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Sumas torcidas de espacios $C(K)$ y dónde encontrarlas

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PHD DISSERTATION

$C(K)$–*twistings* *and where to find them*

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To my parents and to Nazaret.

"Even with these awkward wings,
I'm sure we can fly".

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Contents

Abstract

This dissertation focuses on spaces of continuous functions, or $C(K)$-spaces, and especially on the *twisted sums* they produce. The study of such spaces is mainly performed by combining the usual techniques from Banach space theory with a great deal of topology, plus some homological and categorical ideas. For instance, Chapter 2 describes a good deal of twisted sums of $C(K)$-spaces making use of both homological and topological tools. On the other hand, Chapter 3 studies a number of remarkable properties of twisted sums of $c_0(I)$-spaces, all of which come as a consequence of a representation theorem for such spaces, in the spirit of category theory. In some places, the topological approach relies on descriptive set theory or infinite combinatorics. The perfect example of this phenomenon is Chapter 4, where a counterexample for the longstanding complemented subspace problem for $C(K)$-spaces is constructed. Finally, $C(K)$-spaces possess module structures which should not be ignored, and so Chapter 5 explores the possibility of obtaining twisted sums with $C(K)$-spaces that are also endowed with such structures.

Resumen

La presente tesis se centra en los espacios de funciones continuas, o espacios $C(K)$, y más concretamente, en el estudio de las *sumas torcidas* que producen. El análisis de dichas sumas torcidas se lleva a cabo combinando ideas de la teoría clásica de espacios de Banach con técnicas propias de la topología, la homología y la teoría de categorías. Concretamente, en el Capítulo 2 se describen sumas torcidas de espacios $C(K)$ que surgen de construcciones topológicas y homológicas, mientras que en el Capítulo 3 se estudian propiedades de las sumas torcidas de espacios tipo $c_0(I)$ que se desprenden de un teorema de representación al más puro estilo de la teoría de categorías. En algunos casos, además, tales construcciones requieren el uso de elementos de teoría descriptiva de conjuntos y combinatoria infinita. Prueba de ello es el Capítulo 4, donde se construye un contrajemplo al clásico problema del subespacio complementado en espacios $C(K)$. Por último, dado que los espacios $C(K)$ están dotados de ciertas estructuras de módulo, el Capítulo 5 está dedicado a explorar la posibilidad de construir sumas torcidas de espacios $C(K)$ que también posean dichas estructuras.

Introduction

Everyone most surely knows what $C(K)$-spaces of continuous functions on compact spaces are. And everyone interested in Banach spaces certainly knows that $C(K)$-spaces play a fundamental role in Banach space theory.

But perhaps not everyone knows what a twisted sum of Banach spaces is.

The study of twisted sums of Banach spaces includes, but is not limited to, the study of complemented and uncomplemented subspaces, when a Banach space property passes to subspaces and quotients, or under which circumstances an operator can be extended or lifted. In other words, twisted sums of Banach spaces exist just because Banach spaces do. The interested reader may have a look at Chapter 1 for some background on twisted sums of Banach spaces.

There are, at least in the author's opinion, two good reasons to explore the topic of twisted sums of $C(K)$-spaces. First: in order to understand $C(K)$-spaces, one must understand twisted sums of $C(K)$-spaces. Second: just as $C(K)$-spaces play a fundamental role on the theory of Banach spaces, we expect twisted sums of $C(K)$-spaces to play an equally important role on the theory of twisted sums of Banach spaces. It is our belief that these pages contribute to substantiate such claims.

The title of this dissertation is self-explanatory: here we will deal with twisted sums of $C(K)$-spaces and explain in which contexts they arise. This idea is most patent in Chapter 2, especially at the beginning, where we collect several instances of “classical" twisted sums of $C(K)$-spaces. Such examples appeared before any attempt of building a theory of twisted sums of Banach spaces, and so we now consider them in the appropriate context. We also display new examples featuring a number of diverse techniques coming from descriptive set theory, topology and infinite combinatorics. This chapter is essentially based on the papers

[18] F. CABELLO, J. M. F. CASTILLO, W. MARCISZEWSKI, G. PLEBANEK AND A. SALGUERO-ALARCÓN, *Sailing over three problems of Koszmider*, Journal of Functional Analysis, 279 (2020), no. 4, 108571, 22 pp.

[87] G. PLEBANEK AND A. SALGUERO-ALARCÓN, *On the three-space property for C(K)-spaces*, Journal of Functional Analysis, 281 (2021), 109193, 15 pp.

While Chapter 2 focuses on examples, Chapter 3 initiates the study of the general structure and properties of twisted sums of $C(K)$-spaces. It is rather natural that the first step of such a study deals with twisted sums of $c_0(I)$-spaces, and so Chapter 3 is entirely devoted to them. In particular, it is described how any twisted sum of $c_0(\kappa)$ and X can be obtained by means of certain compact spaces which are made by adjoining a discrete subspace to (B_{X^*}, weak^*). The material of Chapter 3 borrows from the paper

[27] J. M. F. CASTILLO AND A. SALGUERO-ALARCÓN, *Twisted sums of* $c_0(I)$, to be published in Quaestiones Mathematicae (2022)

Most of the tools and techniques displayed in Chapters 2 and 3 somehow crystallize in Chapter 4, where a solution to the longstanding open problem concerning complemented subspaces of $C(K)$-spaces is provided. Precisely, we construct a $C(K)$-space containing a 1-complemented subspace which is not a $C(K)$-space. The contents from this chapter are based on the paper

[88] G. PLEBANEK AND A. SALGUERO-ALARCÓN, *The complemented subspace problem: A counterexample*, to be published (2022)

Chapter 5 tells quite a different story. For a start, it features twisted sums of $C(K)$-spaces and other Banach spaces. Moreover, the techniques employed are brought from the general theory of twisted sums of Banach spaces, contrarily to those used in the previous chapters, which are mostly specific of twisted sums of $C(K)$-spaces. There are two results in Chapter 5 deserving special attention. The first one constitutes the main result of the paper

[19] F. CABELLO, J. M. F. CASTILLO AND A. SALGUERO-ALARCÓN, *The behaviour of quasi-linear maps on C(K)-spaces*, Journal of Mathematical Analysis and Applications, 475 (2019), pp. 1714–1719

and it analyzes the behavior of twisted sums of a Banach space with a $C(K)$-space. The other main result in Chapter 5 features the construction of a very special twisted sum of

L_1 and $C(K)$ using another very special twisted sum of Hilbert spaces. This result is contained in the paper

[21] F. Cabello and A. Salguero-Alarcón, *When Kalton and Peck met Fourier*, to be published in Annales de l'Institut Fourier (2022)

whose purpose is to explore the construction of twisted sums of classical Banach L_1-modules by means of Fourier analysis. Generally speaking, we could say that Chapter 5 constitutes an attempt of showing the interaction between twisted sums with $C(K)$-spaces and twisted sums of other Banach spaces. Be as it may, the full extent of such interaction has yet to be uncovered.

Chapter 1

Fundamentals of $C(K)$-spaces

This chapter is merely introductory, and contains all the neccesary background to proceed. It gathers material from essentially two areas: homology and category theory applied to Banach spaces, and topological spaces with some mention to Martin's axiom. It has been our intention to make the dissertation reasonably self-contained and so we have sketched a number of proofs. If, however, the reader feels terribly hungry for something new, then we encourage them to just go for Chapter 2.

1.1 A categorical introduction

A *category* $\mathscr{C}$ consists of:

- A class of *objects*.
- For every pair of objects X and Y of $\mathscr{C}$, a class $\mathrm{Hom}_{\mathscr{C}}(X,Y)$, whose elements are called *morphisms* and are denoted by $f\colon X \to Y$.
- For every objects X, Y, Z of $\mathscr{C}$, a map

  $$\mathrm{Hom}_{\mathscr{C}}(X,Y) \times \mathrm{Hom}_{\mathscr{C}}(Y,Z) \to \mathrm{Hom}_{\mathscr{C}}(X,Z) \quad , \quad (f,g) \mapsto g \circ f$$

  which is called *composition of morphisms* and satisfies the following properties:

  - It is *associative*; namely, for morphisms $f \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$, $g \in \mathrm{Hom}_{\mathscr{C}}(Y,Z)$ and $h \in \mathrm{Hom}_{\mathscr{C}}(Z,W)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

- For every object X of $\mathscr{C}$, there is a morphism $1_X \in \mathrm{Hom}(X,X)$, called the *identity morphism*, such that $f \circ 1_X = f$ for every $f \in \mathrm{Hom}_{\mathscr{C}}(Y,X)$ and $1_X \circ g = g$ for every $g \in \mathrm{Hom}_{\mathscr{C}}(X,Z)$.

A *covariant functor* F between two categories $\mathscr{C}$ and $\mathscr{D}$ assigns:

- To every object X in $\mathscr{C}$, an object FX in $\mathscr{D}$.
- To every morphism $f \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$, a morphism $Ff \in \mathrm{Hom}_{\mathscr{D}}(FX,FY)$ such that F respects the associativity; namely, $F(g \circ f) = Fg \circ Ff$, and $F1_X = 1_{FX}$ for every object X of $\mathscr{C}$.

On the other hand, a *contravariant functor* F between two categories $\mathscr{C}$ and $\mathscr{D}$ assigns:

- To every object X in $\mathscr{C}$, an object FX in $\mathscr{D}$.
- To every morphism $f \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$, a morphism $Ff \in \mathrm{Hom}_{\mathscr{D}}(FY,FX)$, such that the associativity is now respected in the form $F(g \circ f) = Ff \circ Fg$, and again $F1_X = 1_{FX}$ for every object X of $\mathscr{C}$.

We will denote $F\colon \mathscr{C} \rightsquigarrow \mathscr{D}$ for a functor (covariant or contravariant) acting between $\mathscr{C}$ and $\mathscr{D}$.

The categorical point of view will only be explicitly used a few times in this dissertation, but the whole of it is pervaded by these notions. Our interest here resides mostly within two categories:

1. The category Ban of Banach spaces and (linear, continuous) operators. We follow the standard notation: $\mathrm{Hom}_{Ban}(X,Y)$ will be denoted as the customary $\mathscr{L}(X,Y)$, and the composition of operators $S \circ T$ will be denoted simply as ST. On some occasions, it will be necessary to work in Ban_1, the category of Banach spaces and *contractive* operators; that is, operators with norm at most 1.

2. The category $CHaus$ of compact Hausdorff spaces, or *compacta*, and continuous mappings. We will write $C(K,L)$ instead of $\mathrm{Hom}_{CHaus}(K,L)$.

The interaction between both categories is mainly performed by means of the following two functors:

1. The contravariant functor $\bigcirc^*\colon Ban_1 \rightsquigarrow CHaus$ which takes any Banach space X to its dual unit ball B_{X^*} endowed with the weak* topology and every contractive operator $T\colon X \to Y$ to the continuous mapping $T^*|_{B_{Y^*}}\colon (B_{Y^*}, w^*) \to (B_{X^*}, w^*)$.

2. The contravariant functor $C(\cdot)\colon \mathit{CHaus} \rightsquigarrow \mathit{Ban}_1$ taking any compactum K to the Banach space $C(K)$ and any continuous mapping $\varphi\colon K \longrightarrow L$ to the operator $\varphi^\circ\colon C(L) \longrightarrow C(K)$ given by $\varphi^\circ(f) = f \circ \varphi$.

In fact, there is a deep relation between these two functors –the proper word for it is *adjointness*. Two contravariant functors $F\colon \mathscr{C} \rightsquigarrow \mathscr{D}$ and $G\colon \mathscr{D} \rightsquigarrow \mathscr{C}$ are said to be *adjoint* if for every objects X in $\mathscr{C}$ and Y in $\mathscr{D}$, there is a natural bijection

$$\mathrm{Hom}_{\mathscr{D}}(X, FY) = \mathrm{Hom}_{\mathscr{C}}(Y, GX)$$

In our particular case, this means $C(K, B_{X^*}) = \mathscr{L}(X, C(K))$. Precisely, the relation between some continuous mapping $\varphi\colon K \longrightarrow B_{X^*}$ and some contractive operator $T\colon X \longrightarrow C(K)$ is given by the equality $\langle\varphi(t), x\rangle = Tx(t)$ for every $t \in K$ and $x \in X$.

It is no secret that this dissertation deals with $C(K)$-spaces. To avoid explicit mention of the underlying compactum, we will use the term *C-space* for a Banach space which is isomorphic to a space of the form $C(K)$ for some compactum K. In general, we will write $X \simeq Y$ to mean that the Banach spaces X and Y are isomorphic. Let us remark a particularly interesting example of C-space:

Proposition 1.1.1. *Every hyperplane of a C-space is a C-space. Moreover, every C-space containing a complemented copy of c_0 is isomorphic to its hyperplanes.*

Proof. Any two hyperplanes of a given Banach space are isomorphic, so it is sufficient to consider two arbitrary different points t_1, t_2 of a compactum K and show that $X = \{f \in C(K) : f(t_1) = f(t_2)\}$ is a C-space. Now, X is readily seen to be isomorphic to $C(\tilde{K})$, where $\tilde{K}$ is the quotient space of K obtained by identifying the points t_1 and t_2. If, moreover, $C(K)$ contains a complemented copy of c_0, then $C(K) \simeq X \oplus c_0 \simeq X \oplus c_0 \oplus \mathbb{R} \simeq C(K) \oplus \mathbb{R}$, and consequently $C(\tilde{K})$ is isomorphic to a hyperplane of $C(K) \oplus \mathbb{R}$. Since $C(K)$ is also a hyperplane of $C(K) \oplus \mathbb{R}$, we conclude. □

We remark the fact that a C-space need not be isomorphic to its hyperplanes. Koszmider provides in [66] an involved construction of a compact space K such that $C(K)$ is not isomorphic to any of its proper subspaces.

The well-known class of $\mathscr{L}_\infty$-spaces provides a natural generalization of C-spaces. Given $1 \le p \le \infty$ and $\lambda \ge 1$, we say a Banach space X is an *$\mathscr{L}_{p,\lambda}$-space* if every finite-dimensional subspace $E \subseteq X$ is contained in a finite-dimensional subspace $F \subseteq X$ for which there is an isomorphism $T\colon F \longrightarrow \ell_p^{\dim F}$ with $\|T\| \cdot \|T^{-1}\| \le \lambda$. Now, we say X is an $\mathscr{L}_p$-space when it is an $\mathscr{L}_{p,\lambda}$-space for some $\lambda \ge 1$. Actually, only the cases $p = 1$

and $p = \infty$ will be used later. A simple argument through partitions of unity shows that C-spaces are $\mathscr{L}_\infty$-spaces.

We now turn to the duals of C-spaces and $\mathscr{L}_\infty$-spaces. The classical theorem of F. Riesz allows us to identify the dual of $C(K)$ with $M(K)$, the space of (finite, regular, Borel, signed) measures on K. The symbol $M_1(K)$ will stand for the dual unit ball of $C(K)$ with the weak* topology. On the other hand, duals of $\mathscr{L}_\infty$-spaces are $\mathscr{L}_1$-spaces; in fact, a Banach space X is an $\mathscr{L}_\infty$-space if and only if X^* is an $\mathscr{L}_1$-space [72, Theorem III]. The Banach spaces which are $\mathscr{L}_{\infty,1+\varepsilon}$-spaces for every $\varepsilon > 0$ deserve special attention: they are called *Lindenstrauss spaces*, and they coincide with the class of isometric $L_1(\mu)$-preduals [72, Theorem II]

1.2 Homological principles in Banach spaces

An *exact sequence* of Banach spaces is a diagram

$$0 \longrightarrow Y \xrightarrow{\ j\ } Z \xrightarrow{\ q\ } X \longrightarrow 0$$

formed by Banach spaces and linear continous operators so that the kernel of each arrow coincides with the image of the preceding one. By virtue of the open mapping theorem, this amounts to saying that j is an into isomorphism, q is a quotient operator and X is isomorphic to $Z/j(Y)$. The middle space is usually called a *twisted sum* of Y and X, or an *extension* of X by Y.

Even if we are only concerned with Banach spaces, the awful truth is that one has to deal with *quasi-Banach spaces* in order to work with twisted sums of Banach spaces. The reason is simple: a twisted sum of two Banach spaces does not need to be a Banach space –cf. [89] or [61, §4]. Hence let us recall that a *quasi-norm* on a vector space X is a map $\|\cdot\|: X \to \mathbb{R}$ satisfying

- $\|x\| = 0$ if and only if $x = 0$.
- $\|\lambda x\| = |\lambda| \cdot \|x\|$.
- There is $\Delta \geq 1$ such that $\|x + y\| \leq \Delta(\|x\| + \|y\|)$.

for all $\lambda \in \mathbb{R}$ and $x, y \in X$. As with norms, every quasi-norm induces a vector topology in X which is generated by the open unit ball $\{x \in X : \|x\| < 1\}$, and when such topology is complete we say X is a *quasi-Banach space*. Of course, when $\Delta = 1$, we recover Banach spaces. The open mapping theorem also works in quasi-Banach spaces [93, 2.11], so the first paragraph in this section also applies to quasi-Banach spaces, word by word.

However, we must admit that in virtually every exact sequence appearing throughout the dissertation in which X and Y are Banach spaces, so will be Z. This is thanks to a deep result of Kalton and Roberts [64, Theorem 6.3] which, in combination with [60, Theorem 4.10], assures that if X is an $\mathscr{L}_\infty$-space and Y is a Banach space, then any twisted sum of Y and X is also a Banach space. Keeping this in mind, the reader is free to think all the time about Banach spaces.

We say two exact sequences

$$0 \longrightarrow Y \longrightarrow Z_k \longrightarrow X \longrightarrow 0 \qquad k = 1, 2$$

are *equivalent* if there is an operator $u\colon Z_1 \to Z_2$ making commutative the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Z_1 \arrow[r] \arrow[d, "u"] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] & Z_2 \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

The next lemma shows that u is in fact an isomorphism, and therefore this is a true equivalence relation.

The three-lemma 1.2.1. *Consider the following diagram diagram with exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, "\alpha"] & Z \arrow[r] \arrow[d, "\beta"] & X \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X' \arrow[r] & 0
\end{tikzcd}$$

If α and γ are injective (respectively, surjective) then so is β.

We say that an exact sequence $0 \longrightarrow Y \xrightarrow{j} Z \xrightarrow{q} X \longrightarrow 0$ is *trivial*, or that *it splits*, if it is equivalent to the sequence

$$0 \longrightarrow Y \xrightarrow{\theta} Y \oplus X \xrightarrow{\rho} X \longrightarrow 0$$

where $\theta(y) = (y, 0)$ and $\rho(y, x) = x$. The following conditions are equivalent to the triviality of a given exact sequence:

- The operator j admits a right-inverse operator, usually called *projection*.
- The operator q admits a left-inverse operator, which will be called a *selection*.

Of course, the existence of a projection for j is equivalent to the fact that $j(Y)$ is *complemented* in Z. We also say that $j(Y)$ is *λ-complemented* in Z when there is a projection for j having norm no greater than λ.

We denote $\operatorname{Ext}(X,Y)$ the set of all short exact sequences of Y and X modulo equivalence. An element of $\operatorname{Ext}(X,Y)$ will be thus represented by

$$0 \longrightarrow Y \xrightarrow{j} Z \xrightarrow{q} X \longrightarrow 0 \qquad [z]$$

In fact, $\operatorname{Ext}(X,Y)$ can be endowed with a vector space structure [14, §7] –cf. also [16]– in which the equivalence class of the trivial exact sequences is the zero element. Therefore, $\operatorname{Ext}(X,Y)=0$ means every twisted sum of Y and X is trivial.

A secondary theme which will appear more often than not throughout these pages is that of 3-space properties. After all, it can be argued that 3-space properties is one of the topics that motivated the study of twisted sums in Banach spaces. Let us recall that a property $\mathscr{P}$ of Banach spaces is a *3-space property* whenever every twisted sum of two spaces having $\mathscr{P}$ also has $\mathscr{P}$. The monograph [23] is enthusiastically devoted to the study of 3-space properties. For what we matter here, it is not difficult to check that "to be an $\mathscr{L}_\infty$-space" is a 3-space property. Also, just in case the impatient reader is wondering, "to be a C-space" is not a 3-space property. In fact, not even "to be a Lindenstrauss space" is a 3-space property. The appropriate place for such considerations is Section 2.3.

Quasi-linear maps

It was a discovering of Kalton [60] that exact sequences can be represented by certain nonlinear maps called *quasi-linear maps*, which are homogeneous maps $\Omega\colon X \to Y$ acting between quasi-Banach spaces for which there exists $M \geq 0$ such that for every $x_1, x_2 \in X$, the following holds:

$$\|\Omega(x_1+x_2)-\Omega(x_1)-\Omega(x_2)\| \leq M\big(\|x_1\|+\|x_2\|\big)$$

The smallest M satisfying the previous inequality will be referred to as the *quasi-linearity constant* of Ω. Every quasi-linear map between two quasi-Banach spaces $\Omega\colon X \to Y$ induces an exact sequence

$$0 \longrightarrow Y \longrightarrow Y \oplus_\Omega X \longrightarrow X \longrightarrow 0$$

where $Y \oplus_\Omega X$ is the vector space $Y \times X$ endowed with the quasi-norm $\|(x,y)\|_\Omega = \|y-\Omega x\| + \|x\|$. In fact, every exact sequence

$$0 \longrightarrow Y \xrightarrow{j} Z \xrightarrow{q} X \longrightarrow 0$$

is equivalent to one induced by a certain quasi-linear map. Indeed, let $L\colon X \to Z$ be a linear (not necessarily continuous) section and $B\colon X \to Z$ a continuous homogeneous section, which exists by the open mapping theorem. The difference $B - L$ takes values in $\ker q = j(Y)$, and it is easily checked that $\Omega = j^{-1}(B - L)$ is a quasi-linear map from X to Y. Finally, there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Y \oplus_\Omega X \arrow[r] \arrow[d, "u"] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r, "j"] & Z \arrow[r, "q"] & X \arrow[r] & 0 \end{tikzcd}$$

where $u(y, x) = j(y) + L(x)$.

We say a quasi-linear map $\Omega\colon X \to Y$ is *trivial* if it induces the trivial twisted sum. This happens if and only if Ω can be written as the the sum of a linear map $L\colon X \to Y$ and a bounded map $B\colon X \to Y$. Precisely, any map u which makes commutative a diagram of the form

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Y \oplus_\Omega X \arrow[r] \arrow[d, "u"] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r] & Y \oplus X \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

is necessarily of the form $u(y, x) = (y - Lx, x)$, where $L\colon X \to Y$ is a linear map. Additionally, one can check that such a u is continuous precisely when $\Omega - L$ is bounded.

Now, we say two quasi-linear maps $\Omega, \Psi\colon X \to Y$ are *equivalent* if $\Omega - \Psi$ is trivial, which is the same as saying that their induced exact sequences are equivalent. If we denote $Q(X, Y)$ the set of equivalence classes of quasi-linear maps from X to Y, it is clear that there is a natural correspondence

$$\operatorname{Ext}(X, Y) \;=\!=\!=\!=\; Q(X, Y)$$

Pullback and pushout

Two basic homological constructions are the *pullback* and the *pushout*. Consider a category $\mathscr{C}$ (which in practice will always be *Ban* or Ban_1) and the following diagram of objects and morphisms in $\mathscr{C}$:

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \\ B & \end{tikzcd}$$

The pushout of this diagram, if it exists, is an object PO in $\mathscr{C}$, together with two morphisms $\alpha_1 : A \to PO$ and $\beta_1 : B \to PO$, enjoying the following *universal property*: given another object X in $\mathscr{C}$ and two morphisms $\alpha' : A \to X$ and $\beta' : B \to X$ such that $\alpha'\alpha = \beta'\beta$, there is a unique morphism $\gamma : PO \to X$ that makes diagram (1.a) commutative:

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[d, "\alpha_1"] \arrow[ddr, bend left, "\alpha'"] & \\ B \arrow[r, "\beta_1"] \arrow[drr, bend right, "\beta'"] & PO \arrow[dr, dashed, "\gamma"] & \\ & & X \end{tikzcd} \tag{1.a}$$

By virtue of the universal property of the pushout, the space PO is unique up to isomorphism. In Ban, or Ban_1, the pushout is

$$PO = \frac{A \oplus_1 B}{\overline{\Delta}}$$

where $\Delta = \{(\alpha c, -\beta c) : c \in C\}$ and the morphisms $\alpha_1 : A \to PO$ and $\beta_1 : B \to PO$ are the canonical quotient operators $\alpha_1(a) = \overline{(a, 0)}$ and $\beta_1(b) = \overline{(0, b)}$.

Now we consider an exact sequence $0 \to Y \xrightarrow{j} Z \xrightarrow{q} X \to 0$ and an operator $S : Y \to V$. If we form the pushout, we end up with a diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, "S"] & Z \arrow[r, "q"] \arrow[d, "S_1"] & X \arrow[r] & 0 \\ & V \arrow[r, "j_1"] & PO & & \end{tikzcd}$$

which is far from telling the whole story. Let us complete it: first, note that if j is an into isomorphism, then Δ is closed and j_1 is also an into isomorphism. Second, we have $jq = 0S$ for the zero morphism $0 : V \to X$, and so the universal property of the pushout yields a quotient operator $q_1 : PO \to X$ satisfying $q_1 S_1 = q$.

Taken all together, this produces the *pushout diagram*

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, "S"] & Z \arrow[r, "q"] \arrow[d, "S_1"] & X \arrow[r] \arrow[d, equal] & 0 & {[\mathrm{z}]} \\ 0 \arrow[r] & V \arrow[r, "j_1"] & PO \arrow[r, "q_1"] & X \arrow[r] & 0 & {[\mathrm{z}S]} \end{tikzcd} \tag{1.b}$$

in which the lower sequence is also exact, since j_1 is into, q_1 is onto and

$$\ker q_1 = \{\overline{(x,b)} : q(x) = 0\} = \{\overline{(jy,b)} : y \in Y\} = \{\overline{(jy,0)} : y \in Y\} = j_1(V)$$

We will call the lower row in (1.b) the *pushout sequence*.

Lemma 1.2.2. *The pushout sequence* $[zS]$ *in (1.b) splits if and only if* S *admits an* extension *through* j*; that is, an operator* $R\colon Z \to V$ *such that* $Rj = S$.

Proof. If the lower row splits, then an extension for S is obtained by composition of S_1 with a projection for j_1. For the converse, note that for every extension R of S, the operator $S_1 - j_1R$ vanishes on $j(Y)$. Hence, there exists an operator $U\colon X \to PO$ such that $Uq = S_1 - j_1R$, and so U is a selection for q_1 because $q_1Uq = q_1(S_1 - j_1R) = q$. □

In fact, diagram (1.b) characterizes the pushout, in the sense that in every diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{j} & Z & \xrightarrow{q} & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & V & \xrightarrow{i} & A & \xrightarrow{p} & X & \longrightarrow & 0 \end{array}$$

the lower row must be equivalent to the pushout sequence $[zS]$ in (1.b). Indeed, since $Tj = iS$, by virtue of the universal property of the pushout we obtain an operator $u\colon PO \to A$ which satisfies $uj_1 = i$ and $uS_1 = T$, thus necessarily making commutative the bottom left square of diagram (1.c). To show commutativity of the bottom right square, we observe that the operator pu satisfies $puS_1 = pT = q$ and $puj_1 = pi = 0$, just like q_1 does. Since q_1 was obtained through the universal property of the pushout, we must have $q_1 = pu$.

$$\begin{array}{cccccccccc} 0 & \longrightarrow & Y & \xrightarrow{j} & Z & \xrightarrow{q} & X & \longrightarrow & 0 & [z] \\ & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle T} & & \| & & & \\ 0 & \longrightarrow & V & \xrightarrow{i} & A & \xrightarrow{p} & X & \longrightarrow & 0 & \\ & & \| & & {\scriptstyle u}\uparrow & & \| & & & \\ 0 & \longrightarrow & V & \xrightarrow{j_1} & PO & \xrightarrow{q_1} & X & \longrightarrow & 0 & [zS] \end{array} \tag{1.c}$$

The pullback deals with the "dual" situation. Precisely, consider the following diagram in $\mathscr{C}$:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & C \\ & & {\scriptstyle \beta}\uparrow \\ & & B \end{array}$$

Its pullback, if it exists, is an object PB of $\mathscr{C}$, together with two morphisms $\alpha_1 : PB \longrightarrow A$ and $\beta_1 : PB \longrightarrow B$ such that the following *universal property* is satisfied: whenever X is another object in $\mathscr{C}$ with morphisms $\alpha' : X \longrightarrow A$ and $\beta' : X \longrightarrow B$ such that $\alpha\alpha' = \beta\beta'$, there is a unique morphism $\gamma : X \longrightarrow PB$ making diagram (1.d) commutative.

$$\begin{tikzcd}
 & A \arrow[r, "\alpha"] & C \\
 & PB \arrow[u, "\alpha_1"] \arrow[r, "\beta_1"] & B \arrow[u, "\beta"] \\
X \arrow[uur, bend left, "\alpha'"] \arrow[ur, "\gamma"] \arrow[urr, bend right, "\beta'"'] & &
\end{tikzcd} \tag{1.d}$$

It is not difficult to see that, if it exists, the universal property implies that PB is unique up to isomorphism. The pullback exists in *Ban* and Ban_1, and it is

$$PB = \{(a, b) \in A \oplus_\infty B : \alpha(a) = \beta(b)\}$$

where $\alpha_1 : PB \longrightarrow A$ and $\beta_1 : PB \longrightarrow B$ are simply $\alpha_1(a, b) = a$ and $\beta_1(a, b) = b$.

Now, let us consider an exact sequence $0 \longrightarrow Y \xrightarrow{j} Z \xrightarrow{q} X \longrightarrow 0$ and an operator $T : W \longrightarrow X$. If we form the pullback

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "j"] & Z \arrow[r, "q"] & X \arrow[r] & 0 \\
 & & PB \arrow[u, "T_1"] \arrow[r, "q_1"] & W \arrow[u, "T"] &
\end{tikzcd}$$

we end up once again with a diagram which is not complete. To fix this, note that q_1 is onto just because so is q, and also that $0T = qj$ for the zero operator $0 : Y \longrightarrow W$, hence the universal property of the pullback yields an into isomorphism $j_1 : Y \longrightarrow PB$ such that $T_1 j_1 = j$. In other words, we obtain the *pullback diagram*

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, equal] & Z \arrow[r, "q"] & X \arrow[r] & 0 & {[z]} \\
0 \arrow[r] & Y \arrow[r, "j_1"] & PB \arrow[u, "T_1"] \arrow[r, "q_1"] & W \arrow[u, "T"] \arrow[r] & 0 & {[Tz]}
\end{tikzcd} \tag{1.e}$$

whose lower sequence is also exact because j_1 is into, q_1 is onto and

$$\ker q_1 = \{(z, w) \in PB : w = 0\} = \{(z, 0) : q(z) = 0\} = \{(jy, 0) : y \in Y\} = j_1(Y)$$

Hence we will refer to the lower row in (1.e) as the *pullback* sequence.

Lemma 1.2.3. *The pullback sequence* $[Tz]$ *in (1.e) splits if and only if* T *admits a* lifting *through* q*; that is, an operator* $R\colon W \longrightarrow Z$ *such that* $qR = T$.

Proof. If the pullback sequence splits, then the composition of T_1 with any selection for q_1 provides the desired lifting. Conversely, if R is a lifting for T through q, then $Rq_1 - T_1$ takes values in $\ker q = j(Y)$, and so the mapping $P = j^{-1}(Rq_1 - T_1)$ is a projection for i_1, since $jPj_1 = (Rq_1 - T)j_1 = j$. □

Diagram (1.e) characterizes the pullback, meaning that if there is a diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, equal] & Z \arrow[r, "q"] & X \arrow[r] & 0 \\ 0 \arrow[r] & Y \arrow[r, "i"] & A \arrow[r, "p"] \arrow[u, "S"] & W \arrow[r] \arrow[u, "T"] & 0 \end{tikzcd}$$

then the lower row is equivalent to the pullback sequence $[Tz]$ in (1.e). Indeed, since $qS = Tp$, let us appeal once more to the universal property of the pullback to obtain an operator $u\colon A \longrightarrow PB$ which satisfies $T_1u = S$ and $q_1u = p$; in particular, it makes commutative the bottom right square in (1.f). Finally, the operator ui satisfies $T_1ui = S_i = j$ and $q_1ui = pi = 0$, and so $ui = j_1$, since j_1 was obtained by virtue of the universal property of the pullback. Hence the bottom left square in (1.f) is also commutative.

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, equal] & Z \arrow[r, "q"] & X \arrow[r] & 0 & {[z]} \\ 0 \arrow[r] & Y \arrow[r, "i"] \arrow[d, equal] & A \arrow[r, "p"] \arrow[u, "S"] \arrow[d, "u"] & W \arrow[r] \arrow[u, "T"] \arrow[d, equal] & 0 & \\ 0 \arrow[r] & Y \arrow[r, "j_1"] & PB \arrow[r, "q_1"] & W \arrow[r] & 0 & {[Tz]} \end{tikzcd} \tag{1.f}$$

From the quasi-linear point of view, the pullback and the pushout have simple realizations. Let us fix an exact sequence induced by some quasi-linear map Ω.

$$0 \longrightarrow Y \longrightarrow Y \oplus_\Omega X \longrightarrow X \longrightarrow 0$$

If $S\colon Y \to V$ is an operator, then the pushout sequence is induced by the quasi-linear map ΩS, as it is witnessed by the diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, "S"] & Y \oplus_\Omega X \arrow[r] \arrow[d, "S \times \mathrm{Id}_V"] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & V \arrow[r] & V \oplus_{\Omega S} X \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

Also, given $T\colon W \to X$ an operator, the pullback sequence is induced by $T\Omega$:

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] & Y \oplus_\Omega X \arrow[r] & X \arrow[r] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[u, equal] & Y \oplus_{T\Omega} W \arrow[r] \arrow[u, "\mathrm{Id}_Y \times T"] & W \arrow[r] \arrow[u, "T"'] & 0
\end{tikzcd}$$

This brings full sense to the notation employed in diagrams (1.e) and (1.b). But, most importantly, we deduce that pullbacks and pushouts are associative and commute with each other. Precisely:

- Making pushout with S and then pushout with S' is the same as making pushout with $S'S$. Similarly, making pullback with T and then pullback with T' is the same as making pullback with $T'T$.

- Making pushout with S and then pullback with T is the same as making first pullback with T and then pushout with S.

Of course, these facts also follow from the universal properties of the pullback and the pushout, but their proofs are not so straightforward.

Diagonal principles

Let us exploit the pushout and the pullback a little further. Assume we have two exact sequences $[z]$ and $[z']$. We can ask:

- When can be assured that $[z']$ is a pushout (or a pullback) of $[z]$?

- If we do know that $[z']$ is a pushout of $[z]$, when can we assure that $[z]$ is *also* a pushout (or a pullback) of $[z']$, and what consequences may that have?

Actually, the answer to both items lie in the very definitions of the pullback and the pushout. We will focus on the pushout, since it is the one we need for later. The first question is easily disposed of with the following result:

Proposition 1.2.4. *Consider the following diagram:*

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "j"] & Z \arrow[r, "q"] & X \arrow[r] \arrow[d, equal] & 0 & {[z]} \\
0 \arrow[r] & Y' \arrow[r, "j'"] & Z' \arrow[r, "q'"] & X \arrow[r] & 0 & {[z']}
\end{tikzcd}$$

$[z']$ is a pushout of $[z]$ if and only if $[z'q] = 0$.

Proof. It is clear that $[\mathrm{z}q] = 0$ because the identity map on Z is a lifting for q. Hence, $[\mathrm{z}'] = [T\mathrm{z}]$ implies $[\mathrm{z}'q] = [(T\mathrm{z})q] = [T(\mathrm{z}q)] = 0$. As for the converse, $[\mathrm{z}q'] = 0$ means q can be lifted to an operator $S\colon Z \to Z'$. The restriction of S to $j(Y)$ takes values on $j'(Y') = \ker q'$ since $q'Sj = qj = 0$. Therefore, there is an operator $T\colon Y \to Y'$ satisfying $j'T = Sj$, and so $[\mathrm{z}'] = [T\mathrm{z}]$, as witnessed by the diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, "T"] & Z \arrow[r, "q"] \arrow[d, "S"] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y' \arrow[r, "j'"] & Z' \arrow[r, "q'"] & X \arrow[r] & 0 \end{tikzcd}$$

□

The answer to the second question is concealed in the so-called *diagonal pushout* sequence. Given a pushout diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, "T"] & Z \arrow[r, "q"] \arrow[d, "T_1"] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & V \arrow[r, "j_1"] & PO \arrow[r, "q_1"] & X \arrow[r] & 0 \end{tikzcd}$$

the diagonal pushout sequence is formed just by letting ourselves go:

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "\Delta"] & Z \times V \arrow[r] & PO \arrow[r] & 0 \end{tikzcd} \tag{1.g}$$

The operators are $\Delta(y) = (jy, -Ty)$ and the quotient map is the one appearing in the definition of PO. In fact, the diagonal pushout sequence is no other thing that the pullback $[\mathrm{z}q_1]$, as witnessed by the following diagram in which $Q(z, v) = z$:

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "j"] & Z \arrow[r, "q"] & X \arrow[r] & 0 \\ 0 \arrow[r] & Y \arrow[r, "\Delta"] \arrow[u, equal] & Z \times V \arrow[r] \arrow[u, "Q"] & PO \arrow[r] \arrow[u, "q_1"] & 0 \end{tikzcd} \tag{1.h}$$

Definition. We say two exact sequences

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] & Z \arrow[r] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

are *semi-equivalent* if each of them is a pushout of the other one.

Right-side diagonal principle 1.2.5. *If the exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "j"] & Z \arrow[r, "q"] & X \arrow[r] \arrow[d, equal] & 0 & {[\mathtt{z}]} \\ 0 \arrow[r] & Y' \arrow[r, "j'"] & Z' \arrow[r, "q'"] & X \arrow[r] & 0 & {[\mathtt{z'}]} \end{tikzcd}$$

are semi-equivalent, then $Y \times Z' \simeq Y' \times Z$.

Proof. If $[\mathtt{z}'] = [T\mathtt{z}]$, consider the diagonal pushout sequence as in diagram (1.h), namely

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "j"] & Z \arrow[r, "q"] & X \arrow[r] & 0 \\ 0 \arrow[r] & Y \arrow[r] \arrow[u, equal] & Z \times Y' \arrow[r] \arrow[u] & Z' \arrow[r] \arrow[u, "q'"] & 0 \end{tikzcd}$$

The lower row splits precisely when there is a lifting for q'. But, since $[\mathtt{z}] = [S\mathtt{z}']$, the pushout diagram

$$\begin{tikzcd} 0 \arrow[r] & Y' \arrow[r, "j'"] \arrow[d, "S"] & Z' \arrow[r, "q'"] \arrow[d, "\bar{S}"] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r, "j"] & Z \arrow[r, "q"] & X \arrow[r] & 0 \end{tikzcd}$$

asserts that $\bar{S}$ is the lifting for q' we were looking for. □

Of course, there is also a diagonal pullback sequence and a "left-side" diagonal principle for the pullback, and they follow from a dualization of the previous arguments. However, since we will not use them, we prefer to refer the reader to [16, Ch. 2] for a fully detailed exposition on the basic techniques of homology applied to Banach spaces.

1.3 Martin's axiom and its consequences

We will now focus on C-spaces, and more precisely, on their underlying compact spaces. A good amount of topological properties of those compacta are under the influx of cardinality axioms. Therefore, we may recall what the standard assumptions regarding cardinal numbers are and what effects they produce. Actually, not much acquaintance of cardinalities is necessary to proceed. We will only work with the following cardinalities:

- $\aleph_0$, the cardinality of the set of natural numbers $\mathbb{N}$.

- $\mathfrak{c}$, the cardinality of the real numbers, which agrees with $2^{\aleph_0}$.
- $\aleph_1$, the first uncountable cardinal.

In general, given a cardinality κ, we write κ^+ for the least cardinal number which is strictly bigger than κ. In particular, $\aleph_1 = \aleph_0^+$, hence $\aleph_1 \leq \mathfrak{c}$. However, it is well known that the equality $\aleph_1 = \mathfrak{c}$ cannot be proven nor disproven within ZFC. The basic assumption to this respect is, of course:

Continuum hypothesis (CH). $\aleph_1 = \mathfrak{c}$.

