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Generalized a-Weyl's Theorem and the Single-Valued Extension Property

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1. Introduction and definitions

Throughout this paper, $\mathcal{L}(X)$ denote the algebra of all bounded linear operators acting on a Banach space X. For $T \in \mathcal{L}(X)$, let T^*, $N(T)$, $R(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ap}(T)$ denote respectively the adjoint, the null space, the range, the spectrum, the point spectrum and the approximate point spectrum of T. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of T defined by

$$\alpha(T) = \dim N(T) \qquad \text{and} \qquad \beta(T) = \operatorname{codim} R(T)\,.$$

If the range $R(T)$ of T is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then T is called an *upper semi-Fredholm* (resp. a *lower semi-Fredholm*) operator. In the sequel $SF_+(X)$ (resp. $SF_-(X)$) will denote the set of all upper (resp. lower) semi-Fredholm operators. If $T \in \mathcal{L}(X)$ is either upper or lower semi-Fredholm, then T is called a *semi-Fredholm* operator, and the *index* of T is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is a *Fredholm* operator. An operator T is called *Weyl* if it is Fredholm of index zero. For $T \in \mathcal{L}(X)$ and $n \in \mathbb{N}$ define $c_n(T)$ and $c'_n(T)$ as follows $c_n(T) = \dim R(T^n)/R(T^{n+1})$ and $c'_n(T) = \dim N(T^{n+1})/N(T^n)$. The *descent* $q(T)$ and the *ascent* $p(T)$ are given by

$$q(T) = \inf\big\{n \,:\, c_n(T) = 0\big\} = \inf\big\{n \,:\, R\big(T^n\big) = R\big(T^{n+1}\big)\big\}\,,$$
$$p(T) = \inf\big\{n \,:\, c'_n(T) = 0\big\} = \inf\big\{n \,:\, N\big(T^n\big) = N\big(T^{n+1}\big)\big\}\,.$$

Key words: semi-B-Fredholm operator, generalized a-Weyl's theorem, single-valued extension property.

(We shall, henceforth, shorten $T-\lambda I$ to $T-\lambda$). A bounded linear operator T is called *Browder* if it is Fredholm of finite ascent and descent. The essential spectrum $\sigma_e(T)$, Weyl spectrum $\sigma_w(T)$, and Browder spectrum $\sigma_b(T)$ of $T \in \mathcal{L}(X)$ are defined by

$$\begin{aligned} \sigma_e(T) &= \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Fredholm}\}, \\ \sigma_w(T) &= \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Weyl}\}, \\ \sigma_b(T) &= \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Browder}\}. \end{aligned}$$

Evidently

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T).$$

For a subset $K \subseteq \mathbb{C}$, we write $\operatorname{acc} K$ (resp. $\operatorname{iso} K$) for the accumulation (resp. isolated) points of K.

We say that *Weyl's theorem* holds for $T \in \mathcal{L}(X)$ if

$$\sigma(T) \setminus \sigma_w(T) = E_0(T),$$

where $E_0(T)$ is the set of isolated point of $\sigma(T)$ which are eigenvalues of finite multiplicity, and that *Browder's theorem* holds for $T \in \mathcal{L}(X)$ if

$$\sigma_w(T) = \sigma_b(T).$$

For $T \in \mathcal{L}(X)$, let $SF_+^-(X)$ be the class of all $T \in SF_+(X)$ with $\operatorname{ind} T \leq 0$. The *essential approximate point spectrum* $\sigma_{SF_+^-}(T)$ and the *Browder essential approximate point spectrum* $\sigma_{ab}(T)$ are defined by

$$\begin{aligned} \sigma_{SF_+^-}(T) &= \left\{\lambda \in \mathbb{C} : T-\lambda \text{ is not in } SF_+^-(X)\right\}, \\ \sigma_{ab}(T) &= \bigcap \{\sigma_{ap}(T+K) : TK = KT \text{ and } K \in \mathcal{K}(X)\}, \end{aligned}$$

where $\mathcal{K}(X)$ is the ideal of compact operators on X. Recall that [25] a complex number λ is not in $\sigma_{ab}(T)$ if and only if $T-\lambda \in SF_+^-(X)$ and $p(T-\lambda) < \infty$. We say that *a-Wey's theorem* holds for $T \in \mathcal{L}(X)$ if

$$\sigma_{ap}(T) \setminus \sigma_{SF_+^-}(T) = E_0^a(T),$$

where $E_0^a(T)$ is the set of isolated points of $\sigma_{ap}(T)$ which are eigenvalues of finite multiplicity, and that *a-Browder's theorem* holds for $T \in \mathcal{L}(X)$ if

$$\sigma_{SF_+^-}(T) = \sigma_{ab}(T).$$

In [9, 26], it is shown that for any $T \in \mathcal{L}(X)$ we have the implications:

$$\text{a-Weyl's theorem} \;\Rightarrow\; \text{Weyl's theorem} \;\Rightarrow\; \text{Browder's theorem}\,,$$

$$\text{a-Weyl's theorem} \;\Rightarrow\; \text{a-Browder's theorem} \;\Rightarrow\; \text{Browder's theorem}\,.$$

