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Sylow 2-Subgroups of Solvable $\mathbb{Q}$-Groups

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Abstract: A finite group whose irreducible characters are rational valued is called a rational or a $\mathbb{Q}$-group. In this paper we obtain various results concerning the structure of a Sylow 2-subgroup of a solvable $\mathbb{Q}$-group.

Key words: $\mathbb{Q}$-groups, Sylow subgroups, extraspecial 2-groups.

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1. Introduction and preliminary results

Let G be a finite group and χ be a complex character of G. Let $\mathbb{Q}(\chi)$ denote the subfield of the complex number $\mathbb{C}$ generated by all the values $\chi(x)$, $x \in G$. By definition χ is called rational if $\mathbb{Q}(\chi) = \mathbb{Q}$, where $\mathbb{Q}$ denotes the field of rational numbers. A finite group G is called a rational group or a $\mathbb{Q}$-group, if every complex irreducible character of G is rational. Equivalently G is a $\mathbb{Q}$-group if and only if every x in G is conjugate to x^m where $m \in \mathbb{N}$ is prime to the order of x. This will imply that for every $x \in G$ of order n we have $\frac{N_G(\langle x\rangle)}{C_G(\langle x\rangle)} \cong \mathrm{Aut}(\langle x\rangle)$ a group of order $\varphi(n)$ where φ is the Euler φ-function. The symmetric group S_n and the Weyl group of the complex Lie algebras are examples of $\mathbb{Q}$-groups. Elementary abelian 2-groups and extra-special 2-groups are also $\mathbb{Q}$-groups. Rational groups have been studied extensively, but their classification is far from being complete. It is proved in [5] that if G is a solvable $\mathbb{Q}$-group, then $\pi(G) \subseteq \{2,3,5\}$. Also by [4] non-abelian composition factors of any finite $\mathbb{Q}$-group can only be either $SP_6(2)$ or $O_8^+(2)$. In [2] the structure of Frobenius $\mathbb{Q}$-groups has been found.

An important problem concerning $\mathbb{Q}$-groups is to classify them through the structure of a Sylow 2-subgroup. Any non-trivial $\mathbb{Q}$-group is of even order and there is a long standing conjecture that a Sylow 2-subgroup of a $\mathbb{Q}$-group is also a $\mathbb{Q}$-group [9, page 13]. The following results determine the structure of $\mathbb{Q}$-groups with a specified Sylow 2-subgroup.

Result 1. ([9, page 21] and [1, page 60]) *Let G be a $\mathbb{Q}$-group with an abelian Sylow 2-subgroup P. Then P is an elementary abelian 2-group and G is a supersolvable $\{2,3\}$-group. Moreover the commutator subgroup G' of G is a normal Sylow 3-subgroup of G and G splits over G' with P as a complement. In other words G is a 2-nilpotent group.*

Let D_{2n} denote the dihedral group of order $2n$. It is proved in [9, page 25] that if G is a $\mathbb{Q}$-group with a Sylow 2-subgroup P isomorphic to D_{2n}, then $n = 1, 2$ or 4. If $n = 1$ or 2, then P is abelian and the structure of G follows from Result 1. But as far as the authors know the structure of G in case $n = 4$ is not mentioned anywhere. However in [1, page 61] it is proved that if G is a solvable group with a Sylow 2-subgroup isomorphic to D_8, then $\pi(G) \subseteq \{2,3\}$.

Result 2. ([9, page 35] and [1, page 62]) *Let G be a $\mathbb{Q}$-group with a Sylow 2-subgroup P isomorphic to the quaternion group Q_8. Then G contains a normal elementary abelian p-group E_p, where $p = 3$ or 5, and $G = E_p : P$, where ":" denotes semi-direct product. In other words G is a 2-nilpotent group. Moreover if G is non-nilpotent, then G is isomorphic to a Frobenius group with complement isomorphic to Q_8, the quaternion group of order 8.*

Motivated by the above results in this paper we obtain some properties of $\mathbb{Q}$-groups having certain Sylow 2-subgroups. We also determine when extensions of certain groups is a $\mathbb{Q}$-group. Finally we find conditions on solvable $\mathbb{Q}$-groups having an extra-special Sylow 2-subgroup.

2. Sylow 2-subgroups of $\mathbb{Q}$-groups

In this section we study $\mathbb{Q}$-groups with certain conditions on their Sylow 2-subgroups.

