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Matrix Representations for Low Dimensional Lie Algebras

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1. Introduction

A well known theorem in the theory of Lie algebras due to Ado asserts that every real Lie algebra g of dimension n has a faithful representation as a subalgebra of $gl(p, \mathbf{R})$ for some p. The theorem does not give much information about the value of p but the leads one to believe that p may be very large in relation to the size of n and consequently it seems to be of limited practical value. In this paper we shall take a different approach and construct representations for all the Lie algebras of dimension five and less. In fact in the interests of simplicity we give a matrix representation of a Lie group corresponding to each Lie algebra. The algebra representation is of course obtained by differentiating and evaluating at the identity. It should be appreciated that the construction of these representations is non-trivial and far from an algorithmic business, even though after the fact the representations may seem to be obvious.

If the Lie algebra g is semi-simple there are well known representations that are associated to the standard A_k,B_k, C_k and D_k series that are of the order of $\sqrt{n}$, where n is the dimension of g. On the other hand it is clear that semi-simple algebras are very much the exception rather than the rule: for example there are only two semi-simples in dimension five or less. More generally if g has a trivial center then the adjoint representation furnishes a faithful representation of g and in the notation used above $p = n$. Many of our representations are constructed in this way. We also develop in Section 3 some theorems that explain how to obtain representations in the case where the algebra has a non-trivial center. In particular we shall show that if g has a codimension one abelian nilradical then it has a faithful representation as a subalgebra of $gl(n, \mathbf{R})$. We refer to the 1976 list given by Patera et al. [6] for a comprehensive list of the *indecomposable* algebras of dimension five and less, which in turn was based on [4] and [5]. We have followed the list given in [6] and made allowance for slight typographical changes. The non-abelian two-dimensional algebra has trivial center and only one three-dimensional algebra, named for Heisenberg, has a non-trivial center. In dimension four there are four algebras that have a non-trivial center, none of which involves a parameter. As the dimension of g increases the algebras form moduli; that is to say there are families of inequivalent algebras depending on several parameters. Of the 40 five-dimensional algebras listed in [6] the following have a non-trivial center and therefore the adjoint representation is not faithful:

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 8c,\ 9bc(b=0),\ 10,\ 14pq,\ 15(a=0),\ 19b(a=0),\ 20(a=0),$$
$$22,\ 25b(p=0),\ 26(\epsilon=\pm 1, p=0),\ 28(a=0),\ 29,\ 30(a=-1),\ 38,\ 39.$$

Thus we have 22 cases to consider of which four depend on one parameter and one of which depends on two.

In Section 2 we state a few results about the matrices that can occur in a faithful representation of a Lie algebra and examine the role played by non-derogatory matrices. The main conclusion of Section 2 is that it is not possible to "lift" the adjoint representation of algebra 4.12 to algebra 5.39. In fact 5.39 caused us a great deal of trouble until we realized that algebra 4.12 can be represented in $gl(3, \mathbf{R})$. In Section 3 we give a few general results about representations of Lie algebras. In Section 4 we list all the representations corresponding to indecomposable Lie algebras of dimension five and less. In fact rather than giving representations for the Lie algebras we give in each case a matrix group whose Lie algebra coincides with a given algebra. It is straightforward then to construct the matrix representation of the algebra by differentiation. We have also given a representation of the algebra in terms of vector fields, in the vast majority of cases the right-invariant vector fields, that can also be obtained from the representation. A point to bear in mind here that there is a trade-off between complexity of the group matrix denoted by S and the form of the right-invariant vector fields. It is not possible in general to have both of them in the simplest possible form. Consider for example algebra 5.36 listed in Section 4. We have given the S-matrix with exponentials in the 12, 13 and 23 entries: as a result the vector fields do not contain exponentials. On the other hand the exponentials in the 12, 13 and 23 entries can be dropped but one obtains much more complicated vector fields.

