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Families of Almost Disjoint Hamel Bases

Lorenz Halbeisen

Institut für Informatik und angewandte Mathematik,
Theoretische Informatik und Logik, Universität Bern,
Neubrückstr. 10, 3012 *Bern, Switzerland. halbeis@iam.unibe.ch*

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For any set S, let $|S|$ denote the cardinality of S and let $\mathscr{P}(S)$ denote the set of all subsets of S. If $|S| = \kappa$, then $|\mathscr{P}(S)|$ is also denoted by 2^κ.

It is known that for any infinite dimensional Banach space X and for any Hamel basis H of X we have $|H| = |X|$, which is at least the cardinality of the continuum (see the remark at the end of this note). By stretching the vectors of some Hamel basis, it is easy to construct $|\mathscr{P}(X)|$ different Hamel bases of X, and even if we consider just normalized Hamel bases, *i.e.*, each vector has norm one, there are still $|\mathscr{P}(X)|$ different normalized Hamel bases of X (see [1, Proposition 2.1]):

Fact. Every Banach space X over a complete field has $|\mathscr{P}(X)|$ different normalized Hamel bases.

Let X be an arbitrary infinite dimensional real or complex Banach space of cardinality κ. Obviously, one cannot aim for more than 2^κ different normalized Hamel bases, but one could try to find a family of 2^κ different normalized Hamel bases such that the cardinality of the intersection of any two of them is less than κ. In the sequel, two such Hamel bases of X will be called almost disjoint. In [1] it is asked whether every infinite dimensional Banach space of cardinality κ admits 2^κ pairwise almost disjoint normalized Hamel bases (see [1, Question 4]). In the following, a complete answer to this question is given by transforming first the problem into a purely set-theoretical statement and then using results by Baumgartner, Sierpiński, and Tarski.

For infinite cardinals κ, λ, μ let $A(\kappa, \lambda, \mu)$ be the following statement (cf. [2, p. 406]): There exists a family $\mathscr{F} \subseteq \mathscr{P}(\kappa)$ such that

- $|\mathscr{F}| = \lambda$,
- for all $S \in \mathscr{F}$, $|S| = \mu$,
- for all $S_1, S_2 \in \mathscr{F}$ with $S_1 \neq S_2$, $|S_1 \cap S_2| < \mu$.

THEOREM. *An infinite dimensional Banach space X of cardinality κ admits λ pairwise almost disjoint normalized Hamel bases if and only if $A(\kappa, \lambda, \kappa)$.*

Proof. ($\Rightarrow$) This follows by the fact that every Hamel basis of X has cardinality κ.

($\Leftarrow$) Assume $A(\kappa, \lambda, \kappa)$ and let $\mathscr{F} = \{S_\alpha : \alpha < \lambda\} \subseteq \mathscr{P}(\kappa)$ be the corresponding family of subsets of κ. By the properties of $\mathscr{F}$, for all $\alpha < \lambda$ we have $|S_\alpha| = |\kappa \setminus S_\alpha| = \kappa$, and so, for each $\alpha < \lambda$ we can define a bijection f_α between S_α and $\kappa \setminus S_\alpha$. Let now $H \subseteq X$ be any normalized Hamel basis of X. Since $|H| = \kappa$, there exists a bijection g between κ and H. For $S \subseteq \kappa$ let $g[S] := \{g(\gamma) : \gamma \in S\}$. Further, for each $\alpha < \lambda$ let

$$\tilde{H}_\alpha = g[S_\alpha] \cup \big\{g(\gamma) + 2 \cdot g\big(f_\alpha(\gamma)\big) : \gamma \in S_\alpha\big\}$$