Lemma 1. *If a generalized quaternion group P is the Sylow 2-subgroup of a $\mathbb{Q}$-group G, then P is isomorphic to the quaternion group of order 8.*

Proof. The generalized quaternion group of order 2^{n+1}, $n \geq 2$, has the following presentation: $P = \left\langle x, y : x^{2^n} = 1,\, y^2 = x^{2^{n-1}},\, y^{-1}xy = x^{-1} \right\rangle$. Suppose G is a $\mathbb{Q}$-group and P is a Sylow 2-subgroup of G. By definition we have $\left[N_G(\langle x\rangle) : C_G(\langle x\rangle)\right] = \varphi(2^n) = 2^{n-1}$. Therefore $|N_G(\langle x\rangle)| = 2^{n-1} \times |C_G(\langle x\rangle)|$, hence the 2-part of $|N_G(\langle x\rangle)|$ is at least $2^{n-1} \times 2^n = 2^{2n-1}$. Since a Sylow 2-subgroup of G has order 2^{n+1}, we must have $2^{2n-1} \leq 2^{n+1}$, hence $n \leq 2$. Thus $|P| = 8$ and $P \cong Q_8$ is the quaternion group of order 8. ∎

PROPOSITION 1. *Let G be a solvable $\mathbb{Q}$-group of even order with exactly one conjugacy class of involutions. Then a Sylow 2-subgroup of G is either elementary abelian or isomorphic to the quaternion group of order* 8.

Proof. Let S be a Sylow 2-subgroup of G. By [9] the center $Z(S)$ of S is a non-trivial elementary abelian 2-group. If x and y are involutions in $Z(S)$, then by assumption x and y are conjugate in G. By a well-known result [10, page 137], x and y are conjugate in $N_G(S)$ the normalizer of S in G. But by [9] we have $N_G(S) = S$. Therefore x and y are conjugate in S implying $x = y$. Hence $|Z(S)| = 2$. Now assume $|S| > 2$. By a result of J. Thompson cited in [8, page 511], S is isomorphic to a homocyclic or a Suzuki 2-group. If S is homocyclic then S is isomorphic to the direct product of cyclic groups of the same order, hence $Z(S) = S$ must be an elementary abelian 2-group. Otherwise if S is a Suzuki 2-group, then by [8, page 311], $S' = \phi(S) = Z(S) = \{x : x \in S, x^2 = 1\}$, implying that S has only one involution. Therefore S must be isomorphic to a generalized quaternion group. Since G is assumed to be a $\mathbb{Q}$-group, hence, by Lemma 1, S is isomorphic to the quaternion group of order 8 and the proposition is proved. ∎

PROPOSITION 2. *Let G be a supersolvable $\mathbb{Q}$-group. Then Sylow 2-subgroups of G are $\mathbb{Q}$-groups.*

Proof. Let G be a non-trivial supersolvable $\mathbb{Q}$-group. Then there is a cyclic normal subgroup $\langle x\rangle$ of prime order p in G where p is the largest prime in $\pi(G)$. Now $\frac{N_G(\langle x\rangle)}{C_G(\langle x\rangle)} = \frac{G}{C_G(\langle x\rangle)} \cong \mathbb{Z}_{p-1}$ is a $\mathbb{Q}$-group, hence $p-1 \le 2$. Therefore $\pi(G) \subseteq \{2,3\}$. By [10, page 158], if $3 \mid |G|$ then a Sylow 3-subgroup P of G is normal in G. Hence $\frac{G}{P}$ is a Sylow 2-subgroup of G which must be a $\mathbb{Q}$-group. ∎

3. EXTENSIONS OF ABELIAN GROUPS AS $\mathbb{Q}$-GROUPS

In this section we will consider split extensions of groups and determine when they are $\mathbb{Q}$-groups. Let a group G act on a group H. The Cartesian product $H \times G$ endowed with the following law of composition: $(g,h)(g',h') = \left(gg', h^{g'}h'\right)$, $g, g' \in G$, $h, h' \in H$, is a group called the semi-direct product of H with G and is denoted by $H \rtimes G$ or $H : G$. The group $L = H \rtimes G$ is also called a split extension of H by G and we may regard H as a normal subgroup of L such that $\frac{L}{H} \cong G$.