For a bounded linear operator T and a nonnegative integer n define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_0 = T$). If for some integer n the range space $R(T^n)$ is closed and T_n is an upper (resp. a lower) semi-Fredholm operator, then T is called an *upper* (resp. *lower*) *semi-B-Fredholm* operator, see [7]. In this case the *index* of T is defined as the index of the semi-Fredholm operator T_n, see [6]. Moreover if T_n is a Fredholm operator, then T is called a *B-Fredholm* operator. A *semi-B-Fredholm* operator is an upper or a lower semi-B-Fredholm operator. An operator $T \in \mathcal{L}(X)$ is said to be a *B-Weyl operator* if it is a B-Fredholm operator of index zero. The *semi-B-Fredholm spectrum* $\sigma_{SBF}(T)$ and the *B-Weyl spectrum* $\sigma_{BW}(T)$ of T are defined by

$$\sigma_{SBF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a semi-B-Fredholm operator}\}\,,$$
$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator}\}\,.$$

We say that *generalized Weyl's theorem* holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T)\,,$$

where $E(T)$ is the set of all isolated eigenvalues of T, and *generalized Browder's theorem* holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = \pi(T)\,,$$

where $\pi(T)$ is the set of all poles of T(see [6, Definition 2.13]). Generalized Weyl's theorem and generalized Browder's theorem has been studied in [5, 6]. Similarly, let $SBF_+(X)$ be the class of all upper semi-B-Fredholm operators, and $SBF_+^-(X)$ the class of all $T \in SBF_+(X)$ such that $\text{ind}(T) \leq 0$. Also let

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not in } SBF_+^-(X)\}\,,$$

called the *semi-essential approximate point spectrum*, see [6]. We say that T obeys *generalized a-Weyl's theorem* if

$$\sigma_{SBF_+^-}(T) = \sigma_{ap}(T) \setminus E^a(T)\,,$$

where $E^a(T)$ is the set of all eigenvalues of T which are isolated in $\sigma_{ap}(T)$ ([6, Definition 2.13]). From [6], we know that

$$\text{generalized a-Weyl's theorem} \;\Rightarrow\; \text{generalized Weyl's theorem} \;\Rightarrow\; \text{Weyl's theorem}\,,$$

$$\text{generalized a-Weyl's theorem} \;\Rightarrow\; \text{a-Weyl's theorem}\,.$$

For $T \in \mathcal{L}(X)$ we say that T is *Drazin invertible*, if there exists $B, U \in \mathcal{L}(X)$ such that U is nilpotent and $TB = BT$, $BTB = B$ and $TBT = T + U$. It is known that T is Drazin invertible if and only if it has finite ascent and descent, which is also equivalent to the fact that $T = T_0 \oplus T_1$, where T_0 is invertible and T_1 is nilpotent, see [14, Proposition A] and [17, Corollary 2.2]. The Drazin spectrum is defined by

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Drazin invertible }\}\,.$$

As in [21], define the set $LD(X)$ by

$$LD(X) = \Big\{T \in \mathcal{L}(X) \,:\, p(T) < \infty \text{ and } R\Big(T^{p(T)+1}\Big) \text{ is closed}\Big\}\,.$$

An operator $T \in \mathcal{L}(X)$ is said to be *left Drazin invertible* if $T \in LD(X)$. The left Drazin spectrum $\sigma_{LD}(T)$ of T is defined by

$$\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not in } LD(X)\}\,.$$

It is known, see [6, Lemma 2.12], that

$$\sigma_{SBF_+^-}(T) \subseteq \sigma_{LD}(T) \subseteq \sigma_{ap}(T)\,.$$

We say that $\lambda \in \sigma_{ap}(T)$ is a *left pole* of T if $T - \lambda \in LD(X)$, and that $\lambda \in \sigma_{ap}(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T - \lambda) < \infty$. We denote by $\pi^a(T)$ the set of all left poles of T, and by $\pi_0^a(T)$ the set of all left poles of finite rank. We say that T obeys *generalized a-Browder's theorem* if

$$\sigma_{SBF_+^-}(T) = \sigma_{ap}(T) \setminus \pi^a(T)\,.$$

It is known [6], that

$$\text{generalized a-Browder's theorem} \;\Rightarrow\; \text{a-Browder's theorem}\,,$$

$$\text{generalized a-Browder's theorem} \;\Rightarrow\; \text{generalized Browder's theorem}\,.$$