Martin's axiom is a weaker (and far more interesting) form of the continuum hypothesis. Roughly speaking, it amounts to say that cardinalities $\aleph_0 \leq \kappa < \mathfrak{c}$ "behave like" $\aleph_0$, whether they exist or not. The usual and more convenient formulation of Martin's axiom is in terms of partially ordered sets. However, since it will not be used in this form throughout the dissertation, we will state Martin's axiom in terms of topological spaces, since then it arises as a natural generalization of Baire's category theorem. To do so, recall that a topological space satisfies the *countable chain condition*, *ccc* for short, if every collection of disjoint open sets is at most countable. Every separable space is necessarily ccc, but the converse is not true [45, 12I]. Now, consider the following statement:

MA(κ). *No ccc compactum can be written as a union of κ-many nowhere dense subsets.*

We know that MA($\aleph_0$) is true: this is simply Baire's category theorem. On the other hand, MA($\mathfrak{c}$) is false, since the unit interval $[0, 1]$ can be written as the disjoint union of its singletons, which are nowhere dense subsets. Hence it is reasonable to ask what can be said for cardinalities between $\aleph_0$ and $\mathfrak{c}$.

Martin's axiom (MA). *For any $\kappa < \mathfrak{c}$,* MA(κ) is true.

There is a weakening of Martin's axiom which we do use later, and it also follows the philosophy of accepting that the behaviour of cardinalities smaller than $\mathfrak{c}$ is similar to that of $\aleph_0$. Given two sets A and B, we will write $A \subseteq^* B$ to indicate that A is *almost contained* in B; that is, when $A \setminus B$ is finite.

Martin's axiom for σ-centered posets, or $\mathfrak{p} = \mathfrak{c}$. *For every family $\mathscr{A} \subseteq \mathscr{P}(\mathbb{N})$ satisfying $|\mathscr{A}| < \mathfrak{c}$ and such that the intersection of every finite subfamily is infinite, there exists an infinite $B \in \mathscr{P}(\mathbb{N})$ such that $B \subseteq^* A$ for all $A \in \mathscr{A}$.*

To examine the relation between our statement of Martin's axiom and $\mathfrak{p} = \mathfrak{c}$, let us appeal to [45, p. 14C] for a translation of the latter into the language of topology:

Theorem 1.3.1. *The following are equivalent:*

i) $\mathfrak{p} = \mathfrak{c}$.

ii) No separable compactum can be written as a union of less than $\mathfrak{c}$*-many nowhere dense sets.*

It is now clear that $\mathsf{CH} \Rightarrow \mathsf{MA} \Rightarrow [\mathfrak{p} = \mathfrak{c}]$. Furthermore, none of these implications can be reversed [45, 11E].

Finally, as an example of a straightforward application of $\mathfrak{p} = \mathfrak{c}$, let us mention the following: $[0, 1]^\kappa$ is sequentially compact for $\kappa < \mathfrak{c}$. This is because what ensures the sequential compactness of $[0, 1]^{\mathbb{N}}$ is the fact that, given any countable decreasing chain of infinite subsets of $\mathbb{N}$, there is another infinite subset which is almost contained in all of them. Hence $\mathfrak{p} = \mathfrak{c}$ allows this argument to work for $\kappa < \mathfrak{c}$.

And now for the elephant in the room:

Why axioms?

These seemingly subtle axiomatic distinctions are absolutely necessary for anyone willing to venture into the realm of twisted sums of C-spaces. There are appropriate examples in several places of this dissertation, especially in Chapter 2, but let us describe another instance of this situation which hopefully will content the impatient reader.

It is a classic fact that if K is metrizable, then $C(K)$ is separable, and therefore Sobczyk's theorem asserts that $\operatorname{Ext}(C(K), c_0) = 0$ –see [20] for a clear exposition of Sobczyk's theorem and several extensions. The paper [17] asks about the converse, which would later become known as the *CCKY problem*: is $\operatorname{Ext}(C(K), c_0) \neq 0$ whenever K is a non-metrizable compacta? A full answer to this problem appeared in [7] and [78]: while it is true that $\operatorname{Ext}(C(K), c_0) \neq 0$ for *most* compacta, the problem is undecidable within the usual set theory. Indeed, the CCKY problem has an affirmative answer under CH, but counterexamples arise in the presence of Martin's axiom. Perhaps one of the most surprising instances features certain compacta of weight $\aleph_1$ in the presence of Martin's axiom, like the Cantor's cube $2^{\aleph_1}$. Corollary 5.2 in [78], in conjunction with [31, Theorem 2.7], reads:

- $[\mathsf{CH}]$ $\operatorname{Ext}(C(2^{\aleph_1}), c_0) \neq 0$.
- $[\mathsf{MA} + \neg\mathsf{CH}]$ $\operatorname{Ext}(C(2^{\aleph_1}), c_0) = 0$.

Another example of this phenomenon is given by the classical Alexandroff-Urysohn spaces generated by almost disjoint families of size $\aleph_1$, which with we will deal in Section 2.1.1.

1.4 Compact spaces in functional analysis

The previous section has hopefully established beyond doubt that cardinalities matter in the study of C-spaces. And it is evident that compact spaces matter in the study of C-spaces. Perhaps the simplest way in which all these three elements interact is by mean of cardinal invariants. Section 2.2.2 clearly illustrates this phenomenon, but one can perceive this interaction even at a rudimentary level, as we now describe.

Let us recall the basics of cardinal invariants: $w(X)$ and $d(X)$ will stand for the *weight* and the *density* of a topological space X. Also, the inequalities $d(X) \leq w(X)$ and $d(X) \leq |X|$ are always true, and if X is metrizable then $d(X) = w(X)$. When working with compacta, we can add one more inequality to the list: $w(K) \leq |K|$ [41, Th. 3.1.21]. Cardinal invariants are particularly useful to relate compact spaces with their corresponding spaces of continuous functions. For instance, a careful read of the Stone-Weierstrass theorem yields $w(K) = d[C(K)] = w[C(K)]$, while the inequality $d[M_1(K)] \leq d(K)$ arises from the universal property of Stone-Čech compactifications of discrete sets: every compactum of density κ is a quotient of βI, where I is a discrete topological space of cardinality κ.

Let us now focus on the interaction between compacta and Banach spaces. We will take a tour through the standard classes of compact spaces which often appear in functional analysis, together with their fundamental properties.

Scattered compacta

Given a compactum K, the *derived set* of K is denoted by K'. In general, given an ordinal α, we inductively define $K^{(\alpha+1)} = (K^{\alpha})'$, and $K^{(\alpha)} = \bigcap_{\beta<\alpha} K^{\beta}$ in the case α is a limit ordinal. We say K is *scattered* if there is some ordinal α such that $K^{(\alpha)} = \emptyset$, and the *height* of K is the least ordinal α satisfying such equality. There are several classical characterizations of scattered compacta. Let us state those that will be later of use.

Theorem 1.4.1. *The following are equivalent:*

i) K is scattered.

ii) Every non-empty subset of K contains an isolated point.

iii) $C(K)$ is Asplund*; that is, every separable subspace of $C(K)$ has separable dual.*

iv) $M(K)$ is isomorphic to $\ell_1(K)$.

We close with the following remark about the weight of scattered compacta. Due to the fact that we could not find a proof for it in the literature, we sketch one.

Proposition 1.4.2. *If K is scattered, then $w(K) = |K|$.*

Proof. It is clear that, if $\mathrm{ht}(K)$ denotes the height of K, then $K = \bigcup_{\alpha<\mathrm{ht}(K)} K^{(\alpha)} \setminus K^{(\alpha+1)}$. Hence given $\alpha < \mathrm{ht}(K)$ and $x \in K^{(\alpha)} \setminus K^{(\alpha+1)}$, there is an open neighbourhood of x, say V_x, such that $V_x \cap K^{(\alpha)} = \{x\}$ and $V_x \cap K^{(\alpha+1)} = \varnothing$. Since no V_x can be written as a union $\bigcup_{y\in Y} V_y$ for any $Y \subseteq K \setminus \{x\}$, and every base for the topology of K must contain some V_x for every $x \in K$, the proposition follows. □

Stone compacta

Stone compacta are those that possess a base of clopen sets; or equivalently, totally disconnected compacta. There is a classical category duality between the category Boole of Boolean algebras (and Boolean homomorphisms) and the category Stone of Stone compacta (and continuous mappings) which is known as Stone duality. It will be useful in the sequel to provide particularly interesting realizations of certain Stone compacta.

We now briefly describe Stone duality. Given $\mathfrak{B}$ a Boolean algebra, the set $\mathrm{ult}(\mathfrak{B})$ of all the ultrafilters on $\mathfrak{B}$ carries a natural topology having as a base the sets $\{\mathfrak{p} \in \mathrm{ult}(\mathfrak{B}) : B \in \mathfrak{p}\}$ where $B \in \mathfrak{B}$. It is not difficult to check that $\mathrm{ult}(\mathfrak{B})$ is a Stone space. On the other hand, if K is a Stone space, the set $\mathrm{clop}(X)$ of all clopen sets of K is readily seen to be a Boolean algebra. Finally, any Boolean algebra $\mathfrak{B}$ is naturally isomorphic to $\mathrm{clop}(\mathrm{ult}(\mathfrak{B}))$, and any Stone space X is naturally isomorphic to $\mathrm{ult}(\mathrm{clop}(X))$. Now we turn to morphisms: if $f\colon \mathfrak{B} \to \mathfrak{C}$ is a Boolean homomorphism, then $f^*\colon \mathrm{ult}(\mathfrak{C}) \to \mathrm{ult}(\mathfrak{B})$ defined by $f^*(\mathfrak{p}) = f^{-1}(\mathfrak{p})$ is continuous. Also, given a continuous mapping $g\colon K \to L$ between Stone spaces, then $g^*(C) = g^{-1}(C)$ defines a Boolean homomorphism from $\mathrm{clop}(L)$ to $\mathrm{clop}(K)$.

In categorical terms, the previous discussion defines two contravariant functors $\mathrm{ult}\colon \mathit{Boole} \rightsquigarrow \mathit{Stone}$ and $\mathrm{clop}\colon \mathit{Stone} \rightsquigarrow \mathit{Boole}$, together with natural isomorphisms $\mathfrak{B} \to \mathrm{clop}(\mathrm{ult}(\mathfrak{B}))$ and $X \to \mathrm{ult}(\mathrm{clop}(X))$ for every Boolean algebra $\mathfrak{B}$ and every Stone space X, so that the following diagrams are commutative:

$$\begin{tikzcd} \mathrm{clop}(\mathrm{ult}(\mathfrak{B})) \arrow[r, "\sim"] \arrow[d, "f^{**}"] & \mathfrak{B} \arrow[d, "f"] \\ \mathrm{clop}(\mathrm{ult}(\mathfrak{C})) \arrow[r, "\sim"] & \mathfrak{C} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{ult}(\mathrm{clop}(X)) \arrow[r, "\sim"] \arrow[d, "g^{**}"] & X \arrow[d, "g"] \\ \mathrm{ult}(\mathrm{clop}(Y)) \arrow[r, "\sim"] & Y \end{tikzcd}$$

In other words, the functors clop $\circ$ ult and ult $\circ$ clop are naturally equivalent to the identity functors in *Boole* and *Stone*, respectively. This is what it truly mean for two categories to be equivalent (or to be more precise, *anti-equivalent*, since the above functors are contravariant).

In this particular setting, we will denote $M(\mathfrak{B})$ the Banach space of *finitely* additive functions on $\mathfrak{B}$ endowed with the *variation norm* $\|\mu\| = |\mu|(1_{\mathfrak{B}})$, where $1_{\mathfrak{B}}$ is the unit element of $\mathfrak{B}$ and $|\mu|$ is the so-called *variation*:

$$|\mu|(A) = \sup\{|\mu(B)| + |\mu(A \setminus B)| : B \in \mathfrak{B}, B \subseteq A\}$$

The space $M(\mathrm{ult}(\mathfrak{B}))$ is canonically isomorphic to $M(\mathfrak{B})$, since every finitely additive measure on $\mathfrak{B}$ defines, by means of integration [39, p. III.2], a functional on the dense subspace of $C(\mathrm{ult}(\mathfrak{B}))$ consisting of simple $\mathfrak{B}$-measurable functions, and all functionals in $C(\mathrm{ult}(\mathfrak{B}))$ arise this way. Under such identification, it is clear that the weak* topology on $M(\mathrm{ult}(\mathfrak{B}))$ becomes the topology of convergence on the elements of $\mathfrak{B}$. We will identify $M(\mathfrak{B})$ with $M(\mathrm{ult}(\mathfrak{B}))$ when it suits us without further mention.

Eberlein and Corson compacta

A compactum is *Eberlein* if it is homeomorphic to a weakly compact set of some Banach space. Thanks to the well-known theorem of Amir and Lindenstrauss [3], every Eberlein compactum can be realised as a weakly compact set of $c_0(I)$ for a certain set I. Two canonical examples of Eberlein compacta are metrizable compacta and the one-point compactification of any discrete space I, which we will denote as αI. Indeed, metrizable compacta can be embedded into $\prod_{n=1}^{\infty}[0, \frac{1}{n}]$, which is a norm-compact subset of ℓ_2; while αI can be realized as the weakly compact subspace $\{1_i : i \in I\} \cup \{0\}$ of $c_0(I)$.

The following well-known characterization theorem is perhaps the most important result concerning Eberlein compacta and Banach spaces –cf. [3, Thm. 2].

Theorem 1.4.3. *For a compactum K, the following are equivalent:*

i) K is Eberlein.

ii) $C(K)$ is weakly compactly generated, *WCG for short; that is, there is a weakly compact subset $S \subset X$ such that $C(K) = \overline{\mathrm{span}(S)}$.*

iii) $M_1(K)$ *is Eberlein.*

The class of Corson compacta constitutes a fruitful generalization of Eberlein compacta. We say that a compactum is *Corson* if it can be embedded into a space of the form

$$\Sigma([0,1]^\Gamma) = \{x \in [0,1]^\Gamma : x(i) \neq 0 \text{ for countably many } i \in I\}$$

for some set I, endowed with the pointwise topology. The following two results are classical, but we sketch their proofs for the reader's convenience:

Proposition 1.4.4. *Every Corson compactum is* Fréchet-Urysohn*; that is, a point x belongs to the closure of some subset A if and only if there is a sequence of points in A converging to x.*

Proof. It clearly suffices to show that the spaces $\Sigma([0,1]^\Gamma)$ are Fréchet-Urysohn. Hence assume a point $x \in \Sigma([0,1]^\Gamma)$ belongs to the closure of some subset A. Write $\operatorname{supp}(x) = \{\gamma_{n,0} : n \in \mathbb{N}\}$ and choose $a_1 \in A$ such that $|a_1(\gamma_{1,0}) - x(\gamma_{1,0})| < 1$. Now consider $\operatorname{supp}(a_1) = \{\gamma_{n,1} : n \in \mathbb{N}\}$ and choose $a_2 \in A$ satisfying $|a_2(\gamma_{n,j}) - x(\gamma_{n,j})| < \frac{1}{2}$ whenever $n \in \{1,2\}$ and $j \in \{0,1\}$. Repeating this process inductively, we produce a sequence $(a_n)_{n\in\mathbb{N}}$ in A such that $a_n(\gamma) \to x(\gamma)$ whenever $\gamma \in \bigcup_{n\in\mathbb{N}} \operatorname{supp}(a_n) \cup \operatorname{supp}(x)$, and $a_n(\gamma) = 0$ otherwise, so actually $x = \lim_n a_n$. □

Proposition 1.4.5. *Every separable Corson compactum is metrizable.*

Proof. We will show a somewhat stronger result: for any subspace $X \subseteq \Sigma([0,1]^\Gamma)$, the equality $w(X) = d(X)$ holds. This implies, in particular, that separable subspaces of $\Sigma([0,1]^\Gamma)$ are second countable, and it is a classical result that second countable compacta are metrizable [41, Th. 4.2.8]. So we assume X contains a dense subset $\{x_\alpha : \alpha < \kappa\}$. Then, the support of any $x \in X$ must lie inside the set $S = \bigcup_{\alpha<\kappa} \operatorname{supp}(x_\alpha)$. Hence X is a subspace of $\Sigma([0,1]^S)$, and the latter has weight $|S| = \kappa$. □

Corson compacta are, however, somewhat sensitive to Martin's axiom. For instance, it cannot be decided in ZFC whether $M_1(K)$ is Corson provided K is. To explain why, we need the following definition:

Definition. A compactum K has *property (M)* if every measure on K has metrizable support.

Under MA + ¬CH, every Corson compactum has property (M), since the support of a measure on a compactum is clearly ccc, and then an argument using Martin's axiom shows that ccc Corson compacta are separable; details are in [45, 43Pi]. However, there are counterexamples assuming CH [83, §5]. The combination of both facts has a fatal effect, in view of the following result:

Theorem 1.4.6. *[5, Thm 3.5] Let K be a Corson compactum. Then $M_1(K)$ is Corson if and only if K has property (M).*

Rosenthal compacta

Let X be a Polish space, that is, a separable completely metrizable space. A function $f\colon X \longrightarrow \mathbb{R}$ is of *Baire class 1*, or *Baire-1* for short, if it is a pointwise limit of continuous functions on X, and we write $B_1(X)$ for the topological space of Baire-1 functions on X endowed with the pointwise topology. We say a compactum is *Rosenthal* if it can be embedded in $B_1(X)$ for some Polish space X.

Every metrizable compactum is Rosenthal, since any such K embeds into the space $C[C(K)]$ of continuous functions on the Polish space $C(K)$ by the formula $x \mapsto \delta_x(f) = f(x)$. Another classical example of a Rosenthal compactum is the so-called *double arrow space*, which we will represent by $\mathbb{S}$, and it is the product space

$$\mathbb{S} = \big((0,1] \times \{0\}\big) \cup \big([0,1) \times \{1\}\big)$$

with the topology induced by the lexicographic order. Indeed, $\mathbb{S}$ can be realised as the subspace of functions $[0,1] \longrightarrow \mathbb{R}$ which are increasing and its range is $\{0,1\}$, and such functions are clearly Baire-1.

It is a consequence of the very definition that Rosenthal compacta have a restriction on its cardinality and its weight:

Proposition 1.4.7. *Every Rosenthal compactum has cardinality and weight no bigger than $\mathfrak{c}$.*

Proof. It suffices to prove that for every Polish space X, the space $B_1(X)$ has weight and cardinality no bigger than $\mathfrak{c}$. The bound on the weight can be deduced using that every point in X is the limit of a sequence contained in a certain dense countable set, so $|X| \leq {\aleph_0}^{\aleph_0} = \mathfrak{c}$, and therefore $w[B_1(X)] \leq w(\mathbb{R}^X) \leq \mathfrak{c}$. As for the cardinality, note that every element in $C(X)$ is determined by its values on a countable dense subset of X, which implies $|C(X)| \leq {\aleph_0}^{\aleph_0} = \mathfrak{c}$. Since every Baire-1 function on X is the pointwise limit of a sequence of elements in $C(X)$, we arrive to $|B_1(X)| \leq |C(X)^{\mathbb{N}}| \leq \mathfrak{c}^{\aleph_0} = \mathfrak{c}$. $\square$

Rosenthal compacta also possess very strong sequential properties, as shown by Bourgain, Fremlin and Talagrand in [13, §3].

Proposition 1.4.8. *Every Rosenthal compactum is Fréchet-Urysohn.*

The class of Rosenthal compacta can be thought as a generalisation of the class of metrizable compacta, but it is very different to the classes of Eberlein or Corson compacta. Indeed, the double arrow space is a Rosenthal compactum which is not Corson, since it is separable but not metrizable. On the other hand, the one-point compactification of a discrete set of size $2^{\mathfrak{c}}$ is an Eberlein compactum which cannot be Rosenthal by Proposition 1.4.7. There are even examples of Corson compacta of weight $\mathfrak{c}$ which are not Rosenthal [96, p.289]. However, this situation cannot happen when considering Eberlein compacta:

Proposition 1.4.9. *Every Eberlein compactum of weight no bigger than $\mathfrak{c}$ is a Rosenthal compactum.*

Proof. The argument in Proposition 1.4.5 shows how an Eberlein compactum K of weight at most $\mathfrak{c}$ can be embedded as a weakly compact subset of $c_0(\mathbb{R})$. Now, the weak topology and the pointwise topology agree on weakly compact subsets of $c_0(\mathbb{R})$, therefore K can be regarded as a compact subset of $c_0(\mathbb{R})$ endowed with the pointwise topology. To finish, note that $c_0(\mathbb{R})$ (with the pointwise topology) is a subspace of $B_1(\mathbb{R})$, since for every $x \in \mathbb{R}$, the characteristic function 1_x is clearly Baire-1, and Baire-1 functions constitute a vector space which is closed under uniform limits. □

We will mostly concern ourselves with separable Rosenthal compacta. For that purpose, some notions of descriptive set theory are needed. Let X be a Polish space and $\mathrm{Bor}(X)$ the σ-algebra of its Borel sets. We can assign to every Borel set a "level of complexity" as follows: write $\Sigma^0_1(X)$ and $\Pi^0_1(X)$ for the classes of open and closed sets in X, respectively. Now, for every countable ordinal α, set

$$\Sigma^0_\alpha(X) = \left\{ \bigcup_{n=1}^{\infty} A_n : A_n \in \Pi^0_{\alpha_n}(X), \alpha_n < \alpha \right\} \quad , \quad \Pi^0_\alpha(X) = \{A : A^c \in \Sigma^0_\alpha(X)\}$$

Then $\mathrm{Bor}(X) = \bigcup_{\alpha<\omega_1} \Sigma^0_\alpha(X) = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha(X)$ where ω_1 is the first uncountable ordinal. Hence we say that $B \in \mathrm{Bor}(X)$ is of *Borel class* α if such α is the minimum ordinal satisfying that $B \in \Sigma^0_{1+\alpha}(X) \cup \Pi^0_{1+\alpha}(X)$.

A simple inductive argument shows that homeomorphisms between Polish spaces preserve the class of Borel sets. Since a function $f : X \to \mathbb{R}$ on a Polish space X is of

Baire class 1 if and only if $f^{-1}(G)$ is an F_σ-set for every open subset $G \subseteq \mathbb{R}$ [65, §24], then by the same token we infer that Baire-1 bijections increment the class of any Borel set in at most 1.

We will have the necessity of going a bit the Borel hierarchy:

Definition. A subset of a Polish space is *analytic* if it is a continous image of a Polish space.

It is well-known that every Borel subset of a Polish space is a continuous image of the Polish space $\mathbb{N}^{\mathbb{N}}$ (endowed with the product topology) [65, Theorem 7.9]. Therefore, every Borel subset of a Polish space is analytic, but the reciprocal is not true [65, Theorem 14.2]. The class of analytic sets of a Polish space X is denoted by $\Sigma^1_1(X)$ and, in a similar fashion to the Borel hierarchy, $\Pi^1_1(X)$ is the class of *co-analytic* sets; that is to say, those whose complement in X is analytic.

The following characterization due to Godefroy [47, Th. 4], and its consequences, will be of paramount importance. Let us introduce some notation first: given D any countable dense set in a separable Rosenthal compactum K, $C_D(K)$ stands for the space $C(K)$ with the topology of pointwise convergence on D. Properly speaking, we are identifying every $f \in C(K)$ with $(f(d))_{d \in D} \in \mathbb{R}^D$.

Theorem 1.4.10. (Godefroy) *A separable compactum K is Rosenthal if and only if for every countable dense set $D \subseteq K$ the space $C_D(K)$ is an analytic subset of $\mathbb{R}^D$.*

In light of this, it makes perfect sense to study the Borel class of the subsets $C_D(K) \subseteq \mathbb{R}^D$, assuming they are Borel. This idea presumably led Marciszewski to consider the following definition in [75] –cf. also [18, §6]:

Definition. Given K a separable Rosenthal compactum, we define its *Rosenthal index* $\mathrm{ri}(K)$ as the minimum ordinal α with the property that there is a countable dense set $D \subset K$ such that $C_D(K)$ is a Borel set of $\mathbb{R}^D$ of class α. In the case that no $C_D(K)$ is Borel, we set $\mathrm{ri}(K) = \omega_1$.

In general, $\mathrm{ri}(K) \geq 2$ whenever K is infinite [75, Th. 2.1]. This value can be attained; for instance, $\mathrm{ri}(\alpha\mathbb{N}) = 2$ and also $\mathrm{ri}(\mathbb{S}) = 2$ [37, 5.6 and 5.7]. The main utility of the Rosenthal index is that it is almost preserved by isomorphisms of C-spaces, as it is witnessed by the following theorem [18, Th. 6.3]:

Theorem 1.4.11. *Assume K and L are separable Rosenthal compacta such that $C(K) \simeq C(L)$. Then $\mathrm{ri}(K) \leq 1 + \mathrm{ri}(L)$ –and by symmetry, $\mathrm{ri}(L) \leq 1 + \mathrm{ri}(K)$.*

During the proof it will be convenient to use the following notation. Let M be a subset of $M(K)$ separating points in $C(K)$ and denote $C_M(K)$ the space of continous real-valued functions on K endowed with the weak topology of M; that is to say, we identify $f \in C(K)$ with $\big(\langle \mu, f\rangle\big)_{\mu\in M} \in \mathbb{R}^M$. This extends the notation $C_D(K)$ for any dense subset $D \subseteq K$ because if we let $\Delta(D) = \{\delta_d : d \in D\}$, then $C_{\Delta(D)}(K) = C_D(K)$.

Proof of Theorem 1.4.11. It is clearly enough to work with infinite compacta. Assume $T\colon C(K) \to C(L)$ is an isomorphism of norm one, and let D and E be dense countable subsets of K and L realizing the values of $\mathrm{ri}(K)$ and $\mathrm{ri}(L)$, respectively. Let us look at the map T acting between $C_D(K)$ and $C_E(K)$: it is obviousy bijective, but it need not be continuous anymore. However, we will show that both T and T^{-1} are Baire-1. In order to do so, we need the following observation:

Claim. *There exists countable sets of measures $M \subseteq M_1(K)$ and $N \subseteq M_1(L)$ which contain $\Delta(D)$ and $\Delta(E)$, respectively, such that $T\colon C_M(K) \to C_N(L)$ is an isomorphism.*

Proof of the claim. Note that $T^*(M_1(L)) \subseteq M_1(K)$ and that, if we let $S = T/\|T^{-1}\|$, then $S^*(M_1(K)) \subseteq M_1(L)$. Relying on this fact, we construct the desired sets by a back-and-forth argument. Let $M(0) = \Delta(D)$, $N(0) = \Delta(E)$ and

$$M(n+1) = T^*[N(n)] \quad , \quad N(n+1) = S^*[M(n)]$$

The sets $M = \bigcup_{n=0}^{\infty} M(n)$ and $N = \bigcup_{n=0}^{\infty} N(n)$ satisfy our purposes, as we now show. Let us see, for example, that $T\colon C_M(K) \to C_N(L)$ is continuous. Assume that a sequence $(g_n)_{n\in\mathbb{N}}$ converges to some function g in $C_M(K)$. The point is that if $\nu \in N$, then there is $n \in \mathbb{N}$ such that $\nu \in N(n)$, and therefore $T^*\nu \in M(n+1)$. Since $\langle T^*\nu, g_n\rangle \to \langle T^*\nu, g\rangle$, then also $\langle \nu, Tg_n\rangle \to \langle \nu, Tg\rangle$. ■

Now, to prove that $T\colon C_D(K) \to C_E(L)$ and its inverse are Baire-1, we consider the following diagram:

$$\begin{tikzcd} C_M(K) \arrow[r, "T"] \arrow[d, "\mathrm{Id}"] & C_N(L) \arrow[d, "\mathrm{Id}"] \\ C_D(K) \arrow[r, dashed, "T"] & C_E(L) \end{tikzcd}$$

Both identities are continuous, and their respective inverses are of Baire class 1. Indeed, let us show it for $\mathrm{Id}^{-1}\colon C_D(K) \to C_M(K)$. Given $\mu \in M$, there is a sequence $(\alpha_n^{\mu})_{n\in\mathbb{N}}$ of convex combinations of Δ_D converging to μ in the weak* topology. Therefore, the maps

$\varphi_n\colon C(D) \longrightarrow C_M(K)$ defined by $\langle \mu, \varphi_n g \rangle = \left\langle \alpha_n^\mu, g \right\rangle$ for any $\mu \in M$ are continuous and converge pointwise to Id^{-1}.

In order to finally compare the Borel classes of $C_D(K)$ and $C_E(L)$, we appeal to a classical extension theorem of Kuratowski [69, Ch. 3, VII] to choose Borel sets $A \subset \mathbb{R}^D$ and $B \subset \mathbb{R}^E$ of class 2 which contain $C_D(K)$ and $C_E(L)$, respectively, together with a Baire-1 bijection $\widehat{T}\colon A \longrightarrow B$ that extends T. The equality $C_E(L) = \widehat{T}[C_D(K)] \cap B$ and the fact that both $C_D(L)$ and $C_E(L)$ cannot be of Borel class lower than 2 yields $\mathrm{ri}(L) \leq 1 + \mathrm{ri}(K)$, and analogously, $\mathrm{ri}(K) \leq 1 + \mathrm{ri}(L)$. □

Let us also record the following observation, which is essentially contained in the previous proof.

Corollary 1.4.12. *Let K be a separable Rosenthal compactum and $D, E \subset K$ countable dense subsets. If $C_D(K)$ is of Borel class α, then $C_E(K)$ is of Borel class not greater than $1 + \alpha$.*

We finish with two important results, both due to Godefroy. The first one deals with the Rosenthal character of $M_1(K)$, while the second ensures that the class of separable Rosenthal compacta is stable under isomorphisms of function spaces.

Theorem 1.4.13. *[47, Prop. 7] If K is a Rosenthal compactum, then so is $M_1(K)$.*

Theorem 1.4.14. *[47, Prop. 11] Assume that K is a separable Rosenthal compactum. If $C(K) \simeq C(L)$, then L is also separable and Rosenthal.*

So, enough with prolegomena. It is time for some action.

Chapter 2

Twisted sums of $C(K)$-spaces on display

As the title suggests, this chapter contains many of the twisted sums of C-spaces that are known up to the present day. Some of them are classical, others have been just recently constructed. We could say that the majority of the chapter revolves around two objects:

- The so-called *Alexandroff-Urysohn compacta*, which arise from almost disjoint families $\mathscr{A}$ of subsets of natural numbers and are in turn denoted $K_{\mathscr{A}}$. They give rise to the Nakamura-Kakutani exact sequences

  $$0 \longrightarrow c_0 \longrightarrow C_0(K_{\mathscr{A}}) \longrightarrow c_0(\mathscr{A}) \longrightarrow 0$$

  which we develop in Section 2.1.1.

- The *double arrow space* $\mathbb{S}$, whose space of continuous funcions $C(\mathbb{S})$ will be presented via the Aharoni-Lindenstrauss sequence:

  $$0 \longrightarrow C[0,1] \longrightarrow C(\mathbb{S}) \longrightarrow c_0(\mathfrak{c}) \longrightarrow 0$$

  Similar exact sequences appear when we consider variations of $\mathbb{S}$ in which only points of a fixed subset of $[0,1]$ become "split" –see Section 2.1.2 for details.

Section 2.2 is based on the paper [18]. First, we show that there exists non-trivial twisted sums $0 \longrightarrow c_0 \longrightarrow Z \longrightarrow X \longrightarrow 0$ where X is either $c_0(\mathfrak{c})$ or $C(\mathbb{S})$. However, such a result is obtained through a counting argument –see Theorem 2.2.1– and so it does not provide us with any examples. Therefore, we will employ some techniques from descriptive set theory to construct an uncountable collection of pairwise non-isomorphic elements of $\operatorname{Ext}(c_0(\mathfrak{c}), c_0)$ (Theorem 2.2.7) as well as several explicit twisted sums of c_0 and $C(\mathbb{S})$ (Theorems 2.2.10 and 2.2.11).

Section 2.2.2 deserves special mention. It is devoted to the study of $C(K_{\mathscr{A}})$-spaces when $|\mathscr{A}| < \mathfrak{c}$ and their subsequent implications on twisted sums of c_0 and $c_0(I)$ when $|I| < \mathfrak{c}$. We solve several questions posed by Koszmider in [67] concerning non-isomorphic $C(K_{\mathscr{A}})$ spaces, and provide a partial classification of spaces of the form $C(K)$ when K is a scattered compacta of weight smaller than $\mathfrak{c}$, assuming Martin's axiom.

Finally, Section 2.3 contains a method to consistently obtain twisted sums of c_0 and $C(K)$ which are not C-spaces provided K has weight $\mathfrak{c}$ and $M_1(K)$ satisfies some additional sequential properties. Such method was first described in [87] and improved in [27]. As an application, we obtain a twisted sum of c_0 and $c_0(\mathfrak{c})$ which is not a C-space, which solves a problem posed by J. M. F. Castillo. The core of this construction is, again, the space of continuous functions on a carefully knitted Alexandroff-Urysohn compactum.

2.1 Classical examples

Almost all constructions which are usually tagged as "folklore" can be obtained by considering a continuous embedding or surjection between compacta and then applying the functor $C(\cdot)\colon CHaus \rightsquigarrow Ban_1$.

If $\iota\colon K \longrightarrow L$ is a continuous embedding, then the restriction ι° is an isometric quotient operator, and Tiezte's extension theorem assures that the following sequence is exact:

$$0 \longrightarrow \cdot \longrightarrow C(L) \xrightarrow{\iota^\circ} C(K) \longrightarrow 0$$

The subspace $\ker \iota^\circ$ is always a C-space. Precisely, if we write $L/\iota(K)$ for the quotient space obtained by identifying every point in $\iota(K)$, we clearly can assume that its underlying set is $\{\iota(K)\} \cup \{t : t \in L \setminus \iota(K)\}$. Now $\ker \iota^\circ$ can be identified with the subspace $\{f \in C(L/\iota(K)) : f(\iota(K)) = 0\}$, which is a hyperplane of a C-space, hence a C-space –see Proposition 1.1.1. In this context, it is customary to speak of (linear) *extension* operators rather than of bounded selections for ι°. The classical Borsuk-Dugundji theorem says that such extension operators always exist provided L is metrizable. A more general criterion for triviality is given by Pełczyński in [84, Prop. 4.2]:

Proposition 2.1.1. *Given a continuous embedding $\iota\colon K \longrightarrow L$ between compacta, $\iota^\circ[C(L)]$ is a λ-complemented subspace of $C(K)$ if and only if there is a continuous mapping $\varphi\colon L \longrightarrow (\lambda \cdot M_1(K), weak^*)$ such that $\varphi(\iota t) = \delta_t$ for every $t \in K$.*

Proof. It is a consequence of the fact that the functors $\bigcirc^*\colon Ban_1 \rightsquigarrow CHaus$ and $C(\cdot)\colon CHaus \rightsquigarrow Ban_1$ are adjoint –see Section 1.1. □

We are now concerned with the "dual" situation. If $\pi\colon K \to L$ is a continuous surjection between compacta, then π° is an isometric embedding, and so we obtain the exact sequence:

$$0 \longrightarrow C(L) \xrightarrow{\pi^\circ} C(K) \longrightarrow \cdot \longrightarrow 0$$

A projection for π° often receives the name of (linear) *averaging* operator, simply because $\pi^\circ[C(L)]$ is the subspace of all functions in $C(K)$ which are constant on the *fibers* of π; that is, the sets $\pi^{-1}(t)$ for $t \in K$. Little is known about the nature of the quotient space $C(K)/\pi^\circ[C(L)]$: it need not be a C-space, and we will display the proper counterexample in Section 4.3, but it is open to decide if it is a Lindenstrauss space.

Regarding triviality, there is a companion to Proposition 2.1.1, also due to Pełczyński [84, Prop. 4.1]:

Proposition 2.1.2. *Let $\pi\colon K \to L$ be a continuous surjection between compacta. Then $\pi^\circ[C(L)]$ is λ-complemented in $C(K)$ if and only if there is a continuous mapping $\varphi\colon L \to (\lambda \cdot M_1(K), weak^*)$ such that for every $t \in L$, $\operatorname{supp}\varphi(t) \subseteq \pi^{-1}(t)$ and $\varphi(t)[\pi^{-1}(t)] = 1$.*

2.1.1 Nakamura-Kakutani exact sequences

There are several constructions in topology which produce twisted sums of Banach spaces. For instance, if $\gamma\mathbb{N}$ is a compactification of $\mathbb{N}$, and we call $\gamma\mathbb{N}^* = \gamma\mathbb{N} \setminus \mathbb{N}$ its *remainder*, then the natural inclusion $\iota\colon \gamma\mathbb{N}^* \to \gamma\mathbb{N}$ induces a twisted sum

$$0 \longrightarrow c_0 \longrightarrow C(\gamma\mathbb{N}) \xrightarrow{\iota^\circ} C(\gamma\mathbb{N}^*) \longrightarrow 0$$

which is trivial when, by definition, $\gamma\mathbb{N}$ is a *tame* compactification. For example, by virtue of Sobczyk's theorem, every metrizable compactification is tame. On the opposite side of the spectrum, the maximal compactification $\beta\mathbb{N}$ is not tame, and this fact produces the very famous non-trivial exact sequence

$$0 \longrightarrow c_0 \longrightarrow \ell_\infty \longrightarrow \ell_\infty/c_0 \longrightarrow 0$$

With more generality, given a compactum K we can produce the exact sequence

$$0 \longrightarrow C_0(K \setminus K') \longrightarrow C(K) \xrightarrow{\iota^\circ} C(K') \longrightarrow 0$$

where $K \setminus K'$ is the set of isolated points in K. A particularly fruitful case of these ideas is the following:

Definition. Let us fix an infinite cardinal number κ and a compactum K. A *κ-discrete extension* of K is a compactum L which contains a subspace homeomorphic to K such that $L \setminus K$ is a discrete set of size κ.

We will denote $K \cup \kappa$ to refer to a particular discrete extension of K, hence identifying $L \setminus K$ with κ. In the particular case that κ is countable, we will write $K \cup \omega$. Clearly every κ-discrete extension $K \cup \kappa$ of K produces an exact sequence

$$0 \longrightarrow c_0(\kappa) \longrightarrow C(K \cup \kappa) \xrightarrow{\iota^\circ} C(K) \longrightarrow 0$$

where $\iota\colon K \to K \cup \kappa$ is the canonical inclusion. However, it is not always the case that such exact sequences are non-trivial. For instance, every countable discrete extension of $[0,1]$ is metrizable, and therefore every exact sequence of the form

$$0 \longrightarrow c_0 \longrightarrow C\big([0,1] \cup \omega\big) \xrightarrow{\iota^\circ} C[0,1] \longrightarrow 0$$

splits. Of course, we already knew this, since $\operatorname{Ext}(C[0,1], c_0)$ by Sobczyk's theorem.

The paramount example of twisted sums of C-spaces obtained using discrete extensions can be traced back to Nakamura and Kakutani [82]. Let us say that a family of infinite subsets of $\mathbb{N}$ is *almost disjoint* if the intersection of every two of its members is finite. There are almost disjoint families in $\mathscr{P}(\mathbb{N})$ of size $\mathfrak{c}$, and this can be seen by considering, for each irrational number, a Cauchy sequence of rational numbers converging to it. Of course, since every subfamily of an almost disjoint family is also almost disjoint, there are almost disjoint families of every size below $\mathfrak{c}$.

Our interest in almost disjoint families is that they generate the so-called *Alexandroff-Urysohn compacta*. Given an almost disjoint family $\mathscr{A}$, let us write $K_{\mathscr{A}}$ for the Stone space of the Boolean algebra generated by $\mathscr{A}$ and all finite subsets of $\mathbb{N}$. $K_{\mathscr{A}}$ has three types of points:

- The *principal* ultrafilters $\mathfrak{p}_n$, where $\mathfrak{p}_n$ consists of all the sets containing the natural number n.
- Given $A \in \mathscr{A}$, there is only one ultrafilter $\mathfrak{p}_A$ containing A and no finite set.
- Finally, there is only one ultrafilter not containing any $A \in \mathscr{A}$, which it is usually denoted as ∞.

In view of this, it is reasonable to take as the underlying set of $K_{\mathscr{A}}$ the set $\mathbb{N} \cup \{p_A : A \in \mathscr{A}\} \cup \{\infty\}$. With this identification, let us specify the (Stone) topology in $K_{\mathscr{A}}$: points in $\mathbb{N}$ are isolated, a basic neighbourhood of any p_A is of the form $\{p_A\} \cup A \setminus F$, where F is a finite subset of $\mathbb{N}$, and $K_{\mathscr{A}}$ is the one-point compactification of $\mathbb{N} \cup \{p_A : A \in \mathscr{A}\}$. We will call $K_{\mathscr{A}}$ the Alexandroff-Urysohn space associated to $\mathscr{A}$. It is clear that $K_{\mathscr{A}}$ is a countable discrete extension of $\alpha\mathscr{A}$ in which $\mathbb{N}$ is dense, and it is a separable scattered compactum of height 3. Also note that $K_{\mathscr{A}}$ is metrizable precisely when $\mathscr{A}$ is countable.