LEMMA 2. *Split Extension of an elementary abelian* 2*-group by another elementary abelian* 2*-group is a* $\mathbb{Q}$*-group.*

Proof. Let E_1 and E_2 be elementary abelian 2-groups and $G = E_1 \rtimes E_2$ be their semi-direct product. Operations of E_1 and E_2 will be written additively. Since $\frac{G}{E_1} \cong E_2$, every non-identity element of G is of order 2 or 4. To prove that G is a $\mathbb{Q}$-group it is enough to prove that every element of order 4 in G is conjugate to its inverse. Let $x = (g, v) \in G$, where $g \in E_2$ and $v \in E_1$. If x is of order 4, then $v + v^g \neq 0$ and $(g, v)^{-1} = (g, v^g)$. Now $(1, v)^{-1}(g, v)(1, v) = (g, v^g)$, proving that x and x^{-1} are conjugate in G and the lemma is proved. ∎

Let V be a vector space over a finite field on which the group G acts. Then we can form the usual semi-direct product $V \rtimes G$ with the operation $(g, v)(h, u) = (gh, v^h + u)$, where $g, h \in G$ and $u, v \in V$. In the following we will assume G is a certain group and find necessary and sufficient conditions such that $V \rtimes G$ is a $\mathbb{Q}$-group.

Let p be an odd prime and V be a 2-dimensional vector space over the Galois field $GF(p)$. It is a well-known fact that there are $a, b \in GF(p)$ such that $a^2 + b^2 = -1$. If we set

$$i = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \qquad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad k = \begin{pmatrix} -b & a \\ a & b \end{pmatrix},$$

then it is easy to see that $Q_8 = \langle i, j, k \rangle$ is isomorphic to the quaternion group of order 8. Therefore V is an irreducible module for Q_8 and we can form the semi-direct product $V \rtimes Q_8$. Our next result is the following.

PROPOSITION 3. *Let V be a* 2*-dimensional irreducible module over the field $GF(p)$, p an odd prime, for the quaternion group Q_8. Then $V \rtimes Q_8$ is a $\mathbb{Q}$-group if and only if $p = 3$ or* 5.

Proof. First we will prove that the order of elements of the group $V \rtimes Q_8$ is one of the numbers $1, 2, 4$ or p. Elements of $V \rtimes Q_8$ are of the forms (g, v) where $g \in Q_8$ and $v \in V$. It is obvious that for $n \in N$ we have $(g, v)^n = (g^n, vg^{n-1} + vg^{n-1} + \cdots + vg + v)$ and hence $O(I, o) = 1$, $O(-I, o) = 2$, $O(x, v) = 4$ for all $x \in Q_8 - \{\pm I\}$ and $v \in V$, finally $O(I, v) = p$ for all $v \in V - \{0\}$. Now for elements (g, v) and (h, u) of $V \rtimes Q_8$ it can be verified that $(h, u)^{-1}(g, v)(h, u) = (h^{-1}gh, -uh^{-1}gh + vh + u)$.

Now if we consider (x, v), $v \in Q_8 - \{\pm I\}$, $v \in V$, then x and $x^3 = -x$ are conjugate in Q_8 and hence there exists $y \in Q_8$ such that $y^{-1}xy = -x$.

Therefore from $(y,u)^{-1}(x,v)(y,u) = (y^{-1}xy, -uy^{-1}xy + vy + u) = (-x, ux + vy+u) = (x,v)^3 = (x^3, vx^2+vx+v) = (-x, vx)$ we will obtain $ux+vy+u = vx$, thus $u(x+I) = v(x-y)$ from which we will obtain $u = \frac{1}{2}v(I+x+yx-y)$. Hence (x,v) and $(x,v)^3$ for all $x \in Q_8 - \{\pm I\}$ and $v \in V$ are conjugate in $V \rtimes Q_8$.

Now we will consider elements of order p, say (I,v), where $v \in V - \{0\}$. Let m be an integer such that $0 < m < p$ and $(x,u)^{-1}(I,v)(x,u) = (I,v)^m$, then $vx = mv$. Hence m is an eigenvalue of $x \in G$. But it is easy to see that eigenvalues of elements of Q_8 are either ± 1 or roots of the equation $t^2+1 = 0$ in $GF(p)$. If the only eigenvalues occurring are ± 1, then $p = 3$ and if rots of $t^2+1 = 0$ occur we must have $p = 5$. The converse is obviously true, i.e., if $p = 3$ or 5, then (I,v) is conjugate to $(I,v)^m$ for all $0 < m < p$. The proposition is proved now. ∎

Next we consider the symmetric group S_n of degree n. In this case we assume V is a vector space of dimension n over the Galois field $GF(q)$ where q is a power of the prime p. We assume S_n as the symmetric group of the set $\{1,2,\ldots,n\}$ and V has basis $\{e_1,\ldots,e_n\}$. Therefore the action of S_n on V is as follows: $e_i\pi = e_{(i)\pi}$ for all $1 \le i \le n$ and $\pi \in S_n$. We consider the semi-direct product $V \rtimes S_n$ called the hyperoctahedral group and prove the following result.

