] Paouris, G., Concentration of mass on convex bodies, *Geom. Funct. Anal.* **16** (2006), 1021 – 1049.

[14] Sodin, S., Tail-sensitive Gaussian asymptotics for marginals of concentrated measures in high dimension, to appear in *Geom. Funct. Anal.*

[15] Sudakov, V.N., Typical distributions of linear functionals in finite dimensional spaces of higher dimensions, *Soviet Math. Dokl.* **19** (6) (1978), 1578 – 1582.

[16] Voigt, J., A concentration of mass property for isotropic convex bodies in high dimensions, *Israel J. Math.* **115** (2000), 235 – 251.