We now examine the space $C(K_{\mathscr{A}})$. For a start, it is clear that $C(K_{\mathscr{A}})$ contains complemented copies of c_0; for instance, the closed span of the set $\{1_n : n \in A\}$ for any fixed $A \in \mathscr{A}$. Hence, by Proposition 1.1.1, $C(K_{\mathscr{A}})$ is isomorphic to its hyperplanes and in particular, it is isomorphic to the hyperplane $C_0(K_{\mathscr{A}})$ of functions vanishing at the point at infinity.

Proposition 2.1.3. *If $\mathscr{A}$ is an uncountable almost disjoint family of subsets of $\mathbb{N}$, the* Nakamura-Kakutani *exact sequence*

$$0 \longrightarrow c_0 \longrightarrow C_0(K_{\mathscr{A}}) \longrightarrow c_0(\mathscr{A}) \longrightarrow 0 \tag{2.a}$$

is not trivial.

Proof. Since $\mathbb{N}$ is a dense countable set in $K_{\mathscr{A}}$, the assignment $f \mapsto (f(n))_{n\in\mathbb{N}}$ places $C(K_{\mathscr{A}})$ as a subspace of ℓ_∞. A closer look may reveal that its range is actually the closed span of the set $\{1_n : n \in \mathbb{N}\} \cup \{1_A : A \in \mathscr{A}\} \cup \{1\}$ inside ℓ_∞. In any event, $C(K_{\mathscr{A}})$ cannot be even isomorphic to $c_0 \oplus c_0(\mathscr{A})$, since the latter is not a subspace of ℓ_∞. □

In principle, one should be able to extend the previous construction to higher cardinalities. Let us pick an infinite set I such that $|I| = \kappa$ and say, following [38], that a family $\mathscr{A}$ of countably infinite subsets of I is *almost disjoint* if, again, the intersection of every two members of $\mathscr{A}$ is finite. The ultrafilter space of the Boolean algebra generated by $\mathscr{A}$ and all finite subsets of I will be denoted $K_{\mathscr{A}}$, as before, and it produces a twisted sum

$$0 \longrightarrow c_0(\kappa) \longrightarrow C_0(K_{\mathscr{A}}) \longrightarrow c_0(\mathscr{A}) \longrightarrow 0 \tag{2.b}$$

which is non-trivial under the assumption that $|\mathscr{A}| > \kappa$. The argument is identical to that of Proposition 2.1.3: $C(K_{\mathscr{A}})$ is a subspace of $\ell_\infty(\kappa)$, hence it cannot be isomorphic to $c_0(\kappa) \oplus c_0(\mathscr{A})$, which is not. However, this construction only works in its full generality for $\kappa < \mathfrak{c}$. This is because, when $\kappa < \mathfrak{c}$, the size of $\mathscr{A}$ is bounded by $\mathfrak{c}$, and therefore we have access to almost disjoint families having any size between κ and $\mathfrak{c}$. On the other hand, if $\kappa \geq \mathfrak{c}$, then $\mathfrak{c} \leq |\mathscr{A}| \leq \kappa$ and we cannot decide whether sequences (2.b) split or not.

Banach spaces of the form $C(K_{\mathscr{A}})$ play a major part in a number of constructions related to twisted sums of C-spaces. Two clear instances of this fact can be found in

Section 2.3, where twisted sums of c_0 and $c_0(\mathfrak{c})$ which are not C-spaces are constructed using Alexandroff-Urysohn compacta, and in Chapter 4, in which a very special almost disjoint family $\mathscr{A}$ is built in such a way that $C(K_{\mathscr{A}})$ contains a complemented subspace which is not a C-space. $C(K_{\mathscr{A}})$-spaces also play a major role in the CCKY problem, which asks whether $\operatorname{Ext}(C(K), c_0) \neq 0$ when K is non-metrizable. In fact, by combining [7, Theorem 6.2] with [78, Corollary 5.3] we obtain:

Theorem 2.1.4. *Let $\mathscr{A}$ be an uncountable almost disjoint family. Then:*

i) $\operatorname{Ext}(C(K_{\mathscr{A}}), c_0) \neq 0$ *whenever* $|\mathscr{A}| = \mathfrak{c}$.

ii) [MA] $\operatorname{Ext}(C(K_{\mathscr{A}}), c_0) = 0$ *whenever* $|\mathscr{A}| < \mathfrak{c}$.

Finally, we cannot resist to reproduce here how Nakamura-Kakutani sequences can be used to produce a non-trivial twisted sum of c_0 and ℓ_∞. This construction is due to Cabello and Castillo [15, §2.1] –cf. also [6, §2.2.5].

Theorem 2.1.5. $\operatorname{Ext}(\ell_\infty, c_0) \neq 0$.

Proof. Let us look at the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & c_0 \arrow[r] & C(K_{\mathscr{A}}) \arrow[r] & c_0(\mathfrak{c}) \arrow[r] & 0 \\
0 \arrow[r] & c_0 \arrow[r] \arrow[u, equal] & PB \arrow[r] \arrow[u, hook, "j_1"] & \ell_2(\mathfrak{c}) \arrow[r] \arrow[u, hook, "j"] & 0 \\
0 \arrow[r] & c_0 \arrow[r] \arrow[u, equal] & CC \arrow[r] \arrow[u, "Q_1"] & \ell_\infty \arrow[r] \arrow[u, "Q"] & 0
\end{tikzcd}$$

The middle row is obtained by making pullback with the canonical inclusion $j\colon \ell_2(\mathfrak{c}) \to c_0(\mathfrak{c})$, which has dense range. This forces j_1 to also have dense range, and therefore PB cannot be isomorphic to $c_0 \oplus \ell_2(\mathfrak{c})$ because in such a case $C(K_{\mathscr{A}})$ would be WCG and every copy of c_0 in a WCG space is complemented. The space PB is known as the *Johnson-Lindenstrauss space*. It appeared first in [58], and [23] shows how it can be used as a counterexample to decide if certain properties are 3-space properties.

Now, to obtain the lower row, we make pullback with a quotient operator $Q\colon \ell_\infty \to \ell_2(\mathfrak{c})$. Such an operator can be obtained as follows: pick a quotient operator $q\colon \ell_1(\mathfrak{c}) \to \ell_2(\mathfrak{c})$ and apply [35, Th. 4.15] to ensure that, since $\ell_1(\mathfrak{c})$ is a subspace of ℓ_∞ and q is 2-summing, it can be extended to a quotient operator $Q\colon \ell_\infty \to \ell_2(\mathfrak{c})$. It remains to show that the lower row is non trivial. Note that, if $CC \simeq c_0 \oplus \ell_\infty$, the operator $Q_1|_{\ell_\infty}$

is either weakly compact or it is an isomorphism in some copy of ℓ_∞ [2, Thm. 5.5.5]. It is clear that the second possibility cannot happen: PB does not contain ℓ_∞ because "not containing ℓ_∞" is a 3-space property [23, 3.2.f]. Hence we assume that $Q_1|_{\ell_\infty}$ is weakly compact. Since $C(K_{\mathscr{A}})$ is a subspace of ℓ_∞, and weakly compact subsets of ℓ_∞ are norm-separable, $j_1 Q_1|_{\ell_\infty}$ has separable range, and so the same is true for $Q_1|_{\ell_\infty}$. But $Q_1|_{c_0}$ clearly has separable range, hence the range of Q_1 must be separable, which is not possible. □

2.1.2 Variations on the double arrow space

Let us now present what can be considered the most classical example of a twisted sum of C-spaces induced by a continuous surjection between compacta. We introduce the following versions of the double arrow space appearing in [77]. Given $A \subset (0, 1)$, consider

$$\mathbb{S}_A = \big([0,1] \times \{0\}\big) \cup \big(A \times \{1\}\big)$$

with the topology induced by the lexicographic order; that is, $(x, i) \prec (j, i)$ if either $x < y$ or $x = y$ and $i < j$. This can be seen a version of the double arrow space where only the points in A are "split". In particular, $\mathbb{S}_{(0,1)}$ is homeomorphic to $\mathbb{S}$. The space $C(\mathbb{S}_A)$ can be easily recognised:

Proposition 2.1.6. *$C(\mathbb{S}_A)$ is isometrically isomorphic to the closed subspace of $\ell_\infty[0, 1]$ of functions which are continuous except at points of A, where they are left-continuous and have right-sided limits.*

Let us mention a classical example of a collection of twisted sums closely related to these double arrow spaces. For a fixed subset $A \subset (0, 1)$, the natural surjection $\pi\colon \mathbb{S}_A \to [0, 1]$ defined as $\pi(t, i) = t$ for every $(t, i) \in \mathbb{S}_A$, produces the twisted sum

$$0 \longrightarrow C[0,1] \xrightarrow{\pi^\circ} C(\mathbb{S}_A) \longrightarrow \cdot \longrightarrow 0$$

The quotient norm is easily computed as

$$\|\overline{f}\| = \inf_{g \in C[0,1]} \|f - \pi^\circ(g)\| = \tfrac{1}{2} \max_{a \in A} |f(a,1) - f(a,0)|$$

Therefore, we can identify the quotient $C(\mathbb{S}_A)/\pi^\circ(C[0, 1])$ with the range of the *jump* operator

$$J\colon C(\mathbb{S}_A) \to \ell_\infty(A) \quad , \quad Jf(a) = \tfrac{1}{2}\big(f(a,1) - f(a,0)\big)$$

Now, let us observe that for any $f \in C(\mathbb{S}_A)$ and $\varepsilon > 0$, the fact that the bounded set $\{a \in A : |f(a,0) - f(a,1)| > \varepsilon\}$ has cluster points violates the continuity of f. Hence $J[C(\mathbb{S}_A)]$ is isomorphic to $c_0(A)$, and we have an exact sequence

$$0 \longrightarrow C[0,1] \xrightarrow{\pi^\circ} C(\mathbb{S}_A) \xrightarrow{J} c_0(A) \longrightarrow 0 \tag{2.c}$$

whose splitting depends on the size of A and its position inside $[0,1]$.

Proposition 2.1.7. *Let A be any subset of* $[0,1]$. *In any of the following cases, the exact sequence (2.c) is not trivial:*

i) A is countable and dense.

ii) A is uncountable.

Proof. Concerning (i), we will actually show that if $J(f_n) = e_n$, where $(e_n)_{n=1}^\infty$ denotes the canonical basis of c_0, then $(f_n)_{n=1}^\infty$ cannot be weakly Cauchy. The argument is essentially due to Aharoni and Lindenstrauss [1, Remark (ii)]. Let us write $A = \{a_n : n \in \mathbb{N}\}$ and assume towards a contradiction that $(f_n)_{n=1}^\infty$ is weakly Cauchy. The following observation is the key: for any open interval $I \subset (0,1)$, any $\alpha \in \mathbb{R}$ and any $\nu \in \mathbb{N}$, there are $n > \nu$, an open interval I_1 with $\overline{I_1} \subseteq I$ and $\beta \in \mathbb{R}$ such that $|\beta - \alpha| > 1$ and $|f_n(t) - \beta| < \frac{1}{4}$ for all $t \in I_1$. Indeed, let $n > \nu$ such that $a_n \in I$ and recall that $J(f_n) = e_n$ means that $f_n(a_n,1) - f_n(a_n,0) = 2$. Therefore we choose β either as $f_n(a_n,1)$ or $f_n(a_n,0)$, and then I_1 using the left-side or right-side limit condition accordingly.

Now we use the above observation inductively to produce a subsequence $(f_{n_k})_{k=1}^\infty$, a sequence of real numbers $(\alpha_k)_{k=1}^\infty$ such that $|\alpha_{k+1} - \alpha_k| > 1$ and a sequence of open intervals $(I_k)_{k=1}^\infty$ such that $\overline{I_{k+1}} \subseteq I_k$ and $|f_{n_k}(t) - \alpha_k| < \frac{1}{4}$ for $t \in I_k$. In particular, if $t \in \bigcap_{k=1}^\infty I_k$, then $|f_{n_{k+1}}(t) - f_{n_k}(t)| \geq \frac{1}{2}$ for every $k \in \mathbb{N}$, in contradiction with $(f_n)_{n=1}^\infty$ being weakly Cauchy.

The proof of (ii) is easy recalling that, since $\mathbb{S}_A$ is separable, $C(\mathbb{S}_A)$ is a subspace of ℓ_∞, but $C[0,1] \oplus c_0(A)$ is not whenever A is uncountable. □

Some remarks are in order. First, in the case when A is countable, the very existence of the exact sequence (2.c), or rather the fact that $\mathbb{S}_A$ is second countable whenever A is countable, implies that $\mathbb{S}_A$ is metrizable, and so $C(\mathbb{S}_A) \simeq C[0,1]$. Therefore, (2.c) can be disguised as an exact sequence

$$0 \longrightarrow C[0,1] \longrightarrow C[0,1] \longrightarrow c_0 \longrightarrow 0$$

witnessing the classical fact that $C[0,1]$ contains uncomplemented copies of itself [84, §9]. There is a similar construction working with the Cantor set, which was independently carried out by Foiaş and Singer in [43]. The existence of the non-trivial exact sequence

$$0 \longrightarrow C[0,1] \longrightarrow C(\mathbb{S}) \longrightarrow c_0(\mathfrak{c}) \longrightarrow 0$$

is also mentioned by Lindenstrauss in his survey paper [71], but it can be traced back to Corson [33, Example 2], who used such an example to illustrate that "to be weakly Lindelöf" is not a 3-space property. Finally, Marciszewski sheds some light in [77, §4] concerning the isomorphic classification of the spaces $C(\mathbb{S}_A)$ for $A \subseteq (0,1)$, but such a problem remains essentially open. In fact, it was only recently that Michalak [80] proved that $C(\mathbb{S})$ and $C(\mathbb{S}_A)$ for $A = \mathbb{I} \cap (0,1)$ are not isomorphic.

2.2 Counting twisted sums of c_0 and $C(K)$

We will now pursue a more detailed study of twisted sums of C-spaces, focusing in two particular types: twisted sums of c_0 and $c_0(\mathfrak{c})$, which can be considered as the simplest non-trivial case of twisted sums of C-spaces; and twisted sums of c_0 and $C(\mathbb{S})$, where the double arrow space $\mathbb{S}$ acts as the most representative member of the class of separable linearly ordered compacta. Note that in the latter case we have not provided any concrete example of a non-trivial twisted sum yet.

The first attempt on a detailed description of twisted sums of c_0 and $c_0(\mathfrak{c})$ was to observe that there are "too many" non-isomorphic of them. This fact is mentioned in [79, §7.4], and an explicit proof, which we will develop here, was provided in [18, §5]. It all boils down to the following result:

Theorem 2.2.1. *Fix an infinite cardinal number κ, and let $\mathscr{K}$ be a family of pairwise non-homeomorphic compacta such that*

i) For every $K \in \mathscr{K}$, $d(K) = \kappa$ and $|M(K)| \leq 2^\kappa$.

ii) $C(K) \simeq C(L)$ for every $K, L \in \mathscr{K}$.

Then $|\mathscr{K}| \leq 2^\kappa$.

Proof. Let $\pi_K : \beta\kappa \to K$ denote a continuous mapping onto K, which in turn gives an embedding $\pi_K^\circ : C(K) \to \ell_\infty(\kappa)$. Given any two compacta $K, L \in \mathscr{K}$, the following compatibility condition

$$\pi_K(x) = \pi_K(x') \iff \pi_L(x) = \pi_L(x')$$

cannot hold, or else the universal property of the quotient space would claim that K and L are homeomorphic. Therefore, we can assume that there are two different points $x, x' \in \beta\kappa$ such that $\pi_K(x) = \pi_K(x')$ but $\pi_L(x) \neq \pi_L(x')$. This implies that $\pi_K^\circ[C(K)] \neq \pi_L^\circ[C(L)]$ in $\ell_\infty(\kappa)$, since there is a function $g \in C(L)$ such that $\pi_L(g)$ separates $\pi_L(x)$ and $\pi_L(x')$, but g certainly cannot belong to $\pi_K^\circ[C(K)]$.

Now let us fix some $K_0 \in \mathscr{K}$ and consider an isomorphism $T_K \colon C(K_0) \to C(K)$. By our previous reasoning, the operators $S_K \colon C(K_0) \to \ell_\infty(\kappa)$ given by $S_K = \pi_K^\circ \circ T_K$, are all different, so the assignment $K \to S_K$ defines an injective map from $\mathscr{K}$ to $\mathscr{L}(C(K_0), \ell_\infty(\kappa))$. Finally,

$$|\mathscr{L}(C(K_0), \ell_\infty(\kappa))| \leq |M(K_0)^\kappa| = 2^\kappa$$

so $|\mathscr{K}| \leq 2^\kappa$, as we wanted. □

Corollary 2.2.2. *There are $2^\mathfrak{c}$ non-isomorphic twisted sums of c_0 and $c_0(\mathfrak{c})$.*

Proof. It is enough to show, thanks to Proposition 2.1.3, that there are $2^\mathfrak{c}$ non isomorphic spaces of the form $C(K_\mathscr{A})$ where $\mathscr{A}$ is an almost disjoint family of subsets of $\mathbb{N}$ having cardinality $\mathfrak{c}$. For every such $\mathscr{A}$, the space $K_\mathscr{A}$ is separable and scattered, hence $|M(K_\mathscr{A})| = |K_\mathscr{A}| = \mathfrak{c}$ (see Theorem 1.4.1). Moreover, every homeomorphism $K_\mathscr{A} \to K_\mathscr{B}$ is determined by a permutation of $\mathbb{N}$, and there are $\mathfrak{c}$ many of these. Since there are $2^\mathfrak{c}$ different spaces $K_\mathscr{A}$, we can select a subfamily $\{A_\eta : \eta < 2^\mathfrak{c}\}$ such that the corresponding compacta $K_{\mathscr{A}_\eta}$ are pairwise non-homeomorphic. Finally, a direct application of Theorem 2.2.1 informs us there is a subset $E \subseteq 2^\mathfrak{c}$ of cardinality $2^\mathfrak{c}$ such that $C(K_{\mathscr{A}_\eta}) \not\simeq C(K_{\mathscr{A}'_\eta})$ whenever $\eta, \eta' \in E$ and $\eta \neq \eta'$. □

The previous corollary may very well be extended to other cardinalities, except for the fact that one must assume that $2^{\kappa^{\aleph_0}} > 2^\kappa$. For example, if $\kappa = \mathfrak{c}$, then the previous inequality is clearly not true. But, under Martin's axiom, we have that for every $\kappa < \mathfrak{c}$, $2^\kappa = \mathfrak{c}$ while $2^{\kappa^{\aleph_0}} = 2^\mathfrak{c}$ [45, p. 21C]. Therefore, we can state:

Corollary 2.2.3. [MA(κ)] *There are $2^\mathfrak{c}$ non-isomorphic twisted sums of $c_0(\kappa)$ and $c_0(\mathfrak{c})$.*

In order to produce non-trivial twisted sums of c_0 and $C(\mathbb{S})$ by means of Theorem 2.2.1, we construct a large family of compactifications of $\mathbb{N}$ with remainder homeomorphic to $\mathbb{S}$. First, let us see $\mathbb{S}$ as a Stone space of a certain algebra of subsets of an infinite countable set. Let us write $Q = \mathbb{Q} \cap (0, 1)$ and for each $x \in (0, 1)$, we call $P_x = \{q \in Q : q \leq x\}$

and consider $\mathfrak{A}$ the algebra generated by $\{P_x : x \in (0,1)\}$. Now we look at $\mathrm{ult}(\mathfrak{A})$: since $P_x \subseteq P_y$ whenever $x \leq y$, every ultrafilter $\mathfrak{p}$ on $\mathfrak{A}$ is determined by the set

$$A(\mathfrak{p}) = \{x \in (0,1) : P_x \in \mathfrak{p}\}$$

which, in addition, is a subinterval of $(0,1)$ of the form $[y,1]$ or $(y,1]$ for some $y \in [0,1]$. Hence, we define $\mathfrak{p}_y^+$ and $\mathfrak{p}_y^-$ via the equalities $A(\mathfrak{p}_y^+) = [y,1)$ and $A(\mathfrak{p}_y^-) = (y,1]$, respectively. The mapping

$$h\colon \mathrm{ult}(\mathfrak{A}) \longrightarrow \mathbb{S} \quad , \quad \begin{cases} h(\mathfrak{p}_y^+) = (y,0) \\ h(\mathfrak{p}_y^-) = (y,1) \end{cases}$$

is readily checked to be an isomorphism.

Now, for each $x \in (0,1)$ choose $(q_n^x)_{n\in\mathbb{N}}$ a strictly increasing sequence of points in Q converging to x, and call $S_x = \{q_n^x : n \in \mathbb{N}\}$. Select any function $\theta\colon (0,1) \longrightarrow 2$ and define

$$R_x^\theta = \begin{cases} P_x & \text{if } x \in \theta^{-1}(0) \\ P_x \setminus S_x & \text{if } x \in \theta^{-1}(1) \end{cases} \tag{2.d}$$

Note that $R_x^\theta \subseteq^* R_y^\theta$ whenever $x \leq y$, because in such a case $P_x \cap S_y$ is finite. Finally, let us consider the subalgebra $\mathfrak{B}^\theta$ of $\mathscr{P}(Q)$ generated by all the sets $\{R_x^\theta : x \in [0,1]\}$, together with all finite subsets of Q. It is clear that there are two types of ultrafilters in $\mathrm{ult}(\mathfrak{B}^\theta)$:

- the principal ultrafilters, which clearly form a countable dense set.
- the ultrafilters not containing finite sets; every such ultrafilter $\mathfrak{p}$ is completely determined by the set $\{x \in (0,1) : R_x^\theta \in \mathfrak{p}\}$.

This is enough to prove that $\mathrm{ult}(\mathfrak{B}^\theta)$ is a compactification of $\mathbb{N}$ whose remainder is homeomorphic to $\mathbb{S}$. To simplify the notation, we will write L^θ for the space $\mathrm{ult}(\mathfrak{B}^\theta)$.

Corollary 2.2.4. *There are $2^\mathfrak{c}$ non-isomorphic twisted sums of c_0 and $C(\mathbb{S})$.*

Proof. Let us fix $\theta \in 2^{(0,1)}$. Then L^θ is clearly separable, and to show that $|M(L^\theta)| = \mathfrak{c}$ it is enough to prove that $|M(\mathbb{S})| = \mathfrak{c}$, since L^θ is a countable discrete extension of $\mathbb{S}$. The size of $M(\mathbb{S})$ can be computed from the fact that $C(\mathbb{S})$ is a twisted sum of $C[0,1]$ and $c_0(\mathfrak{c})$ –see (2.c)–, which implies $M(\mathbb{S}) \simeq M[0,1] \oplus \ell_1(\mathfrak{c})$. Now, every homeomorphism $L^\theta \longrightarrow L^{\theta'}$ extends a permutation of the countable set Q, hence we can single out $2^\mathfrak{c}$ many non-homeomorphic spaces L^θ. An application of Theorem 2.2.1 finishes the proof. □

2.2.1 The hall of descriptive set-theoretic twisted sums

Although we showed in Corollary 2.2.2 that twisted sums of c_0 with either $c_0(\mathfrak{c})$ or $C(\mathbb{S})$ are abundant, the techniques we employed do not provide explicit examples of non-isomorphic spaces. We now indicate, by means of some descriptive set-theoretic techniques, a more effective way of constructing the desired twisted sums.

First, we focus on twisted sums of c_0 and $c_0(\mathfrak{c})$. Let us denote $2^{<\omega} = \bigcup_{n=0}^{\infty} 2^n$ the space of finite sequences taking values in $\{0, 1\}$, which is the so-called full dyadic tree. Every element in the Cantor space 2^ω will be identified with a "branch" of $2^{<\omega}$ as we now indicate. For any $x \in 2^\omega$, we say $s \in 2^n$ is an *initial segment* of x if $s(k) = x(k)$ for all $1 \le k \le n$, and write $B(x)$ for the subset of $2^{<\omega}$ of initial segments of x. Under the standard identification $\mathscr{P}(2^{<\omega}) = 2^{2^{<\omega}}$, the mapping $B\colon 2^\omega \to 2^{2^{<\omega}}$ defined as $x \mapsto B(x)$ is a continuous embedding.

Our compacta arise from almost disjoint families of sets of $2^{<\omega}$ in the following way. Fix Z a dense Borel subset of 2^ω of class $\alpha \ge 2$ and call

$$\mathscr{A}_Z = B(Z) = \{B(z) : z \in Z\}$$

which is clearly an almost disjoint family. In the sequel, let us simply write K_Z for the Alexandroff-Urysohn compactum produced by $\mathscr{A}_Z$. The next results show that K_Z is a Rosenthal compactum and its Rosenthal index is closely related to the Borel class of Z.

Proposition 2.2.5. *If Z is Borel, then K_Z is a Rosenthal compactum.*

Proof. Since Z is a Borel set of the Polish space 2^ω, it is analytic, and therefore it suffices to find a realization of K_Z inside $B_1(Z)$. Given $z \in Z$, let us write 1_z for the characteristic function of the singleton $\{z\}$, and for every $s \in 2^{<\omega}$, consider the function

$$1_s : Z \to 2 \quad , \quad 1_s(z) = \begin{cases} 1 & \text{if } s \in B(z) \\ 0 & \text{otherwise} \end{cases}$$

It is now straightforward to check that the subspace $\{1_s : s \in 2^{<\omega}\} \cup \{1_z : z \in Z\} \cup \{0\}$ provides the desired realization. $\square$

Theorem 2.2.6. [75, Th. 4.2] *If Z is a dense set of 2^ω of Borel class $\alpha \ge 2$, then K_Z satisfies*

$$\alpha \le \mathrm{ri}(K_Z) \le 1 + \alpha + 1$$

Proof. We will show two assertions; namely:

(i) for any countable dense set $D \subseteq K_Z$, the Borel class of $C_D(K_Z)$ in $\mathbb{R}^D$ is at least α.

(ii) for $S = 2^{<\omega}$, $C_S(K_Z)$ is of Borel class $1 + \alpha + 1$.

Let us concern ourselves with (i). Pick a countable dense set $D \subseteq K_Z$ and note that necessarily $S \subseteq D$. The set $\widetilde{Z} = Z \setminus \{B(z) : z \in D\}$ is again of Borel class α, because D is countable. We now consider $\widetilde{Z}$ as a subset of $C_D(K_Z)$ via the embedding

$$h\colon \widetilde{Z} \to C_D(K_Z) \quad , \quad h(z) = 1_{B(z)}$$

which has the additional property that $h(\widetilde{Z})$ is a closed subset of $C_D(K_Z)$. This implies the existence of a closed subset $C \subseteq \mathbb{R}^D$such that $h(\widetilde{Z}) = C_D(K_Z) \cap C$, hence $C_D(K_Z)$ has Borel class at least α.

The proof for (ii) is much longer. First, let us observe that if C is a clopen subset of K_Z, then $C \cap S$ is either of the form $\left(\bigcup_{i=1}^n B(z_i)\right) \triangle F$ or $\left(\bigcap_{i=1}^n B(z_i)^c\right) \triangle F$, where $\triangle$ denotes symmetric difference. We will rewrite this fact in an appropriate "descriptive" form: given any finite set $F \subset S$ and $n \in \mathbb{N}$, we define the continuous mapping

$$\varphi(F,n)\colon (2^\omega)^n \to 2^S \quad , \quad \varphi(F,n)(z_1,\ldots,z_n) = \left(\bigcup_{i=1}^n B(z_i)\right) \triangle F$$

and write $H(F,n) = \varphi(F,n)(Z^n) = \{C \subset S : C = \left(\bigcup_{i=1}^n B(z_i)\right) \triangle F\}$. Then clearly $\varphi(F,n)^{-1}[H(F,n)] = Z^n$, hence we infer from [65, 24.20] that $H(F,n)$ is a Borel subset of class α, and so is

$$H = \bigcup\{H(F,n) : F \subset S \text{ finite}, n \in \mathbb{N}\}$$

On the other hand, note that $H' = \{S \setminus C : C \in H\}$ is homeomorphic to H and therefore

$$H \cup H' = \{C \cap S : C \text{ is a clopen in } K_Z\}$$

is a Borel subset of 2^S of Borel class α.

Now the crux of the argument is that such a description of the "traces" in S of clopen subsets of K_Z allows to show that for a function $f\colon S \to \mathbb{R}$, the following facts are equivalent:

(*) $f \in C_S(K_Z)$.

(**) f is bounded and for every $p, q \in \mathbb{Q}$ with $p < q$ there exists $r \in \mathbb{Q}$ with $p < r < q$ such that $f^{-1}(r, +\infty) \in H \cup H'$.

In other words, if for a fixed $r \in \mathbb{Q}$ we define $\psi_r : \mathbb{R}^S \to 2^S$ as $\psi_r(f) = f^{-1}(r, +\infty)$, the previous equivalence can be rewritten as the following equality:

$$C_S(K_Z) = \left(\bigcup_{m=1}^{\infty}[-m,m]^S\right) \cap \left(\bigcap_{\substack{p,q\in\mathbb{Q}\\ p<q}} \bigcup_{\substack{r\in\mathbb{Q}\\ p<r<q}} \psi_r^{-1}(H \cup H')\right) \tag{2.e}$$

This is useful because ψ_r is easily seen to be a Baire-1 function, and so (2.e) actually says that $C_S(K_Z)$ is a Borel subset of $\mathbb{R}^S$ of class $1 + \alpha + 1$. Therefore, the proof of (ii) is finished by showing the aforementioned equivalence between (*) and (**).

So we now assume that $f \in C_S(K_Z)$. Since K_Z is a scattered compactum, so is $f(K_Z)$, and in particular it is a totally disconnected subset of $\mathbb{R}$. Hence given two rational numbers $p < q$, there is another rational number r with $p < r < q$ such that

$$f(K_Z) = \big(f(K_Z) \cap (r, +\infty)\big) \cup \big(f(K_Z) \cap (-\infty, r)\big)$$

that is, $f^{-1}(r, +\infty)$ is a clopen set in K_Z. This shows (*) implies (**). For the converse, we recall that K_Z is Rosenthal, hence a Fréchet-Urysohn space –see Proposition 1.4.8. Hence a bounded function $f : S \to \mathbb{R}$ can be extended to a continuous function on K_Z if and only if for every sequence $(s_n)_{n=1}^{\infty}$ of points in S converging in K_Z, the sequence $(f(s_n))_{n=1}^{\infty}$ is convergent. This implies that if a bounded function $f : S \to \mathbb{R}$ does not belong to $C_S(K_Z)$, there exist two sequences $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ of points in S converging to the same point $z \in K_Z$ and verifying that $f(s_n) < p < q < f(t_n)$ for suitable $p, q \in \mathbb{Q}$. Then for every rational number r with $p < q < r$, the set $f^{-1}(r, +\infty)$ cannot belong to $H \cup H'$, because for every clopen set $C \subset K_Z$, either $z \in C$, and so $C \cap S$ contains all but finitely many points of $(s_n)_{n=1}^{\infty}$, or $z \notin C$, which means $C \cap S$ contains only finite many points of the sequence $(t_n)_{n=1}^{\infty}$. □

The previous theorems indicate how to construct, for every ordinal $1 < \xi < \omega_1$, a Borel set $Z_\xi \subseteq 2^\omega$ such that the spaces $C(K_{Z_\xi})$ are pairwise non-isomorphic. This is enough to produce the desired twisted sums:

Corollary 2.2.7. *There is a family of separable Rosenthal compacta $\{K_\xi : 1 < \xi < \omega_1\}$ such that*

i) $C(K_\xi)$ is a non-trivial twisted sum of c_0 and $c_0(\mathfrak{c})$.

ii) For any $\xi \neq \xi'$, $C(K_\xi)$ and $C(K_{\xi'})$ are not isomorphic.

We do not know how to produce the corresponding version of Corollary 2.2.7 for twisted sums of c_0 and $C(\mathbb{S})$, but there are some partial results in [18, §7] which we now develop. The first and most alarming fact should be that no explicit example of a twisted sum of c_0 and $C(\mathbb{S})$ has been produced, even if we showed in Corollary 2.2.4 that there are $2^{\mathfrak{c}}$ many of them. In our way to construct such space, we will need the following result, which may be interesting in itself.

Proposition 2.2.8. *Let K and L be separable Rosenthal compacta. If the twisted sum*

$$0 \longrightarrow C(K) \longrightarrow C(M) \longrightarrow C(L) \longrightarrow 0$$

is trivial, then M is a Rosenthal compacta with $\mathrm{ri}(M) \leq 1 + \max\{\mathrm{ri}(K), \mathrm{ri}(L)\}$.

Proof. The triviality of the exact sequence yields $C(M) \simeq C(K) \times C(L) \simeq C(K \sqcup L)$, and since $K \sqcup L$ is Rosenthal, so is M, thanks to Theorem 1.4.14. To finish the proof, we only need to show that $\mathrm{ri}(K \sqcup L) \leq \max\{\mathrm{ri}(K), \mathrm{ri}(L)\}$ and then appeal to Theorem 1.4.11. But this is not a complicated matter, for if $D \subseteq K$ and $E \subseteq L$ are countable dense sets such that $C_D(K)$ is of Borel class α in $\mathbb{R}^D$ and $C_E(L)$ is of Borel class β in $\mathbb{R}^E$, then $C_{D\cup E}(K \sqcup L)$ is a Borel subset of $\mathbb{R}^{D\cup E} = \mathbb{R}^D \times \mathbb{R}^E$ of class no bigger than $\max\{\alpha, \beta\}$. Indeed, write $\pi_1 : \mathbb{R}^{D\cup E} \to \mathbb{R}^D$ and $\pi_2 : \mathbb{R}^{D\cup E} \to \mathbb{R}^E$ for the canonical projections and use the following string of equalities:

$$C_{D\cup E}(K \sqcup L) \simeq C_D(K) \times C_E(L) = \pi_1^{-1}[C_D(K)] \cap \pi_2^{-1}[C_E(L)] \qquad \square$$

The following lemma contains all the technicalities we may need to construct a non-trivial twisted sum of c_0 and $C(\mathbb{S})$. For this purpose, let us recover some notation: we previously constructed a family $\{L^\theta : \theta \in 2^{(0,1)}\}$ of compactifications of $\mathbb{N}$ with remainders homeomorphic to $\mathbb{S}$ –see just after equation (2.d). In particular, each L^θ is a certain space of ultrafilters over a certain algebra $\mathfrak{B}^\theta \subseteq \mathscr{P}(Q)$, where $Q = \mathbb{Q} \cap (0,1)$. It will be now convenient to represent θ as the characteristic function of a certain set $Z \subseteq (0,1)$. Hence we will simply write $L^{1_Z} = L(Z)$.

Lemma 2.2.9. *Given $Z \subseteq (0,1)$, the space $C_Q(L(Z))$ contains a G_δ-subset homeomorphic to* $[0,1] \setminus (Z \cup Q)$.

Proof. We will look for our G_δ-subset inside the closed subset $C_Q(L(Z), 2)$ consisting of the functions taking only the values 0 and 1. In other words, $C_Q(L(Z), 2) = C_Q(L(Z)) \cap 2^Q$. It is convenient to note that every function in $C_Q(L(Z), 2)$ defines a clopen of $L(Z)$; that is, an element of the algebra $\mathfrak{B}^{1_Z}$.

First, we identify $\mathscr{P}(Q)$ with 2^Q and consider the following subsets of 2^Q:

$$P_1 = \{(x,1) \cap Q : x \in [0,1] \setminus Q\}$$
$$P_2 = \{(q,1) \cap Q : q \in Q\}$$
$$P_3 = \{[q,1) \cap Q : q \in Q\}$$

We claim that the set P_1 is a G_δ-set in 2^Q. Indeed, $P_2 \cup P_3$ is a countable set, while $P = P_1 \cup P_2 \cup P_3$ is closed, since a function g belongs to $2^Q \setminus P$ whenever there are $q < q'$ in Q such that $g(q) = 1$ and $g(q') = 0$. Next, we observe that the mapping $x \mapsto (x,1) \cap Q \subseteq 2^Q$ establishes a homeomorphism between $[0,1] \setminus Q$ and P_1. Finally, the set $(x,1) \cap Q$ defines a continuous function on $C(L(Z))$ if and only if it belongs to $\mathfrak{B}^{1_Z}$, and by the very definition of such algebra this happens precisely when $x \in [0,1] \setminus (Z \cup Q)$. Putting all together, the result is that the set $X = P_1 \cap C_Q(L(Z))$ is a G_δ-set homeomorphic to $[0,1] \setminus (Z \cup Q)$. □

Theorem 2.2.10. *Let $Z \subseteq (0,1)$ be a Borel set of class 5 or greater. Then $C(L(Z))$ is a non-trivial twisted sum of c_0 and $C(\mathbb{S})$.*

Proof. Assume that $C(L(Z))$ is trivial as a twisted sum of c_0 and $C(\mathbb{S})$. Since $\mathrm{ri}(\alpha\mathbb{N}) = \mathrm{ri}(\mathbb{S}) = 2$, by Proposition 2.2.8 we conclude that $\mathrm{ri}(L(Z)) \leq 3$. Now, an application of Corollary 1.4.12 yields that $C_Q(L(Z))$ is a Borel set of $\mathbb{R}^Q$ of class no bigger than 4, and in particular, the Borel class of every G_δ-subset of $C_Q(L(Z))$ is also bounded by 4. Therefore, Lemma 2.2.9 assures that Z is of Borel class at most 4. □

We can further exploit Lemma 2.2.9 to show that "to be isomorphic to a $C(K)$-space for K a Rosenthal compactum" is not a 3-space property.

Theorem 2.2.11. *If $Z \subset (0,1)$ is not co-analytic, then $L(Z)$ is not a Rosenthal compactum. Therefore, there is a non-trivial twisted sum*

$$0 \longrightarrow c_0 \longrightarrow C(L(Z)) \longrightarrow C(\mathbb{S}) \longrightarrow 0$$

in which $C(L(Z))$ cannot be isomorphic to any $C(K)$-space for K a Rosenthal compactum.

Proof. Assume Z is not co-analytic. Then $[0,1] \setminus (Z \cup Q)$ is not analytic, and so Lemma 2.2.9 implies that $C_Q(L(Z))$ is not analytic either. Now we call on Theorem 1.4.10 to conclude that $L(Z)$ is not a Rosenthal compactum, from which it follows, thanks to Proposition 2.2.8, that $C(L(Z))$ cannot be trivial as a twisted sum of c_0 and $C(\mathbb{S})$. □

2.2.2 One space to rule them all

This section is devoted to analyze the situation concerning the number of non-isomorphic spaces $C(K_{\mathscr{A}})$ for $|\mathscr{A}| < \mathfrak{c}$ under Martin's axiom. Or, rather, why Corollary 2.2.2 cannot be transported to lower cardinalities.

First of all, let us observe that none of our previous arguments can be adapted for cardinalities $\kappa < \mathfrak{c}$. Indeed, the counting argument featuring Theorem 2.2.1 relies on the fact that every homeomorphism between $K_{\mathscr{A}}$ and $K_{\mathscr{B}}$ is defined by a bijection of $\mathbb{N}$, and this is why we need the inequality $2^\kappa > \mathfrak{c}$ to obtain a large class of non-homeomorphic compacta. However, such inequality is false for cardinalities $\kappa < \mathfrak{c}$ under Martin's axiom. On the other hand, the reason why the descriptive methods of the previous section cannot provide twisted sums of c_0 and $c_0(I)$ for $|I| < \mathfrak{c}$ is because every Borel set in a Polish space is either countable or has cardinality $\mathfrak{c}$ [65, Th. 13.6].

In any case, the isomorphic classification of $C(K_{\mathscr{A}})$-spaces when $\aleph_0 < |\mathscr{A}| < \mathfrak{c}$ under Martin's axiom is radically different:

Theorem 2.2.12. [$\mathsf{MA}(\kappa)$] *Let $\mathscr{A}$ and $\mathscr{B}$ be almost disjoint families such that $|\mathscr{A}| = |\mathscr{B}| = \kappa$. Then:*

i) $C(K_{\mathscr{A}})$ and $C(K_{\mathscr{B}})$ are isomorphic.

ii) $C(K_{\mathscr{A}})$ is isomorphic to its square.