Proposition 4. *$V \rtimes S_n$ is a $\mathbb{Q}$-group if and only if $p = 2$.*

Proof. With regard to the above explanation we consider the element $(1,e_i)$, $1 \le i \le n$, of order p in $V \rtimes S_n$. This element must be conjugate to $(1,e_i)^m$, where $0 < m < p$. Therefore there exists $(\pi,v) \in V \rtimes S_n$ such that $(\pi,v)^{-1}(1,e_i)(\pi,v) = (1,e_i)^m$ from which we obtain $e_i\pi = me_i$ and therefore $e_{(i)\pi} = me_i$ which implies $m = 1$. Therefore $p = 2$. By [9, Corollary 96A, page 96] the hyperoctahedral group B_n is a $\mathbb{Q}$-group and this is the group $V \rtimes S_n$ in the case $p = 2$, the proof is complete now. ∎

Now let V be a vector space of dimension n over the Galois field $GF(q)$, q a power of the prime p. Let $G = GL_n(q)$ be the group of automorphisms of V. Then G acts on V and we can form the semi-direct product $V \rtimes G$. Our next result is concerned with the above consideration.

Lemma 3. *Let q and n be positive integers. Then $\varphi(q^n - 1) = n$ if and only if $(n,q) = (1,2)$, $(1,3)$ or $(2,2)$, where φ denotes the Euler φ-function.*

Proof. If $n = 1$, then $\varphi(q-1) = 1$ and obviously $q - 1 = 1$ or 2 implying $q = 2$ or 3. Therefore we will assume $n \geq 2$. It can be proved that for any positive integer m if $q \geq 3$, then $q^m \geq m^2$ and in the case of $m \geq 4$ we have $2^m \geq m^2$. Now for any integer t it is easy to prove that $\varphi(t) \geq \frac{1}{2}\sqrt{t}$. Hence if $\varphi(q^n - 1) = n$, then $n \geq \frac{1}{2}\sqrt{q^n - 1}$ which implies $q^{\frac{n}{2}} < 2n + 1$. First we assume $q \geq 3$. Since $n \geq 2$ we obtain $2n+1 > q^{\frac{n}{2}} \geq \frac{n^2}{4}$ implying $n^2 < 8n+4$, hence $n \leq 8$. If $n \geq 4$, then from $q^{\frac{n}{2}} < 2n + 1$ we obtain $q = 2$ which is not the case. Therefore $n = 3$ or 2. If $n = 3$, then $q = 3$ and if $n = 2$, then $q = 3$ or 4, and in both cases $\varphi(q^n - 1) \neq n$. Now we will assume $q = 2$. If $\frac{n}{2} \geq 4$, then $2n + 1 > 2^{\frac{n}{2}} \geq \frac{n^2}{4}$ implies $n \leq 8$, hence $n = 8$. But $\varphi(2^8 - 1) \neq 8$, so we assume $n < 8$. Now case by case examination of the Euler φ-function yields $\varphi(2^2 - 1) = 2$ as the only possibility. The Lemma is proved now. ∎

PROPOSITION 5. *$V \rtimes GL_n(q)$, $n \geq 2$, is a $\mathbb{Q}$-group if and only if $(n, q) = (2, 2)$.*

Proof. If $H = V \rtimes GL_n(q)$ is a $\mathbb{Q}$-group, then by [9] the group $\frac{H}{N} \cong GL_n(q)$ is also a $\mathbb{Q}$-group. Now for any $\lambda \in GF(q)^*$ the matrices λI and $\lambda^{-1} I$ must be conjugate in $GL_n(q)$ from which we will obtain $\lambda^2 = 1$ or $\lambda = \pm 1$. Therefore $q = 2$ or 3. Now by [7, page 187] the group $GL_n(q)$ has an element h of order $q^n - 1$ such that $\frac{N(\langle h \rangle)}{G(\langle h \rangle)} \cong Z_n$. Therefore $\varphi(q^n - 1) = n$. Now by Lemma 3 we obtain $(n, q) = (2, 2)$. The converse of the proposition is obvious and the Proposition is proved now. ∎

4. SOLVABLE $\mathbb{Q}$−GROUPS WITH EXTRASPECIAL SYLOW 2-SUBGROUP

As we mentioned in the introduction an extraspecial 2-group is a $\mathbb{Q}$-group and it may appear as a Sylow 2-subgroup of a $\mathbb{Q}$-group. In of [1, problem 83, page 301] part 2 asks to classify rational $\mathbb{Q}$-groups with an extra-special Sylow 2-subgroup. Now we recall the definition of an extra-special p-group and its structure from [3].

