

Multifractals and Projections

FADHILA BAHROUN, IMEN BHOURI

Faculté des Sciences de Monastir, 5000 Monastir, Tunisie
e-mail: fadhila_bah@yahoo.fr, bhour_iimen@yahoo.fr

(Presented by M. Mbekhta)

AMS *Subject Class.* (2000): 28A20, 28A75, 31A15

Received September 10, 2005

1. INTRODUCTION AND PRELIMINARIES

Let m be an integer with $0 < m < n$ and $G_{n,m}$ stand for the Grassmann manifold of all m -dimensional linear spaces of \mathbb{R}^n . For $V \in G_{n,m}$, we denote $p_V : \mathbb{R}^n \rightarrow V$ the orthogonal projection onto V , then $\{p_V; V \in G_{n,m}\}$ is compact in the space of all linear maps from \mathbb{R}^n to \mathbb{R}^m , and the identification of V with p_V induces a compact topology for $G_{n,m}$. Fixing $V_0 \in G_{n,m}$, we can define an orthogonally invariant Radon probability measure $\gamma_{n,m}$ on $G_{n,m}$ by

$$\gamma_{n,m}(A) = v_n \{g \in O(n) ; g(V_0) \in A\} \quad \text{for } A \in G_{n,m}$$

where v_n denotes the unique Haar measure on the orthogonal group $O(n)$ of \mathbb{R}^n normalized so that $v_n(O(n)) = 1$. The uniqueness implies that $\gamma_{n,m}$ is independent of V_0 (see [6]). In other words,

$$\gamma_{n,m} = f_{V_0*}v_n \quad \text{with } f_{V_0}(g) = g(V_0) \quad \text{for } g \in O(n)$$

and $f_{V_0*}v_n$ is the image of the measure v_n under the map f_{V_0} .

For a Borel probability measure μ on \mathbb{R}^n , supported on the compact set S_μ , and for $V \in G_{n,m}$, we define $\tilde{\mu}$, the projection of μ onto V by

$$\tilde{\mu}(E) = \mu(p_V^{-1}(E))$$

for all $E \subseteq V$. Thus if $f : V \rightarrow \mathbb{R}$ is continuous then

$$\int_V f(u) d\tilde{\mu}(u) = \int_{\mathbb{R}^n} f(p_V(x)) d\mu(x).$$

If ν is a Borel probability measure on S_μ , one defines, for $p > 0$

$$T_{\mu,\nu}(p) = \liminf_{r \rightarrow 0} \frac{1}{\log r} \log \int_{S_\mu} \mu(B(x, r))^p d\nu(x)$$

the lower generalized p -spectral dimension of μ . It is closely related to the Rényi dimension in its integral version (see [9]) and if ν is a Gibbs measure for the measure μ , i. e there exists a measure ν on S_μ and constants $\underline{K} > 0, \overline{K} > 0$ and $t_q \in \mathbb{R}$ such that for every $x \in S_\mu$ and every $0 < r < \lambda$

$$\underline{K}\mu(B(x, r))^q (2r)^{t_q} \leq \nu(B(x, r)) \leq \overline{K}\mu(B(x, r))^q (2r)^{t_q}.$$

$T_{\mu,\nu}$ represents the C_μ function of Olsen's multifractal formalism [8]. This quantity appears as a generalization of the lower p -spectral dimension defined in [5]. For $p \geq 0$,

$$D_p(\mu) = \liminf_{r \rightarrow 0} \frac{1}{p \log r} \log \int_{S_\mu} \mu(B(x, r))^p d\mu(x).$$

In particular, in the case $\mu = \nu$ one has

$$T_{\mu,\mu}(p) = pD_p(\mu)$$

In [5], Hunt and Kaloshin proved the following statement:

THEOREM (HK). *Let $0 < p \leq 1$. If $D_p(\mu) \leq m$, one has*

$$D_p(\tilde{\mu}) = D_p(\mu) \quad \text{for } \gamma_{n,m}\text{-almost all } V \in G_{n,m}.$$

In this paper, we investigate the relationship between $T_{\mu,\nu}(p)$ and $T_{\tilde{\mu},\tilde{\nu}}(p)$. Which allows us to compare the multifractal spectra of the measure μ and that of its projections.

2. PROJECTION RESULTS

In this section, we show that the generalized p -spectral dimension is preserved under almost every orthogonal projection.

