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A New Proof of James' Sup Theorem

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1. Preliminaries and main results

1.1. Introduction. A normed space E is said to be reflexive if the canonical mapping j_E from E to the second continuous dual E'' is onto. We say that E is sup-reflexive if every f in the continuous dual E' reaches its upper bound on the closed unit ball B_E of E. It is an immediate consequence of the Tychonov-Alaoglu Theorem that reflexivity implies sup-reflexivity. That, conversely, every sup-reflexive Banach space is reflexive is the famous James' sup Theorem (see [12], [13]). James' original proof was rather long and involved but there exist simpler proofs (see [19], [21], [22], [5], and [3, Theorem 3.2] in the separable case).

We say that E is J-reflexive (James-reflexive) if B_E does not contain any sequence $(a_n)_{n\in\mathbb{N}}$ satisfying

$$\inf_{n\in\mathbb{N}} \operatorname{dist}(\operatorname{span}\{a_i : i < n\}, \operatorname{conv}\{a_i : i \geq n\}) > 0$$

Indeed James proves that every sup-reflexive normed space is J-reflexive, and that every J-reflexive Banach space is reflexive. The second implication has short proofs (see for example [18] and [15, Theorem 3.9]). In this paper, we provide a new proof of the first implication. Notice that, along the way, we give a short proof of the J-reflexivity of sup-reflexive separable normed spaces (Corollary 2 of Section 2).

The key point in the general case is the following dichotomy result (see Theorem 2 below), which extends a result due to Hagler and Johnson (see [6]): given a normed space E, either E contains an asymptotically isometric copy of

$\ell^1(\mathbb{N})$, or every bounded sequence of the continuous dual E' admits a sequence of normalized blocks pointwise converging to 0.

1.2. Presentation of the results. Given a real vector space E and a sequence $(x_n)_{n\in\mathbb{N}}$ in E, a sequence $(b_n)_{n\in\mathbb{N}}$ in E is a block sequence of $(x_n)_n$ if there exists a sequence $(F_n)_{n\in\mathbb{N}}$ of pairwise disjoint finite subsets of $\mathbb{N}$ and a sequence $(\lambda_i)_{i\in\mathbb{N}}$ of real numbers such that for every $n \in \mathbb{N}$, $b_n = \sum_{i\in F_n} \lambda_i x_i$. If for each $n \in \mathbb{N}$, $\sum_{i\in F_n} |\lambda_i| = 1$, the block sequence $(b_n)_{n\in\mathbb{N}}$ is said to be normalized. If in addition, for every $i \in \cup_{n\in\mathbb{N}} F_n$, $\lambda_i \geq 0$, we say that the block sequence is convex. In particular, every infinite subsequence is a convex block sequence. The sequence $(F_n)_n$ is called a sequence of supports of the block sequence $(b_n)_n$. We say that a topological space is sequentially compact if every sequence of this space has an infinite subsequence which converges. We say that a subset C of a topological vector space E is block compact if every sequence of C has a normalized block sequence $(b_n)_{n\in\mathbb{N}}$ which converges in E; notice that without loss of generality one may assume that it converges to 0: consider the normalized block sequence $(\frac{b_{2n}-b_{2n+1}}{2})_{n\in\mathbb{N}}$. We say that C is convex block compact if every sequence of C has a convex block sequence which converges in E.

Using Simons' inequality and Rosenthal's ℓ^1-theorem, we prove the following result (see Section 2):

Theorem 1. *Given a sup-reflexive normed space E, if its dual ball $B_{E'}$ is weak* block compact, then E is J-reflexive.*

We then prove (Section 3):

Theorem 2. *If a normed space E does not contain any asymptotically isometric copy of $\ell^1(\mathbb{N})$, then its dual ball $B_{E'}$ is weak* block compact.*

Here, we say that E contains an asymptotically isometric copy of $\ell^1(\mathbb{N})$ if there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in B_E, and some sequence $(\delta_n)_{n\in\mathbb{N}}$ in $]0,1[$ converging to 1 satisfying the following inequality, for every finite sequence $(\lambda_i)_{0\leq i\leq n}$ in $\mathbb{R}$:

$$\sum_{0\leq i\leq n} \delta_i |\lambda_i| \leq \left\| \sum_{0\leq i\leq n} \lambda_i a_i \right\|$$

Our Theorem 2 generalizes a result due to Hagler and Johnson ([6] or [16]), where the normed space E contains no isomorphic copy of $\ell^1(\mathbb{N})$.