Generalized a-Weyl's theorem has been studied in [6]. In particular it is shown that generalized a-Weyl's theorem implies generalized a-Browder's theorem. It has been established for operator T on a Hilbert space for which the adjoint T^* is p-hyponormal or M-hyponormal [8]. In this paper, we study generalized a-Weyl's theorem and generalized a-Browder's theorem for operator T acting on a Banach space such that T or T^* has the SVEP. In section 2, we prove that the spectral mapping theorem holds for the semi-essential approximate point spectrum $\sigma_{SBF_+^-}(T)$, for every $f \in H(\sigma(T))$, where $H(\sigma(T))$ denotes the set of all analytic functions defined on an open neighbourhood U of $\sigma(T)$. In section 3, we show that if T is a bounded linear operator such that T^* has the SVEP, then T satisfies generalized a-Weyl's theorem if and only if it satisfies generalized Weyl's theorem, and we show that generalized a-Browder's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$, also we give a necessary and sufficient condition for T to obey generalized Weyl's theorem. One class of operators which was introduced in [23], is the class $\mathcal{P}(X)$ of all operators $T \in \mathcal{L}(X)$ for which for every complex number λ there exists an integer $d_\lambda \geq 1$ such that the following condition holds

$$H_0(T-\lambda) = N(T-\lambda)^{d_\lambda}.$$

In section 4, we give an application for the class $\mathcal{P}(X)$.

2. Spectral mapping theorem for the semi-essential approximate point spectrum

We say that $T \in \mathcal{L}(X)$ has the *single-valued extension property* at λ_0, (SVEP for short) if for every open neighbourhood U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation

$$(T-\lambda)f(\lambda) = 0$$

for all $\lambda \in U$ is the function $f \equiv 0$. $T \in \mathcal{L}(X)$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$ (see [16]).

Recall that the Drazin spectrum $\sigma_{LD}(T)$, $T \in \mathcal{L}(X)$ satisfies the spectral mapping theorem for analytic functions on an open neighbourhood of $\sigma(T)$ which is non-constant on each component of its domain of definition, see [22] page 194. In this section we will show that under the hypothesis T or T^* has the SVEP; the spectral mapping theorem holds for the semi-essential approximate point spectrum $\sigma_{SBF_+^-}(T)$, for every analytic functions on an open neighbourhood of $\sigma(T)$.

We start with the following:

Proposition 2.1. *Let $S, T, A, B \in \mathcal{L}(X)$ be mutually commuting operators, satisfying $TA + BS = I$. Then $TS \in SBF_+(X)$ if and only if $T, S \in SBF_+(X)$.*

Proof. The result follows from [21, Lemma 1] and [21, Lemma 8]. ∎

As an immediate consequence of the previous proposition we have the following:

Corollary 2.1. *Let $P(X) = (X - \lambda_1)^{n_1} \cdots (X - \lambda_m)^{n_m}$ be a polynomial with complex coefficients. Then $P(T) = (T - \lambda_1)^{n_1} \cdots (T - \lambda_m)^{n_m}$ is a upper semi-B-Fredholm operator if and only if $T - \lambda_i$ is a upper semi-B-Fredholm operator for all $i \in \{1, \ldots, m\}$.*

Proof. Since for all relatively prime polynomials P, Q there exist polynomials P_1, Q_1 such that $PP_1 + QQ_1 = 1$, we have $P(T)P_1(T) + Q(T)Q_1(T) = I$. From Proposition 2.1 applied inductively for a relatively prime polynomials $(X - \lambda_i)$ and $(X - \lambda_j)$ where $i, j \in \{1, \ldots, m\}$, we get the desired result. ∎

Let $A(X)$ be the set of all $T \in \mathcal{L}(X)$ such that

$$\operatorname{ind}(T - \lambda) \operatorname{ind}(T - \mu) \geq 0 \qquad \text{for all } \lambda, \mu \in \mathbb{C} \setminus \sigma_{SBF_+}(T)\,.$$

Theorem 2.1. *If $T \in A(X)$, then*

$$f(\sigma_{SBF_+^-}(T)) = \sigma_{SBF_+^-}(f(T)) \qquad \textit{for every } f \in H(\sigma(T))\,.$$