Proof. The crux of the argument lies in Theorem 2.1.4, plus some homological magic. To show (i), let us consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & c_0 \arrow[r] & C(K_{\mathscr{A}}) \arrow[r, "q_a"] & c_0(\kappa) \arrow[r] \arrow[d, equal] & 0 & {[\mathsf{a}]} \\
0 \arrow[r] & c_0 \arrow[r] & C(K_{\mathscr{B}}) \arrow[r, "q_b"] & c_0(\kappa) \arrow[r] & 0 & {[\mathsf{b}]}
\end{tikzcd}$$

Since $\operatorname{Ext}(C(K_{\mathscr{B}}), c_0) = 0$ thanks to Theorem 2.1.4, the lower row of the pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & c_0 \arrow[r] & C(K_{\mathscr{A}}) \arrow[r, "q_a"] & c_0(\kappa) \arrow[r] & 0 & {[\mathsf{a}]} \\
0 \arrow[r] & c_0 \arrow[r] \arrow[u, equal] & PB \arrow[r] \arrow[u] & C(K_{\mathscr{B}}) \arrow[r] \arrow[u, "q_b"] & 0 & {[\mathsf{a}q_b]}
\end{tikzcd}$$

necessarily splits; that is, $[\mathsf{a}q_b] = 0$. Analogously, $[\mathsf{b}q_a] = 0$. Hence, an appeal to Proposition 1.2.4 yields that the exact sequences $[\mathsf{a}]$ and $[\mathsf{b}]$ are semi-equivalent. We now apply the Diagonal Principle 1.2.5 to obtain $c_0 \oplus C(K_{\mathscr{B}}) \simeq c_0 \oplus C(K_{\mathscr{A}})$, which in turn gives $C(K_{\mathscr{A}}) \simeq C(K_{\mathscr{B}})$ because every $C(K_{\mathscr{A}})$-space contains a complemented copy of c_0.

On the other hand, (ii) is a consequence of the well-known identity

$$\operatorname{Ext}(C(K_{\mathscr{A}}) \oplus C(K_{\mathscr{A}}), c_0) = \operatorname{Ext}(C(K), c_0) \times \operatorname{Ext}(C(K), c_0)$$

together with the fact that every twisted sum X of c_0 and $c_0(\kappa)$ such that $\operatorname{Ext}(X, c_0) = 0$ is isomorphic to $C(K_{\mathscr{A}})$, which we now prove. The same reasoning as before yields that the exact sequences

$$\begin{tikzcd} 0 \arrow[r] & c_0 \arrow[r] & C(K_{\mathscr{A}}) \arrow[r, "q_a"] & c_0(\kappa) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & c_0 \arrow[r] & X \arrow[r, "q"] & c_0(\kappa) \arrow[r] & 0 \end{tikzcd}$$

are semi-equivalent, hence $C(K_{\mathscr{A}}) \simeq c_0 \oplus C(K_{\mathscr{A}}) \simeq c_0 \oplus X$. In order to conclude we need to prove that X contains a complemented copy of c_0. By virtue of [28, §4, Proposition], X has Pełczyński's property (V), and since the quotient operator q cannot be weakly compact, there is a subspace X_0 of X isomorphic to c_0 such that $q|_{X_0}$ is an isomorphism. Since $q(X_0)$ is complemented in $c_0(\kappa)$ [50], X_0 must also be complemented in X. Hence $X \simeq c_0 \oplus X$, and the proof concludes. □

Note that if two Alexandroff-Urysohn compacta $K_{\mathscr{A}}$ and $K_{\mathscr{B}}$ are such that $C(K_{\mathscr{A}}) \simeq C(K_{\mathscr{B}})$, then the relations between cardinalities and weights for infinite scattered compacta (see Proposition 1.4.2) imply $|\mathscr{A}| = |\mathscr{B}|$. Hence, Theorem 2.2.12 states that, under Martin's axiom, the situation concerning the isomorphic classification of $C(K_{\mathscr{A}})$-spaces whenever $|\mathscr{A}| < \mathfrak{c}$ is as trivial as possible. This answers affirmatively two questions posed by Koszmider in [67]. On another matter, it is worth mentioning that even part (ii) of the previous corollary fails for almost disjoint families of size $\mathfrak{c}$. In [68], Koszmider and Laustsen construct such a family $\mathscr{A}$ such that the only decompositions of the space $C(K_{\mathscr{A}})$ are $C(K_{\mathscr{A}}) \simeq c_0 \oplus C(K_{\mathscr{A}})$. In particular, such $C(K_{\mathscr{A}})$ is not isomorphic to its square.

In light of the results of this section, it is clear that the general philosophy of Martin's axiom –that is, every cardinality between $\aleph_0$ and $\mathfrak{c}$ "behaves like" $\aleph_0$– is also witnessed by the spaces $C(K_{\mathscr{A}})$. Let us summarize the situation:

- If $|\mathscr{A}| = \aleph_0$, then $K_{\mathscr{A}}$ is clearly homeomorphic to ω^2, so $C(K_{\mathscr{A}}) \simeq c_0$ and there is only one space $C(K_{\mathscr{A}})$.

- However, if $|\mathscr{A}| = \mathfrak{c}$, then Theorem 2.2.2 says that there are $2^{\mathfrak{c}}$ non-isomorphic spaces $C(K_{\mathscr{A}})$.
- Finally, for a family $\mathscr{A}$ with cardinality $\aleph_0 < \kappa < \mathfrak{c}$, Theorem 2.2.12 claims that, under Martin's axiom, there is only one space $C(K_{\mathscr{A}})$ up to isomorphism.

The powerful result that $\operatorname{Ext}(C(K_{\mathscr{A}}), c_0) = 0$ when $|\mathscr{A}| < \mathfrak{c}$ under Martin's axiom also has consequences in the structure of twisted sums of c_0 and $c_0(\kappa)$:

Proposition 2.2.13. [$\mathsf{MA}(\kappa)$] *Every twisted sum of c_0 and $c_0(\kappa)$ is a quotient of $C(K_{\mathscr{A}})$, where $\mathscr{A}$ is an almost disjoint family of cardinality κ.*

Proof. Assume that we have an exact sequence

$$0 \longrightarrow c_0 \longrightarrow X \longrightarrow c_0(\kappa) \longrightarrow 0 \qquad [\mathsf{z}]$$

Since $\operatorname{Ext}(C(K_{\mathscr{A}}), c_0) = 0$, Proposition 1.2.4 informs us that we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & c_0 \arrow[r] \arrow[d] & C(K_{\mathscr{A}}) \arrow[r] \arrow[d] & c_0(\kappa) \arrow[r] \arrow[d, equal] & 0 & {[\mathsf{a}]} \\
0 \arrow[r] & c_0 \arrow[r] & X \arrow[r] & c_0(\kappa) \arrow[r] & 0 & {[\mathsf{z}]}
\end{tikzcd}$$

that is, $[\mathsf{z}]$ is a pushout of $[\mathsf{a}]$. Now, the diagonal pushout sequence –see equation (1.g)–

$$0 \longrightarrow c_0 \longrightarrow c_0 \oplus C(K_{\mathscr{A}}) \longrightarrow X \longrightarrow 0$$

witnesses X as a quotient of $c_0 \oplus C(K_{\mathscr{A}}) \simeq C(K_{\mathscr{A}})$. □

We now provide a partial classification theorem for spaces of continuous functions on scattered compacta in presence of Martin's axiom, which is essentially a generalization of Theorem 2.2.12.

Theorem 2.2.14. [$\mathsf{MA}(\kappa)$] *Let K and L be separable scattered compacta of finite height and weight κ such that $C(K') \simeq C(L')$. Then $C(K) \simeq C(L)$.*

Proof. Since K is separable, $K \setminus K'$ is countably infinite, so there is an exact sequence

$$0 \longrightarrow c_0 \longrightarrow C(K) \xrightarrow{\iota^\circ} C(K') \longrightarrow 0$$

Of course, the same works for L. Therefore, if $T\colon C(K') \to C(L')$ is an isomorphism, we have the following diagram

$$\begin{tikzcd} 0 \arrow[r] & c_0 \arrow[r] & C(K) \arrow[r, "T\iota^\circ"] & C(L') \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & c_0 \arrow[r] & C(L) \arrow[r, "J^\circ"] & C(L') \arrow[r] & 0 \end{tikzcd}$$

To carry on with the usual arguments, we need a stronger form of Theorem 2.1.4: under $\mathsf{MA}(\kappa)$, every separable scattered compactum K of finite height and weight κ satisfies $\operatorname{Ext}(C(K), c_0)$. This is precisely what Correa and Tausk show in [32, Corollary 4.2]. Therefore, the Diagonal Principle 1.2.5 can be applied and we obtain $c_0 \oplus C(L) \simeq c_0 \oplus C(K)$. All that is needed to conclude is the fact that, whenever K is scattered, $C(K)$ contains a complemented copy of c_0. We prove this right away –see Lemma 2.2.15 just below. □

Lemma 2.2.15. *Every scattered compactum K contains a convergent sequence which is not eventually constant. Therefore, $C(K)$ contains a complemented copy of c_0.*

Proof. Assume K is a scattered compactum. Choose inductively points $x_n \in K$ such that for every $n \in \mathbb{N}$, x_n is an isolated point in $K \setminus \{x_1, \ldots, x_{n-1}\}$. The resulting subspace $A = \{x_n : n \in \mathbb{N}\}$ is discrete, but it cannot be a closed subset of K. Hence there is a point $x \in K$ which is isolated in ∂A, together with two disjoint open sets U and V separating x from the closed set $\partial A \setminus \{x\}$. The set $B = A \cap U$ is countably infinite, and it is where our convergent sequence lies: let us write $B = \{y_n : n \in \mathbb{N}\}$ and check that $(y_n)_{n\in\mathbb{N}}$ converges to x. Indeed, B is discrete and $\overline{B} = B \cup \{x\}$, because $\partial B \subseteq \partial A$ but no point of $\partial A \setminus \{x\}$ can be in U. □

Finally, let us point out two consequences:

Corollary 2.2.16. [$\mathsf{MA}(\kappa)$] *Let K and L be separable scattered compacta of height* 3 *and weight κ. Then $C(K) \simeq C(L)$.*

Proof. It is deduced from Theorem 2.2.14 and two well-known facts about scattered compacta. First, every scattered compactum K of height 2 is a finite sum of one-point compactifications of discrete spaces, hence $C(K)$ is isomorphic to $c_0(K)$. Second, for scattered compacta, its weight coincides with its cardinality (Proposition 1.4.2). □

Corollary 2.2.17. [$\mathsf{MA}(\kappa)$] *Let K be a separable scattered compacta of finite height and weight κ. Then $c_0\big(C(K)\big)$ –the c_0-direct sum of $C(K)$– is isomorphic to $C(K)$. In particular, $C(K)$ is isomorphic to its square.*

Proof. Note that $c_0\big(C(K)\big) \simeq C(\alpha\mathbb{N}\times K)$, and $\alpha\mathbb{N}\times K$ has the weight of K and finite height. Therefore, we appeal once again to [32, Corollary 4.2] to obtain that $\operatorname{Ext}\big(C(\alpha\mathbb{N}\times K), c_0\big) = 0$. Applying the diagonal principles to the semi-equivalent sequences

$$\begin{tikzcd}
0 \arrow[r] & c_0 \arrow[r] & C(K) \arrow[r] & c_0(\kappa) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & c_0 \arrow[r] & c_0\big(C(K)\big) \arrow[r] & c_0(c_0(\kappa)) \arrow[r] & 0
\end{tikzcd}$$

and using the fact that $C(K)$, as well as $c_0\big(C(K)\big)$, contain complemented copies of c_0, the proof is finished. □

2.3 The 3-space property for C-spaces

.

It has been known for some time that "to be a C-space" fails to be a 3-space property. In fact, there are a number of interesting counterexamples in the literature, which we will briefly describe.

The first counterexample of which we are aware is based on a clever construction by Benyamini [11] which was later treated in [23] –see also [6, §2.2.6]. Fix $0 < \tau < 1$ and consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & c_0 \arrow[r] & C(\beta\mathbb{N}) \arrow[r] & C(\beta\mathbb{N}^*) \arrow[r] & 0 \\
0 \arrow[r] & c_0 \arrow[r] \arrow[u, equal] & PB_\tau \arrow[r] \arrow[u] & C(\beta\mathbb{N}^*) \arrow[r] \arrow[u, "\tau\cdot\mathrm{Id}"] & 0
\end{tikzcd}$$

where $\beta\mathbb{N}^* = \beta\mathbb{N}\setminus\mathbb{N}$. The pullback space is

$$PB_\tau = \{(f,g) \in C(\beta\mathbb{N})\times C(\beta\mathbb{N}^*) : f(t) = \tau g(t)\ \forall t \in \beta\mathbb{N}^*\}$$

which can be seen as a renorming of ℓ_∞. However, it has the singular property that it cannot be $(1/\tau)$-complemented in any C-space. Therefore, the twisted sum space

$$\begin{tikzcd}
0 \arrow[r] & c_0(c_0) \arrow[r] \arrow[d, equal] & c_0(PB_{1/n}) \arrow[r] \arrow[d, equal] & c_0(C(\beta\mathbb{N}^*)) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & c_0 \arrow[r] & c_0(PB_{1/n}) \arrow[r] & C(\beta\mathbb{N}^*\times\alpha\mathbb{N}) \arrow[r] & 0
\end{tikzcd}$$

cannot be a complemented space of a C-space.

Later, two remarkable examples appear in [17], making clear that by replacing c_0 with a more elaborated separable C-space one can produce quite untypical twisted sums. Precisely, the authors construct twisted sums

$$0 \longrightarrow C(\omega^\omega) \longrightarrow Z \xrightarrow{q} c_0 \longrightarrow 0$$

$$0 \longrightarrow C[0,1] \longrightarrow Z \xrightarrow{q} c_0 \longrightarrow 0$$

with *strictly singular* quotient operator. This means that no restriction of q to an infinite-dimensional subspace can be an isomorphism, and in particular, X cannot have even Pełczyński's property (V), let alone be a Lindenstrauss space.

Hence it was conjectured for some time that a twisted sum of C-spaces which is not a C-space needed some "complex" C-space either on the subspace or the quotient, and therefore, that every twisted sum of $c_0(I)$-spaces is a C-space. A counterexample appeared in [87], whose main purpose is to consistently prove that there is a twisted sum of c_0 and $C(K)$ which is not a C-space provided K is an Eberlein compactum of weight $\mathfrak{c}$. The list of compacta K for which such a twisted sum exists was substantially enlarged in [27, Theorem 2.4]:

Theorem 2.3.1. $[\mathfrak{p} = \mathfrak{c}]$ *There exists a twisted sum*

$$0 \longrightarrow c_0 \longrightarrow X \longrightarrow C(K) \longrightarrow 0 \tag{2.f}$$

in which X is not a C-space provided K has weight $\mathfrak{c}$ and belongs to one of the following classes of compacta:

i) Corson compacta with property (M).

ii) Separable Rosenthal compacta.

iii) Scattered compacta of finite height.

To be honest, what we will actually prove is that exact sequences (2.f) exist when K has weight $\mathfrak{c}$ and $M_1(K)$ is a sequentially compact space of size $\mathfrak{c}$ containing a copy of $\alpha\mathfrak{c}$. In particular, these properties are satisfied by compacta of weight $\mathfrak{c}$ belonging to any of the three classes mentioned in the previous theorem, as we now show.

We will deal with Corson compacta with property (M) first. A quick glance to Theorem 1.4.6 is enough to conclude that $M_1(K)$ is also Corson. On the other hand, since separable Corson compacta are metrizable, property (M) implies that every measure on K has metrizable support. But metrizable compacta can only carry $\mathfrak{c}$ many measures and K, having weight $\mathfrak{c}$, is the union of at most $\mathfrak{c}$ many metrizable subspaces. Hence $|M_1(K)| = \mathfrak{c}$. Finally, we show that $M_1(K)$ contains $\alpha\mathfrak{c}$ using a variation of [87, Lemma 4.2]. Consider an embedding of K inside $\Sigma([0,1]^{\mathfrak{c}})$ and pick, for each $\alpha < \mathfrak{c}$ a pair of points $x_\alpha, y_\alpha \in K$ and a set $I_\alpha \subseteq \mathfrak{c}$ such that

- $I_\alpha = \bigcup_{\beta<\alpha}(\operatorname{supp} x_\beta \cup \operatorname{supp} y_\beta)$.
- $x_\alpha \neq y_\alpha$ but $x_\alpha(i) = y_\alpha(i)$ for every $i \in I_\alpha$.

This construction can be carried out thanks to the fact that $w(K) = \mathfrak{c}$, for if at some step $\alpha < \mathfrak{c}$ it were not possible to choose appropriate $x_\alpha, y_\alpha \in K$, this would mean that K can be embedded inside $\Sigma([0,1]^{I_\alpha})$, which has weight $|I_\alpha| < \mathfrak{c}$. All that is left is to check that, if we define

$$\mu_\alpha = \frac{1}{2}(\delta_{x_\alpha} - \delta_{y_\alpha})$$

then the set $\Sigma = \{\mu_\alpha : \alpha < \mathfrak{c}\} \cup \{0\}$ is homeomorphic to $\alpha\mathfrak{c}$ with 0 as the point at infinity. For this purpose it suffices to note that the functions of the form $f = \prod_{i\in F}\pi_i$ where F is a finite subset of Γ, spans a subalgebra of $C(K)$ which separates points, and it is clear that for any such f, the value $\langle \mu_\alpha, f\rangle$ can only be non-zero for at most one $\alpha < \mathfrak{c}$.

Let us turn to case (ii), when K is a separable Rosenthal compactum. Then $M_1(K)$ is also separable Rosenthal, and in particular, its very definition assures that $M_1(K)$ is sequentially compact and has size $\mathfrak{c}$. It also contains $\alpha\mathfrak{c}$ by virtue of a result of Todorčević [97, Th. 9]. We must point out that Todorčević's result contains the additional requirement that $M_1(K)$ has a non-G_δ point, but if $0 \in M_1(K)$ were a weak* G_δ-point then K would be metrizable, in contradiction with the fact that $w(K) = \mathfrak{c}$. Indeed, if one could write $\{0\} = \bigcap_{n=1}^\infty V_n$, where $V_n = \{\mu \in M_1(K) : |\langle \mu, f_{(i,n)}\rangle| < \varepsilon_n\}$ for certain functions $f_{(i,n)} \in C(K)$ with $1 \le i \le k_n$ and a certain $\varepsilon_n > 0$, then the collection $\bigcup_{n=1}^\infty \{f_{(1,n)}, \dots, f_{(k_n,n)}\}$ separates points in $C(K)$, thus making $C(K)$ separable.

Finally, assertion (iii) concerning scattered compacta of finite height is proved by noting that $C(K)$ is Asplund whenever K is scattered, and in particular, it has weak* sequentially compact dual ball [53, Corollary 2]. In addition, $M(K) \simeq \ell_1(K)$, which implies that $|M_1(K)| = \mathfrak{c}$ provided K has weight $\mathfrak{c}$. We now appeal to [7, Lemma 6.4] to infer that K, and therefore $M_1(K)$, contains a copy of $\alpha\mathfrak{c}$. Such a result only works under the hypothesis that $\mathfrak{c}$ is a regular cardinal, but this is the case under $\mathfrak{p} = \mathfrak{c}$ as indicated by [45, 21K].

The plan

We will now discuss the proof of Theorem 2.3.1 at a non-technical level. The heart of it lies on the following definition and the extraordinarily obvious characterization of C-spaces it provides.

Definition. Let X be a Banach space. Given any weak* compact subspace $F \subseteq B_{X^*}$, there is a natural operator of norm 1 defined by

$$T: X \to C(F) \quad , \quad Tx(x^*) = \langle x^*, x\rangle$$

- Given $0 < c < 1$, we say F is *c-norming* if T is an into isomorphism with $\|T^{-1}\| < \frac{1}{c}$.
- We say F is *free* is T is onto.

Proposition 2.3.2. *A Banach space X is a C-space if and only if there is a weak* compact subset of B_{X^*} which is c-norming and free.*

This previous proposition marks the way: we will construct a twisted sum of c_0 and $C(K)$ whose dual unit ball contains no norming and free sets. In this line, the next observation allows us to recognize c-norming sets which are not free.

Lemma 2.3.3. *Let F be a weak* compact subset of B_{X^*} which is c-norming for some $0 < c \leq 1$. If there are different points x_0^*, x_1^*, $x_2^* \in F$ such that $\|x_0^* - \frac{1}{2}(x_1^* + x_2^*)\| < c$, then F cannot be free.*

Proof. It stems from the idea that any three points in a compactum can be separated by a continuous function, but not necessarily by a linear and continuous one. Precisely, consider a function $g \in C(F)$ satisfying $g(x_0^*) = 1$, $g(x_1^*) = g(x_2^*) = 0$ and $\|g\| = 1$. If we assume that F is free, there must be a point $x \in X$ of norm at most $\frac{1}{c}$ such that $g(x^*) = \langle x^*, x\rangle$ for every $x^* \in F$. Then

$$1 = |x_0^*(x) - \tfrac{1}{2}(x_1^* + x_2^*)(x)| \leq \tfrac{1}{c}\|x_0^* - \tfrac{1}{2}(x_1^* + x_2^*)\|$$

and so the inequality $\|x_0^* - \frac{1}{2}(x_1^* + x_2^*)\| < c$ cannot occur. □

Once the basic elements have been established, let us first describe the general setting of the construction before delving deeper into too many technicalities. In light of the previous lemmas, our objective is to prevent any c-norming set in B_{X^*} from being free.

Regardless of the nature of X, its dual is isomorphic to $\ell_1 \oplus M(K)$, so every $\mu \in X^*$ decomposes as $\mu = \mu^{\mathbb{N}} + \mu^K$, that is, a sum of two measures supported in $\mathbb{N}$ and K, respectively. It is clear that every closed c-norming set F in B_{X^*} necessarily contains a bounded sequence $(\mu_n)_{n=1}^\infty$ such that $|\langle \mu_n, e_n \rangle| = |\mu_n^{\mathbb{N}}\{n\}| \geq c$. On the other hand, the sequential compactness of $M_1(K)$ assures that $(\mu_n^K)_{n=1}^\infty$ has a convergent subsequence. Therefore, for *every* possible sequence of measures $(\mu_n^{\mathbb{N}})_{n=1}^\infty$ satisfying the above condition and *every* possible family of infinite sets $\mathscr{D} \subset \mathscr{P}(\mathbb{N})$ with the property that $(\mu_n^K)_{n\in D}$ converges whenever $D \in \mathscr{D}$, we will find a privileged $I \in \mathscr{D}$ such that the closure of $(\mu_n^{\mathbb{N}})_{n\in I}$ (and hence F) contains three points satisfying the condition of Lemma 2.3.3. These ideas are materialised within the following definitions.

Definition. Let K be a compactum. Given $0 < c \leq 1$, we say that three measures of $M(K)$ of the form $c'\delta_{x_i} + \nu_i$, $i \in \{0, 1, 2\}$ constitute a *c-fork* if

i) $|c'| > c$.

ii) x_i are pairwise distinct points of K.

iii) $\|\nu_i - \nu_j\| < \frac{c}{2}$ for $i \neq j$.

In the final stage of the proof, c-forks will be conveniently identified to give rise to the desired forbidden triples. So we may think of c-forks as a form of "destroying" norming free sets.

Definition. Let $[\mathbb{N}]^\omega$ denote the family of all infinite subsets of $\mathbb{N}$. We say that $\mathscr{D} \subseteq [\mathbb{N}]^\omega$ is *dense* if every $A \in [\mathbb{N}]^\omega$ contains some $D \in \mathscr{D}$.

The paramount example of a dense family arises by fixing any sequence $(x_n)_{n\in\mathbb{N}}$ in a sequentially compact space and considering the subsets $I \subseteq \mathbb{N}$ with the property that $(x_n)_{n\in I}$ is convergent. Consequently, dense families will be needed to keep track of the convergent subsequences of certain sequences of measures in $M_1(K)$.

The main technical result is the upcoming Theorem 2.3.4. It provides us with an almost disjoint family which, in a sense, book-keeps all the "information" mentioned above which we need in order to prevent any norming set from being free. From now on, we will tacitly identify ℓ_1 with the subspace of $M(\mathscr{P}(\mathbb{N}))$ consisting of σ-additive measures. Therefore, it should be clear what do we mean when we consider some $\mu \in \ell_1$ restricted to a certain subalgebra of $\mathscr{P}(\mathbb{N})$.

Theorem 2.3.4. $[\mathfrak{p} = \mathfrak{c}]$ *Let $\{\mathscr{D}_\gamma : \gamma < \mathfrak{c}\}$ be a list of dense families in $[\mathbb{N}]^\omega$. There exists an almost disjoint family $\mathscr{A}$ such that the Boolean algebra $\mathfrak{A}$ generated by $\mathscr{A}$ and all finite subsets of $\mathbb{N}$ satisfies the following property:*

For every $0 < c \leq 1$, for every sequence $(\mu_n)_{n\in\mathbb{N}}$ in the unit ball of ℓ_1 satisfying $|\mu_n(n)| \geq c$ and for every $\gamma < \mathfrak{c}$, there is a set $I \subset \mathbb{N}$ such that:

- *$I \subseteq D$ for some $D \in \mathscr{D}_\gamma$.*
- *The closure of $\{\mu_n : n \in I\}$ in $M(\mathfrak{A})$ contains a c-fork.*

We postpone its proof; first, let us indicate how to deduce Theorem 2.3.1 from Theorem 2.3.4 by appealing to our two previous results.

Proof of Theorem 2.3.1. Assume that K has weight $\mathfrak{c}$ and that $M_1(K)$ is a sequentially compact space of size $\mathfrak{c}$ containing a copy of $\alpha\mathfrak{c}$. In particular, we can enumerate all sequences from $M_1(K)$ as $(\lambda_n^\gamma)_{n\in\mathbb{N}}$, for $\gamma < \mathfrak{c}$. We apply Theorem 2.3.4 with $\mathscr{D}_\gamma$ the family of subsets $D \subseteq \mathbb{N}$ such that $(\mu_n^\gamma)_{n\in D}$ is weak* convergent. It will be convenient to write $\mathscr{A} = \{A_\xi^i : \xi < \mathfrak{c}, i = 0, 1, 2\}$ and simply put $p_{A_\xi^i} = p_\xi^i$; that is, p_ξ^i stands for the ultrafilter on $\mathfrak{A}$ containing A_ξ^i and no finite set. Hence given any $0 < c \leq 1$, any bounded sequence $(\mu_n)_{n\in\mathbb{N}}$ in ℓ_1 such that $|\mu_n(n)| > c$ regardless of $n \in \mathbb{N}$, and any $\mathscr{D}_\gamma$, there is $\xi < \mathfrak{c}$ and $|c'| > c$ such that $\{c'\delta_{x_\xi^i} + \nu_i,\ i = 0, 1, 2\}$ is the c-fork whose existence is guaranteed by Theorem 2.3.4.

We now construct a countable discrete extension L of $M_1(K)$ by attaching the compactum $K_{\mathscr{A}}$ to $M_1(K)$ in a convenient way. Once this is done, the desired twisted sum will be the pullback sequence in the following diagram, where e is just the natural evaluation map $e(f)(\mu) = \langle \mu, f\rangle$:

$$\begin{tikzcd}
0 \arrow[r] & c_0 \arrow[r] \arrow[d, equal] & C(L) \arrow[r] & C(M_1(K)) \arrow[r] & 0 \\
0 \arrow[r] & c_0 \arrow[r] & X \arrow[r] \arrow[u, hook] & C(K) \arrow[r] \arrow[u, hook, "e"] & 0
\end{tikzcd}$$

We can assume without loss of generality that the copy of $\alpha\mathfrak{c}$ that lies inside $M_1(K)$ is of the form $\Sigma = \{\sigma_\xi, -\sigma_\xi : \xi < \mathfrak{c}\} \cup \{0\}$, where 0 is the "point at infinity": it suffices to pick $\{x_\xi : \xi < \mathfrak{c}\} \cup \{x\}$ any copy of $\alpha\mathfrak{c}$ inside $M_1(K)$ and define $\sigma_\xi = \frac{1}{2}(x_\xi - x)$. Consider the mapping $\psi : K'_{\mathscr{A}} \to \Sigma \subseteq M_1(K)$ defined by

$$\psi(p_\xi^0) = 0 \quad , \quad \psi(p_\xi^1) = \sigma_\xi \quad , \quad \psi(p_\xi^2) = -\sigma_\xi \tag{2.g}$$

and let L be the adjunction space of $M_1(K)$ and $K_{\mathscr{A}}$ by ψ; in other words, we consider the quotient space of $M_1(K) \sqcup K_{\mathscr{A}}$ obtained by the equivalence relation which identifies the points in every set $\psi^{-1}(\lambda) \cup \{\lambda\}$ for $\lambda \in M_1(K)$. Let us note that L is clearly a countable discrete extension of $M_1(K)$. We write $\pi\colon M_1(K) \sqcup K_{\mathscr{A}} \to L$ for the quotient mapping.

Finally, we show that the pullback space X cannot be a C-space, for which it suffices to check that no norming set inside B_{X^*} can be free. Indeed, assume $F \subseteq B_{X^*}$ is c-norming for some $0 < c \leq 1$. In particular, F must contain a sequence $(\mu_n)_{n\in\mathbb{N}}$ that satisfies $|\mu_n(n)| > c$ for every $n \in \mathbb{N}$. Let us decompose $\mu_n = \mu_n^{\mathbb{N}} + \mu_n^K$ into measures supported in $\mathbb{N}$ and K, respectively. The key point is that, by virtue of Theorem 2.3.4, there are $I \subset \mathbb{N}$ and $\xi < \mathfrak{c}$ verifying:

- The closure of $\{\mu_n^{\mathbb{N}} : n \in I\}$ contains the c-fork $c'\delta_{p_i^\xi} + \nu_i$, for $i = 0, 1, 2$.
- $(\mu_n^K)_{n\in I}$ is convergent to some μ.

Therefore, if we denote $\nu_i' = \pi^*(\nu_i)$ the image measure of ν under π, the following triple lies in the closure of $\{\mu_n : n \in I\}$ in $M(L)$:

$$\nu_0' + \mu, \quad c'\sigma_\xi + \nu_1' + \mu, \quad -c'\sigma_\xi + \nu_2' + \mu$$

Now, since $\|\nu_i - \nu_j\| < \frac{c}{2}$ for $i, j \in \{0, 1, 2\}$, we also have $\|\nu_i' - \nu_j'\| < \frac{c}{2}$. Hence

$$\left\|\tfrac{1}{2}\left[(c'\sigma_\xi + \nu_1' + \mu) + (-c'\sigma_\xi + \nu_2' + \mu)\right] - (\nu_0' + \mu)\right\| < \tfrac{c}{2}$$

which informs us that we have found a forbidden triple inside F. Lemma 2.3.3 tells us that F is not a free set, and so an application of Proposition 2.3.2 yields X is not a C-space. □

The details

It is now the moment to develop the proof of Theorem 2.3.4. It will essentially come off as a consequence of the *Iterative Lemma* 2.3.8, which in turn relies on some mildly simple observations and well-known results. Let us describe the three ingredients needed for the Iterative Lemma.

Lemma 2.3.5. *Assume $(\mu_n)_{n\in\mathbb{N}}$ is a bounded sequence in ℓ_1. Then for every infinite subset $T \subset \mathbb{N}$, there are infinite disjoint subsets $N, S \subset T$ such that $(\mu_n)_{n\in N}$ is norm-convergent on the set S.*

Proof. It follows a standard gliding jump argument. There is no loss of generality if we assume $T = \mathbb{N}$ and $(\mu_n)_{n\in\mathbb{N}}$ to be weak* null. We construct increasing sequences of natural numbers $\{n_k : k \in \mathbb{N}\}$ and $\{s_k : k \in \mathbb{N}\}$ such that

- $\sum_{j=1}^{s_{k-1}} |\mu_n(j)| < 2^{-(k+1)}$ for every $n \geq n_k$.
- $\sum_{j=s_k}^{\infty} |\mu_{n_k}(j)| < 2^{-(k+1)}$.

Set $N = \{n_k : k \in \mathbb{N}\}$ and $S = \{s_k : k \in \mathbb{N}\}$. Then $(\mu_n)_{n\in N}$ is norm-convergent to 0 on S, since for any $k \in \mathbb{N}$

$$\sum_{j\in S} |\mu_{n_k}(j)| \leq \sum_{j=1}^{k-1} |\mu_{n_k}(s_j)| + \sum_{j=s_k}^{\infty} |\mu_{n_k}(j)| < 2^{-k} \qquad \square$$

We will need the well-known Rosenthal's lemma in the following form [36, p. 18]:

Lemma 2.3.6. (Rosenthal) *Let $(\mu_n)_{n\in\mathbb{N}}$ be a bounded sequence in ℓ_1. For every $\varepsilon > 0$ and every infinite subset $N \subseteq \mathbb{N}$, there is an infinite subset $R \subset N$ such that $|\mu_n|(R \setminus \{n\}) < \varepsilon$ whenever $n \in R$.*

Our last lemma provides some simple conditions to guarantee a nice behavior of cluster points of a sequence $(\mu_n)_{n\in\mathbb{N}}$ in ℓ_1 when one passes from a certain subalgebra to a bigger one. The following notation will be appropriate: let us write $\|\mu\|_{\mathfrak{B}}$ for the norm of the restriction $\mu|_{\mathfrak{B}}$ of a measure $\mu \in \ell_1$ to a certain subalgebra $\mathfrak{B} \subseteq \mathscr{P}(\mathbb{N})$. Recall that we are working under the identification $M(\mathrm{ult}(\mathfrak{B})) = M(\mathfrak{B})$ described in Section 1.4, and therefore $M(\mathfrak{B})$ is endowed with the topology of pointwise convergence on elements of $\mathfrak{B}$.

Lemma 2.3.7. *Fix an infinite subset $I \subset \mathbb{N}$. Consider $\mathfrak{B}$ and $\mathfrak{C}$ two subalgebras of $\mathscr{P}(\mathbb{N})$ such that $\mathfrak{C}$ is contained in the subalgebra generated by $\mathfrak{B} \cup \mathscr{P}(I)$, and $(\mu_n)_{n\in\mathbb{N}}$ a sequence in ℓ_1.*

i) If $(\mu_n)_{n\in\mathbb{N}}$ is convergent on $\mathfrak{B}$ and $|\mu_n|(n) < \varepsilon$ for every $n \in \mathbb{N}$, then any two cluster points μ', μ'' of $\{\mu_n : n \in \mathbb{N}\}$ in $M(\mathfrak{C})$ satisfy $\|\mu' - \mu''\|_{\mathfrak{C}} < 6\varepsilon$.

ii) If $(\mu_n)_{n\in\mathbb{N}}$ is norm-convergent on I, then any two cluster points μ', μ'' of the set $\{\mu_n : n \in \mathbb{N}\}$ in $M(\mathfrak{C})$ satisfy $\|\mu' - \mu''\|_{\mathfrak{B}} = \|\mu' - \mu''\|_{\mathfrak{C}}$.

Proof. Let us assume with no loss of generality that $\mathfrak{C}$ is the subalgebra generated by $\mathfrak{B} \cup \mathscr{P}(I)$, and so elements of $\mathfrak{C}$ are of the form $C = (B_1 \cap E_1) \cup (B_2 \setminus E_2)$, for some

$B_1, B_2 \in \mathfrak{B}$ and $E_1, E_2 \in \mathscr{P}(I)$. To show (i), first note that $|\mu' - \mu''|(I) < 2\varepsilon$ by hypothesis. On the other hand, $\mu'(B) = \mu''(B)$ for every $B \in \mathfrak{B}$ and so

$$\begin{aligned}|\mu'(B \setminus E) - \mu''(B \setminus E)| &= |\mu'(B) - \mu'(B \cap E) - \mu'(B) + \mu''(B \cap E)| \\ &= |\mu''(B \cap E) - \mu'(B \cap E)| < 2\varepsilon\end{aligned}$$

A quick reference to the definition of variation shows this is enough to conclude that $|\mu' - \mu''|(\mathbb{N} \setminus I) < 4\varepsilon$, hence $\|\mu' - \mu''\|_{\mathfrak{C}} < 6\varepsilon$, as desired.

We now deal with (ii). In this case, the hypotheses imply that $|\mu' - \mu''|(E) = 0$ for every $E \in \mathscr{P}(I)$, so in particular $|\mu' - \mu''|(I) = 0$. A similar argument as in (i) allows us to obtain

$$|\mu'(B \setminus E) - \mu''(B \setminus E)| \leq |\mu'(B) - \mu''(B)|$$

which yields $\|\mu' - \mu''\|_{\mathfrak{C}} \leq \|\mu' - \mu''\|_{\mathfrak{B}}$, and the reverse inequality always holds. □

The Iterative Lemma works roughly as follows: given a subalgebra of $\mathscr{P}(\mathbb{N})$, together with a new sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ and a new dense family $\mathscr{D}$, it enlarges the subalgebra in such a way that some appropriate subsequence $(\mu_n)_{n\in I}$ with indices in some $I \in \mathscr{D}$ now has a c-fork in its closure. The following definition will be needed throughout it.

Definition. A subalgebra $\mathfrak{B} \subseteq \mathscr{P}(\mathbb{N})$ is *trivial* on an infinite set $S \subseteq \mathbb{N}$ if for every $B \in \mathfrak{B}$ either $B \cap S$ or $B \cap S^c$ is finite.

Iterative Lemma 2.3.8. $[\mathfrak{p} = \mathfrak{c}]$ *Consider:*

- *A subalgebra $\mathfrak{B} \subseteq \mathscr{P}(\mathbb{N})$ containing all finite sets of $\mathbb{N}$ and such that $|\mathfrak{B}| < \mathfrak{c}$.*
- *An infinite subset $T \subset \mathbb{N}$ in which $\mathfrak{B}$ is trivial.*
- *A number $0 < c \leq 1$, and a sequence $(\mu_n)_{n\in\mathbb{N}}$ in the unit ball of ℓ_1 such that $|\mu_n(n)| > c$ for all $n \in T$.*
- *A dense family $\mathscr{D}$ in $[\mathbb{N}]^\omega$.*

Then there are infinite disjoint sets $I, S \subset T$ and pairwise disjoint sets $A_0, A_1, A_2 \subset S$ such that:

i) $I \in \mathscr{D}$.

ii) The sequence $(\mu_n)_{n\in I}$ is norm-convergent on S.

iii) If we write $\mathfrak{B}'$ for the subalgebra generated by $\mathfrak{B} \cup \{A_0, A_1, A_2\}$, then the closure of $\{\mu_n : n \in I\}$ in $M(\mathfrak{B}')$ contains a c-fork.

Proof. It will be convenient to decompose $\mu_n = b_n\delta_n + \alpha_n$, where $\alpha_n(n) = 0$ for every $n \in \mathbb{N}$. We recall the fact that every measure lying on $M(\mathfrak{B})$ can be seen as an element in $[-1, 1]^{\mathfrak{B}}$, and for that reason we can assume, applying $\mathfrak{p} = \mathfrak{c}$ and passing to a subsequence, that $(\mu_n)_{n\in T}$ is convergent on $\mathfrak{B}$. Let us further suppose that the sequence $(b_n)_{n\in T}$ converges to some b, which of course satisfies $|b| > c$; without loss of generality we treat the case $b > c$. In that case, $(\delta_n)_{n\in T}$ is also convergent, since $\mathfrak{B}$ is trivial on T, and therefore $(\alpha_n)_{n\in T}$ is convergent on $\mathfrak{B}$.

Now it is time to suitably apply our trio of lemmas. First, we use Lemma 2.3.5 to obtain disjoint sets $N, S \subseteq T$ such that $(\mu_n)_{n\in N}$ is norm-convergent on S. Next, apply Rosenthal's lemma 2.3.6 to localize an infinite subset $R \subseteq N$ with the property that $|\alpha_n|(R) < \frac{c}{12}$ for every $n \in R$. Last, consider an infinite subset $I \subseteq R$ such that $I \in \mathscr{D}$. Split I in three infinite sets, $I = A_0 \cup A_1 \cup A_2$ and call $\mathfrak{B}'$ the subalgebra generated by $\mathfrak{B} \cup \{A_0, A_1, A_2\}$. Then (i) and (ii) are clearly satisfied. Regarding (iii), let us write x_i for the only ultrafilter in $\mathrm{ult}(\mathfrak{B}')$ containing A_i but not containing any finite set, where $i \in \{0, 1, 2\}$, and consider ν_i any cluster point of the sequence $(\alpha_n)_{n\in A_i}$. It is now a simple matter to check that, since $(\delta_n)_{n\in A_i}$ converges to δ_{x_i}, the measures $b\delta_{x_i} + \nu_i$, $i = 0, 1, 2$ constitute a c-fork inside the weak* closure of $\{\mu_n : n \in I\}$ in $M(\mathfrak{B})$. In particular, $\|\nu_i - \nu_j\| < \frac{c}{2}$ thanks to Lemma 2.3.7. □

With the Iterative Lemma in mind, the proof of Theorem 2.3.4 is by no means difficult. Let us consider an enumeration

$$\{(c_\beta, (\mu_n^\beta)_{n\in\mathbb{N}}, \mathscr{D}_\gamma) : \beta, \gamma < \mathfrak{c}\}$$

where $0 < c_\beta \leq 1$, $(\mu_n^\beta)_{n\in\mathbb{N}}$ is a sequence such that $|\mu_n^\beta(n)| > c_\beta$ and $\mathscr{D}_\gamma$ is a dense family of $[\mathbb{N}]^\omega$. We will do an inductive processes over the pairs $\xi = (\beta, \gamma)$ ordered in type $\mathfrak{c}$. Precisely, we will find almost disjoint sets I_ξ (where step ξ will performed) and S_ξ (where subsequent steps of the process will be done), together with pairwise disjoint sets $A_\xi^0, A_\xi^1, A_\xi^2 \subseteq I_\xi$, such that

$$\mathscr{A} = \{A_\xi^0, A_\xi^1, A_\xi^2 : \xi < \mathfrak{c}\}$$

will be our desired algebra.