Remark 1. Notice that (see [4]) there is an equivalent norm N on $\ell^1(\mathbb{N})$ forbidding any asymptotically isometric copy of $\ell^1(\mathbb{N})$ in the renormed space $(\ell^1(\mathbb{N}), N)$. Thus, there are normed spaces having isomorphic copies of $\ell^1(\mathbb{N})$, but having no asymptotically isometric copies of $\ell^1(\mathbb{N})$.

We finally prove (Section 4):

THEOREM 3. *A sup-reflexive normed space does not contain any asymptotically isometric copy of* $\ell^1(\mathbb{N})$.

This result is a straightforward generalization of a short theorem due to James (see [11, Theorem 2 p. 209]), which asserts that there is a linear continuous functional $f : \ell^1(\mathbb{N}) \to \mathbb{R}$ such that, for every normed space E containing $\ell^1(\mathbb{N})$, there is a norm-preserving linear extension of f that does not attain its norm on B_E.

Thus we get the following result that yields a new proof of James' sup theorem:

COROLLARY 1. *Every sup-reflexive normed space is J-reflexive.*

Proof. Let E be a sup-reflexive normed space. According to Theorem 3, the space E does not contain any asymptotically isometric copy of $\ell^1(\mathbb{N})$; so, with Theorem 2, $B_{E'}$ is weak* block compact; whence, by Theorem 1, E is J-reflexive. ∎

Remark 2. James (see [10]) gave an example of a sup-reflexive normed space which is not complete (hence which is not reflexive).

2. Spaces with a weak* block compact dual ball

2.1. Spaces with a weak* convex block compact dual ball.

THEOREM. (Simons' inequality [20]) *Let S be a set and $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of $\ell^\infty(S)$. Denote by Λ the set of sequences $(\lambda_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$ satisfying $\sum_{n\in\mathbb{N}} \lambda_n = 1$. Assume that for every $(\lambda_n)_{n\in\mathbb{N}} \in \Lambda$, the infinite convex combination $\sum_{n\in\mathbb{N}} \lambda_n f_n$ reaches its upper bound on S. Then*

$$\inf\{\sup_S \sum_{n\in\mathbb{N}} \lambda_n f_n \ : \ (\lambda_n)_{n\in\mathbb{N}} \in \Lambda\} \leq \sup_S \limsup_{n\in\mathbb{N}} f_n$$

Given a normed space E, and a real number $\vartheta > 0$, we say that a sequence $(a_n)_{n\in\mathbb{N}}$ in E is a ϑ-sequence if $\inf_{n\in\mathbb{N}} d(\mathrm{span}\{a_i : i < n\}, \mathrm{conv}\{a_i : i \geq n\}) \geq \vartheta$. Given a ϑ-sequence $(a_n)_{n\in\mathbb{N}}$ in B_E, the Hahn-Banach theorem implies a sequence $(f_n)_{n\in\mathbb{N}}$ in $B_{E'}$ satisfying $f_n(a_i) = 0$ if $i < n$ and $f_n(a_i) \geq \vartheta$ if $n \leq i$. Indeed, since the distance between the convex sets $\mathrm{span}\{a_i : i \leq n\}$ and $\mathrm{conv}(\{a_i : i > n\})$ is $\geq \vartheta$, there exists some $f \in \Gamma_{E'}$ satisfying $\sup_{\mathrm{span}\{a_i : i \leq n\}} f + \vartheta \leq \inf_{\mathrm{conv}\{a_i : i > n\}} f$. Since the linear functional f is bounded on the vector subspace $\mathrm{span}\{a_i : i \leq n\}$, f is null on this subspace. Call such a sequence $(a_n, f_n)_{n\in\mathbb{N}}$ in $B_E \times B_{E'}$ a ϑ-triangular sequence of E. Thus, a normed space E is J-reflexive if and only if it has no ϑ-triangular sequence for any $\vartheta > 0$.