THEOREM 2.1. *Let p be a real number.*

1. *If $0 < p \leq 1$ and $T_{\mu,\nu}(p) \leq pm$, then*

$$T_{\tilde{\mu},\tilde{\nu}}(p) = T_{\mu,\nu}(p) \quad \text{for } \gamma_{n,m}\text{-almost all } V \in G_{n,m}.$$

2. If $p > 1$, one has

$$\inf(T_{\mu,\nu}(p), m) \leq T_{\tilde{\mu},\tilde{\nu}}(p) \leq T_{\mu,\nu}(p) \text{ for } \gamma_{n,m}\text{-almost all } V \in G_{n,m}.$$

Remark. The first assertion of this theorem is a generalization of that of Hunt and Kaloshin. In fact, in the case $\mu = \nu$, assertion 1 is the main theorem of Hunt and Kaloshin. Assertion 2 extends the result of Hunt and Kaloshin to the case $p > 1$ untreated in their work. Indeed, considering the relation $D_p(\mu) = \frac{1}{p}T_{\mu,\mu}(p)$, the equality $D_p(\mu) = D_p(\tilde{\mu})$ remain valid for $\gamma_{n,m}$ -almost all $V \in G_{n,m}$, if $p > 1$ and $D_p(\mu) < \frac{m}{p}$.

Proof of Theorem 2.1. The first assertion is proved in the same way as Theorem (HK). The second property results from the following lemma.

LEMMA 2.2. *If $p > 0$, then*

1. $T_{\mu,\nu}(p) = \inf\{s \geq 0 : I_{s,p}(\mu, \nu) = \infty\}$,
2. $T_{\mu,\nu}(p) = \sup\{s \geq 0 : I_{s,p}(\mu, \nu) < \infty\}$,

where

$$I_{s,p}(\mu, \nu) = \int_{S_\mu} \left(\int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{s/p}} \right)^p d\nu(x).$$

Proof. Let s be a number such that $s < T_{\mu,\nu}(p)$. One has

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{s/p}} &\leq 1 + \sum_{n \geq 0} \int_{2^{-n-1} < |x-y| \leq 2^{-n}} \frac{d\mu(y)}{|x-y|^{s/p}} \\ &\leq 1 + \sum_{n \geq 0} 2^{(n+1)s/p} \mu(B(x, 2^{-n})) \end{aligned}$$

thus

$$I_{s,p}(\mu, \nu) \leq \int_{S_\mu} \left(1 + \sum_{n \geq 0} 2^{(n+1)s/p} \mu(B(x, 2^{-n})) \right)^p d\nu(x).$$

In the case $0 < p \leq 1$, we have

$$I_{s,p}(\mu, \nu) \leq 1 + \sum_{n \geq 0} 2^{(n+1)s} \int_{S_\mu} \mu(B(x, 2^{-n}))^p d\nu(x).$$

Then for a suitable choice of $\delta > 0$ such that $s + \delta < T_{\mu,\nu}(p)$, one has

$$I_{s,p}(\mu, \nu) \leq 1 + 2^s \sum_{n \geq 0} 2^{-n\delta} < \infty.$$

Now consider the case $p > 1$, remember that for every $a, b \geq 0$,

$$(a + b)^p \leq 2^{p-1}(a^p + b^p).$$

It results that

$$\left(\int_{\mathbb{R}^n} \frac{d\mu(y)}{|x - y|^{s/p}} \right)^p \leq 2^{p-1} + 2^{p-1} \left(\sum_{n \geq 0} 2^{(n+1)s/p} \mu(B(x, 2^{-n})) \right)^p$$

For $\alpha > 0$, Hölder inequality implies

$$\begin{aligned} & \left(\int_{\mathbb{R}^n} \frac{d\mu(y)}{|x - y|^{s/p}} \right)^p \\ & \leq 2^{p-1} + 2^{p-1} \left(\sum_{n \geq 0} 2^{(n+1)s+n\alpha p} \mu(B(x, 2^{-n}))^p \right) \left(\sum_{n \geq 0} 2^{\frac{-n\alpha p}{p-1}} \right)^{p-1} \end{aligned}$$

Further, if $\delta > 0$ such that $s + \delta < T_{\mu,\nu}(p)$ we obtain

$$I_{s,p}(\mu, \nu) \leq 2^{p-1} + C \sum_{n \geq 0} 2^{n(\alpha p - \delta)}$$

where C is constant depending only of p . Statement 1 follows by considering $\alpha < \delta/p$. To establish statement 2, fix $p > 0$. For $\varepsilon > 0$, one has

$$I_{s,p}(\mu, \nu) \geq \varepsilon^{-s} \int_{S_\mu} (\mu(B(x, \varepsilon)))^p d\nu(x)$$