Proof. Let $\mu \in \sigma_{SBF_+^-}(f(T))$. Since $f - \mu$ has only a finite number of zeros $\lambda_1, \lambda_2, \ldots, \lambda_m$ in $\sigma(T)$, it can be written as $f(z) - \mu = (z - \lambda_1)^{n_1} \cdots (z - \lambda_m)^{n_m} g(z)$, where g is a function analytic on a neighbourhood of $\sigma(T)$ and $g(z) \neq 0$ for $z \in \sigma(T)$. Then by the spectral mapping theorem for the ordinary spectrum we have $f(T) - \mu = (T - \lambda_1)^{n_1} \cdots (T - \lambda_m)^{n_m} g(T)$, and $g(T)$ is invertible. If $\mu \notin f(\sigma_{SBF_+^-}(T))$, then $\lambda_i \notin \sigma_{SBF_+^-}(T)$ for all $i \in \{1, \ldots, m\}$, because if there exists $\lambda_j \in \sigma_{SBF_+^-}(T)$, $j \in \{1, \ldots, \}$, then $f(\lambda_j) - \mu = 0$, hence $\mu = f(\lambda_j) \in f(\sigma_{SBF_+^-}(T))$. By Corollary 2.1, this implies that $(T - \lambda_1)^{n_1} \cdots (T - \lambda_m)^{n_m}$ is upper semi-B-Fredholm. Hence, from Proposition 2.1 applied for $(T - \lambda_1)^{n_1} \cdots (T - \lambda_m)^{n_m}$ and $g(T)$, we get that $f(T) - \mu$ is upper semi-B-Fredholm. Since $\operatorname{ind}(f(T) - \mu) = \sum_{i=1}^{m} \operatorname{ind}(T - \lambda_i)^{n_i} \leq 0$, then μ is not in $\sigma_{SBF_+^-}(f(T))$. Which is a contradiction. Thus

$$\sigma_{SBF_+^-}(f(T)) \subseteq f(\sigma_{SBF_+^-}(T))\,.$$

Conversely, suppose that $\mu \notin \sigma_{SBF_+^-}(f(T))$. That is $f(T) - \mu$ is upper semi-B-Fredholm and $\operatorname{ind}(f(T) - \mu) \leq 0$. By Proposition 2.1 applied for $f(T) - \mu$ and $g(T)^{-1}$, we conclude that $(T - \lambda_1)^{n_1} \cdots (T - \lambda_m)^{n_m}$ is upper semi-B-Fredholm. Hence, by Corollary 2.1, we get that $T - \lambda_i$ is upper semi-B-Fredholm for $i \in \{1, \ldots, m\}$. Since $\operatorname{ind}(f(T) - \mu) = \sum_{i=1}^{m} \operatorname{ind}(T - \lambda_i)^{n_i} \leq 0$ and $T \in A(X)$, then $\operatorname{ind}(T - \lambda_i) \leq 0$ for $i \in \{1, \ldots, m\}$. So $T - \lambda_i \in SBF_+^-(X)$. Thus $\mu \notin f(\sigma_{SBF_+^-}(T))$. ∎

For $T \in \mathcal{L}(X)$, let $\rho_{SBF}(T) = \mathbb{C} \setminus \sigma_{SBF}(T)$. In the following proposition we prove that if T or T^* has the SVEP, then $T \in A(X)$.

Proposition 2.2. *Let T be a bounded linear operator on X.*

(i) *If T has the SVEP, then* $\operatorname{ind}(T - \lambda) \leq 0$ *for all* $\lambda \in \rho_{SBF}(T)$.

(ii) *If T^* has the SVEP, then* $\operatorname{ind}(T - \lambda) \geq 0$ *for all* $\lambda \in \rho_{SBF}(T)$.

Proof. (i) Let $\lambda \in \rho_{SBF}(T)$, then $T - \lambda$ is semi-B-Fredholm. By [7, Corollary 3.2], for $\mu \in \mathbb{C}$ such that $|\lambda - \mu|$ is small enough we have $T - \mu$ is semi-Fredholm and $\operatorname{ind}(T - \lambda) = \operatorname{ind}(T - \mu)$. If T has the SVEP, then from [3, Corollary 2.7] we deduce that $\operatorname{ind}(T - \mu) \leq 0$, and hence $\operatorname{ind}(T - \lambda) \leq 0$. Which proves (i).

(ii) Suppose that T^* has the SVEP, then from [3, Corollary 2.7] we get that $\operatorname{ind}(T - \mu) \geq 0$, and hence $\operatorname{ind}(T - \lambda) \geq 0$. ∎

As an immediate consequence of Theorem 2.1 and Proposition 2.2 we obtain the following result.

Corollary 2.2. *Let $T \in \mathcal{L}(X)$. If T or T^* has the SVEP, then*

$$f(\sigma_{SBF_+^-}(T)) = \sigma_{SBF_+^-}(f(T)) \qquad \text{for every } f \in H(\sigma(T))\,.$$

3. Generalized a-Weyl's theorem and the SVEP.

Let $T \in \mathcal{L}(X)$ and $d \in \mathbb{N}$. Then T has a *uniform descent* for $n \geq d$ if

$$R(T) + N(T^n) = R(T) + N(T^d) \qquad \text{for all } n \geq d\,.$$

If in addition, $R(T) + N(T^d)$ is closed then T is said to have a topological uniform descent for $n \geq d$, see [12].

For an operator $T \in \mathcal{L}(X)$, we denote by $F^a(T)$ the set of all isolated points λ of $\sigma_{ap}(T)$ for which $T - \lambda$ is semi-B-Fredholm.