LEMMA 1. *Given a sup-reflexive space E, $\vartheta > 0$, and a ϑ-triangular sequence $(a_n, f_n)_{n\in\mathbb{N}}$ of E, no convex block sequence of $(f_n)_{n\in\mathbb{N}}$ pointwise converges.*

Proof. Seeking for a contradiction, assume that some convex block sequence $(b_n)_{n\in\mathbb{N}}$ pointwise converges to some f. Without loss of generality, we may assume that the sequence of supports $(F_n)_{n\in\mathbb{N}}$ satisfies $F_0 < F_1 < \cdots < F_n < \ldots$. Observe that for every $n \in \mathbb{N}$, $f(a_n) = 0$. Then denoting by h_n the mapping $\frac{b_n - f}{2}$, and by d_n the last element of F_n, the sequence $(a_{d_n}, h_n)_{n\in\mathbb{N}}$ is $\frac{\vartheta}{2}$-triangular. Using Simons' inequality and the sup-reflexivity of E, there exists some finite convex combination $g := \sum_{n\in F} \lambda_n h_n$ of $(h_n)_{n\in\mathbb{N}}$ such that $\sup_{B_E} g \leq \frac{\vartheta}{4}$; but for any integer $N > \max F$, $g(a_N) = \sum_{i\in F} \lambda_i h_i(a_N) \geq \frac{\vartheta}{2}$: contradiction! ∎

COROLLARY 2. *A sup-reflexive normed space with a weak* convex block compact dual ball is J-reflexive. In particular, a sup-reflexive separable normed space is J-reflexive.*

Proof. Indeed, given a separable normed space E, the closed unit ball of E', being homeomorphic with a closed subset of $[-1, 1]^{\mathbb{N}}$, is weak* sequentially compact. ∎

2.2. SPACES WITH WEAK* BLOCK COMPACT DUAL BALL. Recall that a bounded sequence $(f_n)_{n\in\mathbb{N}}$ in a normed space E is equivalent to the canonical basis of $\ell^1(\mathbb{N})$ if there exists some real number $m > 0$ satisfying $m \sum_{n\in\mathbb{N}} |\lambda_n| \leq \left\| \sum_{n\in\mathbb{N}} \lambda_n f_n \right\|$ for every $(\lambda_n)_n \in \ell^1(\mathbb{N})$: if in addition, for every $n \in \mathbb{N}$, $\|f_n\| \leq 1$, we say that $(f_n)_{n\in\mathbb{N}}$ is m-equivalent to the canonical basis of $\ell^1(\mathbb{N})$.

Theorem. (Rosenthal's ℓ^1-theorem) *Given a set X and a bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $\ell^\infty(X)$, there exists a subsequence of $(f_n)_{n\in\mathbb{N}}$ which pointwise converges, or there exists a subsequence which is equivalent to the canonical basis of $\ell^1(\mathbb{N})$.*

Proof. (of Theorem 1) Seeking for a contradiction, assume that some supreflexive normed space E is not J-reflexive, though $B_{E'}$ is weak* block compact. Non J-reflexivity of E yields some ϑ-triangular sequence $(a_n, f_n)_{n\in\mathbb{N}}$ of E, with $\vartheta > 0$. Then, with Lemma 1, no infinite subsequence of $(f_n)_{n\in\mathbb{N}}$ pointwise converges, so, using Rosenthal's ℓ^1-theorem, there exists some infinite subsequence $(f_n)_{n\in A}$ and some $m > 0$ such that the bounded sequence $(f_n)_{n\in A}$ is m-equivalent to the canonical basis of $\ell^1(\mathbb{N})$. Now, by the weak* block compactness of $B_{E'}$, $(f_n)_{n\in A}$ has a normalized block sequence $(b_n)_{n\in\mathbb{N}}$ weak* converging to 0. Using Simons' inequality, there exists some finite convex combination $g := \sum_{i\in F} \lambda_i b_i$ of $(b_n)_{n\in\mathbb{N}}$ such that $\|g\| = \sup_{B_E} g \leq \frac{m}{2}$; but, since the block sequence $(b_n)_{n\in\mathbb{N}}$ is normalized, it is also m-equivalent to the canonical basis of $\ell^1(\mathbb{N})$, hence $\|g\| \geq m \sum_{i\in F} |\lambda_i| = m$: the contradiction! ∎