Let us write $\mathfrak{B}_\xi$ for the algebra generated by all finite sets of $\mathbb{N}$ and $\{A_\eta^0, A_\eta^1, A_\eta^2 : \eta < \xi\}$. The process will be carried out guaranteeing that $I_\eta, S_\eta \subseteq^* S_\xi$ for every $\eta < \xi$ and $I_\eta \cap S_\xi =^* \varnothing$ for every $\eta \geq \xi$, as well as that for every $\xi = (\beta, \gamma)$:

i) The sequence $(\mu_n^\beta)_{n\in I_\xi}$ is norm-convergent on S_ξ.

ii) The sequence $(\mu_n^\beta)_{n\in I_\xi}$ is convergent on $\mathfrak{B}_\xi$.

iii) $I_\xi \in \mathscr{D}_\gamma$.

iv) The closure of $\{\mu_n^\beta : n \in I_\xi\}$ in $M(\mathfrak{B}_{\xi+1})$ contains a c_β-fork.

Our starting point will be $\mathfrak{B}_0$ the finite-cofinite subalgebra in $\mathscr{P}(\mathbb{N})$. Let $\xi = (\beta, \gamma) < \mathfrak{c}$, and assume that the construction has already been done for every $\eta < \xi$. In order to apply the Iterative Lemma 2.3.8, we need to create the appropriate conditions. First, an appeal to $\mathfrak{p} = \mathfrak{c}$ yields an infinite set $Y \subset \mathbb{N}$ such that $Y \subseteq^* S_\eta$ for $\eta < \xi$. Next we look for some infinite subset $T \subseteq Y$ in which the sequence $(\mu_n^\beta)_{n\in T}$ is convergent on $\mathfrak{B}_\xi$; this is possible thanks to $\mathfrak{p} = \mathfrak{c}$ and to the fact that $|\mathfrak{B}_\xi| < \mathfrak{c}$. Note that now $\mathfrak{B}_\xi$ is trivial on T, hence the Iterative Lemma 2.3.8 takes action and provides sets $I_\xi, S_\xi \subseteq T$ as well as pairwise disjoint sets $A_\xi^0, A_\xi^1, A_\xi^2 \subseteq I_\xi$ satisfying conditions (i)–(iv) above. Lemma 2.3.7(ii) gives the final blow: since the final algebra $\mathfrak{A} = \bigcup_{\xi<\mathfrak{c}} \mathfrak{B}_\xi$ is contained in $\mathfrak{B}_\xi \cup \mathscr{P}(S_\xi)$, the c-fork created at step ξ will certainly not be destroyed in future steps. Precisely, if such c-fork is denoted as $c'\delta_i + \nu_i$ for $i \in \{0, 1, 2\}$, then we can assure that $\|\nu_i - \nu_j\|_{\mathfrak{A}} = \|\nu_i - \nu_j\|_{\mathfrak{B}_{\xi+1}} < \frac{c}{2}$. $\square$

Some variations

We finish this section with a few comments and further extensions of the above result. It is clear that the construction of the twisted sum depends on how one selects the compactum Σ defined in the proof of Theorem 2.3.1, just above equation (2.g). Therefore, we will use the following notation:

$$0 \longrightarrow c_0 \longrightarrow Z(K,\Sigma) \longrightarrow C(K) \longrightarrow 0$$

Choosing the copy of Σ with some delicacy may lead to some interesting spaces, as the following result shows. Recall that a *Markuševič basis* for a Banach space X is a biorthogonal system $\{(x_\alpha, x_\alpha^*) \in X \times X^* : \alpha \in A\}$ such that the span of $\{x_\alpha : \alpha \in A\}$ is dense in X and the span of $\{x_\alpha^* : \alpha \in A\}$ is weak* dense in X^*. Furthermore, we say that a Markuševič basis is *norming* if there exists $C \geq 0$ such that $\|x\| \leq C \cdot \sup_{\alpha\in A} |\langle x_\alpha^*, x\rangle|$.

Theorem 2.3.9. $[\mathfrak{p} = \mathfrak{c}]$. *Let K be a compactum of weight $\mathfrak{c}$ which belongs to any of the classes in Theorem 2.3.1 and such that $C(K)$ admits a norming Markuševič basis. Then there exists a twisted sum*

$$0 \longrightarrow c_0 \longrightarrow Z(K,\Sigma) \longrightarrow C(K) \longrightarrow 0$$

in which $Z(K,\Sigma)$ is isomorphic to a subspace of ℓ_∞ and it is not a C-space.

Proof. Let us write $\{(f_\xi,\mu_\xi)\in C(K)\times M(K):\xi<\mathfrak{c}\}$ for the norming Markušević basis. We can easily assume $\|\mu_\xi\|\le 1$ for all $\xi<\mathfrak{c}$. As a consequence, the set $\Sigma=\{\mu_\xi,-\mu_\xi:\xi<\mathfrak{c}\}\cup\{0\}$ is a copy of $\alpha\mathfrak{c}$ inside $M_1(K)$. Therefore, the countable discrete extension $L=M_1(K)\cup\omega$ which produces the space $Z(K,\Sigma)$, as in the proof of Theorem 2.3.1, verifies that the closure of ω in L is $\omega\cup\{\mu_\xi,-\mu_\xi:\xi<\mathfrak{c}\}\cup\{0\}$. We now consider the functionals $(\delta_n)_{n\in\omega}$, where $\delta_n(f,g)=f(n)$, and examine the operator $T\colon Z(K,\Sigma)\to\ell_\infty$ given by $T(f,g)=(\delta_n(f,g))_{n\in\omega}$. It is clear that T is injective, since $(\delta_n)_{n\in\omega}$ separates points. To show that it is an into isomorphism, pick $(f,g)\in Z(K,\Sigma)$ and observe that f must attain its norm at some point in L. If such a point belongs to ω, then clearly $\|f\|=\|(\delta_n(f))\|_\infty$. Otherwise, there is some $\mu\in M_1(K)$ such that

$$\|f\|=|f(\mu)|=|\langle\mu,g\rangle|\le C\cdot\sup_{\xi<\mathfrak{c}}|\langle\mu_\xi,g\rangle|=C\|(\delta_n(f))\|_\infty$$

and so the proof concludes. □

The problem with the previous theorem is that we do not know if it can be applied to a large family of compacta. Indeed, a $C(K)$-space with K a Corson compacta satisfying property (M) admits a Markušević basis [98, Theorem 1.1], but it may not be norming. Therefore, the best result we can provide for such spaces is the existence of an exact sequence

$$0\longrightarrow c_0\longrightarrow Z(K,\Sigma)\longrightarrow C(K)\longrightarrow 0$$

in which $Z(K,\Sigma)$ is not a C-space and has weak* separable dual. In general, the requirement that $C(K)$ admits a norming Markušević basis is quite strong: their existence is not assured even for a $C(K)$-space with K Eberlein [54]. On the other hand, it is a classical result [42, Theorem 11.23] that a compactum K is scattered and Eberlein if and only if $C(K)$ admits a Markušević basis $\{(f_\alpha,\mu_\alpha)\in C(K)\times M(K):\alpha\in A)\}$ which is *shrinking*, meaning that $\{\mu_\alpha:\alpha\in A\}$ is norm-dense in $M(K)$. Finally, we do not know if a $C(K)$-space where K is a separable Rosenthal compacta admits a norming Markušević basis. However, every twisted sum of c_0 and $C(K)$ where K is separable is automatically a subspace of ℓ_∞, since "being a subspace of ℓ_∞" is a 3-space property in the category of Banach spaces.

Be as it may, a particularly interesting example appears when one lets K be the one-point compactification of a discrete set I of size $\mathfrak{c}$. In such a case it is clear that the choice $\Sigma=\{e_\xi^*,-e_\xi^*:\xi<\mathfrak{c}\}\cup\{0\}$, where $(e_\xi^*)_{\xi<\mathfrak{c}}$ is the canonical basis of $\ell_1(\mathfrak{c})$, produces a twisted sum $Z(\Sigma, K)$ which is a subspace of ℓ_∞. The space $Z(\Sigma, K)$ was baptised as PS in [27]. It is also true that PS $\oplus\, c_0(\mathfrak{c})$ is not a C-space either: this can be seen considering a subset $J \subset I$ such that $|J| = |I \setminus J| = \mathfrak{c}$ and showing that the choice $\Sigma_J = \{e_j^*, -e_j^* : j \in J\} \cup \{0\}$ produces a twisted sum space $Z(\Sigma_J, K)$ isomorphic to PS $\oplus\, c_0(\mathfrak{c})$. In the next chapter we show that PS –as well as all twisted sums of c_0 and $c_0(\mathfrak{c})$– is isomorphic to a Lindenstrauss space. But it is not known if PS is a complemented subspace of a C-space, let alone a quotient of a C-space.

On another direction, let us observe that the argument for Theorem 2.3.1 relies heavily on the sequential compactness of $M_1(K)$, and in particular, it cannot be applied when $K = \beta\omega$. Therefore it is still open whether a twisted sum of c_0 and ℓ_∞ which is not a C-space can be constructed. However, $\mathrm{Ext}(\ell_\infty, c_0) \neq 0$, as we showed in Theorem 2.1.5.

Chapter 3

The structure of twisted sums with $c_0(I)$

Twisted sums of $c_0(\kappa)$ and $c_0(I)$ constitute the simplest examples of twisted sums of C-spaces. When $\kappa = \aleph_0$, the best known examples in the literature have already been described in the dissertation:

- Banach spaces of the form $C(K_{\mathscr{A}})$, where $K_{\mathscr{A}}$ is an Alexandroff-Urysohn compactum.
- Under $\mathfrak{p} = \mathfrak{c}$, the middle spaces in the sequences

  $$0 \longrightarrow c_0 \longrightarrow Z(\Sigma) \longrightarrow c_0(\mathfrak{c}) \longrightarrow 0$$

  constructed in Section 2.3, whose main feature is that they are not C-spaces. The space PS is perhaps its most representative member.
- Products and c_0-sums of these.

The present chapter contains some results which point towards a full classification of twisted sums of $c_0(I)$-spaces, following the structure of the recent paper [27]. The greatest achievement in this direction is Theorem 3.2.7, which shows that every twisted sum X of $c_0(\kappa)$ and $c_0(I)$ is either a subspace of $\ell_\infty(\kappa)$ or there is a complemented copy of $c_0(\kappa^+)$ inside X on which the quotient operator $X \to c_0(I)$ becomes an isomorphism. In addition, we also complete the list of properties of twisted sums of $c_0(I)$-spaces showing that they are isomorphically Lindenstrauss and isomorphically polyhedral. All these results stem from a representation theorem –see Theorem 3.2.1– which suggests that every element in $\mathrm{Ext}(X, c_0(\kappa))$ arises from a κ-discrete extension of B_{X^*} endowed with the weak* topology.

3.1 A general perspective

For a start, let us summarize all properties of twisted sums of $c_0(\kappa)$ and $c_0(I)$ which were known prior to the appearance of [27]. We first need to recall a couple of well-known properties of $c_0(I)$:

Lemma 3.1.1. *$c_0(I)$ satisfies the following properties:*

i) Every copy of $c_0(J)$ inside $c_0(I)$ is complemented.

ii) Every subspace of $c_0(I)$ of density κ contains a further subspace isomorphic to $c_0(\kappa)$.

Proof. The first assertion can be found in [50]. As for the second, which is most surely folklore, we will consider the argument given in [81, Lemma 2.7]: for every subspace X of $c_0(I)$ having density κ there is a decomposition $I = \bigcup_{j\in J} I_j$ where $|J| = |I|$ with the property that, if we call $X_j = \{x \in X : \operatorname{supp} x \subseteq I_j\}$ then there are exactly κ many non-zero subspaces X_j, and X is isomorphic to

$$c_0(J, X_j) = \left\{ (x_j) \in \prod_{j\in J} X_j : \forall\varepsilon > 0, \{j \in J : \|x_j\| > \varepsilon\} \text{ is finite} \right\}$$

In particular, X contains a copy of $c_0(\kappa)$. □

Theorem 3.1.2. *Suppose X is a twisted sum of $c_0(\kappa)$ and $c_0(I)$. Then:*

i) X is c_0-saturated and c_0-uppersaturated.

ii) X is Asplund. Consequently, it has weak sequentially compact dual ball and has the Gelfand-Phillips property.*

iii) X has Pełczyński's property (V).

iv) X contains a complemented copy of c_0. In particular, $X \simeq X \oplus c_0$.

v) X is separably injective but not universally separably injective.

Proof. Recall that a Banach space X is c_0-*saturated* if every infinite-dimensional closed subspace of X contains a closed subspace isomorphic to c_0, and that X is c_0-*uppersaturated* if every separable closed subspace of X is contained in a closed subspace of X isomorphic to c_0. Both properties are 3-space properties: c_0-saturation is in [23, Theorem 3.2.e] while c_0-uppersaturation can be found in [6, Prop. 6.2].

Concerning (ii), the Asplund character readily follows from (i). Now, every Asplund space has weak* sequentially compact dual ball, by virtue of [53, Corollary 2]. In its turn, spaces with weak* sequentially compact dual ball have the Gelfand-Phillips property, as shown in [23, Prop. 6.8.c].

Assertion (iii) is in [28, §4, Proposition]. Later, in Theorem 3.2.3 we will show that twisted sums of $c_0(\kappa)$ and $c_0(I)$ are even isomorphically Lindenstrauss spaces. In particular, every twisted sum of $c_0(\kappa)$ and $c_0(I)$ are *Lindenstrauss-Pełczyński* spaces in the sense of [24].

Now, (iv) essentially appeared in the proof of Theorem 2.2.12, and can be deduced from (iii): since the quotient operator $q\colon X \longrightarrow c_0(I)$ cannot be weakly compact, it is an isomorphism in a subspace $X_0 \subseteq X$ isomorphic to c_0. But, by the previous lemma, $q(X_0)$ is necessarily complemented in $c_0(I)$, hence we infer that X_0 is complemented in X. In particular, we have $X \simeq Y \oplus X_0 \simeq Y \oplus X_0 \oplus X_0 \simeq X \oplus X_0$.

Finally, a Banach space X is *separably injective* if every operator $T\colon Y \longrightarrow X$, where Y is a separable Banach space, can be extended to any separable superspace of Y. If moreover T can be extended to *any* superspace of Y, then we say X is *universally separably injective*. Now, to dispose of (v), we need to recall that "to be separably injective" is a 3-space property [6, Prop. 2.11], and that $c_0(I)$ is separably injective, but no twisted sum of $c_0(\kappa)$ and $c_0(I)$ can be universally separably injective because they are never Grothendieck spaces –cf. [6, Prop. 2.8]–. □

The results in this chapter may prompt the reader to presume that twisted sums of $c_0(\kappa)$ and $c_0(I)$ have the same behaviour regardless of the size of κ. However, there is a major difference: twisted sums of c_0 and $c_0(I)$ are never WCG, unless they are trivial, as Sobczyk's theorem asserts –see [20, §4.1] for a list of spaces in which every copy of c_0 is known to be complemented. This is no longer true for twisted sums of $c_0(\kappa)$ and $c_0(I)$ where κ is uncountable: Bell and Marciszewski produce in [9] an Eberlein compactum K of weight $\aleph_\omega$ and height 3 such that the canonical inclusion $\iota\colon K' \hookrightarrow K$ induces a non-trivial twisted sum

$$0 \longrightarrow c_0(\aleph_\omega) \longrightarrow C_0(K) \xrightarrow{\iota^\circ} c_0(\aleph_\omega) \longrightarrow 0$$

As for the seemingly strange choice of the weight $\aleph_\omega$, the explanation lies on [4, Theorem 1.1]: if $|I| < \aleph_\omega$, then every copy of $c_0(I)$ in a WCG space is complemented. A nice improvement of this result appears in [49, Th. 4.8 plus remark on p.800]: for compacta K of weight strictly less than $\aleph_\omega$, $C(K) \simeq c_0(I)$ precisely when K is scattered Eberlein compacta of finite height. Marciszewski goes even further in [76] and characterizes the compacta K for which $C(K)$ is isomorphic to some $c_0(J)$. Be as it may, Bell and Marciszewski's construction is optimal.

3.2 A representation theorem

We arrive to the main result of this chapter, which asserts that all twisted sums with $c_0(\kappa)$ and another Banach space X arise from discrete extensions of (B_{X^*}, weak^*) in a particular way. The mentioned result appeared implicitly in [7], and fully detailed in [27].

Theorem 3.2.1. *Given an exact sequence of* Banach *spaces*

$$0 \longrightarrow c_0(\kappa) \xrightarrow{j} Z \xrightarrow{q} X \longrightarrow 0$$

there is a κ-discrete extension L of $(B_{X^*}, weak^*)$ such that $[\mathtt{z}]$ *is equivalent to the lower sequence of the following diagram:*

$$\begin{tikzcd} 0 \arrow[r] & c_0(\kappa) \arrow[r] & C(L) \arrow[r, "\iota^\circ"] & C(B_{X^*}) \arrow[r] & 0 \\ 0 \arrow[r] & c_0(\kappa) \arrow[r] \arrow[u, equal] & Z(L) \arrow[r] \arrow[u, hook] & X \arrow[r] \arrow[u, hook, "e"] & 0 \end{tikzcd} \tag{3.a}$$

where $\iota\colon B_{X^*} \hookrightarrow L$ is the natural inclusion and $e\colon X \to C(B_{X^*})$ is the canonical evaluation map $e(x)(x^*) = \langle x^*, x\rangle$.

Proof. The dual of the original sequence, namely

$$0 \longrightarrow X^* \xrightarrow{q^*} Z^* \xrightarrow{j^*} \ell_1(\kappa) \longrightarrow 0$$

splits, because $\ell_1(\kappa)$ has the lifting property. Therefore, there is a bounded collection $\{z^*_\alpha : \alpha < \kappa\}$ in Z^* such that $j^*(z^*_\alpha) = e^*_\alpha$ for every $\alpha < \kappa$, where $\{e^*_\alpha : \alpha < \kappa\}$ denotes the canonical basis of $\ell_1(\kappa)$. This implies that any weak* cluster point of $\{z^*_\alpha\}_{\alpha<\kappa}$ belongs to $j(c_0(\kappa))^\perp = q^*(X^*)$, since for every $x \in c_0(\kappa)$, we have $\langle z^*_\alpha, j(x)\rangle = \langle e^*_\alpha, x\rangle \to 0$. On the other hand, it is clear that no z^*_α belongs to $j(c_0(\kappa))^\perp$. Hence consider $r > 0$ satisfying $\|q^*\|\|z^*_\alpha\| \leq r$ for all $\alpha < \kappa$, such that the subset

$$L = q^*(r \cdot B_{X^*}) \cup \{z^*_\alpha : \alpha < \kappa\}$$

endowed with the weak* topology of Z^*, becomes a κ-discrete extension of $q^*(r \cdot B_{X^*})$, which is homeomorphic to B_{X^*}. Then, we can identify L with a κ-discrete extension of

B_{X^*}, which we call again L. It only remains to check that the operator $u\colon Z \to Z(L)$ defined by $uz(x^*) = \langle q^*(x^*), z\rangle$ and $uz(z^*_\alpha) = \langle z^*_\alpha, z\rangle$ makes the following diagram commutative:

$$\begin{tikzcd}
0 \arrow[r] & c_0(\kappa) \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, "u"] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & c_0(\kappa) \arrow[r] & Z(L) \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

□

In the sequel, a twisted sum Z of $c_0(\kappa)$ and X obtained via the κ-discrete extension L, just as Theorem 3.2.1 indicates, will be denoted as $Z(L)$. In particular, Z is isomorphic to the pullback space $\{(f, x) \in C(L) \oplus_\infty X : f|_{B_{X^*}} = e(x)\}$, hence we can identify Z with the subspace of $C(L)$ consisting of the functions whose restriction to B_{X^*} lies in X, endowed with the subspace norm.

We now reformulate the standard criterion for splitting in the language of discrete extensions.

Definition. Let X be a Banach space. A discrete extension L of B_{X^*} is *realizable inside* X^* if the canonical embedding $(B_{X^*}, \text{weak}^*) \hookrightarrow (X^*, \text{weak}^*)$ can be extended to an embedding $L \hookrightarrow (X^*, \text{weak}^*)$.

Proposition 3.2.2. *The lower row of the diagram (3.a) is trivial if and only if L is realizable inside X^*.*

Proof. It is a direct consequence of the fact that the functors $\bigcirc^*\colon Ban_1 \rightsquigarrow CHaus$ and $\mathscr{C}(\cdot)\colon CHaus \rightsquigarrow Ban_1$ are adjoint –see Section 1.1–, plus the pullback splitting criterion 1.2.3. □

Although it is mildly simple, Theorem 3.2.1 already solves several problems concerning twisted sums of $c_0(I)$-spaces:

3.2.1 Twisted sums of $c_0(I)$-spaces are isomorphically Lindenstrauss

Let us recall that a Banach space is *isomorphically Lindenstrauss* if it can be renormed so that it is a Lindenstrauss space. At the beginning of Section 2.3 we provided examples of twisted sums of $\mathscr{C}$-spaces that are not isomorphically Lindenstrauss spaces. We now show that this situation cannot happen when the subspace is a $c_0(I)$-space.

Theorem 3.2.3. *If X is a Lindenstrauss space, then every twisted sum of $c_0(\kappa)$ and X is isomorphically Lindenstrauss.*

Proof. Consider Z a twisted sum of $c_0(\kappa)$ and X, and observe that, until we say so, no specific norm has been asigned to Z. To find the appropriate norm, let us invoke Theorem 3.2.1 to obtain some κ-discrete extension L of B_{X^*} such that $Z \simeq Z(L)$. The corresponding diagram (3.a), once completed, yields

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & & C(L)/Z(L) \arrow[u] \arrow[r, equal] & C(B_{X^*})/X \arrow[u] & \\
0 \arrow[r] & c_0(\kappa) \arrow[r] & C(L) \arrow[u] \arrow[r] & C(B_{X^*}) \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & c_0(\kappa) \arrow[u, equal] \arrow[r] & Z(L) \arrow[u, hook, "u"] \arrow[r] & X \arrow[u, hook, "e"] \arrow[r] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

and, in particular, $Z \simeq Z(L)$ now carries the subspace norm inherited from $C(L)$. We remark that, in this context, e^* satisfies that for every $\mu \in M(B_{X^*})$ and every $x \in X$,

$$\int_{B_{X^*}} e(x)\, d\mu = e(x)(e^*\mu)$$

which is reason enough to call $e^*\mu$ the *barycenter* of μ.

Now, the dual sequence of the middle row, namely,

$$0 \longrightarrow Z(L)^{\perp} \longrightarrow M(L) \xrightarrow{u^*} Z(L)^* \longrightarrow 0$$

asserts that every functional on $Z(L)$ is of the form $u^*(\mu)$ for some $\mu \in M(L)$, and its norm can be computed as

$$\|u^*(\mu)\| = \|\mu|_\kappa\| + \inf_{\nu \in Z(L)^{\perp}} \|(\mu - \nu)|_{B_X^*}\|$$

because $\nu|_\kappa = 0$ for every $\nu \in Z(L)^{\perp}$. We claim that the second term is actually equal to $\|e^*(\mu|_{B_{X^*}})\|$; in fact, the inequality $\|e^*(\mu|_{B_{X^*}})\| \le \|(\mu - \nu)|_{B_X^*}\|$ follows from the fact that e^* is a norm-one operator and $e^*(\nu|_{B_{X^*}}) = 0$ whenever $\nu \in Z(L)^{\perp}$. To see that the converse inequality is also true, we resort to the canonical embedding $\phi\colon B_{X^*} \longrightarrow M_1(L)$ defined by $\phi(x^*) = \delta_{x^*}$ whenever $x^* \in B_{X^*}$ and homogeneously extend it to the whole X^*. Then the choice $\nu = \mu - \phi e^*(\mu|_{B_{X^*}})$ yields $\|(\mu - \nu)|_{B_{X^*}}\| = \|e^*(\mu|_{B^*_X})\|$. As a consequence, $Z(L)^*$ is isometrically isomorphic to $\ell_1(\kappa) \oplus_1 X^*$, hence $Z(L)$ is a Lindenstrauss space provided X is. □

3.2.2 Twisted sums of $c_0(I)$-spaces are isomorphically polyhedral

A Banach space X is said to be *polyhedral* if the unit ball of every finite-dimensional subspace of X has finitely many extreme points. Polyhedrality is an isometric notion: for example, c_0 is polyhedral [48, §III] but c is not –see [51] for a short and beautiful proof–. In fact, no infinite-dimensional C-space can be polyhedral in the supremum norm, since every C-space contains an isometric copy of c. Therefore, we will use the term *isomorphically polyhedral* to refer to a Banach space admitting a renorming under which it is polyhedral.

We now state one of the most effective criteria for obtaining isomorphically polyhedral spaces, which can be found in [44]. Recall that, given a Banach space X, a subset B of its dual unit sphere S_{X^*} is called a *boundary* if for every $x \in X$ there is $x^* \in B$ such that $\langle x^*, x\rangle = \|x\|$.

Definition. A boundary B of a Banach space X is *σ-discrete* if it can be written as a countable union of relatively discrete subsets in the weak*-topology.

Proposition 3.2.4. *If a Banach space X can be renormed to have a σ-discrete boundary, then X is isomorphically polyhedral.*

Proof. This is essentially contained in [44, Theorem 11], where the authors show that $C(K)$ is isomorphically polyhedral whenever K is σ-discrete. To extend this result for arbitrary Banach spaces, note that if B is a σ-discrete boundary for X, then X is isometrically isomorphic to a subspace of $C(\alpha B)$, where αB is the one-point compactification of a σ-discrete set, hence σ-discrete. □

Concerning twisted sums, it is still open whether "to be isomorphically polyhedral" is a 3-space property. We provide a partial affirmative answer to such question, and in particular, we conclude that any twisted sum of $c_0(I)$-spaces is isometrically polyhedral, which is another related question repeatedly posed by Castillo and Papini [25, 26]:

Theorem 3.2.5. *Let X be a Banach space with a σ-discrete boundary. Then every twisted sum of $c_0(\kappa)$ and X can be renormed so that it has a σ-discrete boundary. In particular, it is isomorphically polyhedral.*

Proof. Consider $Z(L)$ a twisted sum of c_0 and X, once again obtained through diagram (3.a). In particular, $Z(L)$ is endowed with the subspace norm of $C(L)$. Let B be a σ-discrete boundary for X, and set

$$\widehat{B} = \{\delta_\alpha, -\delta_\alpha : \alpha < \kappa\} \cup \{\delta_\beta : \beta \in B\} \subseteq Z(L)^*$$

We claim that $\widehat{B}$ constitutes the desired σ-discrete boundary for $Z(L)$. It is indeed a boundary, since given $f \in Z(L)$, there is a point $t \in L$ such that $\|f\| = |f(t)|$. If $t \in \kappa$, then $\langle \pm\delta_t, f\rangle = \|f\|$ choosing the sign judiciously; otherwise, $\|f\| = \|f|_{B_{X^*}}\|$ and so there is $\beta \in B$ such that $\left\langle \delta_\beta, f\right\rangle = \left\langle \beta, f|_{B_{X^*}}\right\rangle = \|f|_{B_{X^*}}\| = \|f\|$. Finally, to check that $\widehat{B}$ is σ-discrete, we just need to observe that $\{\delta_\alpha, -\delta_\alpha : \alpha < \kappa\}$ is relatively discrete in the weak* topology of $Z(L)^*$ and that $\{\delta_\beta : \beta \in B\}$ is σ-discrete, for it is the image of B under the continuous embedding $\phi\colon B_{X^*} \hookrightarrow Z(L)^*$ given by $\phi(x^*) = \delta_{x^*}$. □

Corollary 3.2.6. *Every twisted sum of $c_0(\kappa)$ and $c_0(I)$ is isomorphically polyhedral.*

Knowledgeable people on polyhedral spaces also employ the so-called *Talagrand operators* to provide examples of such spaces. An operator $T\colon X \to c_0(S_{X^*} \times M)$ is Talagrand if for every $x \in X$ there is a pair $(f, m) \in S_{X^*} \times M$ such that $f(x) = \|x\|$ and $T(x, m)(f) \neq 0$. Every Banach space admitting a Talagrand operator is isomorphically polyhedral [44, Prop. 7]. It is, however, not true that every twisted sum of $c_0(I)$-spaces admits a Talagrand operator, despite being isomorphically polyhedral. The counterexample is given by a peculiar family of compact spaces known as *Ciesielski-Pol compacta* [30] –cf. also [34, Chapter VI, 8]– with the following property: for every such K, the inclusion $\iota\colon K' \hookrightarrow K$ induces a twisted sum

$$0 \longrightarrow c_0(\mathfrak{c}) \longrightarrow C(K) \xrightarrow{\iota^\circ} c_0(\mathfrak{c}) \longrightarrow 0$$

such that no injective operator from $C(K)$ into $c_0(J)$ can exist, regardless of the set J.

3.2.3 A dichotomy for twisted sums of $c_0(I)$-spaces

Theorem 3.2.7. *Every twisted sum of $c_0(\kappa)$ and $c_0(I)$ is either a subspace of $\ell_\infty(\kappa)$ or it is trivial on a complemented copy of $c_0(\kappa^+)$.*

Proof. Let $Z(L)$ be a twisted sum of $c_0(\kappa)$ and $c_0(I)$, again appearing in the context of diagram (3.a). The key is to look at the subspace

$$\kappa_\perp = \{f \in Z(L) : f(\alpha) = 0 \; \forall \alpha < \kappa\}$$

and appeal to the fact that the restriction of the quotient operator $q\colon Z(L) \to c_0(I)$ to $\kappa_\perp$ is an isomorphism. Now, if $\kappa_\perp$ is a subspace of $\ell_\infty(\kappa)$, then it possesses a norming subset $\{z^*_\beta : \beta < \kappa\}$, and therefore the set $\{\delta_\alpha : \alpha < \kappa\} \cup \{z^*_\beta : \beta < \kappa\}$ is a norming subset for $Z(L)$ of cardinality κ. On the other hand, if $\kappa_\perp$ is not a subspace of $\ell_\infty(\kappa)$, then $q(\kappa_\perp)$ is a subspace of $c_0(I)$ of density bigger than κ. Consequently, $q(\kappa_\perp)$ contains a copy X_0 of $c_0(\kappa^+)$ in which the quotient operator q is an isomorphism. Finally, since X_0 is necessarily complemented in $c_0(I)$ by virtue of Lemma 3.1.1, then $q^{-1}(X_0)$ is a copy of $c_0(I)$ complemented in $Z(L)$. □

Corollary 3.2.8. *Every twisted sum of c_0 and $c_0(I)$ is either a subspace of ℓ_∞ or is trivial on a copy of $c_0(\aleph_1)$.*

We finish with the following remark. It is actually very simple to construct a twisted sum of c_0 and $c_0(\mathfrak{c})$ which is not a subspace of ℓ_∞: pick one such twisted sum X which is a subspace of ℓ_∞ and then consider $X' = X \oplus c_0(\mathfrak{c})$:

$$\begin{tikzcd}
0 \arrow[r] & c_0 \arrow[r] \arrow[d, equal] & X \oplus c_0(\mathfrak{c}) \arrow[r] \arrow[d, equal] & c_0(\mathfrak{c}) \oplus c_0(\mathfrak{c}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & c_0 \arrow[r] & X' \arrow[r] & c_0(\mathfrak{c}) \arrow[r] & 0
\end{tikzcd}$$

However, this method seems to be essentially the only available, as far as we know. Indeed, the only "preexisting" twisted sums of c_0 and $c_0(\mathfrak{c})$ of which we are aware are spaces of the form $C(K_{\mathscr{A}})$. On the other hand, to construct a twisted sum of c_0 and $c_0(\mathfrak{c})$ of the PS-type from Section 2.3, it can be argued that the natural choice of Σ leads to PS, which is a subspace of ℓ_∞. In consequence, one might venture to think that every twisted sum X of $c_0(\kappa)$ and $c_0(I)$ can be decomposed as $X = X_1 \oplus c_0(I_1)$, where X_1 is a twisted sum of $c_0(\kappa)$ and $c_0(I_1)$ which is a subspace of $\ell_\infty(\kappa)$ and I_1 is an appropriate set of indices with $\kappa < |I_1| \le |I|$. This statement is, in principle, different from the one in Theorem 3.2.7, and we do not know if it is true.

Chapter 4

A counterexample to the complemented subspace problem

The problem of classifying the complemented subspaces of a certain class of Banach spaces has been on the table since the very foundations of Banach space theory. In the particular case of spaces of continuous functions, it was proposed that complemented subspaces of C-spaces are again C-spaces, and such problem became known as the Complemented Subspace Problem (for $C(K)$-spaces). Five powerful results suggested that the answer to the Complemented Subspace Problem could be affirmative:

- A subspace of $c_0(I)$ is complemented if and only if it is isomorphic to $c_0(J)$ for some set J; see Granero [50].
- Complemented subspaces of $C(\omega^\omega)$ are isomorphic either to c_0 or to $C(\omega^\omega)$, as Benyamini showed in [12, Theorem 3].
- Complemented subspaces of ℓ_∞ are isomorphic to ℓ_∞; this is the classical result of Lindenstrauss [70].
- Every complemented subspace of $C[0,1]$ with non-separable dual is isomorphic to $C[0,1]$; this is the very well-known theorem of Rosenthal [90], see also [91] for a survey-like exposition and some related problems.
- Every complemented subspace of $\ell_\infty^c(I)$ is either isomorphic to ℓ_∞ or to $\ell_\infty^c(J)$ for some uncountable subset J verifying $|J| \leq |I|$; this result has been recently obtained by Johnson, Kania and Schechtman [57]. Recall that $\ell_\infty^c(I)$ denotes the closed subspace of $\ell_\infty(I)$ consisting of functions with countable support.

In general, complemented subspaces of C-spaces have been extensively studied, especially in the separable case [10, 73, 85, 91]. The most general result we are aware of asserts that 1-complemented subspaces of separable C-spaces are always C-spaces. Such a result is mentioned in [91, §5], and it can be obtained as a consequence of [10, Lemma 5] and [73, Thm. 4]. The recent paper [88] shows that the situation is different concerning non-separable C-spaces. Indeed, it provides a 1-complemented subspace of a non-separable C-space which is not a C-space, thus solving the Complemented Subspace Problem in the negative after more than 50 years. In this chapter, we will develop the counterexample featuring [88] and use it to solve some other natural questions in the context of twisted sums of C-spaces which are closely related to the material exposed in the previous chapters.

4.1 An overall description

The counterexample witnessing that not every complemented subspace of a C-space is a C-space is the following:

Theorem 4.1.1. *There are two almost disjoint families $\mathscr{A}$ and $\mathscr{B}$, together with a continuous surjection $\pi\colon K_{\mathscr{B}} \longrightarrow K_{\mathscr{A}}$, so that*

i) $C(K_{\mathscr{B}}) \simeq \pi^\circ[C(K_{\mathscr{A}})] \oplus X$.

ii) *Both* $\pi^\circ[C(K_{\mathscr{A}})]$ *and* X *are* 1*-complemented subspaces of* $C(K_{\mathscr{B}})$.

iii) X *is not a* C*-space.*

The majority of the chapter is dedicated to the proof of Theorem 4.1.1. We now set the scene and describe the main ingredients, and leave all the technicalities for Section 4.2. Let us start by providing the framework for the construction. It will be more convenient to work in the set $\mathbb{N} \times 2$ rather than in $\mathbb{N}$. Given $A \subseteq \mathbb{N}$, we denote $\widehat{A}$ for the *cylinder* $A \times 2$, and in particular, C_n stands for the cylinder $\{(n, 0), (n, 1)\}$.

The families $\mathscr{A}$ and $\mathscr{B}$ appearing on Theorem 4.1.1 are of the form:

- $\mathscr{A} = \{A_\xi : \xi < \mathfrak{c}\} \subseteq \mathscr{P}(\mathbb{N})$,
- $\mathscr{B} = \{B^0_\xi, B^1_\xi : \xi < \mathfrak{c}\} \subseteq \mathscr{P}(\mathbb{N} \times 2)$,

and satisfy that for every $\xi < \mathfrak{c}$, $B^0_\xi \cup B^1_\xi = \widehat{A_\xi}$ and $B^0_\xi \cap B^1_\xi = \varnothing$. Let us now move to the Boolean setting and consider

- $\mathfrak{A}$ the Boolean algebra generated by $\mathscr{A}$ and all finite subsets of $\mathbb{N}$,
- $\mathfrak{B}$ the Boolean algebra generated by $\mathscr{B}$ and all finite subsets of $\mathbb{N}\times 2$.

It is clear that $\widehat{\mathfrak{A}} = \{\widehat{A} : A \in \mathfrak{A}\}$ is a Boolean subalgebra of $\mathfrak{B}$ isomorphic to $\mathfrak{A}$, so there is a canonical homomorphism $\mathfrak{A} \hookrightarrow \mathfrak{B}$ which, by Stone duality –see Section 1.4–, induces a quotient map $\pi\colon K_{\mathscr{B}} \to K_{\mathscr{A}}$. Therefore, we have an exact sequence

$$0 \longrightarrow C(K_{\mathscr{A}}) \xrightarrow{\pi^\circ} C(K_{\mathscr{B}}) \longrightarrow X \longrightarrow 0 \tag{4.a}$$

Of course, the proof deals with how to obtain such an exact sequence so that $\pi^\circ[C(K_{\mathscr{A}})]$ is complemented in $C(K_{\mathscr{B}})$ and its complement X is not a C-space. We now give an outline of the strategy.

Complementation of $C(K_{\mathscr{A}})$ inside $C(K_{\mathscr{B}})$.

It is not a difficult matter to produce almost disjoint families as above so that $\pi^\circ[C(K_{\mathscr{A}})]$ is a complemented subspace of $C(K_{\mathscr{B}})$. In fact, we can give a complete explanation of how to do it right now. The key lies in the following definition:

Definition. Given $A \subseteq \mathbb{N}$, a subset $B \subseteq \widehat{A}$ *splits* the cylinder $\widehat{A}$ if the sets $B \cap C_n$ are singletons whenever $n \in A$. In such a case, $\widehat{A} \setminus B$ also splits $\widehat{A}$, and so we say that $\widehat{A}$ is *split into* two sets B and $\widehat{A} \setminus B$.

Complementation Lemma 4.1.2. *If $\widehat{A_\xi}$ is split into B_ξ^0 and B_ξ^1 for every $\xi < \mathfrak{c}$, then both $C(K_{\mathscr{A}})$ and X are 1-complemented subspaces of $C(K_{\mathscr{B}})$.*

Proof. Let us first simplify the notation: we will write $p_\xi \in K_{\mathscr{A}}$ for the only ultrafilter on $\mathfrak{A}$ containing the set A_ξ and no finite subsets of $\mathbb{N}$. In such a case, $\pi^{-1}(p_\xi)$ consists of exactly two points $q_\xi^0, q_\xi^1 \in K_{\mathscr{B}}$ determined by the conditions of containing B_ξ^0 and B_ξ^1, respectively, and no finite subsets of $\mathbb{N}\times 2$. Also, if $p \in K_{\mathscr{A}}$ is the "point at infinity"; that is, the only ultrafilter on $K_{\mathscr{A}}$ not containing finite subsets of $\mathbb{N}$ nor elements of $\mathscr{A}$, then $\pi^{-1}(p) = q$, where q is the only ultrafilter on $\mathfrak{B}$ containing no finite subsets of $\mathbb{N}\times 2$ nor elements of $\mathscr{B}$. We also remark the fact that $\pi^{-1}(n) = C_n$.