DEFINITION 1. A finite p-group P is called extra-special if $P' = Z(P) \cong \mathbb{Z}_p$ and $\frac{P}{P'}$ is an elementary abelian p-group.

Every extra-special p-group is the central product of non-abelian p-groups of order p^3. The dihedral group D_8 and the quaternion group Q_8 are extra-special 2-groups of order 8. If P is an extra-special 2-group, then there is an $m \in \mathbb{N}$ such that $|P| = 2^{2m+1}$. Moreover either $P \cong D_8 \circ D_8 \circ \cdots \circ D_8$ or

$P \cong Q_8 \circ D_8 \circ \cdots \circ D_8$, where $\circ$ denotes the central product and in both cases m different groups are involved.

First we will prove the following two results about a general $\mathbb{Q}$-group. We recall that if G is a finite group, then the largest normal subgroup of odd order in G is denoted by $O(G)$.

LEMMA 4. *Let G be a $\mathbb{Q}$-group with extra-special Sylow 2-subgroup P. If G has a non-trivial center and $O(G) = 1$, then $G = P$.*

Proof. Since $Z(G) \subseteq Z(P) = \langle x \rangle$ is a group of order 2 and $Z(G)$ is assumed to be non-trivial, hence $Z(G) = \langle x \rangle$. Now $\frac{G}{\langle x \rangle}$ is a $\mathbb{Q}$-group with $\frac{P}{\langle x \rangle}$ as a Sylow 2-subgroup. But $\frac{P}{\langle x \rangle}$ is an elementary abelian 2-group, hence, by Result 1, $\frac{G}{\langle x \rangle}$ is a supersolvable $\{2,3\}$-group. Therefore there is a normal 3-subgroup $\overline{N}$ of $\frac{G}{\langle x \rangle}$ such that $\frac{G}{\langle x \rangle} = \overline{N}\left(\frac{P}{\langle x \rangle}\right)$. Let N be the pre-image of N in G and S be a Sylow 3-subgroup of G. Then $\overline{N} = \frac{N\langle x \rangle}{\langle x \rangle}$ and since x has order 2 we have $x \notin S$. But $x \in Z(G)$, hence $x \in C_G(S)$ implying $N = S\langle x \rangle \cong S \times \langle x \rangle$. Now S is a characteristic subgroup of N and hence $S \trianglelefteq G$. Therefore $S \leq O(G) = 1$ which implies $S = 1$ and hence $N = \langle x \rangle$. Consequently $\overline{N} = 1$ which gives the result $G = P$ and the Lemma is proved. ∎

PROPOSITION 6. *Let G be a $\mathbb{Q}$-group with an extra-special Sylow 2-subgroup P. If $Z(G) \neq 1$, then G is a solvable group and there is a normal subgroup N of G with $\pi(N) \subseteq \{3,5\}$ such that $G = NP$ and $N \cap P = 1$.*

Proof. We use induction on $O(G)$. If $O(G) = 1$, then by Lemma 4 we have $G = P$ and $N = 1$ will work in the proposition. Therefore we may assume $O(G) \neq 1$. We know that $\frac{G}{O(G)}$ is a $\mathbb{Q}$-group with a Sylow 2-subgroup isomorphic to P. Since $Z(G)$ is always an elementary abelian 2-group we obtain $Z(G) \neq O(G)$ from which we deduce that $Z\left(\frac{G}{O(G)}\right) \neq 1$. Hence by induction we have $\frac{G}{O(G)} = \overline{N}P$ where $\overline{N} \trianglelefteq \frac{G}{O(G)}$ and $\overline{N} \cap P = 1$. But $\overline{N} = O\left(\frac{G}{O(G)}\right) = 1$ and therefore $G = O(G)P$. Now we set $N = O(G)$, hence $G = NP$. Since $\frac{G}{N}$ is a solvable group and N has odd order we deduce that G is a solvable group. Now, by [5], G is a $\{2,3,5\}$-group and hence $\pi(N) \subseteq \{3,5\}$ and the proposition is proved. ∎

Next we turn to solvable $\mathbb{Q}$-groups with an extra-special Sylow 2-subgroup. First of all let us determine the structure of the solvable $\mathbb{Q}$-groups with Sylow 2-subgroups isomorphic to the dihedral group D_8.