3. Extension of a theorem by Hagler and Johnson

Notation 1. ([6]) If $(b_n)_{n\in\mathbb{N}}$ is a normalized block sequence of a sequence $(x_n)_{n\in\mathbb{N}}$ of a real vector space, we write $(b_n)_n \prec (x_n)_n$.
Given a set X, for every bounded sequence $(f_n)_{n\in\mathbb{N}}$ of $\ell^\infty(X)$, and every subset K of X, let

$$\delta_K(f_n)_n := \sup_K \limsup_n f_n$$

$$\varepsilon_K(f_n)_n := \inf\{\delta_K(h_n)_n : \ (h_n)_n \prec (f_n)_n\}$$

If $(h_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\ell^\infty(X)$, and if for some $n_0 \in \mathbb{N}$, $(h_n)_{n\geq n_0}$ is a normalized block sequence of $(f_n)_n$, then $\delta_K(h_n)_n \leq \delta_K(f_n)_n$ and $\varepsilon_K(f_n)_n \leq \varepsilon_K(h_n)_n$. When K is a symmetric subset of a real vector space X, and when each f_n is linear, then $(f_n)_n$ pointwise converges to 0 on K if and only if $\delta_K(f_n)_n = 0$.

Given a metric space (X, d), for every $x \in X$ and every real number $r > 0$ we denote by $B(x, r)$ the open ball $\{y \in X : \ d(x, y) < r\}$.

Lemma 2. (Quantifier permuting) *Let (K, d) be a precompact metric space, $\lambda > 0$, and $(f_n)_{n\in\mathbb{N}}$ be a sequence of λ-Lipschitz real mappings on K. If $\delta_K(f_n)_n \leq 1$ then, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ satisfying for every $n \geq N$, $\sup_K f_n \leq 1 + \varepsilon$.*

Proof. Let $\eta \in]0, \varepsilon[$. Given $x \in K$, there exists some finite subset F_x of $\mathbb{N}$ satisfying for every $n \in \mathbb{N} \backslash F_x$, $f_n(x) < 1 + \eta$; thus, denoting by ρ the positive number $\frac{\varepsilon - \eta}{\lambda}$, for every $n \in \mathbb{N} \backslash F_x$ and for every $y \in B(x, \rho)$, $f_n(y) < 1 + \varepsilon$. Now the precompact set K is contained in a finite union of the form $\bigcup_{1 \le k \le N} B(x_k, \rho)$. Let F be the finite set $\bigcup_{1 \le k \le N} F_{x_k}$. Then, for every $y \in K$, and for every $n \in \mathbb{N} \backslash F$, $f_n(y) < 1 + \varepsilon$. ∎

LEMMA 3. ([6, Proof of Theorem 1]) *Given a set X and a bounded sequence $(f_n)_{n \in \mathbb{N}}$ in $\ell^\infty(X)$, there exists a normalized block sequence $(b_n)_{n \in \mathbb{N}}$ of $(f_n)_n$ such that $\varepsilon_X(b_n)_n = \delta_X(b_n)_n$.*

Proof. Diagonalization. Choose some normalized block sequence $(h_n^0)_{n \in \mathbb{N}}$ of $(f_n)_n$ such that $\delta_X(h_n^0)_n \le \varepsilon_X(f_n)_n + \frac{1}{2^0}$, and then, for every $i \in \mathbb{N}$, inductively choose some normalized block sequence $(h_n^{i+1})_{n \in \mathbb{N}}$ of $(h_n^i)_n$ such that $\delta_X(h_n^{i+1})_n \le \varepsilon_X(h_n^i)_n + \frac{1}{2^{i+1}}$. For every $n \in \mathbb{N}$, let $b_n := h_n^n$: then $(b_n)_n \prec (f_n)_n$; moreover, given a normalized block sequence $(k_n)_{n \in \mathbb{N}}$ of $(b_n)_n$, for every $i \in \mathbb{N}$, $\delta_X(b_n)_n \le \delta_X(h_n^{i+1})_n \le \varepsilon_X(h_n^i)_n + \frac{1}{2^{i+1}} \le \delta_X(k_n)_n + \frac{1}{2^{i+1}}$ whence $\delta_X(b_n)_n \le \varepsilon_X(b_n)_n$. ∎

Notation 2. We denote by $\mathcal{S}$ be the set of all finite sequences in $\{0, 1\}$.

