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A New Proof of Gabriel's Lemma

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1. Introduction

The technical result [2, Lemma 3.2] of Gabriel, called often Gabriel's Lemma (for the precise formulation see Section 2), played a crucial role in proofs of two famous theorems in representation theory of algebras: the theorem of Gabriel on openness of the set of finite representation type algebras in a variety of all algebras with a fixed dimension (see [2]) and the Geiss Theorem saying that degeneration of a wild algebra is also wild (see [3]). The original proof of Gabriel's Lemma is rather involved and uses geometry of schemes. An alternative proof, proposed by H. Kraft for the case of characteristic 0, applies essentially invariant theory and geometric quotients (see [5]). We present here a new, quite simple proof, which uses only basic projective geometry and adapts some arguments presented in [1].

For basic information concerning algebraic geometry and algebraic groups we refer to [4, 6].

We fix now some notations. Let K be an algebraically closed field, $d, z \in \mathbb{N}$ natural numbers, $\mathrm{Gl}(z)$ a connected affine algebraic group of K-linear automorphisms of the vector space K^z. We consider the set

$$\mathrm{alg}(d) = \{c = (c_{ij}^k) \in K^{d^3} \mid c \text{ satisfies (i) and (ii) below}\}$$

where

$$\text{(i)} \qquad \sum_{s=1}^{d} c_{ij}^{s} c_{sk}^{l} = \sum_{s=1}^{d} c_{is}^{k} c_{jl}^{s}$$

$$\text{(ii)} \qquad \exists\ \alpha_1, \dots, \alpha_d \in K \quad \sum_{i=1}^{d} \alpha_i c_{ij}^{k} = \delta_{jk} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

for $1 \leq i, j, k, l \leq d$. It is known that $\mathrm{alg}(d)$ is an affine variety called a variety of algebras. For any element $c \in \mathrm{alg}(d)$ corresponds a unique associative algebra structure $A(c)$ on K^d admitting a unit element, where multiplication on basic elements $e_1, \dots, e_d \in K^d$ is determined by the equations $e_i \cdot_{A(c)} e_j = \sum_{k=1}^{d} c_{ij}^{k} \cdot e_k$ for $1 \leq i, j \leq d$.

For any $c \in \mathrm{alg}(d)$ we set

$$\mathrm{mod}_c(z) = \{M = (M_i) \in \mathrm{End}_K(K^z)^d \mid M \text{ satisfies (a) and (b) below}\}$$

$$\text{(a)} \qquad \forall\ i, j\ M_i \cdot M_j = \sum_{l=1}^{d} c_{ij}^{l} M_l$$

$$\text{(b)} \qquad \alpha_1 M_1 + \alpha_2 M_2 + \ldots + \alpha_d M_d = I_d$$

where $\alpha_1, \dots, \alpha_d \in K$ are obtained from condition (ii) in definition of the variety $\mathrm{alg}(d)$. It is clear that any sequence (M_i) defines an $A(c)$-module structure on K^z, where the matrix M_i corresponds to the action of the base element e_i of algebra $A(c)$ on vector space K^z. This mapping yields a bijection between $\mathrm{mod}_c(z)$ and all $A(c)$-module structures on K^z. The set $\mathrm{mod}_c(z)$ is an affine variety. The algebraic group $\mathrm{Gl}(z)$ acts regularly on the variety $\mathrm{mod}_c(z)$ by simultaneous conjugation of all matrices defining a given $A(c)$-module structure, two elements in $\mathrm{mod}_c(z)$ belong to the same orbit if and only if the respective $A(c)$-modules are isomorphic.

Suppose that we have an $A(c)$-module W of dimension z with K-linear ordered base $\underline{w} = (w_1, \dots, w_z) \in W^z$. By $M(W, \underline{w})$ we mean the element of $\mathrm{mod}_c(z)$ corresponding to $A(c)$-module K^z with the structure transferred by the isomorphism of vector spaces W and K^z, which maps the consecutive elements of base $\underline{w}$ to adequate elements of standard base of K^z.

We define also a variety

$$\mathrm{algmod}(d, z) = \{(c, M) \mid c \in \mathrm{alg}(d)\ M \in \mathrm{mod}_c(z)\}.$$

On $\mathrm{algmod}(d, z)$ we have the induced $\mathrm{Gl}(z)$-action.

2. The Proof of Gabriel's Lemma

Before we present the announced proof we recall the precise formulation of Gabriel result.