Let $s > T_{\mu,\nu}(p)$ and $\delta > 0$ such that $s - \delta > T_{\mu,\nu}(p)$, then for small ε we have $I_{s,p}(\mu, \nu) \geq \varepsilon^{-\delta}$, hence $I_{s,p}(\mu, \nu)$ is infinite. ■

In order to prove the second statement of Theorem 2.1, it is sufficient to prove this implication

$$T_{\mu,\nu}(p) \leq m \implies T_{\mu,\nu}(p) \leq T_{\tilde{\mu},\tilde{\nu}}(p). \quad (1)$$

Let $s < T_{\mu,\nu}(p)$. One has

$$\int_{G_{n,m}} I_{s,p}(\tilde{\mu}, \tilde{\nu}) d\gamma_{n,m}(v) = \int_{S_\mu} \int_{G_{n,m}} \left(\int_{\mathbb{R}^n} \frac{d\mu(y)}{|p_v(x-y)|^{s/p}} \right)^p d\gamma_{n,m}(v) d\nu(x).$$

By Minkowski inequality [10],

$$\begin{aligned} \int_{G_{n,m}} \left(\int_{\mathbb{R}^n} \frac{d\mu(y)}{|p_v(x-y)|^{s/p}} \right)^p d\gamma_{n,m}(v) \\ \leq \left(\int_{\mathbb{R}^n} \left(\int_{G_{n,m}} \frac{d\gamma_{n,m}(v)}{|p_v(x-y)|^s} \right)^{1/p} d\mu(y) \right)^p. \end{aligned}$$

Since $s < m$,

$$\int_{G_{n,m}} \frac{d\gamma_{n,m}(v)}{|p_v(x-y)|^s} = \frac{C}{|x-y|^s},$$

(See [7], Corollary 3.12) where C is a constant depending only on m , n and s . Hence

$$\int_{G_{n,m}} I_{s,p}(\tilde{\mu}, \tilde{\nu}) d\gamma_{n,m}(v) \leq C I_{s,p}(\mu, \nu),$$

which shows that $I_{s,p}(\tilde{\mu}, \tilde{\nu})$ is finite for $\gamma_{n,m}$ -almost all $v \in G_{n,m}$ and implication (1) follows from Lemma 2.2.

3. APPLICATION

In the following section, we compare the multifractal spectrum of a measure μ and its projections $\tilde{\mu}$ more precisely. In fact, Theorem 2.1 allows us to establish a relationship between the Hausdorff dimension of the singularity spectrum of $\tilde{\mu}$ and the Legendre transform of the generalized prepacking dimensions Λ_μ introduced by Olsen [9]. Before detailing our results, let us recall the multifractal formalism introduced by Olsen.

For $E \subset \mathbb{R}^n$, $q, t \in \mathbb{R}$ and $\delta \geq 0$, we denote

$$\bar{P}_\mu^{q,t}(E) = \limsup_{\delta \rightarrow 0} \sum_i \mu(B(x_i, r_i))^q (2r_i)^t$$

where the supremum is taken over all centered δ -packing of E

$$P_\mu^{q,t}(E) = \inf_{E \subseteq \bigcup_i E_i} \sum_i \bar{P}_\mu^{q,t}(E_i)$$

It is the multifractal generalization of packing measure. The premeasures $\overline{P}_\mu^{q,t}$ assign in the usual way a dimension to each subset E of \mathbb{R}^n

$$\overline{P}_\mu^{q,t}(E) = \begin{cases} \infty & \text{for } t < \Delta_\mu^q(E), \\ 0 & \text{for } \Delta_\mu^q(E) < t. \end{cases}$$

The number $\Delta_\mu^q(E)$ is an extension of the prepacking dimension $\Delta(E)$ of E . Then we are able to define the function $\Lambda_\mu(q) = \Delta_\mu^q(S_\mu)$. Remark that Λ_μ is convex and decreasing (see [8]). This function is related to the multifractal spectrum of the measure μ . More precisely, if $f^*(x) = \inf_y (xy + f(y))$ denotes the Legendre transform of the function f , and if

$$\overline{X}_\mu(\alpha) = \left\{ x \in S_\mu : \limsup_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \leq \alpha \right\}.$$

is the set of singularity, it has been proved in [8] and [1] a lower bound estimate of the singularity spectrum using the Legendre transform of the function Λ_μ . Let recall that the singularity spectrum or the multifractal spectrum of a measure μ is the Hausdorff dimension of the set $\overline{X}_\mu(\alpha)$. Theorem 2.1 allows us to prove a similar inequality for $\tilde{\mu}$ and to compare the multifractal spectrum of μ and that of its projection.