We say that a family $(A_\sigma)_{\sigma \in \mathcal{S}}$ of infinite subsets of $\mathbb{N}$ is a tree (of subsets of $\mathbb{N}$) if for every $\sigma \in \mathcal{S}$, $A_{\sigma \frown 0}$ and $A_{\sigma \frown 1}$ are disjoint subsets of A_σ, where $A_{\sigma \frown i} \in \mathcal{S}$ is the sequence obtained from σ by adding a last term equal to i.

Notation 3. Given an infinite subset A of $\mathbb{N}$, we denote by $i \mapsto i_A$ the increasing mapping from $\mathbb{N}$ onto A.

Proof. (of Theorem 2) We essentially follow the proof of Hagler and Johnson, extending it with the help of Lemma 2. Assuming that E is a normed space, and that $(g_n)_{n \in \mathbb{N}}$ is a bounded sequence in E' without any normalized block sequence pointwise converging to 0, we have to show that E contains an asymptotically isometric copy of $\ell^1(\mathbb{N})$. Using Lemma 3, the sequence $(g_n)_{n \in \mathbb{N}}$ has a normalized block sequence $(f_n)_{n \in \mathbb{N}}$ satisfying $\varepsilon_{B_E}(f_n)_n = \delta_{B_E}(f_n)_n > 0$. We may assume that $\varepsilon_{B_E}(f_n)_n = \delta_{B_E}(f_n)_n = 1$. Let $(u_n)_{n \in \mathbb{N}}$ be a sequence in $]0, 1/3[$ decreasing to 0; for every $n \in \mathbb{N}$, let $\varepsilon_n := \frac{u_n}{2^n}$ and let $\delta_n := 1 - \varepsilon_n$. We build a tree $(A_\sigma)_{\sigma \in \mathcal{S}}$ with $A_\varnothing = \mathbb{N}$, and a sequence $(a_n)_{n \ge 1}$ in B_E satisfying, for every $n \ge 1$, $\sigma = (\alpha_1, \dots, \alpha_n) \in \{-1, 1\}^n$, and $i \in A_\sigma$,

$$\|f_i \restriction \operatorname{span}\{a_1, \dots, a_n\}\| \le 1 + \varepsilon_n \text{ and } \begin{cases} f_i(a_n) \ge 1 - 3u_n & \text{if } \alpha_n = 1 \\ f_i(a_n) \le -1 + 3u_n & \text{if } \alpha_n = 0 \end{cases}$$

Then, with

$$P_n := \{f \in E' : f(a_n) \geq 1 - 3u_n \text{ and } \|f\restriction \operatorname{span}\{a_1, \dots, a_n\}\| \leq 1 + \varepsilon_n\}$$
$$Q_n := \{f \in E' : f(a_n) \leq -1 + 3u_n \text{ and } \|f\restriction \operatorname{span}\{a_1, \dots, a_n\}\| \leq 1 + \varepsilon_n\}$$

it will follow that $(P_n, Q_n)_{n\geq 1}$ is independent (for every disjoint finite subsets F, G of $\mathbb{N}\backslash\{0\}$, $\bigcap_{n\in F} P_n \cap \bigcap_{n\in G} Q_n$ is non-empty). This will imply that the sequence $(a_n)_{n\geq 1}$ in B_E is asymptotically isometric to the canonical basis of $\ell^1(\mathbb{N})$: indeed, given real numbers $\lambda_1, \dots, \lambda_n$, letting $f \in \bigcap_{\{i:\lambda_i>0\}} P_i \cap \bigcap_{\{i:\lambda_i<0\}} Q_i$,

$$\begin{aligned}\|f\restriction \operatorname{span}\{a_1, \dots, a_n\}\| \Big\|\sum_{1\leq i\leq n} \lambda_i a_i\Big\| &\geq f\big(\sum_{1\leq i\leq n} \lambda_i a_i\big)\\ &\geq \big(\sum_{\{i:\lambda_i>0\}} \lambda_i(1 - 3u_i) + \sum_{\{i:\lambda_i<0\}} \lambda_i(-1 + 3u_i)\big) \geq \sum_{1\leq i\leq n} |\lambda_i|(1 - 3u_i)\end{aligned}$$