The following Proposition was established in [6], however the arguments used are different.

Proposition 3.1. *Let $T \in \mathcal{L}(X)$. The following assertions hold.*

(i) *$F^a(T) = \pi^a(T)$, and hence $\sigma_{LD}(T) = \sigma_{ap}(T) \setminus F^a(T) = \operatorname{acc}\sigma_{ap}(T) \cup \sigma_{SBF}(T)$.*

(ii) *If generalized a-Weyl's theorem holds for T, then so does generalized a-Browder's theorem.*

(iii) *If T satisfies generalized a-Browder's theorem, then T satisfies generalized a-Weyl's theorem if and only if $F^a(T) = E^a(T)$.*

Proof. (i) If $\lambda \in F^a(T)$, then $T - \lambda$ is semi-B-Fredholm and λ is isolated in $\sigma_{ap}(T)$, in particular $T-\lambda$ is an operator of topological uniform descent for $n \geq d$. Hence from [12, Theorem 4.7], if $|\beta - \lambda|$ is sufficiently small, then $c'_n(T-\beta) = c'_d(T-\lambda)$ for $n \geq d$. Since λ is isolated in $\sigma_{ap}(T)$, then we can choose β such that $\beta \notin \sigma_{ap}(T)$, and hence $T-\beta$ is injective. So $c'_d(T-\lambda) = 0$, that is $p(T-\lambda)$ is finite. On the other hand, by [21, Lemma 12], $R((T-\lambda)^{p(T-\lambda)+1})$ is closed. This implies that $\lambda \notin \sigma_{LD}(T)$. That is $\lambda \in \pi^a(T)$. Thus $F^a(T) \subseteq \pi^a(T)$. For the reverse inclusion suppose that $\lambda \in \pi^a(T)$, then λ is isolated in $\sigma_{ap}(T)$, see [6, Remark 2.7]. Also, $\lambda \in \pi^a(T)$ implies that $\lambda \notin \sigma_{LD}(T)$, and hence $\lambda \notin \sigma_{SBF_+^-}(T)$. So $T-\lambda$ is semi-B-Fredholm. This implies that, $\lambda \in F^a(T)$. So $\pi^a(T) \subseteq F^a(T)$, and hence $\pi^a(T) = F^a(T)$. Since $\sigma_{LD}(T) = \sigma_{ap}(T) \setminus \pi^a(T)$, then $\sigma_{LD}(T) = \sigma_{ap}(T) \setminus F^a(T) = \operatorname{acc}\sigma_{ap}(T) \cup \sigma_{SBF}(T)$. This gives the proof of (i).

(ii) Suppose that generalized a-Weyl's theorem holds for T, that is $\sigma_{SBF_+^-}(T) = \sigma_{ap}(T) \setminus E^a(T)$. Since

$$E^a(T) \cap \sigma_{SBF}(T) \subseteq E^a(T) \cap \sigma_{SBF_+^-}(T) = \emptyset\,,$$

then

$$E^a(T) \subseteq \operatorname{iso}\sigma_{ap}(T) \cap \rho_{SBF}(T) = F^a(T)\,.$$

Thus $E^a(T) \subseteq F^a(T)$. Since we have always $F^a(T) \subseteq E^a(T)$, then $E^a(T) = F^a(T)$ and $\sigma_{SBF_+^-}(T) = \sigma_{ap}(T) \setminus F^a(T)$, hence by (i) we conclude that T satisfies generalized a-Browder's theorem.

(iii) Suppose that T satisfies generalized a-Weyl's theorem. If $\lambda \in E^a(T)$, then $\lambda \in \sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T)$. Since T satisfies generalized a-Browder's thoerem then $\lambda \in \pi^a(T)$. Hence by (i) $\lambda \in F^a(T)$. Thus $E^a(T) \subseteq F^a(T)$ and therefore

$F^a(T) = E^a(T)$. Conversely Assume that $E^a(T) = F^a(T)$. Since T satisfies generalized a-Browder's theorem, then

$$\begin{aligned}\sigma_{SBF_+^-}(T) &= \sigma_{ap}(T) \setminus \pi^a(T) \\ &= \sigma_{ap}(T) \setminus F^a(T) \quad \text{(by (i))} \\ &= \sigma_{ap}(T) \setminus E^a(T)\,.\end{aligned}$$

Hence T satisfies generalized a-Weyl's theorem. ∎

As mentioned above, generalized a-Weyl's theorem implies generalized Weyl's theorem ([6]). In the following we give a sufficient condition to get the reverse implication.