Let q be a real number. We consider the following conditions:

(H₁) There exists a probability measure on S_μ such that

$$\nu(B(x, r)) \leq \overline{K} \mu(B(x, r))^q (2r)^{\Lambda_\mu(q)},$$

(H₂) $\Lambda'_\mu(q)$ exists,

(H₃) $T'_{\mu, \nu}(0)$ exists,

(H₄) $\Lambda_\mu^*(-\Lambda'_\mu(q)) \leq m$ and $T'_{\mu, \nu}(0) \leq m$,

(H₅) There exists a probability measure ν in S_μ such that there exist $\overline{K} > 0$ and $\underline{K} > 0$ satisfying

$$\underline{K} \mu(B(x, r))^q (2r)^{\Lambda_\mu(q)} \leq \nu(B(x, r)) \leq \overline{K} \mu(B(x, r))^q (2r)^{\Lambda_\mu(q)}$$

for all $x \in S_\mu$ and $r > 0$ small enough. ν is a Gibbs state at the point q .

THEOREM 3.1. *Under the assumptions (H₁), (H₂), (H₃) and (H₄) one has*

$$\dim \overline{X}_{\tilde{\mu}}(T'_{\tilde{\mu}, \tilde{\nu}}(0)) \geq \Lambda_\mu^*(-\Lambda'_\mu(q)) \quad \text{for } \gamma_{n,m}\text{-almost all } V \in G_{n,m},$$

where \dim denotes the Hausdorff dimension.

Before giving the proof of this theorem let us comment it.

Commentaries: 1) Under the hypothesis of Theorem 3.1 we have a relationship between the multifractal spectrum of the measure μ and that of its projection $\tilde{\mu}$. In fact, we have that $\dim \overline{X}_{\tilde{\mu}}(T'_{\tilde{\mu},\tilde{\nu}}(0)) \geq \dim \overline{X}_{\mu}(T'_{\mu,\nu}(0))$ for $\gamma_{n,m}$ -almost all $V \in G_{n,m}$, which constitute a natural prolongement of the results of Mattila, Howroyd and Falconer ([2], [3] and [6]) about the relationship between the dimension of a set or a measure and those of their projections.

2) Under the hypothesis (H_5) , we have $T_{\mu,\nu} = -C_{\mu}$ where C_{μ} is another scaling μ -function introduced by L. Olsen in [8]. So, if we replace (H_2) by (H_5) in Theorem 3.1, we have

$$\dim \overline{X}_{\tilde{\mu}}(-C'_{\tilde{\mu}}(0)) \geq \begin{cases} -\Lambda'_{\mu_+}(q)q + \Lambda_{\mu}(q) & \text{for } q \geq 0, \\ -\Lambda'_{\mu_-}(q)q + \Lambda_{\mu}(q) & \text{for } q < 0. \end{cases}$$

Let us prove the Theorem 3.1. One of the ingredients to prove this theorem is the following proposition.

PROPOSITION 3.2. *Let q be a real number. One has*

$$\Lambda'_{\mu_-}(q) \leq -T'_{\mu,\nu_-}(0) \leq \Lambda'_{\mu_+}(q).$$

Remark. If we replace the condition (H_1) by (H_5) we have also

$$\Lambda'_{\mu_+}(q) \leq -T'_{\mu,\nu_+}(0).$$

Proof of Proposition 3.2. It results from the lower bound

$$\limsup_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \leq T'_{\mu,\nu_-}(0) \quad \nu\text{-a.e. established in [8],}$$

and the lower bound

$$\liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \geq -\Lambda'_{\mu_+}(q) \quad \nu\text{-a.e. established in [1],}$$

that $-T'_{\mu,\nu_-}(0) \leq \Lambda'_{\mu_+}(q)$. To prove the other inequality, that is

$$\Lambda'_{\mu_-}(q) \leq -T'_{\mu,\nu_-}(0),$$

it is sufficient to show that

$$\Lambda_{\mu}(p + q) \geq \Lambda_{\mu}(q) - T_{\mu,\nu}(p) \tag{2}$$

for all $p < 0$.