With the above notations, let us consider the mapping

$$P\colon C(K_{\mathscr{B}}) \to C(K_{\mathscr{A}}) \quad , \quad \begin{cases} Pf(n) = \frac{1}{2}\big(f(n,0)+f(n,1)\big) \\ Pf(p_\xi) = \frac{1}{2}\big(f(q_\xi^0)+f(q_\xi^1)\big) \\ Pf(p) = f(q) \end{cases}$$

It is clear that P is a projection for π° once we check that it is well-defined; namely, that Pf is a continuous function on $K_{\mathscr{A}}$ for every $f \in C(K_{\mathscr{B}})$. For such a purpose, we remark that Pf is automatically continuous at points in $\mathbb{N}$, as well as at p. Moreover, continuity at points p_ξ for $\xi < \mathfrak{c}$ follow simply from the fact that $B_\xi^0 \cap C_n$ and $B_\xi^1 \cap C_n$ are one-point sets for every $n \in A_\xi$, since in such a case we can assure that $\big(Pf(n)\big)_{n\in A_\xi}$ converges to $Pf(p_\xi)$ for every $\xi < \mathfrak{c}$.

Finally, consider the map $Q\colon C(K_{\mathscr{B}}) \to C(K_{\mathscr{B}})$ defined by $Q = \mathrm{Id}_{C(K_{\mathscr{B}})} - \pi^\circ P$. Explicitely,

$$\begin{cases} Qf(n,0) = -Qf(n,1) = \frac{1}{2}\big(f(n,0) - f(n,1)\big) \\ Qf(q_\xi^0) = -Qf(q_\xi^1) = \frac{1}{2}\big(f(q_\xi^0) - f(q_\xi^1)\big) \\ Qf(q) = 0 \end{cases}$$

Since Q is a norm-one idempotent operator whose kernel is $\pi^\circ[C(K_{\mathscr{A}})]$, we conclude that $X \simeq Q[C(K_{\mathscr{B}})]$ is a 1-complemented subspace of $C(K_{\mathscr{B}})$. □

The complement of $C(K_{\mathscr{A}})$ is not a C-space

Providing an exact sequence (4.a) in which X is not a C-space requires much more effort, regardless of whether or not $C(K_{\mathscr{A}})$ is complemented in $C(K_{\mathscr{B}})$. Hence we will forget about the complementation part for the time being and examine what can be inferred from the very existence of an exact sequence (4.a). First, X^* is necessarily isomorphic to $C(K_{\mathscr{A}})^\perp$, and in particular every element in X^* can be regarded as a measure in $M(K_{\mathscr{B}})$. Let us recall that $M(K_{\mathscr{B}})$ is isomorphic to $\ell_1(\mathbb{N}\times 2) \oplus \ell_1(\mathscr{B})$, which allows to decompose every $\mu \in M(K_{\mathscr{B}})$ as

$$\mu = \mu^{\mathbb{N}\times 2} + \mu^{\mathscr{B}} \quad , \quad \mu^{\mathbb{N}\times 2} \in \ell_1(\mathbb{N}\times 2),\ \mu^{\mathscr{B}} \in \ell_1(\mathscr{B})$$

With these notations, it is now clear that every $\mu \in X^*$ satisfies $\mu(C_n) = \mu^{\mathbb{N}\times 2}(C_n) = 0$ for every $n \in \mathbb{N}$. On the other hand, every closed c-norming set for X, now considered as a subset of $M_1(K_{\mathscr{B}})$, must contain a sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ satisfying $|\langle \mu_n, 1_{(n,0)}\rangle| = |\mu_n^{\mathbb{N}\times 2}(n,0)| \geq c$ for all $n \in \mathbb{N}$. These facts suggest that c-norming free sets for X leave a “trace” in the form of some specific sequences in $\ell_1(\mathbb{N}\times 2)$:

Definition. We say that a bounded sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ in $\ell_1(\mathbb{N}\times 2)$ is *admissible* if

- $\mu_n(C_m) = 0$ for every $n, m \in \mathbb{N}$.
- $\inf_{n\in\mathbb{N}} |\mu_n(n,0)| > 0$.

Note that the previous definition deals only with measures on $\ell_1(\mathbb{N}\times 2)$, which do not depend on the concrete form that X will eventually take. The construction now follows the same idea as in Section 2.3: to prevent any sequence $(\psi_n)_{n\in\mathbb{N}}$ in X^* whose $\ell_1(\mathbb{N}\times 2)$-parts are admissible from lying inside a free set, and then conclude that X cannot be a C-space using Proposition 2.3.2. Let us make it official:

Definition. We say a Boolean algebra $\mathfrak{B}$ *blocks* a sequence of measures $(\psi_n)_{n\in\mathbb{N}}$ if there is no closed free subset of $M_1(\mathfrak{B})$ containing $(\psi_n)_{n\in\mathbb{N}}$.

The process of the proof can be roughly summarised as follows: first, we enumerate all sequences of measures $(\psi_n)_{n\in\mathbb{N}}$ whose ℓ_1-parts are admissible and carry out an inductive process of length $\mathfrak{c}$. At a certain step $\xi<\mathfrak{c}$, we have a certain sequence of measures $(\psi_n)_{n\in\mathbb{N}}$ which we want to block, and a Boolean algebra $\bigcup_{\alpha<\xi}\mathfrak{B}_\alpha$ which has been obtained as a result of blocking the admissible sequences of previous steps. Then we construct a bigger algebra $\mathfrak{B}_\xi$ so that it also blocks the sequence $(\psi_n)_{n\in\mathbb{N}}$. At the end, the final algebra $\mathfrak{B}=\bigcup_{\xi<\mathfrak{c}}\mathfrak{B}_\xi$ must block every such sequence $(\psi_n)_{n\in\mathbb{N}}$. We have already encountered with a similar argument in Section 2.3, where the conclusion that the twisted sum is not a C-space was obtained by producing suitable cluster points. Such an argument required dealing with subsequences, and therefore it was inevitable to work under $\mathfrak{p}=\mathfrak{c}$. However, more sophisticated techniques are required if one wants to carry out such a process in ZFC.

4.2 A detailed description

Once the basic ideas have been discussed, we now take a formal approach and give an exhaustive presentation of all the process leading to the proof of Theorem 4.1.1. First, let us remark a few properties of measures which will be used throughout the construction:

- If $\mathscr{B}$ is an almost disjoint family of subsets of $\mathbb{N}\times 2$ and $\mathfrak{B}$ is the Boolean algebra generated by $\mathscr{B}$ and all finite subsets of $\mathbb{N}\times 2$, every measure $\psi\in M(\mathfrak{B})$ can be decomposed as $\psi=\mu+\nu$, where $\mu\in\ell_1(\mathbb{N}\times 2)$ is the so-called ℓ_1*-part* and $\nu\in M(\mathfrak{B})$ vanishes on finite sets. Actually, the fact that $\mathscr{B}$ is almost disjoint allows us to regard ν as an element in $\ell_1(\mathfrak{B})$, and so $|\nu|(B)$ can be non-zero for countably many $B\in\mathscr{B}$.

- If we now have another almost disjoint family $\mathscr{B}_1$ containing $\mathscr{B}$, and $\mathfrak{B}_1$ is the corresponding algebra generated by $\mathscr{B}_1$ and all finite subsets of $\mathbb{N}\times 2$, then every measure $\psi \in M(\mathfrak{B})$ admits a *natural extension* to a measure $\psi_1 \in M(\mathfrak{B}_1)$. Precisely, if $\psi = \mu + \nu$ for $\mu \in \ell_1(\mathbb{N} \times 2)$ and $\nu \in \ell_1(\mathscr{B})$, then we consider $\psi_1 = \mu + \nu_1$, where $\nu_1 \in \ell_1(\mathscr{B}_1)$ is an extension of ν such that $\nu_1(B) = 0$ for every $B \in \mathscr{B}_1 \setminus \mathscr{B}$.

The basics on $\mathfrak{B}$-separation

Our first task should be to produce a way of blocking a certain sequence of measures in some Boolean algebra $\mathfrak{B}$, and for such a purpose, we will introduce the notion of $\mathfrak{B}$-separation of sets. Throughout the section, we fix a subalgebra $\mathfrak{B}$ of $\mathscr{P}(\mathbb{N} \times 2)$ containing all finite sets and such that $|\mathfrak{B}| < \mathfrak{c}$. We may think that such $\mathfrak{B}$ has been obtained at some step $\xi < \mathfrak{c}$ of the construction, as a result of blocking the admissible sequences of previous steps.

Definition. Let M and M' be subsets of $M_1(\mathfrak{B})$. We say that the pair (M, M') is $\mathfrak{B}$-separated if there exists $\varepsilon > 0$ and a finite collection $B_1, \dots, B_n \in \mathfrak{B}$ so that, whenever $\mu \in M$ and $\mu' \in M$, there is $k \in \{1, \dots, n\}$ satisfying $|\mu(B_k) - \mu'(B_k)| \geq \varepsilon$.

First, let us gather a few observations concerning $\mathfrak{B}$-separation:

Lemma 4.2.1. *If there is a simple $\mathfrak{B}$-measurable function $g\colon \mathbb{N} \times 2 \to \mathbb{R}$ so that $|\langle \mu, g\rangle - \langle \mu', g\rangle| \geq \varepsilon$ for some $\varepsilon > 0$, then the pair (M, M') is $\mathfrak{B}$-separated.*

Proof. Let us assume $g = \sum_{k=1}^n a_k 1_{B_k}$ for certain $a_k \in \mathbb{R}$ and $B_k \in \mathfrak{B}$. In such a case, given $(\mu, \mu') \in M \times M'$, there is $k \in \{1, \dots, n\}$ such that $|\mu(B_k) - \mu'(B_k)| \geq \varepsilon'$, where $\varepsilon' = \varepsilon / (\sum_{k=1}^n |a_k|)$. □

Lemma 4.2.2. *If M is an infinite set of measures on $\mathfrak{B}$, the pair $(M', M \setminus M')$ can be $\mathfrak{B}$-separated for less than $\mathfrak{c}$ many infinite subsets $M' \subseteq M$.*

Proof. Let M' be an infinite subset of M and assume that the pair $(M', M \setminus M')$ is $\mathfrak{B}$-separated by some $\varepsilon > 0$ and $B_1, \dots, B_n \in \mathfrak{B}$. Fix a rational number δ such that $0 < \delta < \varepsilon/2$ and for any $q \in [-1, 1]^n \cap \mathbb{Q}^n$ consider

$$M(q) = \{\mu \in M : |\mu(B_k) - q_k| < \delta \ \forall 1 \leq k \leq n\}$$

The hypothesis on $\mathfrak{B}$-separation implies that if $M' \cap M(q) \neq \varnothing$ for some q, then $(M \setminus M') \cap M(q) = \varnothing$. Appealing to the compactness of $[-1, 1]^n$ we deduce that M' can be covered by a finite union $\bigcup_{j=1}^m M(q^j)$, and therefore $(M \setminus M') \cap \bigcup_{j=1}^m M(q^j) = \varnothing$. In other words, M' and $M \setminus M'$ are "physically separated" by $\bigcup_{j=1}^m M(q^j)$. To finish the proof we only need to observe that there are $|\mathfrak{B}|$ many unions of the form $\bigcup_{j=1}^{m} M(q^j)$, and each of them cannot separate more than one pair $(M', M \setminus M')$. $\square$

The Main Lemma

Actually, the previous two assertions are enough to block a certain sequence of measures in $\mathscr{P}(\mathbb{N} \times 2)$ whose ℓ_1-parts are admissible. The process is explained in the upcoming Main Lemma 4.2.5. Before that, we describe two technical results that will be used in the proof of the Main Lemma. First, we record a suitable restatement of Rosenthal's lemma:

Lemma 4.2.3. (Rosenthal) *Let $(\mu_n)_{n \in \mathbb{N}}$ be a bounded sequence on $\ell_1(\mathbb{N} \times 2)$. For every $\varepsilon > 0$ and every infinite subset $E \subseteq \mathbb{N}$ there is an infinite subset $E' \subseteq E$ so that for every $n \in E'$ we have*

$$|\mu_n|\left(\bigcup_{k \in E' \setminus \{n\}} C_k\right) < \varepsilon$$

The second observation shows exactly how to block admissible sequences, thus preventing a norming set from becoming free:

Lemma 4.2.4. *Let $\mathfrak{B}$ be a Boolean algebra of subsets of $\mathbb{N} \times 2$. If $M \subseteq M_1(\mathfrak{B})$ is contained inside a free set, then for every $Z \in \mathfrak{B}$ and every $\varepsilon > 0$ there is a simple $\mathfrak{B}$-measurable function $h \colon \mathbb{N} \times 2 \to \mathbb{R}$ such that $\big|\langle \mu, h \rangle - |\mu(Z)|\big| \leq \varepsilon$ for every $\mu \in M$.*

Proof. Let us denote $K = \mathrm{ult}(\mathfrak{B})$. The assignment $\mu \mapsto |\mu(Z)|$ defines a continuous function in M (with the weak* topology inherited from $M_1(\mathfrak{B})$), hence by the definition of free set there is $f \in C(K)$ such that $\langle \mu, f \rangle = |\mu(Z)|$ for every $\mu \in M$. Since K is a Stone space, continuous $\mathfrak{B}$-simple functions are dense in $C(K)$, so there is such a function h satisfying $\|h - f\| < \varepsilon$. This function h can be readily identified with a simple $\mathfrak{B}$-measurable function clearly satisfying the desired inequality. $\square$

It is finally the time for the Main Lemma, so let us introduce the appropriate notation. Let $\mathfrak{B}$ be a subalgebra of $\mathscr{P}(\mathbb{N} \times 2)$ and Z an infinite subset of $\mathbb{N} \times 2$. The smallest subalgebra containing $\mathfrak{B}$ and Z will be denoted by $\mathfrak{B}[Z]$. Observe that every element in $\mathfrak{B}[Z]$ is of the form $(B_1 \cap Z) \cup (B_2 \cap Z^c)$ for suitable $B_1, B_2 \in \mathfrak{B}$. We also need to recover a definition from Section 2.3 –see just above Lemma 2.3.8–: $\mathfrak{B}$ is said to be *trivial* on Z whenever for every $B \in \mathfrak{B}$, either $B \cap Z$ or $B \cap Z^c$ is finite. Assuming that $\mathfrak{B}$ contains all finite subsets of $\mathbb{N} \times 2$, triviality of $\mathfrak{B}$ on Z implies that, for any $B \in \mathfrak{B}$, both sets $B \cap Z$ and $B^c \cap Z$ cannot be infinite; otherwise $Z^c = (B \cap Z^c) \cup (B^c \cap Z^c)$ would be finite and so $Z \in \mathfrak{B}$.

Finally, to ease notation, given a sequence $(\psi_n)_{n\in\mathbb{N}}$ and a set $J \subseteq \mathbb{N}$, we write $\psi[J]$ to mean $\{\psi_n : n \in J\}$.

Main Lemma 4.2.5. *Assume $(\mu_n)_{n\in\mathbb{N}}$ is an admissible sequence and E is an infinite subset of $\mathbb{N}$. There exist two infinite subsets $J_2 \subseteq J_1 \subseteq E$ and another infinite subset Z which splits $\widehat{J_1}$ so that whenever we are given*

- *a Boolean algebra $\mathfrak{B} \subseteq \mathscr{P}(\mathbb{N} \times 2)$ which contains all finite subsets of $\mathbb{N} \times 2$ and is trivial on $\widehat{E}$,*
- *a sequence of measures $(\nu_n)_{n\in\mathbb{N}}$ in $M(\mathfrak{B})[\widehat{E}]$ which vanishes on finite subsets of $\mathbb{N} \times 2$ and on $\widehat{E}$,*

the sequence $\psi_n = \mu_n + \nu_n$ verifies the following assertion: If $(\psi_n)_{n=1}^{\infty}$ lies inside a free set in $M_1(\mathfrak{B}[Z])$, then at least one of the following pairs is $\mathfrak{B}$-separated:

- $\big(\psi[J_2], \psi[J_1 \setminus J_2]\big)$,
- $\big(\psi[J_1], \psi[\mathbb{N} \setminus J_1]\big)$.

Proof. Let us call $c = \inf_n |\mu_n(n,0)|$, which is a positive number, and set δ such that $0 < \delta < \frac{c}{16}$. By virtue of Rosenthal's lemma 4.2.3, we can assume, shrinking E if necessary, that

$$|\mu_n|(\widehat{E} \setminus C_n) < \delta \quad \forall n \in E \tag{4.b}$$

Next, thanks to the admissibility of $(\mu_n)_{n\in\mathbb{N}}$, for each $n \in E$ there is a one point set $p_n \in C_n$ such that $\mu_n(p_n) \geq c$. Therefore, we can obtain an infinite subset $J_1 \subseteq E$ such that the sequence $\big(\mu_n(p_n)\big)_{n\in J_1}$ converges to some $a \geq c$. Eliminating a finite number of terms from J_1 we can further assume that

$$\big|\mu_n(p_n) - a\big| < \delta \quad \forall n \in J_1 \tag{4.c}$$

Choose any infinite set $J_2 \subseteq J_1$ such that $J_1 \setminus J_2$ is also infinite and define

$$Z = \Big(\bigcup_{n\in J_2} p_n\Big) \cup \Big(\bigcup_{n\in J_1\setminus J_2} C_n \setminus p_n\Big)$$

which clearly splits $\widehat{J_1}$.

For the rest of the proof, we will use the following notation: given $a, b \in \mathbb{R}$ and $\varepsilon > 0$, we will write $a \approx_\varepsilon b$ to mean $|a - b| < \varepsilon$. Let us recall that since the sequence $(\nu_n)_{n\in\mathbb{N}}$ is identically zero on $\widehat{E}$, we can replace ψ_n by μ_n when working inside $\widehat{E}$.

Now, suppose toward a contradiction that $(\psi_n)_{n\in J_1}$ lies inside a c-norming free set in $M_1(\mathfrak{B}[Z])$. Then, according to Lemma 4.2.4, there exists some simple $\mathfrak{B}[Z]$-measurable function h such that

$$\langle \mu_n, h\rangle \approx_\delta |\psi_n(Z)| = |\mu_n(Z)| \quad \forall n \in \mathbb{N}$$

Such a function h can be assumed to be of the form $h = r \cdot 1_Z + g$, where $r > 0$ and g is a simple $\mathfrak{B}$-measurable function. Indeed, since $\mathfrak{B}$ is trivial on Z, any set of the form $B \cap Z$, for $B \in \mathfrak{B}$, is either finite or can be written as $Z \setminus (B^c \cap Z)$, where the latter is necessarily finite, and the same occurs with sets of the form $B \cap Z^c$. Hence we can write $h = a \cdot 1_Z + b \cdot 1_{Z^c} + g$, where g is a linear combination of step functions on finite and cofinite sets, and so it is $\mathfrak{B}$-measurable. Finally, since $\mu_n(Z) = -\mu_n(Z^c)$ we can assume $h = r \cdot 1_Z + g$ (for $r = a - b$) when h acts on any μ_n. We will further suppose that $r > 0$ by switching to the function $\mu \mapsto -|\mu(Z)|$ if necessary. Under such circumstances, we have

$$|\mu_n(Z)| \approx_\delta r \cdot \mu_n(Z) + \langle \psi_n, g\rangle \quad \forall n \in \mathbb{N}$$

which, thanks to (4.b) and (4.c), yields

$$\left.\begin{array}{rlrl} a & \approx_{(3+2r)\delta} & r \cdot a + \langle \psi_n, g\rangle & \forall n \in J_2 \\ a & \approx_{(3+2r)\delta} & -r \cdot a + \langle \psi_k, g\rangle & \forall k \in J_1 \setminus J_2 \\ 0 & \approx_{(3+2r)\delta} & \langle \psi_l, g\rangle & \forall l \in \mathbb{N} \setminus J_1 \end{array}\right\}$$

The plan is to deduce that these equations violate the $\mathfrak{B}$-separation of the sets in assertion (i) by appealing to Lemma 4.2.1, and so we obtain the desired contradiction. Precisely, if r is small, say $0 < r \leq \frac{1}{2}$, the first two approximate equalities give

$$\langle \psi_n, g\rangle \geq \frac{a}{2} - 4\delta \quad \forall n \in J_1$$

while the third one claims that

$$\langle \psi_l, g\rangle \leq 4\delta \quad \forall l \in \mathbb{N} \setminus J_1$$

The choice of $\delta < \frac{c}{16}$ and the fact that $a \geq c$ are enough to conclude that the sets $\psi[J_1]$ and $\psi[\mathbb{N} \setminus J_1]$ are $\mathfrak{B}$-separated.

On the other hand, if $r \geq \frac{1}{2}$, then we infer from the first two relations above that whenever $n \in J_2$ and $k \in J_1 \setminus J_2$,

$$\langle \psi_n, g\rangle - \langle \psi_k, g\rangle \geq 2ra - 2(3+2r)\delta = 2r(a-2\delta) - 6\delta \geq a - 8\delta$$

so this time the sets $\psi[J_2]$ and $\psi[J_1 \setminus J_2]$ are $\mathfrak{B}$-separated. □

Let us remark that the set J_2 in the proof of the Main Lemma has been chosen freely; we only need the fact that both J_2 and $J_1 \setminus J_2$ is infinite.

The secret weapon of $\mathfrak{B}$-separation

Up to now, we have described a way to enlarge a certain algebra $\mathfrak{B}$ with a suitably chosen set Z so that $\mathfrak{B}[Z]$ blocks a certain sequence of measures fixed beforehand. However, there is a drawback in our approach: since now $\mathfrak{B}[Z]$ is bigger, previous sequences which were blocked by $\mathfrak{B}$ may no longer be blocked by $\mathfrak{B}[Z]$.

Fortunately, $\mathfrak{B}$-separation comes to our help again. There is a way of enlarging a certain collection of algebras while, at the same time, preserving non-separation of a certain family of subsets of measures. The following result builds on a clever lemma of Haydon [55, 1D] and may be useful in an inductive process of length $\mathfrak{c}$. Since the nature of the underlying countable set is irrelevant here, let us work with plain $\mathbb{N}$.

Separation Lemma 4.2.6. *Fix a cardinal number $\xi < \mathfrak{c}$. Suppose we are given:*

i) a list $\{\mathfrak{B}_\alpha : \alpha < \mathfrak{c}\}$ of Boolean subalgebras of $\mathscr{P}(\mathbb{N})$, all of which contain finite subsets of $\mathbb{N}$ and verify $|\mathfrak{B}_\alpha| < \mathfrak{c}$;

ii) a list $\{(M_\alpha, M'_\alpha) : \alpha < \xi\}$ of pairs of sets of measures inside $M_1(\mathscr{P}(\mathbb{N}))$ such that, for every $\alpha < \xi$, the pair (M_α, M'_α) is not $\mathfrak{B}_\alpha$-separated.

Then for every almost disjoint family $\mathscr{Z} \subset \mathscr{P}(\mathbb{N})$ of size $\mathfrak{c}$ there is $Z \in \mathscr{Z}$ such that, for every $\alpha < \xi$, the pair (M_α, M'_α) is not $\mathfrak{B}_\alpha[Z]$-separated.

Proof. The Separation Lemma comes easily as a consequence of the following claim:

Claim. *Fix two subsets $M, M' \subseteq M_1(\mathscr{P}(\mathbb{N}))$, and assume there are*

- *$\varepsilon > 0$ and $n, k \in \mathbb{N}$ satisfying $k > 12n/\varepsilon$;*
- *pairs of sets $A_1, B_1, \ldots, A_n, B_n \in \mathfrak{B}$;*
- *almost disjoint sets $Z_1, \ldots, Z_k \subset \mathbb{N}$.*

so that, for every $j \in \{1, ..., k\}$, the pair (M, M') is $\mathfrak{B}[Z_j]$-separated by ε and the sets

$$C_{i,j} = (A_i \cap Z_j) \cup (B_i \cap Z_j^c),\ 1 \le i \le n$$

Then the pair (M, M') is $\mathfrak{B}$-separated.

Let us provide a proof of the claim first. Consider a pair $(\mu, \mu') \in M \times M'$. By hypothesis, for every $j \in \{1, ..., k\}$ there is at least one $i \in \{1, ..., n\}$ so that $|\mu(C_{i,j}) - \mu(C_{i,j})| \ge \varepsilon$. The fact that $k > 12n/\varepsilon$ implies that there must exist some $i_0 \in \{1, ..., n\}$ so that the set

$$J = \{j \in \{1, ..., k\} : |\mu(C_{i_0,j}) - \mu'(C_{i_0,j})| \ge \varepsilon\}$$

verifies $|J| > 12/\varepsilon$. If we consider the finite set $F = \bigcup\{Z_j \cap Z_{j'} : j, j' \in J, j \ne j'\}$, then the sets $Z_j \setminus F$, $j \in J$ are pairwise disjoint. As a consequence, there exists a privileged $j_0 \in J$ with the property that

$$|\mu|(Z_j \setminus F) < \frac{\varepsilon}{6} \quad , \quad |\mu'|(Z_j \setminus F) < \frac{\varepsilon}{6}$$

Indeed, the set $J_1 = \{j \in J : |\mu|(Z_j \setminus F) \ge \frac{\varepsilon}{6}\}$ satisfies $|J_1| \le \frac{6}{\varepsilon} < \frac{1}{2}|J|$, because

$$\frac{\varepsilon}{6}|J_1| \le \sum_{j \in J_1} |\mu|(Z_j \setminus F) \le \|\mu\| \le 1$$

and the same reasoning works for μ'. Thus the existence of such $j_0 \in J$ follows.

With i_0 and j_0 in our power, we now pick C_{i_0,j_0} and simply denote it by C. Then, it is clear that $|\mu(C) - \mu'(C)| \ge \varepsilon$. Now let us slightly modify C to obtain a new set

$$D = (A_{i_0} \cap Z_{j_0} \cap F) \cup (B_{i_0} \cap (Z_{j_0} \cap F)^c) \tag{4.d}$$

which now satisfies $D \in \mathfrak{B}$, since F is finite, and it is easily checked that both $C \setminus D$ and $D \setminus C$ are contained in $Z_{j_0} \setminus F$. Therefore, from the definition of j_0 we infer that

$$|\mu(C) - \mu(D)| < \frac{\varepsilon}{3} \quad , \quad |\mu'(C) - \mu'(D)| < \frac{\varepsilon}{3}$$

and so $|\mu(D) - \mu'(D)| \ge \frac{\varepsilon}{3}$. In other words, the pair (μ, μ') is separated by D and $\varepsilon/3$. We conclude the proof of the claim by realizing that there are only finitely many sets D defined as in (4.d), and so the pair (M', M) is $\mathfrak{B}$-separated by all of them and the constant $\varepsilon/3$.

We now finish the proof of the Separation Lemma 4.2.6. Note that given any $\alpha < \kappa$, any positive rational number ε and any finite family $A_1, B_1, ..., A_n, B_n \in \mathfrak{B}$, the Claim asserts that the pair (M_α, M'_α) can be $\mathfrak{B}_\alpha[Z]$-separated using the collection of sets $\{A_i \cap Z \cup (B_i \cap Z^c),\ 1 \leq i \leq n\}$ and ε only for finitely many $Z \in \mathscr{Z}$. Therefore, there are at most $|\mathfrak{B}_\alpha|$ many sets $Z \in \mathscr{Z}$ for which the pair (M_α, M'_α) is $\mathfrak{B}_\alpha[Z]$-separated. But $|\bigcup_{\alpha<\xi} \mathfrak{B}_\alpha| < \mathfrak{c}$, and so the Separation Lemma 4.2.6 follows. □

End of the proof

With the Main Lemma 4.2.5 and the Separation Lemma 4.2.6 in our power, we can finally prove Theorem 4.1.1. Our aim is to obtain two almost disjoint families $\mathscr{A} = \{A_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathscr{P}(\mathbb{N})$ and $\mathscr{B} = \{B^0_\alpha, B^1_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathscr{P}(\mathbb{N} \times 2)$ in such a way that $\widehat{A_\alpha}$ is split into B^0_α and B^1_α for every $\alpha < \mathfrak{c}$. Then, we check that the sequence

$$0 \longrightarrow C(K_{\mathscr{A}}) \xrightarrow{\pi^\circ} C(K_{\mathscr{B}}) \longrightarrow X \longrightarrow 0$$

splits and X is not a C-space.

Given $\Lambda \subseteq \mathfrak{c}$, we will denote $\mathscr{B}(\Lambda)$ for the subalgebra of $\mathscr{P}(\mathbb{N} \times 2)$ generated by $\{B^0_\alpha, B^1_\alpha : \alpha \in \Lambda\}$ together with all finite subsets from $\mathbb{N} \times 2$. In this way, given $\xi < \mathfrak{c}$, $\mathfrak{B}(\xi)$ is the subalgebra generated by $\{B^0_\alpha, B^1_\alpha : \alpha < \xi\}$ together with all finite sets of $\mathbb{N} \times 2$, and $\mathfrak{B}(\mathfrak{c})$ becomes our final algebra, which we denote simply as $\mathfrak{B}$. We recall once again that, despite the nature of X and $\mathscr{B}$, any element in X^* can be regarded inside $M(\mathfrak{B})$ as a pair $(\mu, \nu) \in \ell_1(\mathbb{N} \times 2) \oplus \ell_1(\mathfrak{c} \times 2)$ so that $\mu(n, 0) = -\mu(n, 1)$ and $\nu(\xi, 0) = -\nu(\xi, 1)$ for every $n \in \mathbb{N}$ and $\xi < \mathfrak{c}$. This allows to code all the sequences which will eventually lie inside a norming set for X^* even before the construction begins. Indeed, consider the set $\mathscr{W}$ whose elements are

$$w = \big((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\big), \quad x_n \in \ell_1(\mathbb{N} \times 2),\ y_n \in \ell_1(\mathfrak{c} \times 2)$$

satisfying the following conditions:

- The sequence $(x_n)_{n\in\mathbb{N}}$ is admissible.
- For every $n \in \mathbb{N}$ and $\xi < \mathfrak{c}$, $y_n(\xi, 0) = -y_n(\xi, 1)$.
- For every $n \in \mathbb{N}$, $\|x_n\| + \|y_n\| \leq 1$.

Then $\mathscr{W}$ contains all the sequences that will be considered during the construction. We will refer to the elements of $\mathscr{W}$ as *codes*. In particular, since $\mathfrak{B} = \bigcup_{\alpha<\mathfrak{c}} \mathfrak{B}(\alpha)$, we say that $w = \big((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\big)$ *codes* a sequence of measures on some $M(\mathfrak{B}(\xi))$ provided $y_n(\alpha, i) = 0$ whenever $i \in 2$ and $\alpha \geq \xi$. Therefore, one can assume that such code w is a sequence of true measures on $M_1(\mathfrak{B}(\xi))$ for every $\xi \geq \alpha$, once such algebras are constructed.

We now enumerate $\mathscr{W} = \{w^\alpha : \alpha < \mathfrak{c}\}$ in such a way that for every $\xi < \mathfrak{c}$, the code $w^\xi = \big((x_n^\xi)_{n\in\mathbb{N}}, (y_n^\xi)_{n\in\mathbb{N}}\big)$ satisfies $y_n^\xi(\alpha, i) = 0$ for every $\alpha \geq \xi$ and $i \in 2$. This order assumption is just there to prevent us from coding a sequence of measures on an algebra before such algebra exists. We fix $\mathscr{R} = \{R_\xi : \xi < \mathfrak{c}\}$ an auxiliary almost disjoint family of subsets of $\mathbb{N}$. The role of R_ξ is to "make room" for step ξ; precisely, the set A_ξ will be constructed inside R_ξ. For every code ψ^ξ, we will provide the corresponding subset A_ξ with a splitting of $\widehat{A_\xi}$ into B_ξ^0 and B_ξ^1, as well as a book-keeping of two sets of indices $J_2^\xi \subseteq J_1^\xi$, all of which verify what we will call the

Inductive Assumptions 4.2.7. *If $(\psi_n)_{n\in\mathbb{N}}$ is a sequence of measures coded by z^ξ, then:*

i) The pairs of sets of measures

- $(\psi[J_2^\xi], \psi[J_1^\xi \setminus J_2^\xi])$
- $(\psi[J_1^\xi], \psi[\mathbb{N} \setminus J_1^\xi])$

are not $\mathfrak{B}(\alpha \setminus \{\xi\})$-separated for every $\alpha \geq \xi$.

ii) $(\psi_n)_{n\in\mathbb{N}}$ cannot lie inside a free set of $M_1(\mathfrak{B}(\xi+1))$.

We begin by declaring $\mathfrak{A}_0$ and $\mathfrak{B}_0$ the subalgebras generated by all finite subsets of $\mathbb{N}$ and $\mathbb{N} \times 2$, respectively. Suppose the construction has been carried out up to some step $\xi < \mathfrak{c}$, and let us describe how to perform step ξ.

First, we pick an almost disjoint family $\mathscr{E}$ of infinite subsets of R_ξ such that $|\mathscr{E}| = \mathfrak{c}$. Notice that, by construction, $\mathfrak{B}(\xi)$ is trivial on every $\widehat{E} \in \mathscr{E}$. Now, consider the sequence $(\psi_n)_{n\in\mathbb{N}}$ coded by w^ξ, and write $\psi_n = \mu_n + \nu_n$, where $(\mu_n)_{n\in\mathbb{N}}$ is an admissible sequence and $(\nu_n)_{n\in\mathbb{N}}$ can be seen as a sequence of measures in $M(\mathfrak{B}(\xi))$, since $\nu_n(\alpha, i) = 0$ whenever $\alpha > \xi$ and $i \in 2$. Observe that, for every $\widehat{E} \in \mathscr{E}$ the sequence of natural extensions of $(\nu_n)_{n\in\mathbb{N}}$ to $\mathfrak{B}(\xi)[\widehat{E}]$ is identically zero on $\widehat{E}$. This enables us to apply the Main Lemma 4.2.5 to $(\psi_n)_{n\in\mathbb{N}}$ $\mathfrak{c}$-many times, on every $E \in \mathscr{E}$. Let us denote $J_2(E) \subseteq J_1(E) \subseteq E$ and $Z(E) \subseteq \widehat{J_1(E)}$ the corresponding sets obtained by the application of the Main Lemma on $E \in \mathscr{E}$.

Next, we invoke Lemma 4.2.2 twice to obtain a subfamily $\mathscr{E}' \subseteq \mathscr{E}$ of cardinality $\mathfrak{c}$ such that for every $E \in \mathscr{E}'$, both pairs of sets $(\psi[J_2(E)], \psi[J_1(E) \setminus J_2(E)])$ and

$(\psi[J_1(E)], \psi[\mathbb{N} \setminus J_1(E)])$ are not $\mathfrak{B}(\xi)$-separated. Therefore, as a consequence of the Main Lemma 4.2.5, the sequence $(\psi_n)_{n\in\mathbb{N}}$ cannot lie inside a free subset in $M(\mathfrak{B}(\xi)[\widehat{E}])$ regardless of $E \in \mathscr{E}'$.

It is now the time to apply the Separation Lemma 4.2.6 to the family of algebras $\{\mathfrak{B}(\xi \setminus \{\alpha\}) : \alpha < \xi\}$ and the almost disjoint family $\{Z(E) : E \in \mathscr{E}'\}$. Hence we obtain a privileged $E \in \mathscr{E}'$ such that, for every $\alpha < \xi$, the sequence $(\psi_n^\alpha)_{n\in\mathbb{N}}$ coded by w^α satisfy that $(\psi[J_2^\alpha], \psi[J_1^\alpha \setminus J_2^\alpha])$ and $(\psi[J_1^\alpha], \psi[\mathbb{N} \setminus J_1^\alpha])$ are not $\mathfrak{B}(\xi \setminus \{\alpha\})[Z(E)]$-separated. This preserves the Inductive Assumptions 4.2.7 for every $\alpha < \xi$. We conclude step ξ by defining

- $A_\xi = J_1(E)$, $B_\xi^0 = Z(E)$, $B_\xi^1 = \widehat{A_\xi} \setminus B_\xi^0$.
- $J_2^\xi = J_2(E)$, $J_1^\xi = J_1(E)$.

We finally prove that the almost disjoint families $\mathscr{A}$ and $\mathscr{B}$ produced as a result of the previous inductive process satisfy the conditions of Theorem 4.1.1. Assertions (i) and (ii) related to the complementation of $C(K_\mathscr{A})$ in $C(K_\mathscr{B})$ are clearly satisfied by virtue of the Complementation Lemma 4.1.2, since every $\widehat{A_\xi}$ is split into B_ξ^0 and B_ξ^1. As for assertion (iii), which concerns the nature of the complement X, it is clear that every norming set for X, when regarded inside $M_1(K_\mathscr{B})$, must contain a sequence of measures

$$\psi_n = \mu_n + \nu_n \quad , \quad \mu_n \in \ell_1(\mathbb{N} \times 2), \ \nu_n \in \ell_1(\mathscr{B})$$

such that $(\mu_n)_{n\in\mathbb{N}}$ is admissible. Since there is a certain $\xi_0 < \mathfrak{c}$ such that $\nu_n(B_\alpha^i) = 0$ for every $\alpha \geq \xi_0$ and every $i \in 2$, the sequence $(\psi_n)_{n\in\mathbb{N}}$ is necessarily coded by some $w^\xi \in \mathscr{W}$ with $\xi_0 \leq \xi < \mathfrak{c}$. Hence, by virtue of the Inductive Assumptions 4.2.7, $(\psi_n)_{n\in\mathbb{N}}$ cannot lie inside a free set inside $M_1(\mathfrak{B}(\alpha))$ for any $\alpha > \xi$ and therefore it cannot lie inside a free set inside $M_1(\mathfrak{B})$ either. The conclusion is that no norming set for X can be free, and so Proposition 2.3.2 guarantees X is not a C-space.

4.3 Further applications

The space X constructed in the previous section which satisfies the theorem was baptised as PS_2 in [27], so we will use such notation from now on. Aside from the Complemented Subspace Problem, its existence has powerful implications on other questions related to twisted sums of C-spaces, as we now show.

Quotients of C-spaces

The exact sequence (4.a) witnesses the fact that the quotient space in an exact sequence of the form

$$0 \longrightarrow C(K) \xrightarrow{\pi^\circ} C(L) \longrightarrow \cdot \longrightarrow 0$$

does not need to be a C-space, even if the exact sequence splits. A similar construction can be derived from (4.a) with the peculiarity that the subspace is a plain c_0:

Proposition 4.3.1. *There is an almost disjoint family $\mathscr{B}$ and a quotient map $\pi\colon K_{\mathscr{B}} \to \alpha\mathbb{N}$ so that the quotient space in the exact sequence*

$$0 \longrightarrow c_0 \xrightarrow{\pi^\circ} C(K_{\mathscr{B}}) \longrightarrow W \longrightarrow 0 \tag{4.e}$$

is not a C-space.

Proof. Consider the almost disjoint family $\mathscr{B}$ obtained in Theorem 4.1.1, and let us denote by $\mathfrak{B}$ the algebra generated by subsets in $\mathscr{B}$ and finite sets in $\mathbb{N} \times 2$, as before. The subalgebra $\mathfrak{A}_0$ of $\mathfrak{B}$ generated by the sets $C_n = \{(n, 0), (n, 1)\}$ is isomorphic to the finite-cofinite algebra in $\mathscr{P}(\mathbb{N})$, and so $C(\mathrm{ult}(\mathfrak{A}_0))$ is isomorphic to c_0. Furthermore, we have a natural quotient map $\pi\colon K_{\mathscr{B}} \to \mathrm{ult}(\mathfrak{A}_0)$ which produces an exact sequence

$$0 \longrightarrow C(\mathrm{ult}(\mathfrak{A}_0)) \xrightarrow{\pi^\circ} C(K_{\mathscr{B}}) \longrightarrow W \longrightarrow 0$$

The space W remains not a C-space since the basic idea of the main construction still works. In fact, every functional on W can be identified with a measure on $K_{\mathscr{B}}$ vanishing on every C_n, and this allows to mimic the argument given in the proof of Theorem 4.1.1 to conclude there cannot be any norming free set in the dual unit ball of W. □

Note, however, that the exact sequence (4.e) is no longer trivial: since W contains complemented copies of c_0, the triviality of (4.e) is equivalent to the fact that $W \simeq C(K_{\mathscr{B}})$, which clearly is not the case. We thank Antonio Avilés for pointing out this fact. In order to find an instance of the mentioned complemented copy of c_0 inside W, consider any $\xi < \mathfrak{c}$ and recall that $\widehat{A_\xi}$ is split into B^0_ξ and B^1_ξ. This allows us to write $B^0_\xi = \{(n, k_0(n)) : n \in A_\xi\}$ and $B^1_\xi = \{(n, k_1(n)) : n \in A_\xi\}$, and so the subspace of W spanned by the functions

$$g_n = 1_{(n,k_0(n))} - 1_{(n,k_1(n))}, \quad n \in A_\xi$$

is isomorphically isomorphic to c_0 and complemented in W, with the projection being just $P(f) = \sum_{n\in\mathbb{N}} \big(f(n, k_0(n)) - f(q^0_\xi)\big) \cdot g_n$.