Theorem 3.1. *Let $T \in \mathcal{L}(X)$.*

(i) *If T^* has the SVEP, then T satisfies generalized a-Weyl's theorem if and only if it satisfies generalized Weyl's theorem.*

(ii) *If T has the SVEP, then T^* satisfies generalized a-Weyl's theorem if and only if it satisfies generalized Weyl's theorem.*

Proof. (i) Suppose that T^* has the SVEP, then by [16, Proposition 1.3.2], we have $\sigma_{ap}(T) = \sigma(T)$, and hence $E(T) = E^a(T)$. If T satisfies generalized Weyl's theorem, then $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$. To prove that generalized a-Weyl theorem holds for T, it suffices to show that $\sigma_{BW}(T) = \sigma_{SBF_+^-}(T)$. For this, suppose that $\lambda \notin \sigma_{SBF_+^-}(T)$, then $T - \lambda$ is an upper semi-B-Fredholm operator and $\text{ind}(T - \lambda) \leq 0$. By Proposition 2.2, we have that $\text{ind}(T - \lambda) \geq 0$. So $\text{ind}(T - \lambda) = 0$. Which implies that $T - \lambda$ is semi-B-Fredholm of index 0, hence $T - \lambda$ is B-Fredholm of index 0, that is $\lambda \notin \sigma_{BW}(T)$. This gives $\sigma_{SBF_+^-}(T) \supseteq \sigma_{BW}(T)$. The other inclusion is always true. So $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$. Since the revers implication is known[6], then the equivalence between Weyl's theorem and a-Weyl's theorem holds for T.

(ii) Outlines the proof of the first statement. ∎

In general, we cannot expect that generalized a-Weyl's theorem holds for operators satisfying the SVEP. Let T defined on l_2 by

$$T(x_1, x_2, \dots) = \left(\frac{1}{2}\, x_2, \frac{1}{3}\, x_3, \dots\right).$$

Then T has the SVEP and $\sigma(T) = \sigma_{SBF_+^-}(T) = E^a(T) = \{0\}$. Thus T does not obey generalized a-Weyl's theorem. However, generalized a-Browder's theorem holds for T whenever T or T^* has the SVEP as shown by the followingf:

Theorem 3.2. *If T or T^* has the SVEP, then generalized a-Browder's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. Suppose that $\lambda \in \sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T)$, then $T - \lambda$ is upper semi-B-Fredholm and $\text{ind}(T-\lambda) \leq 0$. The operator $T - \lambda$ has a topological uniform descent, so from [7, Corollary 3.2], if β is in $\sigma_{ap}(T)$ such that $|\beta - \lambda|$ is sufficiently small, then $T-\beta$ is upper semi-Fredholm operator and $\text{ind}(T-\beta) \leq 0$. Hence $\beta \in \sigma_{ap}(T) \setminus \sigma_{SF_+^-}(T)$. By [24, Proposition 2.4], The SVEP for T or T^* implies that a-Browder's theorem holds for T that is $\sigma_{SF_+^-}(T) = \sigma_{ap}(T) \setminus \pi_0^a(T)$, hence $\beta \in \pi_0^a(T)$. This implies that $p(T-\beta) < \infty$, by [12, Theorem 4.7] $p(T-\lambda) < \infty$. Now, since $T-\lambda$ is semi-B-Fredholm, then there exists an integer n such that $R((T-\lambda)^n)$ is closed and $(T-\lambda)\mid_{R((T-\lambda)^n)}$ is Fredholm. We can assume that $n \geq p(T-\lambda)$, see the proof of Proposition 2.1 of [6]. Since we have $R(T-\lambda) + N((T-\lambda)^{i+1}) = R(T-\lambda) + N((T-\lambda)^i)$ for every $i \geq p(T-\lambda)$ and $R((T-\lambda)^n)$ is closed, then by [22, Lemma 17], we get that $R((T-\lambda)^{p(T-\lambda)+1})$ is closed. So $\lambda \in \pi^a(T)$. Thus

$$\sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T) \subseteq \pi^a(T)\,.$$

For the reverse inclusion. If $\lambda \in \pi^a(T)$, then by Proposition 3.1 (i), $\lambda \in F^a(T)$, that is λ is isolated in $\sigma_{ap}(T)$ and $T-\lambda$ is semi-B-Fredholm. From [7, Corollary 3.2], if we choose $\beta \in \mathbb{C}$ such that $|\lambda - \beta|$ is small enough and $\beta \notin \sigma_{ap}(T)$, then $T-\beta$ is upper semi-Fredholm with $\text{ind}(T-\beta) \leq 0$. So $T-\lambda$ is an upper semi-B-Fredholm operator and $\text{ind}(T-\lambda) \leq 0$, that is $\lambda \notin \sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T)$. Finally, $\sigma_{ap}(T) \setminus \sigma_{SBF_+^-} = \pi^a(T)$. Thus generalized a-Browder's theorem holds for T. To complete the proof, if $f \in H(\sigma(T))$, then by [16, Theorem 3.3.6], $f(T)$ or $f(T^*)$ has the SVEP. Similarly we get the result. ∎

Corollary 3.1. *If T or T^* has the SVEP, then generalized a-Weyl's theorem holds for T if and only if $F^a(T) = E^a(T)$.*

Proof. If T or T^* has the SVEP, then by the preceding theorem generalized a-Browder's theorem holds for T, and by (iii) of Proposition 3.1, we deduce the result. ∎

We have noted that $\sigma_{LD}(T)$ satisfies the spectral mapping theorem for analytic functions f on an open neighbourhood of $\sigma(T)$ which is non-constant on each component of its domain of definition. Following we show that under the hypothesis T or T^* has the SVEP; the condition that f is non-constant on each component of its domain of definition can be left out.