Theorem 1. *Let G be a rational solvable group with a Sylow 2-subgroup isomorphic to D_8. Then G contains a normal 3-subgroup N such that $\frac{G}{N}$ is isomorphic to either D_8 or $\mathbb{S}_4$.*

Proof. By [1, page 61] we have $|G| = 8 \cdot 3^n$, where n is a non-negative integer. The number of Sylow 3-subgroups N_3 of G is either 1 or 4. If $N_3 = 1$, then a Sylow 3-subgroup N of G is normal in G and $\frac{G}{N} \cong D_8$. Assume that $N_3 = 4$ and $\Omega = \{Q_1, Q_2, Q_3, Q_4\}$ is the set of distinct Sylow 3-subgroups of G. If N denotes the kernel of the action of G on Ω by conjugation, then $\frac{G}{N}$ is isomorphic to a subgroup of $\mathbb{S}_4$. Since G is assumed to be a $\mathbb{Q}$-group, therefore $\frac{G}{N}$ is also a $\mathbb{Q}$-group. Since $|N_G(Q_i)| = 2 \cdot 3^n$ and $N = \cap_{i=1}^4 N_G(Q_i)$, hence $4 \mid \left|\frac{G}{N}\right|$. Now it is easy to see that the rational subgroups of $\mathbb{S}_4$ with order divisible by 4 are isomorphic to one of the groups $\mathbb{Z}_2 \times \mathbb{Z}_2$, D_8 or $\mathbb{S}_4$.

If $\frac{G}{N} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $|N| = |N_G(Q_i)|$, for all $1 \leq i \leq 4$, which is a contradiction because the Q_i's are distinct. If $\frac{G}{N} \cong D_8$ or S_4, then we are done and the Theorem is proved now. ∎

Theorem 2. *Let G be a solvable $\mathbb{Q}$-group with an extra-special Sylow 2-subgroup. Then one of the following possibilities holds:*

(a) *G is a 2-nilpotent group.*

(b) *There is a proper normal subgroup N of G such that $\frac{G}{N} = P : E(2)$, where P is a 3-group and $E(2)$ is an elementary abelian 2-group.*

Proof. We use induction on $|G|$. Let P be a Sylow 2-subgroup of G which by assumption is extra-special. By [5] we have $\pi(G) \subseteq \{2, 3, 5\}$. Let E be a minimal normal subgroup of G.

Case 1: $|E|$ is even. Therefore E is a proper elementary abelian 2-subgroup of G and we may assume $E \leq P$. Since $1 \neq E \trianglelefteq P$, hence $E \cap Z(P) \neq 1$. But $Z(P) = P'$ is of order 2. Therefore $Z(P) = P' \subseteq E$. Thus $\frac{P}{E}$ is an abelian group and it is a Sylow 2-subgroup of $\frac{G}{E}$. Hence $\frac{G}{E}$ is a $\mathbb{Q}$-group with an abelian Sylow 2-subgroup, hence by Result 1, $\frac{G}{E} = P : E(2)$ where P is a 3-subgroup of $\frac{G}{E}$, and hence of G, and $E(2)$ is an elementary abelian 2-group. Therefore case (b) of the theorem holds.

Case 2: $|E|$ is odd. Hence $\frac{G}{E}$ is a $\mathbb{Q}$-group with an extra-special Sylow 2-subgroup isomorphic to P.

If a minimal normal subgroup $\frac{A}{E}$ of $\frac{G}{E}$ has even order, then by Case 1, $(\frac{G}{E})/(\frac{A}{E}) \cong \frac{G}{A} = P : E(2)$ where P is a 3-group and $E(2)$ is an elementary abelian 2-group as stated on part (b) of the theorem.

If a minimal normal subgroup $\frac{A}{E}$ of $\frac{G}{E}$ has an odd order, then $(\frac{G}{E})/(\frac{A}{E}) \cong \frac{G}{A}$, $|\frac{G}{E}| < |G|$ and $|A|$ is odd. Therefore by induction we reach a point such that there is a normal subgroup N of G with $\frac{G}{N}$ isomorphic to a Sylow 2-subgroup of G. This implies that G is a 2-nilpotent group, and case (a) of the theorem holds. ∎

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