Let's twist... again?

The space PS_2 is actually the middle space of a non-trivial exact sequence

$$0 \longrightarrow c_0 \longrightarrow \mathsf{PS}_2 \longrightarrow c_0(\mathfrak{c}) \longrightarrow 0$$

and therefore our Theorem 2.3.1 also holds in ZFC for $K = \alpha\mathfrak{c}$. We thank Gonzalo Martínez-Cervantes for kindly suggesting this fact.

To show the existence of the mentioned exact sequence, we need to recall a few facts from the construction. Let us denote $Q\colon C(K_{\mathscr{B}}) \to \mathsf{PS}_2$ the quotient map defined in Lemma 4.1.2. If X_0 stands for the canonical copy of c_0 inside $C(K_{\mathscr{B}})$; namely, the subspace of $C(K_{\mathscr{B}})$ consisting of all functions supported in $\mathbb{N}\times 2$, then it is easy to check that $Q(X_0)$ is isometrically isomorphic to c_0, since it is the closed span of the sequence

$$f_n = Q(1_{(n,0)}) = -Q(1_{(n,1)}) = \tfrac{1}{2}\big(1_{(n,0)} - 1_{(n,1)}\big), \quad n \in \mathbb{N}$$

We now describe the quotient $\mathsf{PS}_2/Q(X_0)$. For such a purpose, given $B \in \mathscr{B}$, consider the characteristic function h_B of the set $\{(n,k) : (n,k) \in B\} \cup \{q_B\} \subseteq K_{\mathscr{B}}$, define

$$f_\xi = Q(h_{B_\xi^0}) = -Q(h_{B_\xi^1}) = \tfrac{1}{2}\big(h_{B_\xi^0} - h_{B_\xi^1}\big)$$

and write $\bar{f}_\xi$ for the image of f_ξ under the quotient map $\mathsf{PS}_2 \to \mathsf{PS}_2/Q(X_0)$. Then any element $\bar{f} \in \mathsf{PS}_2/Q(X_0)$ lies in the closed span of $\{\bar{f}_\xi : \xi < \mathfrak{c}\}$, and for every choice of $N \in \mathbb{N}$, $\xi_1, \dots, \xi_n < \mathfrak{c}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, we have

$$\left\| \sum_{j=1}^{N} \lambda_j \bar{f}_{\xi_j} \right\| = \frac{1}{2} \max_{1\le j\le N} |\lambda_j|$$

thanks to the fact that $\mathscr{B}$ is an almost disjoint family. This proves that $\mathsf{PS}_2/Q(X_0)$ is isometrically isomorphic to $c_0(\mathfrak{c})$.

In light of this result, it may be the case that the construction of twisted sums which are not C-spaces featuring Section 2.3 can be transported to ZFC using the language of $\mathfrak{B}$-separation. In any case, it is not known if PS_2 is isomorphic to PS, or if the latter is complemented in some C-space.

Finally, we mention that PS_2 is a Lindenstrauss space, which follows either from its nature as a twisted sum of c_0 and $c_0(\mathfrak{c})$ –recall Theorem 3.2.3– or simply from the fact that it is 1-complemented in a C-space. Hence, it is still open whether every $\mathscr{L}_\infty$-space which is a quotient of a C-space is necessarily a Lindenstrauss space. Recall that a positive partial answer to this question is known: the classical theorem from Johnson and Zippin [59] states that every separable Lindenstrauss space is a quotient of $C[0,1]$.

Chapter 5

Non-locally trivial twisted sums with C-spaces

So far, we have encountered twisted sums of C-spaces of all sorts. But, regardless of their nature, they all share a common feature: they are *locally trivial*. This essentially means that the dual of any sequence

$$0 \longrightarrow C(L) \longrightarrow Z \longrightarrow C(K) \longrightarrow 0$$

must split –the proper definition will be given later in Section 5.1. In view of this fact, we now turn our attention to exact sequences

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow C(K) \longrightarrow 0 \tag{5.a}$$

which are not locally trivial. This inevitably means Y cannot be a C-space, and so we need to open the door to twisted sums of other Banach spaces and C-spaces. Additionally, this chapter exploits the quasi-linear approach to exact sequences, a technique which has been mostly absent throughout the dissertation.

The behaviour of non-locally trivial exact sequences (5.a) is studied in Section 5.2, where we show that a quasi-linear map acting from a C-space is either locally trivial or it can be restricted to a certain copy of c_0 in which it is non-locally trivial. Such a result constitutes the core of the paper [19].

On the other hand, concrete examples of non-locally trivial exact sequences (5.a) appear in Section 5.3, which is based on the paper [21]. The construction of such examples stems from the idea that if G is a compact topological group, then $C(G)$, as well as the spaces $L_p(G)$ for $1 \leq p \leq \infty$, are modules over the convolution algebra

$L_1(G)$. Therefore, it seems natural to examine whether twisted sums of those spaces can also inherit an L_1-module structure. In particular, we will produce a twisted sum of $L_1(G)$ and $C(G)$ which is an L_1-module by means of some Fourier analysis techniques –see Theorem 5.3.5.

During the whole chapter, twisted sums of ℓ_1 and c_0 are lurking in the shadows. This is no coincidence, for the original motivation for most of the material in this chapter was to produce a explicit twisted sum of ℓ_1 and c_0. Sections 5.2.1 and 5.3.3 describe the new facts and the remaining mysteries surrounding twisted sums of ℓ_1 and c_0.

5.1 Generalities on local triviality

Definition. An exact sequence $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ is said to be *locally trivial*, or that it *locally splits*, if $j(Y)$ is *locally complemented* in Z; with the meaning that there exists $\lambda \geq 1$ such that for every finite-dimensional space $E \subseteq Z$, there is a projection $p_E\colon E \to Y$ such that $p_E j = \mathrm{Id}_Y$ and $\|p_E\| \leq \lambda$.

We mention two classical facts about local complementation. First, every Banach space is locally complemented in its bidual; this is the so-called Principle of Local Reflexivity [72, Th. 3.1] –cf. also [95]. Second, $\mathscr{L}_\infty$-spaces are locally complemented in every superspace [72, §4]. Therefore, exact sequences in which the subspace is a C-space are always locally trivial. For the sake of completeness, let us detail the relation between triviality and local triviality:

Theorem 5.1.1. *[61, Th 3.5] Consider an exact sequence*

$$0 \longrightarrow Y \xrightarrow{j} Z \xrightarrow{q} X \longrightarrow 0 \qquad [\mathtt{z}]$$

i) If $[\mathtt{z}]$ *is trivial, then it is locally trivial.*

ii) If $[\mathtt{z}]$ *is locally trivial and* Y *is complemented in its bidual, then* $[\mathtt{z}]$ *is trivial.*

iii) Moreover, if $[\mathtt{z}]$ *is an exact sequence of* Banach *spaces, then it locally splits if and only if its dual sequence splits:*

$$0 \longrightarrow X^* \xrightarrow{q^*} Z^* \xrightarrow{j^*} Y^* \longrightarrow 0 \qquad [\mathtt{z}^*]$$

Proof. Assertion (i) is obvious. Let us concern ourselves with (ii). By hypothesis there is some $\lambda \geq 1$ satisfying that for every finite-dimensional subspace $E \subseteq Z$, there is an operator $p_E\colon E \to Y$ such that $\|p_E\| \le \lambda$ and jp_E agrees with the identity on $E \cap j(Y)$. Consider $\mathscr{E}$ the directed set consisting of all finite subspaces of Z, ordered by inclusion, and let $\mathfrak{U}$ be any ultrafilter on $\mathscr{E}$ refining the order filter. We define an operator $T\colon Z \to Y^{**}$ by the formula

$$\langle T(x), y^*\rangle = \text{weak}^*\text{-}\lim_{\mathfrak{U}} \langle y^*, p_E(x)\rangle$$

The definition makes sense, since for every $x \in X$ the values $\{p_E(x) : E \in \mathscr{E}\}$ are defined in a cofinal set of $\mathscr{E}$ and they are bounded. To finish, observe that Tj agrees with the canonical inclusion $i_Y\colon Y \hookrightarrow Y^{**}$, and therefore the composition of T with a projection for i_Y yiels the desired projection for j.

We now turn to (iii). If $[\mathsf{z}]$ is locally trivial, then there is $\lambda \ge 1$ satisfying that for every finite dimensional subspace $E \subseteq Z$, there is a projection $p_E\colon E \to Y$ such that $\|p_E\| \le \lambda$ and jp_E agrees with the identity on $E \cap j(Y)$. Therefore we consider $S\colon Y^* \to Z^*$ defined as

$$S(y^*) = \text{weak}^*\text{-}\lim_{\mathfrak{U}} y^* P_E$$

and proceed exactly as in (ii) to show that S is a lifting for j^*. As for the converse, note that if $[\mathsf{z}^*]$ splits, so does $[\mathsf{z}^{**}]$, which implies Y^{**} is complemented in Z^{**}. Therefore, we have the diagram

$$\begin{tikzcd} Y \arrow[r, hook, "j"] \arrow[d, hook, "i_Y"] & Z \arrow[d, hook, "i_Z"] \\ Y^{**} \arrow[r, hook, "j^{**}"] & Z^{**} \arrow[l, bend left, "P"] \end{tikzcd}$$

Now, an appeal to the Principle of Local Reflexivity ensures that Y is locally complemented in Z, because for every finite dimensional subspace $E \subseteq Z$, we can produce a "local projection" $p_E\colon Pi_Z(E) \to Y$ for i_Y in such a way that the norms of p_E are uniformly bounded. □

It will be convenient to rephrase the notion of local splitting of an exact sequence in quasi-linear terms. For such a purpose, let us remark that, if $B\colon X \to Y$ is a homogeneous bounded map acting between Banach spaces, we will use the notation $\|B\|$ with the usual meaning: $\|B\| = \sup_{\|x\|=1} \|B(x)\|$.

Definition. We say a quasi-linear map $\Omega\colon X \to Y$ is *locally trivial* if there is a constant $M \ge 0$ such that for every finite-dimensional subspace $F \subseteq X$ there is a linear map $L_F\colon F \to Y$ such that $\|\Omega(x) - L_F(x)\| \le M\|x\|$ whenever $x \in F$.

In other words, locally trivial maps are *uniformly trivial* on finite-dimensional subspaces. As one could expect, exact sequences that locally split correspond to locally trivial quasi-linear maps:

Proposition 5.1.2. *A quasi-linear map $\Omega\colon X \to Y$ is locally trivial if and only if the induced exact sequence*

$$0 \longrightarrow Y \xrightarrow{\ j\ } Y \oplus_\Omega X \xrightarrow{\ q\ } X \longrightarrow 0$$

is locally trivial.

Proof. Let $E \oplus F$ be a finite-dimensional subspace of $Y \oplus_\Omega X$. It is straightforward to check that $p_{E\oplus F}\colon E \oplus F \to Y$ is a linear projection precisely when it is of form $p_{E\oplus F}(y,x) = y - L_F(x)$, where $L_F\colon X \to Y$ is a linear map. The next equivalence, valid for every $\lambda \geq 1$, is sufficient to prove the proposition:

$$\|p_{E\oplus F}\| \leq \lambda \iff \|\Omega|_F - L_F\| \leq \lambda$$

Therefore, we now prove such equivalence. If the norm of $p_{E\oplus F}$ is bounded by some $\lambda \geq 1$, then given $x \in F$

$$\|\Omega(x) - L_F(x)\| = \|p_{E\oplus F}(\Omega(x), x)\| \leq \lambda\|(\Omega(x), x)\|_\Omega = \lambda\|x\|$$

Conversely, if $\|\Omega(x) - L_F(x)\| \leq \lambda\|x\|$ for $x \in F$, then given any $y \in E$ we have

$$\|p_{E\oplus F}(y,x)\| = \|y - L_F(x)\| \leq \|y - \Omega(x)\| + \|\Omega(x) - L_F(x)\| \leq \lambda\|(y,x)\|_\Omega$$

and so $p_{E\oplus F}$ is bounded. □

5.2 A dichotomy for quasi-linear maps on C-spaces

We now delve into the topic of non-locally trivial twisted sums with C-spaces. The aim of this section is to prove the following theorem:

Theorem 5.2.1. *Every quasi-linear map $\Omega\colon C(K) \to Y$ is either locally trivial or there is a copy of c_0 in $C(K)$ on which the restriction is not locally trivial.*

Quantifying the triviality of quasi-linear maps will be essential in the proof of the theorem above. Therefore, we will speak of a λ-*trivial* quasi-linear map in case there exists a linear map $L\colon X \to Y$ such that $\|\Omega - L\| \leq \lambda$. By the same token, one can also speak of λ-locally trivial quasi-linear maps. The following technical lemma puts such considerations into practice.

Lemma 5.2.2. *Let Ω be a quasi-linear map on X with quasi-linearity constant Q. Assume Ω is λ-trivial on some subspace $[x_1, ..., x_m] \subseteq X$. Then there exists $\delta > 0$ such that, whenever $y_1, ..., y_m$ are norm one points in X satisfying $\|x_i - y_i\| \le \delta$ for all $i \in \{1, ..., m\}$, Ω is $(\lambda + Q + 1)$-trivial on $[y_1, ..., y_m]$.*

Proof. Assume that $\|x_i - y_i\| \le \delta$ for $i \in \{1, ..., m\}$. Let $\eta > 0$ denote the Banach-Mazur distance between $[x_1, ..., x_m]$ and ℓ_1^m, and observe that, in such a case, the triangle inequality yields

$$\sum_i |a_i| \le \tfrac{\eta}{1-\delta\eta} \|\sum_i a_i y_i\|$$

By hypothesis, there exists some linear map L on $[x_1, ..., x_m]$ such that $\|\Omega(x) - L(x)\| \le \lambda\|x\|$ for every $x \in [x_1, ..., x_m]$. We define a new linear map L' on $[y_1, ..., y_m]$ by the formula

$$L'(y_i) = L(x_i) + \Omega(x_i - y_i)$$

Then:

$$\begin{aligned}
\|\Omega(\textstyle\sum_i a_i y_i) \; - L'(\textstyle\sum_i a_i y_i)\| \le&\\
&\le \|\Omega(\textstyle\sum_i a_i y_i) - L(\textstyle\sum_i a_i x_i) - \textstyle\sum_i a_i \, \Omega(x_i - y_i)\| \le\\
&\le \|\Omega(\textstyle\sum_i a_i y_i) - \Omega(\textstyle\sum_i a_i x_i) - \textstyle\sum_i a_i \Omega(x_i - y_i)\| +\\
&\quad + \|\Omega(\textstyle\sum_i a_i x_i) - \textstyle\sum_i L(a_i x_i)\| \le\\
&\le \|\Omega(\textstyle\sum_i a_i y_i) - \Omega(\textstyle\sum_i a_i x_i) - \Omega(\textstyle\sum_i a_i(x_i - y_i))\| +\\
&\quad + \|\Omega(\textstyle\sum_i a_i(x_i - y_i)) - \textstyle\sum_i a_i \, \Omega(x_i - y_i)\| + \lambda \, \|\textstyle\sum_i a_i x_i\| \le\\
&\le Q\big(\|\textstyle\sum_i a_i y_i\| + \|\textstyle\sum_i a_i(x_i - y_i)\|\big) + Q(m-1)\textstyle\sum_i |a_i|\|x_i - y_i\| +\\
&\quad + \lambda\big(\|\textstyle\sum_i a_i y_i\| + \|\textstyle\sum_i a_i(x_i - y_i)\|\big) \le\\
&\le (Q + \lambda)\|\textstyle\sum_i a_i y_i\| + (mQ + \lambda)\textstyle\sum_i |a_i|\|x_i - y_i\|\\
&\le (Q + \lambda)\|\textstyle\sum_i a_i y_i\| + (mQ + \lambda)\delta\textstyle\sum_i |a_i| \le\\
&\le \big(Q + \lambda + (mQ + \lambda)\tfrac{\delta\eta}{1-\delta\eta}\big)\|\textstyle\sum_i a_i y_i\|
\end{aligned}$$

and so we need to choose δ such that $(mQ + \lambda)\frac{\delta\eta}{1-\delta\eta} \le 1$. □

Proof of Theorem 5.2.1. First, let us observe that it is enough to deal with separable C-spaces. This is because if $\Omega\colon C(K) \to Y$ is not locally trivial, then for every $n \in \mathbb{N}$ there is a finite-dimensional subspace $F_n \subseteq C(K)$ on which Ω is not n-trivial. Now the closed subalgebra $H \subseteq C(K)$ spanned by $\bigcup_{n=1}^{\infty} F_n$ is a separable C-space on which Ω

cannot be locally trivial. Then, it clearly suffices to find a copy of c_0 inside H in which $\Omega|_H$ is not locally trivial. This amounts to proving Theorem 5.2.1 in the following two cases: when K is the Cantor space, and when K is a countable ordinal.

We will first do the proof on the Cantor space, which we realize as $\Delta = \{\pm 1\}^{\mathbb{N}}$. Given $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$, we denote

$$\Delta_{\varepsilon_1,\ldots,\varepsilon_n} = \{t \in \Delta : t(k) = \varepsilon_k,\ k \in \{1, \ldots, n\}\}$$

and consider $C(\Delta_{\varepsilon_1,\ldots,\varepsilon_n})$ as a subspace of $C(\Delta)$ via the identification

$$C(\Delta_{\varepsilon_1,\ldots,\varepsilon_n}) = \{f \in C(\Delta) : \operatorname{supp} f \subseteq \Delta_{\varepsilon_1,\ldots,\varepsilon_n}\}$$

Furthermore, we denote $\Omega_{\varepsilon_1,\ldots,\varepsilon_n}$ the restriction of Ω to $C(\Delta_{\varepsilon_1,\ldots,\varepsilon_n})$. Let us split $\Delta = \Delta_+ \cup \Delta_-$. The restrictions Ω_+ and Ω_- cannot be both locally trivial, in view of the following claim:

Claim 1. *Let $\Omega\colon X \to Y$ be a quasi-linear map and $X = X_1 \oplus X_2$. If $\Omega|_{X_1}$ and $\Omega|_{X_2}$ are both locally trivial, then so is Ω.*

Proof of Claim 1. By hypothesis, there is $M > 0$ such that whenever $E_1 \subseteq X_1$ and $E_2 \subseteq X_2$ are finite-dimensional spaces, there exist linear maps $L_{E_1}\colon E_1 \to Y$ and $L_{E_2}\colon E_2 \to Y$ such that $\|\Omega|_{E_i} - L_{E_i}\| \leq \lambda$ for $j = 1, 2$. Therefore, given any finite-dimensional space $E \subseteq X$, let us write $E = E_1 \oplus E_2$ and consider the linear map $L_E\colon E \to Y$ given by

$$L_E(x_1, x_2) = L_{E_1}(x_1) + L_{E_2}(x_2)$$

where $x = (x_1, x_2) \in E$. Then, if $x \in E$, we have

$$\begin{aligned}\|\Omega(x) - L_E(x)\| &\leq \|\Omega(x_1, x_2) - \Omega(x_1, 0) - \Omega(0, x_2)\| + \\ &+ \|\Omega(x_1, 0) - L_{E_1}(x_1, 0)\| + \|\Omega(0, x_2) - L_{E_2}(0, x_2)\| \leq \\ &\leq (Q + M)(\|x_1\| + \|x_2\|) \leq 2(Q + M)\|x\|\end{aligned}$$

■

Therefore we can assume that Ω_+ is not locally trivial. Now we iterate the argument: split $\Delta_+ = \Delta_{+-} \cup \Delta_{++}$ and note that at least one of the restrictions Ω_{++} and Ω_{+-} is not locally trivial. Proceeding in such a way we obtain some element $t \in \Delta$ such that, for every $n \geq 1$, $\Omega_{t(1),\ldots,t(n)}$ is not locally trivial, but $\Omega_{t(1),\ldots,-t(n)}$ may be. In any case, for any natural number $n \in \mathbb{N}$, let us pick $\lambda_n \in \mathbb{R}$ according to the following:

- If $\Omega_{t(1),\ldots,t(n-1),-t(n)}$ is locally trivial, then choose λ_n such that $\Omega_{t(1),\ldots,-t(n)}$ is λ_n-locally trivial but not $(\lambda_n - 1)$-locally trivial.

- If $\Omega_{t(1),\dots,t(n-1),-t(n)}$ is not locally trivial, simply let $\lambda_n = n$.

Claim 2. *If $(\lambda_n)_{n\in\mathbb{N}}$ is bounded above, then Ω is locally trivial.*

Proof of Claim 2. Suppose toward a contradiction that $(\lambda_n)_{n\in\mathbb{N}}$ is bounded above and call $\lambda = \sup_{n\in\mathbb{N}} \lambda_n$. Given any finite collection of functions $f_1, \dots, f_m \in C(\Delta)$, pick $\delta > 0$ from Lemma 5.2.2 and $N \in \mathbb{N}$ such that $|f_i(s) - f_i(t)| < \delta$ whenever $s \in \Delta_{t(1),\dots,t(N)}$. Then each function f_i is at distance at most δ of the function

$$\widehat{f_i} = f_i|_A + f_i(\xi) \cdot 1_{A^c}$$

where $A = \Delta_{-t(1)} \cup \Delta_{t(1),-t(2)} \cup \cdots \Delta_{t(1),\dots,t(n-1),-t(N)}$. Now, Ω is λ-locally trivial on $C(A)$ and 1-trivial on $[1_{A^c}]$, so by Claim 1 it is λ'-trivial (with $\lambda' = \lambda + 1 + 2Q$) on the finite-dimensional subspace spanned by $\widehat{f_1}, \dots, \widehat{f_m}$ and 1_{A^c}. Consequently, Lemma 5.2.2 assures Ω must be $(\lambda' + Q + 1)$-trivial on $[f_1, \dots, f_m]$. ■

We are almost done. For each $n \in \mathbb{N}$, choose a finite-dimensional subspace of $C(\Delta_{t(1),\dots,t(n-1),-t(n)})$ in which Ω is not $(\lambda_n - 1)$-trivial, which in turn must be contained in a subspace of the form $\ell_\infty^{k_n}$ inside $C(\Delta_{t(1),\dots,t(n-1),-t(n)})$ in which Ω is again not $(\lambda_n - 1)$-trivial. Since the clopen sets $\Delta_{t(1),\dots,t(n-1),-t(n)}$ are disjoint, the closed span of $\bigcup_{n=1}^{\infty} \ell_\infty^{k_n}$ is a copy of c_0 inside $C(\Delta)$ in which Ω cannot be locally trivial. This finishes the proof for $C(\Delta)$.

The proof when K is a countable ordinal, say $K = [0, \alpha]$, is almost identical, so let us just sketch the argument. Without loss of generality it can be assumed that α is a limit ordinal, and so we work on the hyperplane $C_0(\alpha)$ of functions in $C[0, \alpha]$ vanishing at α. Now we write $[0, \alpha]$ as a union of two clopen sets A_+ and A_- such that $A_+ \cap A_- \subseteq \{\alpha\}$ and note that at least one of the restrictions of Ω to $C(A_+)$ and $C(A_-)$ is non-locally trivial thanks to Claim 1. Iterating the argument, a copy of c_0 in which Ω is non-locally trivial can be obtained just as we did above. □

The dichotomy we have just obtained can be restated as: every quasi-linear map $\Omega\colon C(K) \to Y$ is locally trivial precisely when *every* restriction to any copy of c_0 is locally trivial. Actually, such result is the only possible one, in the sense we now make precise. First, the word 'locally' is absolutely necessary: it is not true that a quasi-linear map $\Omega\colon C(K) \to Y$ must be trivial when every restriction to c_0 is trivial. In the previous chapters we have encountered a great deal of non-trivial sequences of the form

$$0 \longrightarrow c_0 \longrightarrow \cdot \longrightarrow C(K) \longrightarrow 0$$

all of which serves as counterexamples since, by Sobczyk's theorem, every restriction to any separable subspace of $C(K)$ must be trivial.

It is also false that a locally trivial quasi-linear map must have some *trivial* restriction to some copy of c_0, since there are exact sequences

$$0 \longrightarrow C[0,1] \longrightarrow \cdot \longrightarrow c_0 \longrightarrow 0$$

in which the quotient is strictly singular. We have encountered these sequences at the beginning of Section 2.3.

On the other hand, let us observe that non-locally trivial but strictly singular quasi-linear maps $\Omega\colon c_0 \to Y$ do exist, and therefore it is false that a non-locally trivial quasi-linear map must be trivial on some copy of c_0. For example, consider a quotient map $q\colon \ell_1 \to c_0$ and form the exact sequence

$$0 \longrightarrow \ker q \longrightarrow \ell_1 \xrightarrow{q} c_0 \longrightarrow 0$$

which is not locally trivial since its dual cannot split. However, it is strictly singular because c_0 and ℓ_1 do not have infinite-dimensional subspaces in common.

5.2.1 An application to twisted sums of ℓ_1 and c_0

Twisted sums of ℓ_1 and c_0 are somewhat mysterious objects. For a start, among the classical ℓ_p-spaces, ℓ_1 is the least injective and c_0 is the least projective. Besides, the standard fact that $\operatorname{Ext}(\ell_2, \ell_2) \neq 0$ [40, §4] –cf. also [63]– implies $\operatorname{Ext}(\ell_p, \ell_q) \neq 0$ whenever $1 < p, q < \infty$. Indeed, combine [15, Theorem 2] with the fact that reflexive ℓ_p spaces contain ℓ_2^n uniformly complemented. However, this is no longer true for c_0 or ℓ_1, hence the subsequent argument cannot be applied to shed any light on whether $\operatorname{Ext}(c_0, \ell_1) = 0$ or not.

These facts somehow point out that the twisting of c_0 and ℓ_1 is an extreme case. The existence of such twisted sums was first shown in [15, §4.3] and [17, Theorem 5.1]. Both approaches are different but share a common core: first, one constructs a non-trivial twisted sum of an $\mathscr{L}_1$-space and an $\mathscr{L}_\infty$-space; then, an appeal to [15, Theorem 2] ensures that $\operatorname{Ext}(c_0, \ell_1) \neq 0$ by the use of certain local techniques in Banach spaces. However, this approach does not provide us with any example of a twisted sum of ℓ_1 and c_0, nor any process to construct them.

Theorem 5.2.1 gives a much more explicit way of producing such twisted sums. Consider a non-trivial exact sequence

$$0 \longrightarrow L_1[0,1] \longrightarrow Z \longrightarrow C[0,1] \longrightarrow 0 \qquad [z]$$

–for example, the one in [17, Theorem 5.1]– which additionally is non-locally trivial, since $L_1[0,1]$ is complemented in its bidual [2, Prop. 6.3.10]. Hence, by virtue of Theorem 5.2.1, there is an embedding $j\colon c_0 \to C[0,1]$ such that the lower row of the following diagram is not locally trivial:

$$\begin{tikzcd} 0 \arrow[r] & L_1[0,1] \arrow[r] & Z \arrow[r] & C[0,1] \arrow[r] & 0 & {[\mathrm{z}]} \\ 0 \arrow[r] & L_1[0,1] \arrow[r] \arrow[u, equal] & PB \arrow[r] \arrow[u] & c_0 \arrow[r] \arrow[u, "j"] & 0 & {[\mathrm{z}j]} \end{tikzcd}$$

Now, we consider the dual sequence of $[\mathrm{z}j]$, which is necessarily non-trivial:

$$0 \longrightarrow \ell_1 \longrightarrow PB^* \longrightarrow L_\infty[0,1] \longrightarrow 0 \qquad [j^*\mathrm{z}^*]$$

and since $L_\infty[0,1]$ is a C-space, we apply Theorem 5.2.1 to obtain another embedding $i\colon c_0 \to L_\infty[0,1]$ that finally gives our desired non-trivial twisted sum of ℓ_1 and c_0:

$$\begin{tikzcd} 0 \arrow[r] & \ell_1 \arrow[r] & PB^* \arrow[r] & L_\infty[0,1] \arrow[r] & 0 & {[j^*\mathrm{z}^*]} \\ 0 \arrow[r] & \ell_1 \arrow[r] \arrow[u, equal] & Z \arrow[r] \arrow[u] & c_0 \arrow[r] \arrow[u, "i"] & 0 & {[j^*\mathrm{z}^*i]} \end{tikzcd}$$

We will have to wait until next section, where a very special twisted sum of $L_1(\Delta)$ and $C(\Delta)$ is produced, to see an explicit twisted sum of ℓ_1 and c_0.

5.3 Centralizers on C-spaces

It is fairly common that Banach spaces carry additional structures, and therefore it makes sense to study twisted sums that preserve such structures. The first instance of such a study was carried out by Kalton [62] in relation to module structures over a Banach algebra of the form L_∞, and gave birth to the now standard *centralizers*. In general, module structures are associated with a good supply of "symmetries": while "pointwise" L_∞-module structures are usually connected to unconditionality, "convolution" L_1-module structures often appear in relation to translation-invariant properties. The paper [21] constitutes a first approach to the study of twisted sums in the context of L_1-modules. We will focus on the construction of one particular twisted sum

$$0 \longrightarrow L_1(G) \longrightarrow \cdot \longrightarrow C(G) \longrightarrow 0$$

for G a compact abelian group.

The setting in this section is slightly different from the rest of the dissertation. We will fix A a Banach algebra and consider *quasi-Banach A-modules*, which are quasi-Banach spaces together with a continuous multiplication $A \times X \to X$ defining a module structure in the purely algebraic sense. In particular, when A is the field of scalars, quasi Banach A-modules are just quasi-Banach spaces. An operator $T\colon X \to Y$ between quasi-Banach A-modules is an *A-homomorphism* if $T(ax) = aT(x)$ for every $a \in A$ and $x \in X$. Additionally, the dual of an A-module is also an A-module: more precisely, if X is a Banach (left) A-module, then there is a natural (right) A-module structure on X^* given by $\langle ax^*, x\rangle = \langle x^*, xa\rangle$. So, categorically speaking, we are now concerned with the category A-Mod of quasi-Banach A-modules and A-homomorphisms. We remind the reader that in this dissertation it does no harm if we only think about Banach A-modules, but on the outside there exists non-locally convex twisted sums of Banach spaces, and it may be dangerous to go alone.

Twisted sums of quasi-Banach A-modules and related concepts are defined the usual way, except for the fact that spaces and operators must now live in A-Mod. Therefore, an *exact sequence* in A-Mod is a diagram formed by quasi-Banach A-modules and A-homomorphisms

$$0 \longrightarrow Y \xrightarrow{\ j\ } Z \xrightarrow{\ q\ } X \longrightarrow 0 \tag{5.b}$$

such that the kernel of every homomorphism agrees with the image of the preceding one. The notion of equivalence between two exact sequences now requires that the operator u in the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & Z_1 & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle u} & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & Z_2 & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

is an A-homomorphism. In particular, an exact sequence (5.b) is trivial whenever there is a homomorphism $P\colon Z \to Y$ which is a projection for i; or equivalently, if there is a homomorphism $s\colon X \to Z$ which is a selection for q. And so we have arrived to one delicate point: an exact sequence of Banach A-modules can be trivial in the category of quasi-Banach spaces, but not in the category of quasi-Banach A-modules. There are some nice and simple examples in [21, p.17]. Finally, pullbacks and pushouts also work in this context, since the corresponding operators are now A-homomorphisms. We denote

$\mathrm{Ext}_A(X,Y)$ the vector space of extensions of A-modules of X by Y. Again, if A is the ground field, we are speaking of plain exact sequences of quasi-Banach spaces.

Now let us present the analogue of a quasi-linear map in an A-module setting:

Definition. A quasi-linear map $\Omega\colon X \to Y$ acting between quasi-Banach A-modules is called an *A-centralizer* if there is $C > 0$ such that whenever $a \in A$ and $x \in X$, we have

$$\|\Omega(ax) - a\Omega(x)\| < C\|a\|\|x\|$$

The smallest constant C verifying the above inequality is called the *A-centralizer constant* of Ω.

The condition of A-centralizer is exactly what we need for the outer product $a(y,x) = (ay,ax)$ to define an A-module structure in $Y \oplus_\Omega X$. Indeed, if Ω is an A-centralizer, then

$$\begin{aligned}\|(ay,ax)\|_\Omega &= \|ay - \Omega(ax)\| + \|ax\| \leq \|ay - a\Omega(x)\| + \|a\Omega(x) - \Omega(ax)\| + \|ax\| \leq \\ &\leq (C+1)\|a\|\|(y,x)\|_\Omega\end{aligned}$$

On the other hand, the condition $\|(ay,ax)\|_\Omega \leq C\|a\|\|(y,x)\|_\Omega$ implies that Ω is an A-centralizer just by replacing $y = \Omega(x)$.

One cannot but dream of defining centralizers explicitely on the whole domain space. And this is not a particularity of centralizers, but of general quasi-linear maps. The obstacle is that it is not possible to explicitely define even a linear non-continuous map without appealing to some form of the axiom of choice. However, it is consistent with ZF the fact that every set on the real line is Lebesgue-measurable (which is called *Solovay's axiom*), and under such assumption, every linear map between quasi-Banach spaces is automatically continuous. The latter assertion can be proved by adapting the argument in [46, §4a] –see also the original work of Banach, [8, p. 23]–.

Fortunately, it suffices to define centralizers (and quasi-linear maps) in a dense subset, as we now show. Let us particularize to the case of A-centralizers, since it is the one which concerns us. Assume X and Y are quasi-Banach A-modules and that X_0 is a dense submodule of X. If $\Phi\colon X_0 \to Y$ is an A-centralizer, we consider the twisted sum space $Y \oplus_\Phi X_0$ and denote by $Z(\Phi)$ its completion. Then, the universal property of the completion produces a diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Y \oplus_\Phi X_0 \arrow[r] \arrow[d, hook, "u"] & X_0 \arrow[r] \arrow[d, hook] & 0 & {[\mathsf{z}_0]} \\ 0 \arrow[r] & Y \arrow[r] & Z(\Phi) \arrow[r] & X \arrow[r] & 0 & {[\mathsf{z}]} \end{tikzcd}$$

and it is simple to see that $[\mathtt{z}_\emptyset]$ is trivial whenever $[\mathtt{z}]$ is, since every selection $s_0\colon X_0 \to Y \oplus_\Phi X_0$ extends to a selection $s\colon X \to Z(\Phi)$. We therefore say that $[\mathtt{z}]$ is the extension *induced* by Φ.

5.3.1 Construction of L_1-centralizers

We will now deal only with L_1-modules and their corresponding extensions. Throughout this section, G denotes a compact abelian group and Γ denotes its *dual group* (consisting on continuous group homomorphisms from G to the circle group $\mathbb{T}$). It is well-known that $L_1(G)$ is a Banach algebra under the *convolution product*

$$(f * g)(x) = \int_G f(xy^{-1}) \cdot g(y)\, dy$$

where the integration is computed with respect to the Haar probability measure on G. This operation can be extended to $M(G)$ by letting

$$(\mu * \nu)(A) = \int_G \mu(x^{-1}A) d\nu(x)$$

where $x^{-1}A = \{x^{-1}y : y \in A\}$. So the whole story is that $M(G)$ is a Banach algebra, and $L_1(G)$ is an ideal inside $M(G)$. The classical instances of $L_1(G)$-modules (actually, $M(G)$-modules) are the $L_p(G)$-spaces for $1 \le p \le \infty$, as well as $C(G)$. We will simply write L_p instead of $L_p(G)$ when it is not necessary to specify the group. Standard references for these kind of ideas are the monographs by Rudin [92] and Hewitt and Ross [56]; the first is more classical, the second deals with a more abstract setting.

A basic tool which intertwins the L_∞-module and the L_1-module structure is the *Fourier transform*, and it is one of the key ingredients in our construction of L_1-centralizers. We will denote by $\mathscr{F}$ the Fourier transform from G-objects to Γ-objects. The classical definition for a function $f \in L_1(G)$ is

$$\mathscr{F}(f)(\gamma) = \widehat{f}(\gamma) = \int_G f(x)\overline{\gamma}(x)dx$$

but it can be extended to $M(G)$: given a measure μ on G, then $\mathscr{F}(\mu) = \widehat{\mu}$ is the bounded function on Γ defined by

$$\widehat{\mu}(\gamma) = \int_G \overline{\gamma}(x)d\mu(x)$$

We recall two basic properties of the Fourier transform:

- $\mathscr{F}$ interchanges pointwise product with convolution; that is, $\widehat{\mu * \nu} = \widehat{\mu} \cdot \widehat{\nu}$ whenever $\nu, \mu \in M(G)$.

- The *translation* of a function f on G by some $y \in G$ is defined as $f_y(x) = f(xy^{-1})$. Then $\widehat{f_y} = y^{-1}\widehat{f}$, with the meaning that $\widehat{f_y}(\gamma) = \overline{\gamma(y)}\widehat{f}(\gamma) = \gamma(y^{-1})\widehat{f}(\gamma)$.

Again, [92] or [56] are natural places for a detailed background about the Fourier transform on locally compact groups.

This section aims to obtain a non-trivial L_1-centralizer from $C(G)$ to $L_1(G)$, which will give rise to the desired twisted sum of $L_1(G)$ and $C(G)$. In general, the construction of L_1-centralizers in the context of L_p-spaces –and $C(G)$– can be reduced to defining the corresponding centralizer in a very particular dense submodule, as our next proposition shows. Precisely, there is one natural choice of a dense submodule in the L_p-spaces for finite p and $C(G)$: the space $P(G)$ consisting of the *polynomials* on G; namely, the finite linear combinations of characters. Hence, given X a Banach space of functions on G, we will write $X^0 = X \cap P(G)$, with the norm inherited from X. Recall that L_p^0 is dense in L_p for $1 \leq p < \infty$, while the closure of $P(G)$ in L_∞ is $C(G)$ by the Stone-Weierstrass theorem.

Proposition 5.3.1. *Let $X = L_p(G)$ for $1 \leq p < \infty$ or $C(G)$. Every exact sequence of L_1-modules $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ is equivalent to one induced by an L_1-centralizer $\Phi\colon X^0 \to Y$.*

Proof. The key fact here is that $P(G)$ is a projective L_1-module, from the purely algebraic point of view. Let X be as in the statement of the theorem and consider an exact sequence of L_1-modules

$$0 \longrightarrow Y \xrightarrow{\ j\ } Z \xrightarrow{\ q\ } X \longrightarrow 0$$

The open mapping theorem asserts there is a bounded homogeneous selection B for π. Now let us look for a suitable linear selection. For every $\gamma \in \Gamma$, write $z_\gamma = \gamma * B(\gamma)$, and observe that $\gamma * z_\gamma = (\gamma * \gamma) * B(\gamma) = \gamma * B(\gamma) = z_\gamma$. Then, the mapping $L\colon P(G) \to Z$ defined as $L(\sum_\gamma c_\gamma \gamma) = \sum_\gamma c_\gamma z_\gamma$ is a linear L_1-homomorphism and a selection for q, since we have $q(z_\gamma) = q(\gamma * B(\gamma)) = \gamma * q(B(\gamma)) = \gamma * \gamma = \gamma$.

Consequently, the map $\Phi\colon X = B - L$ takes values in Y and $\Phi\colon X^0 \to Y$ is an L_1-centralizer:

$$\|\Phi(f+g) - \Phi(f) - \Phi(g)\| = \|B(f+g) - B(f) - B(g)\| \leq 2\|B\|\big(\|f\| + \|g\|\big)$$
$$\|\Phi(a * f) - a * \Phi(f)\| = \|B(a * f) - a * B(f)\| \leq 2\|B\|\|a\|\|f\|$$

Finally, the extension generated by Φ is equivalent to the starting one, since the mapping $u\colon X^0 \oplus_\Phi Y \to Z$ makes the following diagram commutative:

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & X^0 \oplus_\Phi Y \arrow[r] \arrow[d, "u"] & X^0 \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & Y \arrow[r] & Z \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

and so extending u to the completion of $X^0 \oplus_\Phi Y$ ends the proof. □

The starting point for our construction of L_1-centralizers is an L_∞-centralizer. Among the most desirable L_∞-centralizers we find the classical Kalton-Peck maps, which we now define. Given I a set and $1 \le p < \infty$, the classical $\ell_p(I)$-spaces are $\ell_\infty(I)$-modules. We will write $\ell_p^0(I)$ for the submodule of $\ell_p(I)$ consisting of finitely supported elements, which is dense in $\ell_p(I)$ provided p is finite. Let also Lip_0 be the set of all Lipschitz functions $\varphi\colon [0, +\infty) \to [0, +\infty)$ vanishing at 0. Given such a φ, the *Kalton-Peck map* $\Omega\colon \ell_p^0(I) \to \ell_p$ is defined as:

$$\Omega(x)(i) = \begin{cases} x(i) \cdot \varphi\left(\log \dfrac{\|x\|_2}{|x(i)|}\right) & \text{if } x(i) \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.c}$$

It can be shown that Ω is an ℓ_∞-centralizer, with quasi-linear constant $8L_\varphi/e$ and centralizer constant $2L_\varphi/e$, where L_φ is the Lipschitz constant of φ. Moreover, Ω is trivial precisely when φ is bounded. This appears in [62] and [63], although not exactly in this form; [16, §3] contains a more homogeneous exposition. Note that the definition of Ω depends not only on φ but also on p. This will not cause any confusion since only the case $p = 2$ is needed later.