THEOREM 3.3. *If T or its adjoint T^* has the SVEP, then*

$$f(\sigma_{LD}(T)) = \sigma_{LD}(f(T)), \text{ for every } f \in H(\sigma(T)).$$

Proof. By Theorem 3.2, generalized a-Browder's theorem holds for T and $f(T)$, for every $f \in H(\sigma(T))$. So $\sigma_{LD}(T) = \sigma_{SBF_+^-}(T)$ and $\sigma_{LD}(f(T)) = \sigma_{SBF_+^-}(f(T))$. If T or T^* has the SVEP then by Corollary 2.2, we have $f(\sigma_{SBF_+^-}(T)) = \sigma_{SBF_+^-}(f(T))$. Hence

$$\begin{aligned} f(\sigma_{LD}(T)) &= f(\sigma_{SBF_+^-}(T)) \\ &= \sigma_{SBF_+^-}(f(T)) \\ &= \sigma_{LD}(f(T)). \end{aligned}$$

∎

4. APPLICATIONS

In this section we will study generalized a-Weyl's theorem and generalized a-Browder's theorem for some classes of operators. For this let us introduce some basic notions which will be used later.

The analytic core of an operator $T \in \mathcal{L}(X)$ is the subspace

$$K(T) := \Big\{ x \in X : \quad Tx_{n+1} = x_n, \ Tx_1 = x, \ \|x_n\| \le c^n \|x\| \\ \text{for some } c > 0 \ (n = 1, 2, \dots), \ x_n \in X \Big\}.$$

The quasi-nilpotent part of T is the subspace

$$H_0(T) := \Big\{ x \in X : \lim_{n \to \infty} \|T^n x\|^{\frac{1}{n}} = 0 \Big\}.$$

The spaces $K(T)$ and $H_0(T)$ are hyperinvariant under T and satisfy $T^{-n}(0) \subset H_0(T)$, $K(T) \subset T^n(X)$ for all $n \in \mathbb{N}$, $TK(T) = K(T)$. For their further properties, see [18, 19].

The class of operators $T \in \mathcal{L}(X)$ for which $K(T) = \{0\}$ was introduced and studied by M. Mbekhta in [19]. It was shown that for such operators, the spectrum is connected and the SVEP holds.

THEOREM 4.1. *If there exists λ such that $K(T-\lambda) = \{0\}$, then $f(T)$ satisfies generalized a-Browder's theorem, for every $f \in H(\sigma(T))$. Moreover, if in addition $N(T-\lambda) = 0$, then generalized a-Weyl's theorem holds for $f(T)$.*

Proof. Since T has the SVEP, then by Theorem 3.2, generalized a- Browder's theorem holds for $f(T)$. Let $\alpha \in \sigma(f(T))$, then $f(z) - \alpha = \prod_{i=1}^{n}(z - \lambda_i)g(z)$, where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \sigma(T)$ and g is an analytic function on an open neighbourhood of $\sigma(T)$, without zeros in $\sigma(T)$. Since $g(T)$ is invertible, then we deduce that

$$N(f(T)-\alpha) = N\left(\prod_{i=1}^{n}(T-\lambda_i)\right) = \bigoplus_{i=1}^{n} N(T-\lambda_i)\,.$$

On the other hand, from [19, Proposition 2.1], we get that $\sigma_p(T) \subseteq \{\lambda\}$. If we suppose that $N(T-\lambda) = \{0\}$, then $\sigma_p(T) = \emptyset$. Which implies that

$$N(f(T)-\lambda) = \{0\}.$$

That is $\sigma_p(f(T)) = \emptyset$. Thus, $F^a(f(T)) = E^a(f(T)) = \emptyset$. Since $f(T)$ satisfies generalized a-Browder's theorem, then by (iii) of Proposition 3.1, generalized a-Weyl's theorem holds for $f(T)$. Which completes the proof. ∎

Let $\mathcal{P}(X)$ be the set of all operators $T \in \mathcal{L}(X)$ such that for every complex number λ there exists an integer $d_\lambda \geq 1$ for which the following condition holds

$$H_0(T-\lambda) = N(T-\lambda)^{d_\lambda}.$$

It is known that if $H_0(T-\lambda)$ is closed for every complex number λ, then T has the SVEP, see [2, 15]. So that, the SVEP is shared by all the operators of $\mathcal{P}(X)$. The class of operators $\mathcal{P}(X)$ is considerably large, it contains, in particular, the classes consisting of generalized scalar, subscalar and algebraically totally paranormal operators on a Banach space, hyponormal, p-hyponormal $(0 < p < 1)$ and M-hyponormal operators on a Hilbert space (see [23]).