whence $\left\|\sum_{1\leq i\leq n} \lambda_i a_i\right\| \geq \frac{1}{1+\varepsilon_n} \sum_{1\leq i\leq n} |\lambda_i|(1 - 3u_i) \geq \sum_{1\leq i\leq n} |\lambda_i| \frac{1-3u_i}{1+\varepsilon_i}$, with the sequence $(\frac{1-3u_i}{1+\varepsilon_i})_{i\in\mathbb{N}}$ in $]0, 1[$ converging to 1.

Building a_{n+1} and $(A_\sigma)_{\sigma\in\{-1,1\}^{n+1}}$ from $(A_\sigma)_{\sigma\in\{-1,1\}^n}$ and $(a_i)_{1\leq i\leq n}$. For every $\sigma \in \{-1, 1\}^n$, we consider two infinite disjoint subsets L_σ and R_σ of A_σ, and we define the following normalized block sequence $(h_i^n)_{i\in\mathbb{N}}$ of $(f_n)_n$:

$$h_i^n := \frac{1}{2^n} \sum_{\sigma\in\{-1,1\}^n} \frac{f_{i_{R_\sigma}} - f_{i_{L_\sigma}}}{2}$$

Since $\delta_{B_E}(h_i^n)_i \geq 1$, there is some $a_{n+1} \in B_E$ satisfying $\limsup_i h_i^n(a_{n+1}) > \delta_{n+1}$ and in particular, the set $J := \{i \in \mathbb{N} : h_i^n(a_{n+1}) > \delta_{n+1}\}$ is infinite. Since the closed unit ball K of the finite dimensional space $\operatorname{span}\{a_1, \dots, a_{n+1}\}$ is compact and $\delta_K(f_i)_i \leq \delta_{B_E}(f_i)_i \leq 1$, Lemma 2 implies the existence of some $N \in \mathbb{N}$ satisfying for every $i \geq N$, $\|f_i\restriction \operatorname{span}\{a_1, \dots, a_{n+1}\}\| \leq 1 + \varepsilon_{n+1}$. Let $J' := \{i \in J : i \geq N\}$. Now, given any $\sigma \in \{-1, 1\}^n$, for every $i \in J'$, since every $\tau \in \{-1, 1\}^n$ satisfies $i_{R_\tau}, i_{L_\tau} \geq i \geq N$,

$$\begin{aligned}\frac{f_{i_{R_\sigma}}(a_{n+1}) - f_{i_{L_\sigma}}(a_{n+1})}{2} &= 2^n h_i^n(a_{n+1}) - \sum_{\substack{\tau\in\{-1,1\}^n\\ \tau\neq\sigma}} \frac{f_{i_{R_\tau}}(a_{n+1}) - f_{i_{L_\tau}}(a_{n+1})}{2}\\ &\geq 2^n \delta_{n+1} - (2^n - 1)(1 + \varepsilon_{n+1})\\ &= 2^n(1 - \varepsilon_{n+1}) - (2^n - 1)(1 + \varepsilon_{n+1})\\ &= 1 - (2^{n+1} - 1)\varepsilon_{n+1} \geq 1 - 2^{n+1}\varepsilon_{n+1} = 1 - u_{n+1}\end{aligned}$$

Thus, $f_{i_{R_\sigma}}(a_{n+1}) \geq 2(1-u_{n+1}) + f_{i_{L_\sigma}}(a_{n+1}) \geq 2(1-u_{n+1}) - (1+\varepsilon_{n+1}) = 1 - u_{n+1}(2+\frac{1}{2^{n+1}}) \geq 1 - 3u_{n+1}$. Likewise, $f_{i_{L_\sigma}}(a_{n+1}) \leq -1 + 3u_{n+1}$. For every $\sigma \in \{-1,1\}^n$, we define $A_{\sigma\frown 0} := \{i_{L_\sigma} : i \in J'\}$ and $A_{\sigma\frown 1} := \{i_{R_\sigma} : i \in J'\}$. ∎