The basic idea of the construction is rather simple: we use the properties of the Fourier transform and the "symmetries" of the Kalton and Peck maps as ℓ_∞-centralizers to obtain the desired L_1-centralizer. Let us consider the composition

$$\mho\colon L_\infty^0(G) \hookrightarrow L_2^0(G) \xrightarrow{\mathscr{F}} \ell_2^0(\Gamma) \xrightarrow{\Omega} \ell_2(\Gamma) \xrightarrow{\mathscr{F}^{-1}} L_2(G) \hookrightarrow L_1(G)$$

where the inclusions are just the natural ones. Then $\mho$ is quasilinear, and its quasilinear constant is no bigger than that of Ω. But there is much more, since this process transfers the "symmetries" of Ω to $\mho$, as we now show:

Proposition 5.3.2. *If Ω is an $\ell_\infty(\Gamma)$-centralizer, then $\mho$ is an $L_1(G)$-centralizer. Moreover:*

i) Ω commutes with characters*; that is, for every $c \in \ell_2^0(\Gamma)$ and every $y \in G$,*

$$\Omega(c \cdot y) = \Omega(f) \cdot y$$

if and only if $\mho$ commutes with translations*; meaning that for every $f \in L_\infty^0(G)$ and every $y \in G$,*

$$\mho(f_y) = \mho(f)_y$$

ii) Ω *commutes with translations if and only if* $\mho$ *commutes with characters.*

Proof. Actually $\mho$ is a centralizer over the convolution algebra $M(G)$. Given $\mu \in M(G)$ and $f \in L_\infty^0(G)$, then

$$\begin{aligned}\|\mho(\mu * f) - \mu * \mho(f)\|_{L_1(G)} &\leq \|\mho(\mu * f) - \mu * \mho(f)\|_{L_2(G)} = \\ &= \|\mathscr{F}^{-1}(\Omega(\widehat{\mu * f})) - \mu * \mathscr{F}^{-1}(\Omega\widehat{f})\|_{L_2(G)} = \\ &= \|\Omega(\widehat{\mu} \cdot \widehat{f}) - \widehat{\mu} \cdot \Omega(\widehat{f})\|_{\ell_2(\Gamma)} \leq \\ &\leq C\|\widehat{\mu}\|_{\ell_\infty(\Gamma)}\|\widehat{f}\|_{\ell_2(\Gamma)} \leq \\ &\leq C\|\mu\|_{M(G)}\|f\|_{L_2(G)} \leq \\ &\leq C\|\mu\|_{M(G)}\|f\|_{C(G)}\end{aligned}$$

Now let us show (i). First assume that Ω commutes with characters, where we regard a element $y \in G$ as $y\colon \Gamma \to \mathbb{C}$, $y(\gamma) = \gamma(y)$. Then for every $f \in L_\infty^0(G)$ and $y \in G$,

$$\mho(f_y) = \mathscr{F}^{-1}\Omega(y^{-1}\widehat{f}) = \mathscr{F}^{-1}(y^{-1} \cdot \Omega\widehat{f}) = \mathscr{F}^{-1}\big(y^{-1} \cdot \widehat{\mho(f)}\big) = \mho(f)_y$$

For the converse, observe that $\Omega(c) = \mathscr{F}\mho\mathscr{F}^{-1}(c)$ for every finitely supported $c\colon \Gamma \to \mathbb{C}$. Then, if $\mho$ commutes with translations and $c = \widehat{f}$, we obtain

$$\Omega(c \cdot y) = \mathscr{F}\mho\mathscr{F}^{-1}(\widehat{f} \cdot y) = \mathscr{F}\mho(f_{y^{-1}}) = \mathscr{F}\big(\mho(f)_{y^{-1}}\big) = \Omega(c) \cdot y$$

The proof of (ii) is identical to that of (i). $\square$

In the case that concerns us, the Kalton-Peck map Ω commutes with both translations and characters. Hence the resulting map $\mho$ possesses an overwhelming collection of symmetries.

The hardest part is, however, to ensure that such $\mho$ is not trivial, and this is what we will take care in the next proposition. Let us first introduce some notation: given $f \in C(G)$ and $\mu \in M(G)$, $f\mu$ stands for the measure on G defined by $(f\mu)(A) = \int_A f\, d\mu$. Note that $\|f\mu\|_{M(G)} \leq \|f\|_{C(G)}\|\mu\|_{M(G)}$.

Proposition 5.3.3. *Let $\Phi\colon L_\infty^0(G)\to L_1(G)$ be a quasi-linear map commuting with characters of G. The following are equivalent:*

i) Φ is trivial.

ii) There is $\mu\in M(G)$ such that $\|\Phi(f)-f\mu\|_1\le M\|f\|_\infty$ for all $f\in L_\infty^0(G)$.

iii) Φ is bounded.

Proof. To show (i) $\Rightarrow$ (ii), we follow a general principle asserting that if a "symmetric" quasi-linear map has a linear map at a finite distance M, then it must also have a "symmetric" linear map at a distance M. So let us assume that Φ is trivial, and let $L\colon L_\infty^0(G)\to L_1(G)$ be a linear map such that $\|\Phi(f)-L(f)\|_{L_1(G)}\le M\|f\|_{C(G)}$ for a certain $M\ge 0$ and every $f\in L_\infty^0(G)$. We now produce a linear L_1-homomorphism with the help of an invariant mean for Γ. Recall that an *invariant mean* for a group Γ is an element $m\in\ell_\infty(\Gamma)^*$ which is positive (i.e., $m(f)\ge 0$ provided $f\ge 0$), $m(1)=1$ and $m(f_\eta)=m(f)$ for every $f\in\ell_\infty(\Gamma)$, $\eta\in\Gamma$. Note that, as a consequence, $\|m\|=1$. Here we will only use the fact that locally compact abelian groups possess invariant means –cf. [94, Example 1.1.7].

Let us put the previous ideas into practice. We isometrically move from $L_1(G)$ to $M(G)$ and define a new linear map $\Lambda\colon L_\infty^0(G)\to M(G)$ by the formula

$$\langle\Lambda(f),h\rangle = m\left[\int_G \gamma^{-1}L(\gamma f)h\,dx\right],\quad h\in C(G)$$

with the meaning that m is applied to the (bounded) function on Γ defined as $\gamma\mapsto\int_G\gamma^{-1}L(\gamma f)hdx$. Considering $\Phi(f)$ as a constant function on $\ell_\infty(\Gamma)$ and using that $\|m\|=1$, we have

$$\begin{aligned}\|\Phi(f)-\Lambda(f)\|_{M(G)} &= \sup_{\|h\|_\infty=1}\left|m\left[\int_G\left(\gamma^{-1}L(\gamma f)-\Phi(f)\right)h\,dx\right]\right|\le\\ &\le\sup_{\gamma\in\Gamma}\|\Phi(f)-\gamma^{-1}L(\gamma f)\|_{L_1(G)}=\\ &=\sup_{\gamma\in\Gamma}\|\gamma^{-1}\Phi(\gamma f)-\gamma^{-1}L(\gamma f)\|_{L_1(G)}=\\ &=\sup_{\gamma\in\Gamma}\|\Phi(\gamma f)-L(\gamma f)\|_{L_1(G)}\le\\ &\le M\|f\|_\infty\end{aligned}$$

On the other hand, $\Lambda(\eta f) = \eta\Lambda(f)$ for every $\eta \in \Gamma$ and every $f \in L^0_\infty(G)$, because given $h \in C(G)$,

$$\langle \Lambda(\eta f), h\rangle = m\left[\int_G \gamma^{-1} L\big((\gamma\eta) f\big) h\, dx\right] = m\left[\int_G \eta^{-1}\gamma^{-1} L\big((\gamma\eta) f\big)(\eta h)\, dx\right] =$$
$$= m\left[\int_G \gamma^{-1} L(\gamma f)(\eta h)\, dx\right] = \langle \Lambda(f), \eta h\rangle = \langle \eta\Lambda(f), h\rangle$$

The previous identity, together with the fact that every polynomial is a finite linear combination of characters, implies $\Lambda(f) = \Lambda(f \cdot 1_G) = f \cdot \Lambda(1_G)$ for every $f \in L^0_\infty(G)$. So assertion (ii) is satisfied letting $\mu = \Lambda(1_G)$.

Finally, the implication (ii) $\Rightarrow$ (iii) is easy, since

$$\|\Phi(f)\|_{L_1} \le \|\Phi(f) - f\mu\|_{M(G)} + \|f\mu\|_{M(G)} \le \big(M + \|\mu\|_{M(G)}\big)\|f\|_\infty$$

and (iii) $\Rightarrow$ (i) is obvious. □

Later, we will need the following "quantitative" version of Proposition 5.3.3. Given a quasi-linear map $\Omega\colon X \longrightarrow Y$, let us write $\delta(\Omega)$ for the infimum of the quantities $\|\Omega - L\|$ where $L\colon X \longrightarrow Y$ is a linear map. Since $\delta(\Omega)$ is finite precisely when Ω is trivial, we can take $\delta(\Omega)$ as a measure of how trivial Ω is.

Corollary 5.3.4. *Let $\Phi\colon L^0_\infty(G) \longrightarrow L_1(G)$ be a quasi-linear map commuting with characters of G. Then $\delta(\Phi) \ge \frac{1}{2}\big(\|\Phi\| - \|\Phi(1_G)\|_{L_1}\big)$.*

Proof. We can clearly assume that both $\delta(\Phi)$ and $\|\Phi\|$ are finite. Given $\varepsilon > 0$, choose a norm-one function $g \in L^0_\infty(G)$ such that $\|\Phi(g)\| \ge \|\Phi\| - \varepsilon$, and a linear map $L\colon L^0_\infty(G) \longrightarrow L_1(G)$ such that $\|\Phi - L\| \le \delta(\Phi) + \varepsilon$. Using the argument from the previous proposition, we obtain a measure $\mu \in M(G)$ that satisfies $\|\Phi(f) - f\mu\|_{M(G)} \le (\delta(\Phi) + \varepsilon)\|f\|_\infty$ for every $f \in L^0_\infty(G)$, and in particular, $\|\Phi(1_G) - \mu\| \le (\delta(\Phi) + \varepsilon)$. All together, these yield

$$\delta(\Phi) + \varepsilon \ge \|\Phi(g) - g\mu\|_{L_1} \ge \|\Phi(g)\|_{L_1} - \|\mu g\|_{M(G)} \ge \|\Phi\| - \varepsilon - \|\mu\| \ \ge$$
$$\ge \|\Phi\| - \varepsilon - \delta(\Phi) - \varepsilon - \|\Phi(1_G)\|_{L_1}$$

that is, $2\delta(\Phi) \ge \|\Phi\| - \|\Phi(1_G)\| - 3\varepsilon$, which is enough to conclude. □

We have finally arrived to the main result of this section. Although we suspect that $\mho$ is non-trivial whenever Ω is, we only have a proof for certain well-behaved Kalton-Peck maps.

Theorem 5.3.5. *Let G be an infinite compact abelian group and Γ its dual group. If $\Omega\colon \ell_2^0(\Gamma) \to \ell_2(\Gamma)$ denotes the Kalton-Peck map associated to an unbounded and concave Lipschitz function, then $\mho\colon L_\infty^0(G) \to L_1(G)$ is not trivial.*

We need some preparation for the proof, so let us first give a sketch of what will be done. Thanks to Proposition 5.3.3, it suffices to show that $\mho$ is unbounded. This amounts to find a bounded sequence of functions $f_n \in L_\infty^0(G)$ so that the L_1-norms of $\mho(f_n)$ are unbounded. For that purpose, we will resort to special sets of characters, namely dissociate and Sidon sets. With those in hand, we will define the suitable functions, which very much resemble the classical Riesz products. We include the necessary background on such concepts just below.

The basics of dissociate and Sidon sets

Let us fix a subset $\Sigma \subseteq \Gamma$. We say Σ is *dissociate* if $1 \notin \Sigma$ and for every finite subsets $D, E \subseteq \Sigma$ and functions $\delta\colon D \to \{\pm 1\}$, $\varepsilon\colon E \to \{\pm 1\}$, the equality

$$\prod_{\gamma \in D} \gamma^{\delta(\gamma)} = \prod_{\gamma \in E} \gamma^{\varepsilon(\gamma)}$$

implies $D = E$ and $\gamma^{\delta(\gamma)} = \gamma^{\varepsilon(\gamma)}$ for all $\gamma \in D$. In particular, this means $\delta(\gamma) = \varepsilon(\gamma)$ provided γ is not of order 2 (and the signs $\delta(\gamma)$ and $\varepsilon(\gamma)$ are irrelevant if γ is of order 2). Actually, this definition makes sense for every topological group, and infinite dissociate sets exists in every infinite group. Also, we say Σ is a *Sidon* set if for every $c \in c_0(\Sigma)$, there is $f \in L_1(G)$ such that $c = \widehat{f}|_\Sigma$, in which case there exists a constant $S(\Sigma)$ so that f can be chosen satisfying $\|f\|_{L_1(G)} \leq S(\Sigma)\|c\|_\infty$. We refer the interested reader to the monograph [74] by López and Ross for a more thorough description of dissociate and Sidon sets as well as their properties. A basic fact that we will later need is that if Σ is dissociate, then $\Sigma \cup \Sigma^{-1}$ is a Sidon set [74, Corollary 2.9].

One of the paramount examples of a Sidon set is the sequence of Rademacher functions $(r_n)_{n=1}^\infty$ as characters on the Cantor group Δ. This is no coincidence, for Sidon sets behave in many aspects very similarly to the Rademacher system, as the following result of Pisier [86, Théorème 2.1] attests: if $(\gamma_n)_{n=1}^\infty$ is a Sidon set, then there exists a constant C depending only on $S(\Sigma)$, such that

$$C^{-1}\left\|\sum_{j=1}^n a_j r_j\right\|_{L_p(\Delta)} \leq \left\|\sum_{j=1}^n a_j \gamma_j\right\|_{L_p(G)} \leq C\left\|\sum_{j=1}^n a_j r_j\right\|_{L_p(\Delta)} \tag{5.d}$$

for every $1 \leq p < \infty$, every $n \in \mathbb{N}$, every scalars $a_1, ..., a_n$ and every elements $\gamma_1, ..., \gamma_n \in \Sigma$. In its turn, the classical Khintchine's inequalities assert that for every $1 \leq p < \infty$, there are constants $A_p, B_p > 0$ such that

$$A_p \Big(\sum_{j=1}^{n} a_j^2 \Big)^{\frac{1}{2}} \leq \Big\| \sum_{j=1}^{n} a_j r_j \Big\|_{L_p(\Delta)} \leq B_p \Big(\sum_{j=1}^{n} a_j^2 \Big)^{\frac{1}{2}} \tag{5.e}$$

for every $n \in \mathbb{N}$ and every scalars $a_1, ..., a_n$. The exact values of A_p and B_p were computed by Haagerup in [52]; we will later use that $A_1 = 1/\sqrt{2}$.

The Riesz products

We now fix an infinite dissociate set of characters $\Sigma = (\gamma_n)_{n=1}^{\infty}$. Let us write Γ_N for the characters that can be written as $\gamma = \prod_{j=1}^{n} \gamma_j^{n_j}$ for some $n_j \in \{0, \pm 1\}$. We define the *length* of a character $\gamma \in \Gamma_N$ as the number $\ell(\gamma)$ of non-zero exponents in the expression $\gamma = \prod_{j=1}^{n} \gamma_j^{n_j}$, which is necessarily unique thanks to the fact that $(\gamma_n)_{n=1}^{\infty}$ is dissociate. Given $\alpha > 1$, which will be fixed later, and $N \in \mathbb{N}$, we define the following *Riesz product* (depending on α and N):

$$f = \prod_{j=1}^{N} \left[1 + \frac{i}{\alpha\sqrt{N}} \left(\frac{\gamma_j + \gamma_j^{-1}}{2} \right) \right] \tag{5.f}$$

To compute the Fourier coefficients of f one must keep track of the elements of order 2. In particular, passing to a subset we only need to distinguish two cases:

(†) No element in Σ has order 2.

(‡) Every element in Σ has order 2.

Assuming (†), we have

$$\widehat{f}(\gamma) = \left(\frac{i}{2\alpha\sqrt{N}} \right)^{\ell(\gamma)}$$

whenever $\gamma \in \Gamma_N$, and 0 otherwise. Note that the Fourier coefficients only depend on γ through $\ell(\gamma)$, and so we can decompose

$$f = 1 + f_1 + \cdots + f_N \quad , \quad f_k = \sum_{\ell(\gamma)=k} \widehat{f}(\gamma) \cdot \gamma = \left(\frac{i}{2\alpha\sqrt{N}} \right)^k \sum_{\ell(\gamma)=k} \gamma$$

If, on the other hand, we assume (‡), then f can be written as

$$f = \prod_{j=1}^{N} \left(1 + \frac{i\gamma_j}{\alpha\sqrt{N}} \right) \tag{5.g}$$

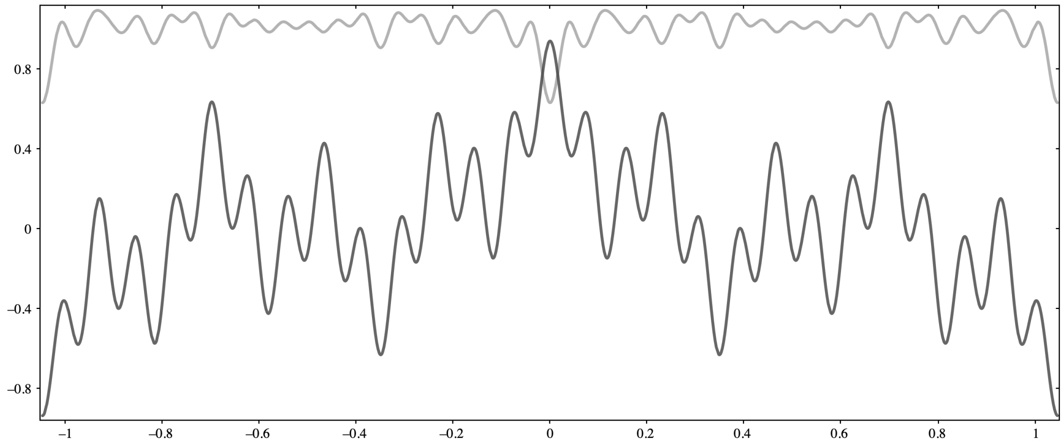

Figure 5.1: Real (above, grey) and imaginary (below, black) part of the Riesz product $\prod_{1\le j\le 4}\big(1 + \frac{i}{4}\cos(3^j t)\big)$ on the interval $[-\pi/3, \pi/3]$.

and so its Fourier coefficients are, for $\gamma \in \Gamma_N$,

$$\widehat{f}(\gamma) = \left(\frac{i}{\alpha\sqrt{N}}\right)^{\ell(\gamma)}$$

and 0 otherwise. In this case, an analogous decomposition is possible:

$$f = 1 + f_1 + \cdots + f_N \quad , \quad f_k = \sum_{\ell(\gamma)=k} \widehat{f}(\gamma)\cdot\gamma = \left(\frac{i}{\alpha\sqrt{N}}\right)^{k} \sum_{\ell(\gamma)=k} \gamma$$

We now collect some useful properties of these functions:

Lemma 5.3.6. *For every $N \in \mathbb{N}$ and $1 \le k \le N$, the following holds:*

a†) If Σ does not contain elements of order 2, then $\|f_k\|_{L_2} \le \frac{1}{\alpha^k}\left(\frac{1}{k!2^k}\right)^{1/2}$, and so

$$1 \le \|f\|_\infty \le \|f\|_{L_2} \le \left(1 + \frac{1}{2\alpha^2 N}\right)^{\frac{N}{2}} \le e^{\frac{1}{4\alpha^2}}$$

a‡) If every element of Σ is of order 2, then $\|f_k\|_{L_2} \le \frac{1}{\alpha^k}\left(\frac{1}{k!}\right)^{1/2}$, and so

$$1 \le \|f\|_\infty \le \|f\|_{L_2} \le \left(1 + \frac{1}{\alpha^2 N}\right)^{\frac{N}{2}} \le e^{\frac{1}{2\alpha^2}}$$

b) In any case, if k is even, then f_k is real; if k is odd, then f_k is purely imaginary.

Proof. $(a\dagger)$ In this case, notice there are exactly $\binom{N}{k}2^{-N}$ characters of length k, and so

$$\|f_k\|_{L_2} = \frac{1}{\alpha^k}\left[\binom{N}{k}(2N)^{-k}\right]^{\frac{1}{2}} = \frac{1}{\alpha^k}\left[\frac{1}{k!2^k}\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{k-1}{N}\right)\right]^{\frac{1}{2}}$$

which, for a fixed k, is an increasing sequence converging to $\frac{1}{\alpha^k}\left(\frac{1}{k!2^k}\right)^{1/2}$. As a consequence, we deduce

$$\|f\|_{L_2}^2 = \sum_{k=0}^{N}\|f_k\|_{L_2(G)}^2 = \sum_{k=0}^{N}\binom{N}{k}\left(\frac{1}{2\alpha^2 N}\right)^k = \left(1+\frac{1}{2\alpha^2 N}\right)^N \leq e^{\frac{1}{2\alpha^2}}$$

Case $(a\ddagger)$ is very similar: now we have $\binom{N}{k}$ characters of length k, which implies

$$\|f_k\|_{L_2} = \frac{1}{\alpha^k}\left[\binom{N}{k}N^{-k}\right]^{\frac{1}{2}} \leq \frac{1}{\alpha^k}\left(\frac{1}{k!}\right)^{\frac{1}{2}} \quad\Rightarrow\quad \|f\|_{L_2}^2 = \left(1+\frac{1}{\alpha^2 N}\right) \leq e^{\frac{1}{\alpha^2}}$$

Finally, (b) is obvious in any case since γ and γ^{-1} have the same length. □

End of the proof

According to the previous lemma, the L_∞-norms of the Riesz products (5.f) are uniformly bounded with respect to N. We now show that the L_1-norms of $\mho(f)$ go to infinity as N goes to infinity. Since $\widehat{f} = \widehat{1} + \widehat{f_1} + \cdots + \widehat{f_N}$ and the functions $\widehat{f_k}$ have disjoint support, we can write

$$\Omega(\widehat{f}) = \widehat{f}\cdot\varphi\left(\log\frac{\|\widehat{f}\|_{\ell_2}}{|\widehat{f}|}\right) = (\widehat{1}+\widehat{f_1}+\cdots+\widehat{f_N})\cdot\varphi\left(\log\frac{\|f\|_{L_2}}{|\widehat{1}+\widehat{f_1}+\cdots+\widehat{f_N}|}\right) =$$

$$= \widehat{1}\cdot\varphi\left(\log\frac{\|f\|_{L_2}}{|\widehat{1}|}\right) + \widehat{f_1}\cdot\varphi\left(\log\frac{\|f\|_{L_2}}{|\widehat{f_1}|}\right) + \cdots + \widehat{f_N}\cdot\varphi\left(\log\frac{\|f\|_{L_2}}{|\widehat{f_N}|}\right)$$

Therefore,

$$\mho(f) = \varphi\left(\log\frac{\|f\|_{L_2}}{|\widehat{1}|}\right) + f_1\cdot\varphi\left(\log\frac{\|f\|_{L_2}}{|\widehat{f_1}|}\right) + \cdots + f_N\cdot\varphi\left(\log\frac{\|f\|_{L_2}}{|\widehat{f_N}|}\right)$$

Now, the idea is to show that the L_1-norms of $\mho(f)$ are large because the L_1-norm of f_1 is much bigger than the L_2-norms of the functions f_k for $k \geq 2$. This argument will serve for both cases (†) and (‡), although the precise calculations are not exactly the same. Details are as follows: if we assume (†), then the imaginary part of $\mho(f)$ is

$$\sum_{1\leq k\leq N \text{ odd}} f_k \cdot \varphi\left(\log \|f\|_{L_2} + k\log(2\alpha\sqrt{N})\right)$$

Since φ is concave, non-negative and vanishes at 0, it is *subadditive*: $\varphi(s+t) \leq \varphi(s)+\varphi(t)$ whenever $s,t>0$. Letting $a = \log \|f\|_{L_2}$ and $b = \log(2\alpha\sqrt{N})$, we have

$$\begin{aligned}
\|\mho(f)\|_{L_1} &\geq \Big\| \sum_{3\leq k\leq N \text{ odd}} f_k \cdot \varphi(a+kb) \Big\|_{L_1} \geq \\
&\geq \varphi(a+b)\cdot\|f_1\|_{L_1} - \sum_{3\leq k\leq N \text{ odd}} \varphi(a+kb)\cdot\|f_k\|_{L_1} \\
&\geq \varphi(a+b)\cdot\|f_1\|_{L_1} - \sum_{3\leq k\leq N \text{ odd}} \big(\varphi(a)+k\varphi(b)\big)\cdot\|f_k\|_{L_2}
\end{aligned}$$

for each $\alpha>1$. Now, $\Sigma\cup\Sigma^{-1}$ is a Sidon set, so applying first Pisier's inequality (5.d) and then Khintchine's inequality (5.e) with $p=1$ to f_1 we obtain:

$$f_1 = \frac{i}{2\alpha\sqrt{N}}\sum_{j=1}^{N}(\gamma_j+\gamma_j^{-1}) \quad\Rightarrow\quad \|f_1\|_{L_1} \geq \frac{1}{2C\alpha}$$

where the constant C depends only on $S(\Sigma\cup\Sigma^{-1})$. To deal with the remainder, observe that $\varphi(a)\leq L_\varphi/(4\alpha^2)$, where L_φ is the Lipschitz constant for L, and so

$$\sum_{3\leq k\leq N \text{ odd}} \big(\varphi(a)+k\varphi(b)\big)\cdot\|f_k\|_{L_2} \leq \left(\frac{L_\varphi}{4\alpha^2}+\varphi(b)\right)\sum_{3\leq k\leq N \text{ odd}} k\|f_k\|_{L_2}$$

It is therefore enough to see that the sum is of order α^{-3}. Using that $k!>2^k$ for $k>3$ and Lemma 5.3.6, we deduce that such sum is majored by

$$\sum_{3\leq k\leq N \text{ odd}} \frac{k}{\alpha^k}\left(\frac{1}{k!2^k}\right)^{\frac{1}{2}} \leq \frac{\sqrt{3}}{4\alpha^3} + \overbrace{\sum_{k=2}^{\infty}\frac{2k+1}{(2\alpha)^{(2k+1)}}}^{(\clubsuit)} = \frac{\sqrt{3}}{4\alpha^3} + \frac{20\alpha^2-3}{8\alpha^3(4\alpha^2-1)^2}$$

Observe that the series in ($\clubsuit$) is, except for a factor of $1/(2\alpha)$, the derivative of the function $\sum_{k=2}^{\infty} t^{2k+1} = \frac{t^5}{1-t^2}$ evaluated at $1/(2\alpha)$. Now, since $\alpha>1$, this yields

$$\sum_{3\leq k\leq N \text{ odd}} k\|f_k\|_{L_2} \leq \frac{\sqrt{3}}{4\alpha^3} + \frac{17}{72\alpha^3} \leq \frac{1}{\alpha^3}$$

Combining,

$$\|\mho(f)\|_{L_1} \geq \frac{\varphi(b)}{2\alpha C} - \left(\frac{L_\varphi}{4\alpha^5} + \frac{\varphi(b)}{\alpha^3}\right) = \frac{\varphi(b)}{\alpha}\overbrace{\left(\frac{1}{2C} - \frac{1}{\alpha^2}\right)}^{(\star)} - \frac{L_\varphi}{4\alpha^5}$$

To finish, we observe that the set $\{\log(2\alpha\sqrt{N}) : N \in \mathbb{N}\}$ constitutes a 1-*net* in $[0, +\infty)$; that is, for every $t \in [0, +\infty)$ there is $N \in \mathbb{N}$ such that $|t - \log(2\alpha\sqrt{N})| < 1$. This, together with the unboundedness of φ, implies that $\varphi(b) \to +\infty$ when $N \to +\infty$. Hence we only need to take $\alpha > 1$ such that $(\star) > 0$ in order to conclude the proof in case $(\dagger)$.

If, however, we work under assumption $(\ddagger)$, then the imaginary part of $\mho(f)$ is

$$\sum_{1\leq k\leq N \text{ odd}} f_k \cdot \varphi\left(\log \|f\|_{L_2} + k\log(\alpha\sqrt{N})\right)$$

Using the estimates in $(\ddagger)$, an entirely analogous reasoning shows unboundedness of $\mho$. First we let $c = \log(\|f\|_{L_2})$, $d = \log(\alpha\sqrt{N})$ and use subadditivity of φ to obtain

$$\|\mho(f)\|_{L_1} \geq \varphi(c+d)\cdot\|f_1\|_{L_1} - \sum_{3\leq k\leq N \text{ odd}} \left(\varphi(c) + k\varphi(d)\right)\cdot\|f_k\|_{L_2}$$

Now, Pisier's inequality in tandem with Khintchine's inequality applied to f_1 yields

$$f_1 = \frac{i}{\alpha\sqrt{N}}\sum_{j=1}^{N}\gamma_j \quad \Rightarrow \quad \|f_1\|_{L_1} \geq \frac{1}{C\alpha}$$

On the other hand, since $k! > k^2$ for $k > 3$, we have the following estimation for the sum:

$$\sum_{3\leq k\leq N \text{ odd}} k\|f_k\|_{L_2} \leq \frac{3}{\sqrt{6}\alpha^3} + \sum_{k<5 \text{ odd}}^{\infty}\frac{1}{\alpha^k} \leq \frac{3}{\sqrt{6}\alpha^3} + \frac{1}{\alpha^3(\alpha^2-1)}$$

Finally,

$$\|\mho(f)\|_{L_1} \geq \frac{\varphi(d)}{\alpha}\overbrace{\left[\frac{1}{C} - \frac{1}{\alpha^2}\left(\frac{3}{\sqrt{6}} - \frac{1}{\alpha^2-1}\right)\right]}^{(\star\star)} - \frac{L_\varphi}{2\alpha^5}\left(\frac{3}{\sqrt{6}} - \frac{1}{\alpha^2-1}\right) \tag{5.h}$$

and so choosing $\alpha > 1$ such that $(\star\star) > 0$ ends the proof.

5.3.2 No L_1-centralizers for C-spaces

In connection with the construction of L_1-centralizers just described, let us mention that no L_1-centralizer acting between C-spaces can exist, unless it is trivial. This fact bears relation with the so-called (Johnson) amenability of Banach algebras –see the book by Runde [94] for more details. Precisely, a Banach algebra A is *amenable* if and only if every extension of quasi-Banach modules $0 \longrightarrow Y^* \longrightarrow Z \longrightarrow X \longrightarrow 0$ which splits as quasi-Banach spaces also splits as quasi-Banach A-modules –cf. [94, Th. 2.3.21]. Now, it is shown in [94, Example 1.1.6] that $L_1(G)$ is amenable whenever G is a compact abelian group. With such considerations in mind, one can finally show:

Theorem 5.3.7. $\mathrm{Ext}_{L_1}\big(C(G), C(G)\big) = 0$.

Proof. It is clear that $\mathrm{Ext}\big(C(G), L_\infty(G)\big) = 0$ thanks to the injectivity of L_∞, and by amenability we deduce that also $\mathrm{Ext}_{L_1}\big(C(G), L_\infty(G)\big) = 0$. Now, every extension of $C(G)$ by itself is induced by an L_1-centralizer $\Phi\colon L_\infty^0(G) \to C(G)$, and so the composition

$$L_\infty^0(G) \xrightarrow{\ \Phi\ } C(G) \hookrightarrow L_\infty(G)$$

must be a trivial centralizer, as a consequence of our previous considerations. Hence there is a morphism $\psi\colon L_\infty^0(G) \to L_\infty(G)$ such that $\|\Phi(f) - \psi(f)\| \le K\|f\|$ for all $f \in L_\infty^0(G)$. To conclude, observe that ψ actually takes values in $L_\infty^0(G)$, because if for $\gamma \in \Gamma$ we denote $h_\gamma = \psi(\gamma)$, then

$$h_\gamma = \psi(\gamma * \gamma) = \gamma * h_\gamma = \widehat{h_\gamma}(\gamma) \cdot \gamma$$

and so h_γ is a polynomial. This implies $\Phi - \psi\colon L_\infty^0(G) \to C(G)$ is bounded, which is enough to conclude. □

In fierce contrast with our previous result, let us mention that there is a variety of L_1-centralizers between L_p-spaces, as [21, §3.2] shows. Precisely, $\mathrm{Ext}_{L_1}(L_q, L_p) \neq 0$ whenever $1 < q \le \infty$ and $1 \le p < \infty$. In fact, a slight variation of the construction featuring Section 5.3.1 produces non-trivial L_1-centralizers $\mho^{pq}\colon L_q(G) \to L_p(G)$ as long as $1 \le p \le 2 \le q \le \infty$. This comes as a consequence of the existence of $\mho$ and the fact that, whenever G is compact, the canonical inclusions $L_r(G) \hookrightarrow L_s(G)$ for $r < s$ factor through $L_t(G)$ for every $r < t < s$. Indeed, the maps

$$\mho^{qp}\colon L_q^0(G) \hookrightarrow L_2^0(G) \xrightarrow{\ \mathscr{F}\ } \ell_2^0(\Gamma) \xrightarrow{\ \Omega\ } \ell_2(\Gamma) \xrightarrow{\ \mathscr{F}^{-1}\ } L_2(G) \hookrightarrow L_p(G)$$

are L_1-centralizers by the same reason $\mho$ is –recall Proposition 5.3.2, and our map $\mho$ becomes $\mho^{\infty 1}$ with this new notation. There is a factorization

$$\mho^{\infty 1}\colon L^0_\infty(G) \hookrightarrow L^0_q(G) \xrightarrow{\mho^{qp}} L_p(G) \hookrightarrow L_1(G)$$

Therefore, the non-triviality of $\mho^{\infty 1} = \mho$ implies that of $\mho^{qp}$.

5.3.3 Twisted sums of ℓ_1 and c_0 (explicit content)

We now apply the construction of the map $\mho$ to produce a non-trivial quasi-linear map $\Phi\colon c_0 \to \ell_1$. Let us work in a less abstract setting: we will denote by Δ, once again, the Cantor group and by D its dual group. The generators in D are the Rademacher functions, which in this context are just projections:

$$r_n\colon \Delta \to \mathbb{T} \quad , \quad r_n(t) = t(n)$$

Actually, every element on D is a finite product of Rademacher functions; that is, a *Walsh* function. The group D can be realised as $\mathrm{fin}(\mathbb{N})$, the group of all finite subsets of $\mathbb{N}$ endowed with the operation of symmetric difference and the discrete topology. Indeed, every $a \in \mathrm{fin}(\mathbb{N})$ defines a Walsh function

$$w_a(t) = \prod_{n\in a} r_n(t)$$

and $w_a \cdot w_b = w_{a\triangle b}$, where $\triangle$ denotes the symmetric difference between a and b. In particular, $w_\varnothing = 1_\Delta$.

Let us define the basic blocks of the construction. Choose $\varphi\colon [0,+\infty) \to [0,+\infty)$ an unbounded concave Lipschitz function vanishing at 0. For each $n \in \omega$, consider the group $\Delta_n = \{\pm 1\}^n$ and denote by D_n its dual group, which can be realised as $\mathrm{fin}\{1,\dots,n\}$. We consider $\Omega_n\colon \ell_2(D_n) \to \ell_2(D_n)$ the corresponding Kalton-Peck map associated to φ –see equation (5.c)– and

$$\mho_n\colon L_\infty(\Delta_n) \to L_1(\Delta_n) \quad , \quad \mho_n(f) = \mathscr{F}^{-1}\Omega_n\mathscr{F}(f)$$

Lemma 5.3.8. *The quasi-linearity constants of $\mho_n$ are uniformly bounded, while the constants $\delta(\mho_n)$ are unbounded.*

Proof. Let us denote by $Q(\Phi)$ the quasi-linearity constant of a quasi-linear map Φ. Then it is a consequence of the definition of $\mho_n$ that $Q(\mho_n) \le Q(\Omega_n)$ for all $n \in \mathbb{N}$. Now, the maps Ω_n can be regarded as "pieces" of the "bigger" quasi-linear map

$$\Omega\colon \ell_2^0(D) \to \ell_2(D) \quad , \quad \Omega(c) = c \cdot \varphi \log\left(\frac{\|c\|_2}{|c|}\right)$$

Indeed, since $\operatorname{supp}(\Omega c) \subseteq \operatorname{supp} c$ for every $c \in \ell_2^0(D)$, we have $\Omega_n = \Omega|_{D_n}$. This automatically implies $Q(\Omega_n) \leq Q(\Omega)$, as we wanted.

As for the triviality constants $\delta(\mho_n)$, it is clear that they are all finite since $\mho_n$ acts between finite-dimensional spaces. Recall that $\delta(\mho_n)$ were defined just above Corollary 5.3.4, and that very same result now yields $\delta(\mho_n) \geq \frac{1}{2}\|\mho_n\|$. To see that the maps $\mho_n$ cannot be uniformly bounded, we appeal to the proof of Theorem 5.3.5 under case (‡): take a Riesz product as in equation (5.g) with the n Rademachers acting on Δ_n and use estimation (5.h). To be more precise, in this case $C = 1/A_1 = \sqrt{2}$, so substituting $\alpha = 2$ and assuming $L_\varphi \leq 1$ gives the bound

$$\|\mho_n(f_n)\|_{L_1} \geq 0.24 \cdot \varphi\big(\log(2\sqrt{n})\big) - 0.02$$

Recalling that $\varphi\big(\log(2\sqrt{n})\big) \to +\infty$ when $n \to +\infty$ and that the L_∞-norms of f_n are uniformly bounded, we are done. □

To finally obtain the desired twisted sum of ℓ_1 and c_0, fix some summable sequence $(c_k)_{k=1}^\infty$ and for each $k \in \mathbb{N}$, choose $n(k) \in \mathbb{N}$ so that the sequence $c_k \cdot \varphi\big(\log 2\sqrt{n(k)}\big)$ goes to infinity. The next result can be shown by appealing to Lemma 5.3.8:

Proposition 5.3.9. *If $\varphi \in \mathrm{Lip}_0$ is concave and unbounded, then the map $\Phi\colon c_0^0\big(L_\infty(\Delta_{n(k)})\big) \to \ell_1\big(L_1(\Delta_{n(k)})\big)$ defined as*

$$\Phi\big((x_k)_{k=1}^\infty\big) = \big(c_k \cdot \mho_{n(k)}(x_k)\big)_{k=1}^\infty$$

is quasi-linear and non-trivial.

But $c_0\big(L_\infty(\Delta_{n(k)})\big)$ is isometrically isomorphic to c_0, and $\ell_1\big(L_1(\Delta_{n(k)})\big)$ is isometrically isomorphic to ℓ_1. Therefore, Φ can be regarded as a quasi-linear map from c_0 to ℓ_1. To be honest, this construction is somewhat disappointing, since Φ does not retain many symmetries from $\mho$. Given the setting of these section, it is unavoidable to ask if there exists a quasi-linear map $\mho\colon c_0 \to \ell_1$ which commutes with translations.

In a different direction, the existence of a strictly singular twisted sum of ℓ_1 and c_0 has been repeatedly asked by Jesús M. F. Castillo [29]. The results displayed in this section suggest that it may be possible to produce a quasi-linear map acting from $C(K)$ to L_1

whose restrictions to any subspace which is spanned by a disjoint sequence is non trivial. Here two functions f and g are *disjoint* if $f \cdot g = 0$ (equivalently, $\operatorname{supp} f \cap \operatorname{supp} g = \varnothing$). This would lead to a strictly singular twisted sum of L_1 an c_0 by virtue of [22, Lemma 4.3], and there is a chance that such a twisted sum could be used in its turn to obtain the desired strictly singular twisted sum of ℓ_1 and c_0 by means of the techniques described in Section 5.2. Despite these considerations, the truth is that we do not know if these strictly singular twisted sums do exist.

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