For p-hyponormal and M-hyponormal operators in Hilbert space, it is shown in [8] that generalized a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$. In the following we will give more for Banach space operators.

THEOREM 4.2. *Let T a bounded operator on X. If there exists a function $h \in H(\sigma(T))$ non constant in any connected component of its domain, and such that $h(T^*) \in \mathcal{P}(X^*)$, then generalized a-Weyl's theorem holds for $f(T)$, for every $f \in H(\sigma(T))$.*

Proof. Suppose that $h(T^*) \in \mathcal{P}(X^*)$, then by [23, Theorem 3.4], we have $T^* \in \mathcal{P}(X^*)$. First, we will show that generalized a-Weyl's theorem holds for T. Since T^* has the SVEP, then by Corollary 3.1, it suffices to show that $F^a(T) = E^a(T)$. For this let $\lambda \in E^a(T)$, then λ is isolated eigenvalue of $\sigma_{ap}(T)$. Since T^* has the SVEP, then $\sigma_{ap}(T) = \sigma(T)$, see [16]. So $X^* = H_0(T^*-\lambda) \oplus K(T^*-\lambda)$, where the direct sum is topological. Since $T^* \in \mathcal{P}(X^*)$, then $H_0(T^* - \lambda) = N(T^* - \lambda)^d$ for some integer d, and hence $X^* = N(T^* - \lambda)^d \oplus K(T^* - \lambda)$. Since

$$(T^* - \lambda)^d(X^*) = (T^* - \lambda)^d K(T^* - \lambda) = K(T^* - \lambda)\,,$$

then $K(T^* - \lambda) = R(T^* - \lambda)^d$, and hence

$$X^* = N(T^* - \lambda)^d \oplus R(T^* - \lambda)^d.$$

So $T - \lambda \mid R(T-\lambda)^d$ is surjective. This implies that $T - \lambda$ is semi-B-Fredholm. So $E^a(T) \subseteq F^a(T)$. Since we have always that $F^a(T) \subseteq E^a(T)$, then $F^a(T) = E^a(T)$. Now, from [23, Theorem 3.4], if $T^* \in \mathcal{P}(X^*)$, then $f(T^*) \in \mathcal{P}(X^*)$ for every $f \in H(\sigma(T))$. Hence, by the same argument we conclude that generalized a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$. ∎

As an easy consequence of the previous theorem, we have the following corollary.

COROLLARY 4.1. *If $T^* \in \mathcal{P}(X^*)$, then generalized a-Weyl's theorem holds for $f(T)$, for every $f \in H(\sigma(T)$.*

Following we give condition for $T \in \mathcal{P}(X)$ which forces $f(T)$ to obey generalized Weyl's theorem for $f \in H(\sigma(T).)$

THEOREM 4.3. *If $T \in \mathcal{P}(X)$ be such that $\sigma(T) = \sigma_{ap}(T)$, then generalized a-Weyl's theorem holds for $f(T)$, for every $f \in H(\sigma(T))$.*

Proof. Suppose that $T \in \mathcal{P}(X)$ and $\sigma(T) = \sigma_{ap}(T)$. First we will prove that generalized a-Weyl's theorem holds for T. Since T has the SVEP, then by Corollary 3.1, it suffices to show that $F^a(T) = E^a(T)$. For this let $\lambda \in E^a(T)$.

Then λ is isolated in $\sigma_{ap}(T) = \sigma(T)$. By [18, Theorem 1.6] $X = H_0(T-\lambda) \oplus K(T-\lambda)$, where the direct sum is topological. Since there exist an integer n such that $H_0(T-\lambda) = N(T-\lambda)^n$, then $X = N((T-\lambda)^n) \oplus K(T-\lambda)$. This implies that $(T-\lambda)^n(X) = (T-\lambda)K(T-\lambda) = K(T-\lambda)$. Thus

$$X = N((T-\lambda)^n) \oplus R((T-\lambda)^n)\,.$$

So $(T-\lambda)^n$ is Fredholm of index 0, and so is $T-\lambda$, see [13]. Hence $T-\lambda$ is B-Fredholm. Finally, $E^a(T) \subseteq F^a(T)$. The other inclusion is clear. Thus $E^a(T) = F^a(T)$. Similarly, we prove that $f(T)$ satisfies generalized Weyl's theorem, because $f(T) \in \mathcal{P}(X)$ and

$$\sigma(f(T)) = f(\sigma(T)) = f(\sigma_{ap}(T)) = \sigma_{ap}(f(T))\,.$$

∎

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