and let $H_\alpha := \big\{h/\|h\| : h \in \tilde{H}_\alpha\big\}$. Firstly, notice that H_α is a normalized Hamel basis of X. Secondly, for all $\alpha_1 < \alpha_2 < \lambda$ we have $|H_{\alpha_1} \cap H_{\alpha_2}| < \kappa$. To see this, notice that for all $h \in (H_{\alpha_1} \cap H_{\alpha_2})$ we either have $h = g(\gamma)$ for some $\gamma \in (S_{\alpha_1} \cap S_{\alpha_2})$, or there are $\gamma_1 \in S_{\alpha_1}$ and $\gamma_2 \in S_{\alpha_2}$ such that the vectors h, $g(\gamma_1) + 2 \cdot g\big(f_{\alpha_1}(\gamma_1)\big)$, and $g(\gamma_2) + 2 \cdot g\big(f_{\alpha_2}(\gamma_2)\big)$ are co-linear, and since H is a Hamel basis, this implies that $\gamma_1 = \gamma_2$ and that this element belongs to $S_{\alpha_1} \cap S_{\alpha_2}$. Therefore, $|H_{\alpha_1} \cap H_{\alpha_2}| \leq 2 \cdot |S_{\alpha_1} \cap S_{\alpha_2}| < \kappa$, which completes the proof.

The answer to the question mentioned above follows now by the following

PROPOSITION. (1) *For all infinite cardinals κ, $A(\kappa, \kappa^+, \kappa)$ holds (where κ^+ denotes the successor cardinal of κ).*

(2) *For all infinite cardinals κ with $\mathrm{cf}(\kappa) > \omega$, it is consistent with* ZFC *that $2^\kappa \geq \kappa^{++}$ and that $A(\kappa, \kappa^{++}, \kappa)$ fails (where $\mathrm{cf}(\kappa)$ denotes the cofinality of κ and* ZFC *are the Zermelo-Fraenkel axioms of set theory including the Axiom of Choice).*

For a proof of (1) see Sierpiński [8, p. 448 f.] (or Tarski [9] and Sierpiński [7]), or Baumgartner [2, Theorem 2.8]. Part (2) follows from Baumgartner [2, Theorem 5.6 (b)] by setting $\nu = \omega$ and $\rho \geq \kappa^{++}$ such that $\mathrm{cf}(\rho) > \kappa$.

Combining the theorem and the proposition we get the following

Corollary. (a) *It is consistent with* ZFC *that for all cardinals κ, each infinite dimensional Banach space of cardinality κ admits 2^κ pairwise almost disjoint normalized Hamel bases.*

(b) *For each cardinal κ it is consistent with* ZFC *that $2^\kappa \geq \kappa^{++}$ and no infinite dimensional Banach space of cardinality κ admits κ^{++} pairwise almost disjoint normalized Hamel bases.*

Proof. The proof of (a) follows from the theorem and part (1) of the proposition, assuming the Generalized Continuum Hypothesis (which says that for every infinite cardinal κ we have $2^\kappa = \kappa^+$).

Part (b) follows from the theorem and part (2) of the proposition, using the fact that for every Banach space X we have $\mathrm{cf}(|X|) > \omega$ (cf. [1, Theorem 2.6]).

Remark. As mentioned above, for any infinite dimensional Banach space X and for any Hamel basis H of X we have $|H| = |X|$, which is at least the cardinality of the continuum $\mathfrak{c}$. Notice that this result — in contrast to the result presented in this note — is provable in ZFC alone without assuming additional axioms. Notice also that the non-trivial part in the proof of $|H| = |X|$ is when $|X| = \mathfrak{c}$, which is for example the case when X is separable: In [6, Theorem I-1] this is proved using a construction in the spirit of the Hahn-Banach Theorem in order to get a basic sequence and then applying an algebraic argument. An alternative proof is given in [3], where the Hahn-Banach construction is replaced with a result of Mazur (cf. [5, Lemma 1.a.6]), and the algebraic argument is replaced with a purely combinatorial one (which is similar to the argument used in [4], where the result is proved for separable spaces). However, all proofs depend on the existence of a basic sequence and we do not know if the result is also valid for more general linear spaces like separable complete linear metric spaces, which leads to the following problem: Is the Hamel dimension of any infinite dimensional separable complete linear metric space always equal to $\mathfrak{c}$?

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