4. No asymptotically isometric copy of $\ell^1(\mathbb{N})$ in sup-reflexive spaces

Proof. (of Theorem 3) Assume the existence of some sequence $(a_n)_{n\in\mathbb{N}}$ in B_E, asymptotically isometric with the canonical basis of $\ell^1(\mathbb{N})$, witnessed by a sequence of coefficients $(\delta_i)_{i\in\mathbb{N}}$ in $]0,1]$ converging to 1. Let $V := \mathrm{span}\{a_n : n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$, consider the linear mapping $g_n : V \to \mathbb{R}$ such that $g_n(a_i) = -\delta_i$ if $i < n$ and $g_n(a_i) = \delta_i$ if $n \leq i$; then, for every sequence $(\lambda_i)_{i\in\mathbb{N}} \in \mathbb{R}$ with $\lambda_i = 0$ for all but finitely many i's, $|g_n(\sum_{j=0}^{\infty} \lambda_j a_j)| = |\sum_{j<n} -\lambda_j\delta_j + \sum_{j\geq n}\lambda_j\delta_j| \leq \sum_{j=0}^{\infty}|\lambda_j|\delta_j \leq \left\|\sum_{j=0}^{\infty}\lambda_j a_j\right\|$ whence g_n is continuous and $\|g_n\| \leq 1$; also for every integer $i \geq n$, $g_n(a_i) = \delta_i$, whence $\lim_{i\to+\infty} g_n(a_i) = 1$; so $\|g_n\| = 1$. With Hahn-Banach, for each $n \in \mathbb{N}$, extend g_n to some $\tilde{g}_n \in S_{E'}$. Let W be the vector subspace of elements $x \in E$ such that $\big(\tilde{g}_n(x)\big)_{n\in\mathbb{N}}$ converges. Then the linear functional $g := \lim_n \tilde{g}_n$ is continuous with norm ≤ 1 on W; extend it to some element $\tilde{g} \in B_{E'}$. Now consider some sequence $(\alpha_i)_{i\in\mathbb{N}}$ in $]0,1[$ such that $\sum_{i\in\mathbb{N}}\alpha_i = 1$ and let $h := \sum_{k\in\mathbb{N}}\alpha_k\tilde{g}_k - \tilde{g}$. Clearly, $\|h\| \leq 2$. Moreover, for every $n \in \mathbb{N}$, $h(a_n) = \sum_{k\leq n}\alpha_k\delta_n - \sum_{k>n}\alpha_k\delta_n + \delta_n = 2\delta_n\sum_{k\leq n}\alpha_k$, thus $\lim_n h(a_n) = 2$. So $\|h\| = 2$. By sup-reflexivity of E, let $u \in B_E$ be such that $h(u) = 2$. Observe that $\tilde{g}(u) = -1$, and for every $k \in \mathbb{N}$, $\tilde{g}_k(u) = 1$ (notice that for each k, $\alpha_k \neq 0$); now $u \in W$ therefore $g(u) = \lim_k \tilde{g}_k(u) = 1$, contradicting $\tilde{g}(u) = -1$! ∎

5. Set-theoretical comments

5.1. Set-theory without choice ZF. Rosenthal's Theorem is a choiceless consequence (see [14, p.135-136]) of the following choiceless result (see for example [1]):

Theorem. (Cohen, Ehrenfeucht, Galvin (1967)) *Every open subset of* $[\mathbb{N}]^{\mathbb{N}}$ *(the set of infinite subsets of* $\mathbb{N}$ *endowed with the topology induced by the product topology on* $\{0,1\}^{\mathbb{N}}$*) is Ramsey.*

The proof of Simons' inequality given in [20] (see also [17]) is choiceless: use convex combinations with finite supports and rational coefficients. Our proofs of Theorems 1 and 3 rely on the Hahn-Banach axiom HB, while our proof of Theorem 2 relies on the axiom of Dependent Choices DC (see [9]), thus our proof of Corollary 1 relies on HB + DC. More generally, for a given Banach space, J-reflexivity, reflexivity, "weak compactness of the unit ball", sup-reflexivity, Smulian-reflexivity, and convex-reflexivity (see [15]) are equivalent in $\mathrm{ZF} + \mathrm{DC} + \mathrm{HB}$.