Theorem. (Gabriel's Lemma) *Projection*

$$\pi : \mathrm{algmod}(d, z) \to \mathrm{alg}(d)\,,\ (c, M) \mapsto c$$

maps closed and $\mathrm{Gl}(z)$*-invariant subsets into closed subsets.*

Proof. We set

$$\mathrm{Sur}(dz, z) = \{\, S \in \mathrm{Hom}_k((K^d)^z, K^z) \mid \mathrm{Im}\, S = K^z\}$$

Clearly, $\mathrm{Sur}(dz, z)$ is a quasi-affine variety which is equipped with the regular action of $\mathrm{Gl}(z)$ induced by the canonical action of $\mathrm{Gl}(z)$ on K^z.

Let Z be the subset of $\mathrm{alg}(d) \times \mathrm{Sur}(dz, z)$ consisting of all pairs (c, S) such that $\mathrm{Ker}\, S$ is an $A(c)$-submodule of a free module $A(c)^z$, where $A(c)^z = (K^d)^z$ as a vector space. In the space $(K^d)^z$ we have a canonical linear base which consists of vectors $\{e_i^j\}$ $(i = 1, \ldots, d;\ j = 1, \ldots, z)$, where the sequence $\{e_i^j\}$ $(i = 1, \ldots, d)$ forms a standard base of j-th coordinate of a product

$$(K^d)^z = \underbrace{K^d \times \cdots \times K^d}_{\text{z times}}$$

for every $j = 1, \ldots, z$.

Any sequence $u = (u_t) \in \{e_i^j\}^z$ consisting of z pairwise different elements, defines an open subset $V_u \subset \mathrm{Sur}(dz, z)$ formed by all elements $S \in \mathrm{Sur}(dz, z)$ for which determinant of the matrix

$$g(u)(S) = [\, S(u_1), \ldots, S(u_z)\,] \in M_z(K)$$

is nonzero. In this way we get open coverings

$$\mathrm{Sur}(dz, z) = \bigcup_u V_u$$

and

$$\mathrm{alg}(d) \times \mathrm{Sur}(dz, z) = \bigcup_u \mathrm{alg}(d) \times V_u$$

of the varieties $\text{alg}(d)$ and $\text{alg}(d) \times \text{Sur}(dz, z)$, respectively. Moreover, we have an action of $\text{Gl}(z)$ on the variety $\text{alg}(d) \times \text{Sur}(dz, z)$ induced by the action of $\text{Gl}(z)$ on $\text{Sur}(dz, z)$. Note that the set Z is $\text{Gl}(z)$-invariant, since for any $S \in \text{Sur}(dz, z)$ the kernel $\ker S$ is a $\text{Gl}(z)$-invariant element of $\text{Gr}(dz - z, dz)$ under the induced action on $\text{Gr}(dz - z, dz)$ which maps the pair $(g, \ker S)$ to element $\ker gS$ for any $g \in \text{Gl}(z)$ (in the other words this map is a trivial action on $\text{Gr}(dz-z, dz)$). We show now that the set Z is closed in $\text{alg}(d) \times \text{Sur}(dz, z)$. It is sufficient to show, that Z is locally closed with respect to above open covering of the set $\text{alg}(d) \times \text{Sur}(dz, z)$.

For any $S \in \text{Sur}(dz, z)$, S belongs to V_u exactly when $S_u = S_{|\,\text{span}(u)}$ is an isomorphism. Therefore we have $(c, S) \in Z \cap (\text{alg}(d) \times V_u)$ if and only if $(c, S) \in \psi^{-1}_{(u,i,v)}((0, \ldots, 0))$ for all $1 \leq i \leq d$, $v \in \{e^j_i\}$, where

$$\psi_{(u,i,v)} : \text{alg}(d) \times V_u \to K^z$$

is given by $\psi_{(u,i,v)}(c, S) = S(e_i \cdot_c (v - S_u^{-1}(S(v))))$ ($\cdot_c$ denotes here an $A(c)$-module multiplication in a free $A(c)$-module $(K^d)^z$). Consequently $Z \cap (\text{alg}(d) \times V_u)$ is closed and Z is locally closed.