Let $\rho > 0$ and $\varepsilon > 0$. It results from the definitions of $T_{\mu,\nu}$ and $\overline{P}_{\mu}^{p+q,\Lambda_{\mu}(q)-T_{\mu,\nu}(p)-\rho}$ that for $r > 0$ small enough,

$$1 \leq \int_{S_{\mu}} \mu(B(x, r))^p r^{-T_{\mu,\nu}(p)-\rho} d\nu(x),$$

and

$$\overline{P}_{\mu, \frac{r}{2}}^{p+q,\Lambda_{\mu}(q)-T_{\mu,\nu}(p)-\rho}(S_{\mu}) < \overline{P}_{\mu}^{p+q,\Lambda_{\mu}(q)-T_{\mu,\nu}(p)-\rho}(S_{\mu}) + \varepsilon.$$

By applying covering Besicovich lemma [4], we have

$$(\overline{K}\xi)^{-1} \leq \overline{P}_{\mu}^{p+q,\Lambda_{\mu}(q)-T_{\mu,\nu}(p)-\rho}(S_{\mu}) + \varepsilon.$$

So

$$\overline{P}_{\mu}^{p+q,\Lambda_{\mu}(q)-T_{\mu,\nu}(p)-\rho}(S_{\mu}) > 0,$$

in other words,

$$\Lambda_{\mu}(p+q) \geq \Lambda_{\mu}(q) - T_{\mu,\nu}(p) - \rho.$$

The arbitrary in ρ implies the inequality (2), which achieves the proof of the proposition. ■

Proof of Theorem 3.1. We have, $\dim \overline{X}_{\tilde{\mu}}(\alpha) \geq \inf(m, \dim \overline{X}_{\mu}(\alpha))$ for $\gamma_{n,m}$ -almost every $V \in G_{n,m}$, since $p_V(\overline{X}_{\mu}(\alpha)) \subset \overline{X}_{\tilde{\mu}}(\alpha)$. In particular, condition (H_4) implies

$$\dim \overline{X}_{\tilde{\mu}}(T'_{\mu,\nu}(0)) \geq \dim \overline{X}_{\mu}(T'_{\mu,\nu}(0)) \quad \text{for } \gamma_{n,m} \text{-almost all } V \in G_{n,m} \quad (3)$$

Hence, the proposition and the assumptions (H_2) and (H_3) give that $\Lambda'_{\mu}(q) = -T'_{\mu,\nu}(0)$. As a consequence, it follows from Theorem 2.1 that

$$\Lambda'_{\mu}(q) = -T'_{\tilde{\mu},\tilde{\nu}}(0) \quad \text{for } \gamma_{n,m} \text{-almost every } V \in G_{n,m}. \quad (4)$$

Thus, the result is a consequence from (3), (4) and the inequality

$$\dim \overline{X}_{\mu}(-\Lambda'_{\mu}(q)) \geq \Lambda_{\mu}^*(-\Lambda'_{\mu}(q)),$$

established in [1]. ■

ACKNOWLEDGEMENTS

The authors thank Jacques Peyrière and Julien Barral for helpful discussions they had with them. And they thanks the anonymous referee for his suggestions and remarks.

REFERENCES

- [1] BEN NASR, F., BHOURI, I., HEURTEAUX, Y., The validity of the multifractal formalism: results and examples, *Adv. Math.*, **165** (2) (2002), 264–284.
- [2] FALCONER, K.J., HOWROYD, J.D., Projection theorems for box and packing dimensions, *Math. Proc. Cambridge Philos. Soc.*, **119** (2) (1996), 287–295.
- [3] FALCONER, K.J., HOWROYD, J.D., Packing dimensions of projections and dimension profiles, *Math. Proc. Cambridge Philos. Soc.*, **121** (2) (1997), 269–286.
- [4] GUZMÁN, M. DE, “Differentiation of Integrals in \mathbb{R}^n ”, Lecture Notes in Mathematics, 481, Springer-Verlag, Berlin-New York, 1975.
- [5] HUNT, B.R., KALOSHIN, V.Y., How projections affect the dimension spectrum of fractal measures, *Nonlinearity*, **10** (5) (1997), 1031–1046.
- [6] MATTILA, P., Hausdorff dimension, orthogonal projections and intersections with planes, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **1** (2) (1975), 227–244.
- [7] MATTILA, P., “Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability”, Cambridge Studies in Advanced Mathematics, 44, Cambridge University Press, Cambridge, 1995.
- [8] OLSEN, L., A multifractal formalism, *Adv. Math.*, **116** (1) (1995), 82–196.
- [9] PESIN, Y.B., “Dimension Theory in Dynamical Systems. Contemporary Views and Applications”, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1997.
- [10] PÓLYA, G., SZEGÖ, G., “Problems and Theorems in Analysis”, Springer-Verlag, New York-Berlin, 1972.