Question 1. Is there some "usual" notion of reflexivity which, for Banach spaces, is not equivalent to "sup-reflexivity" in $\mathrm{ZF} + \mathrm{DC} + \mathrm{HB}$? Is there some "usual" notion of reflexivity which, for separable Banach spaces, is equivalent in ZF, neither to J-reflexivity nor to reflexivity?

5.2. Block compactness. Given a normed space E, obviously,

$$B_{E'} \text{ weak* sequentially compact} \Rightarrow \\ B_{E'} \text{ weak* convex block compact } \Rightarrow B_{E'} \text{ weak* block compact}$$

The first implication is not reversible in set-theory with choice ZFC (see [7]): notice that the construction of the space built there depends on a well-order on $\mathbb{R}$.

Remark 3. The dual ball of a normed space containing an isomorphic copy of $\ell^1(\mathbb{R})$ is not weak* block compact.

Proof. Using the Hahn-Banach Theorem, it is sufficient to prove that the dual ball of $\ell^1(\mathbb{R})$ is not weak* block compact. Let $F := \ell^1(\{0,1\}^{\mathbb{N}})$. We have to show that the closed unit ball of $F' = \ell^\infty(\{0,1\}^{\mathbb{N}})$ is weak* block compact. For every $n \in \mathbb{N}$, denote by $p_n : \{0,1\}^{\mathbb{N}} \to \{0,1\}$ the canonical projection. Of course, $(p_n)_n$ is a bounded sequence in F', but no normalized block sequence of $(p_n)_n$ converges to 0. Indeed, let $(b_n)_{n\in\mathbb{N}}$ be a normalized block sequence of $(p_n)_{n\in\mathbb{N}}$. For every $n \in \mathbb{N}$, b_n is of the form $\sum_{i\in F_n} \lambda_i p_i$ where the finite subsets F_n of $\mathbb{N}$ are pairwise disjoint and the coefficients λ_i are real. For every $n \in \mathbb{N}$, let $F_n^+ := \{i \in F_n : \lambda_i > 0\}$ and $F_n^- := \{i \in F_n : \lambda_i < 0\}$. Consider the subsets $A := \cup_{n\in\mathbb{N}}(F_{2n}^+ \cup F_{2n+1}^-)$ and $B := \cup_{n\in\mathbb{N}}(F_{2n}^- \cup F_{2n+1}^+)$ of $\mathbb{N}$. If $(b_n(A))_{n\in\mathbb{N}}$ converges, then its limit is 0 whence $(b_{2n}(B))_{n\in\mathbb{N}}$ converges to -1 and $(b_{2n+1}(B))_{n\in\mathbb{N}}$ converges to 1, thus $(b_n(B))_{n\in\mathbb{N}}$ does not converge. ∎

Remark 4. Using Theorem 2, it follows that any space isomorphic with $\ell^1(\mathbb{R})$ contains an asymptotically isometric copy of $\ell^1(\mathbb{N})$.

Recall that CH denotes the Continuum Hypothesis and that MA denotes Martin's Axiom.

Remark 5. In $\mathrm{ZFC} + \mathrm{MA} + \neg\mathrm{CH}$, the following conditions are equivalent for a given normed space E:

1. E does not contain any isomorphic copy of $\ell^1(\mathbb{R})$;
2. the ball $B_{E'}$ is weak* convex block compact;
3. the ball $B_{E'}$ is weak* block compact.

Proof. The implication (1) $\Rightarrow$ (2) is due to Haydon, Levy and Odell (see [8]). The implication (2) $\Rightarrow$ (3) is trivial and the implication (3) $\Rightarrow$ (1) is Remark 3. ∎

QUESTION 2. Is the implication (3) $\Rightarrow$ (2) provable in ZFC?

QUESTION 3. According to a theorem due to Bourgain ([2]), "The dual ball of a normed space not containing any isomorphic copy of $\ell^1(\mathbb{N})$ is weak* convex block compact". Does this result persist in ZFC for normed spaces which do not contain asymptotically isometric copies of $\ell^1(\mathbb{N})$? (Using Remarks 4 and 5, the answer is positive in $\mathrm{ZFC} + \mathrm{MA} + \neg\mathrm{CH}$.)

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