We define now a partial function $\varphi_u : Z \dashrightarrow \text{algmod}(d, z)$, which is given by the formula

$$\varphi_u(c, S) = (c, M(W_{(c,S)}, \underline{w}_{(c,S)}))$$

for $(c, S) \in Z \cap (\text{alg}(d) \times V_u)$, where $W_{(c,S)} = (K^d)^z / \text{Ker}\, S$ and $\underline{w}_{(c,S)} = (u_1 + \text{Ker}\, S, \ldots, u_z + \text{Ker}\, S)$. We show now that φ_u is regular on its domain $\mathcal{D}_u$. Clearly, $\mathcal{D}_u = Z \cap (\text{alg}(d) \times V_u)$ forms an open set in Z. Notice also that $\text{Ker}\, S = \text{span}(\{e^j_i - S_u^{-1}(S(e^j_i))\}_{e^j_i \notin \{u_1, \ldots, u_z\}})$ and that

$$\underline{M_i}(u_j + \text{Ker}\, S) = e_i \cdot_c u_j - \sum_{k,l;e^l_k \in u'} \Big[(e_i \cdot_c u_j)^l_k \cdot (e^l_k - S_u^{-1}(S(e^l_k)))\Big] + \text{Ker}\, S$$

where $(\underline{M_i})_{i=1,\ldots,d} = M(W_{(c,S)}, \underline{w}_{(c,S)})$, $u' = \{e^j_i\} \setminus \{u_1, \ldots, u_z\}$ and elements $v^l_k \in K$ satisfy the equation $v = \sum_{i,j} v^j_i \cdot e^j_i$ for $v \in (K^d)^z$. Consequently, $\underline{M_i}(u_j + \text{Ker}\, S)$ depends regularly on $(c, S) \in Z \cap (\text{alg}(d) \times V_u)$ and $\varphi_{u|\mathcal{D}_u} : \mathcal{D}_u \to \text{algmod}(d, z)$ is regular.

Now we can complete our proof. We carry out our reasoning in similarly way as in [1]. Let $\eta : \text{Sur}(dz, z) \to \text{Gr}(dz - z, dz)$ be a map given by $\eta(S) = \text{Ker}\, S$ for $S \in \text{Sur}(dz, z)$, where $\text{Gr}(dz - z, dz)$ is a Grassmann variety (see [4, 6]). The map η is locally a projection (Z is covered by the sets of the form $\mathcal{D}_u$, see also [1, Lemma 2]) and therefore so is the induced product $\eta' = \mathrm{id} \times \eta : \mathrm{alg}(d) \times \mathrm{Sur}(dz, z) \to \mathrm{alg}(d) \times \mathrm{Gr}(dz - z, dz)$. Consequently, η' is an open map. Consider the following family of diagrams

$$\begin{array}{ccc} Z & \hookrightarrow & \mathrm{alg}(d) \times \mathrm{Sur}(dz, z) \\ \Big\downarrow{\scriptstyle \varphi_u} & & \Big\downarrow{\scriptstyle \eta'} \\ \mathrm{algmod}(d, z) & & \mathrm{alg}(d) \times \mathrm{Gr}(dz - z, dz) \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & \mathrm{alg}(d) & \end{array}$$

where the horizontal map is a canonical embedding. The map η' is an open surjection with fibres equal to $\mathrm{Gl}(z)$-orbit. Fix a closed, $\mathrm{Gl}(z)$-invariant subset $X \subseteq \mathrm{algmod}(d, z)$. Then the set $X' = \bigcup_u \varphi_u^{-1}(X)$ is $\mathrm{Gl}(z)$-invariant and closed in Z since X' is locally closed with respect to the open covering $Z = \bigcup_u \mathcal{D}_u$ (for any $(c, S) \in \mathcal{D}_{u_1} \cap \mathcal{D}_{u_2}$ we have an equality $\mathrm{Gl}(z) \cdot \varphi_{u_1}(c, S) = \mathrm{Gl}(z) \cdot \varphi_{u_2}(c, S)$ and therefore $\varphi_{u_2}^{-1}(X) \cap \mathcal{D}_{u_1} \subseteq \varphi_{u_1}^{-1}(X)$). Consequently, X' is closed and $\mathrm{Gl}(z)$-invariant in $P = \mathrm{alg}(d) \times \mathrm{Sur}(dz, z)$, since Z is closed. Hence $\eta'(X')$ is a closed set as complement of the image $\eta'(P \setminus X')$. Finally, by completeness of the variety $\mathrm{Gr}(dz - z, dz)$, we infer that $\pi_1(\eta'(X')) = \pi(X)$ is a closed set. ∎

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