In Section 5 we have singled out several algebras that extend to arbitrary dimensions and constitute particularly nice examples. For standard facts about Lie algebras and Lie groups we refer the reader to [2] and [3]. We use $\langle e_1, e_2, \ldots, e_s \rangle$ to denote the s-dimensional subspace of g generated by $e_1, e_2, \ldots, e_s$.

In the future we plan to apply our techniques to the six and higher dimensional algebras though some new methods will be required. The six-dimensional case will be a mammoth undertaking, comprising as it does, approximately 160 classes of algebra. GT wishes to thank the Mathematics Department of Utah State University for their hospitality and particularly Ian Anderson for help in using his Vessiot routines. Most of the calculations were done with the MAPLE symbolic manipulation program. The authors would also like to thank the referee for a conscientious and extremely constructive report.

2. Non-derogatory matrices

In this section we shall give some results about matrices that are useful in finding Lie algebra representations. We say that a matrix M is *non-derogatory* if, when put into Jordan canonical form, corresponding to each eigenvalue, there is just one Jordan block. An elegant characterization of a non-derogatory matrix is that its characteristic polynomial coincides with its minimal polynomial. An important property of a non-derogatory matrix is that its commuting algebra is commutative and a necessary and sufficient condition for the commuting algebra to be commutative is that M should be non-derogatory. In fact any matrix A that commutes with M must have the same block structure as does M and the commuting algebra is generated by powers of matrices with just one non-zero block taken from M. We shall call an $n \times n$ matrix that is upper triangular with zeroes on the main diagonal and 1's on each entry above the main diagonal, the standard non-derogatory nilpotent matrix.

Now let us consider the matrix equation

$$E = [A, B]$$

where A and B belong to the commuting algebra of E. Such equations occur frequently in the study of Lie algebras.

LEMMA 2.1. *E is nilpotent.*

Proof. Since E is a commutator, it has trace zero. Moreover, since E commutes with A and B we have

$$E^2 = E(AB - BA) = EAB - BEA$$

and more generally,

$$E^{k+1} = (E^k A)B - B(E^k A)$$

for $k = 0, 1, 2, \ldots$. Hence all powers of E are commutators and hence have trace zero. It follows that E is nilpotent. ∎

Notice that in Lemma 2.1 it is necessary to know only that E commutes with either A or B. Also the rank of E is at least two less than n because if it had rank $n-1$ it would be non-derogatory and its commuting algebra would be abelian.

Next we give a companion Lemma that is likely to be useful for Lie algebras that are not nilpotent. Indeed if the reader consults the list of Lie algebras given in Section 4 it will be seen that most of the brackets are of the type occurring in either Lemma 2.1 or Lemma 2.2.

LEMMA 2.2. *If $[E, F] = F$, then $[E, F^n] = nF^n$ for $n = 0, 1, 2, \ldots$ and hence F is nilpotent.*

Proof. Assume by induction that $[E, F^n] = nF^n$. Then we have

$$\begin{aligned}[E, F^{n+1}] &= EF^{n+1} - F^{n+1}E = [E, F]F^n + F[E, F^n] \\ &= F^{n+1} + nF \cdot F^n = (n+1)F^{n+1}\end{aligned}$$

and hence the formula holds for all n. It follows that all powers of F have trace zero and hence F is nilpotent. ∎

Let us see now what can be said when the matrix E has some non-derogatory properties.

LEMMA 2.3. *Consider in $\mathbf{R^k}$ the matrix equation*

$$[A, B] = E$$

where A and B commute with E which is nilpotent. Then if E has a Jordan decomposition of the following form

$$E = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix},$$

where N_1 is a standard non-derogatory nilpotent of rank $m-1$, then

$$m \leq \frac{k+1}{2}.$$

Proof. Suppose that E has the given decomposition and let N_1 and N_2 be of size $m \times m$ and $n \times n$, respectively, where $m + n = k$. Consider a block matrix of the form

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$$

that commutes with E. It follows that

$$[a_1, N_1] = 0, \quad [b_1, N_2] = 0, \quad b_1 N_2 - N_1 b_1 = 0, \quad c_1 N_1 - N_2 c_1 = 0.$$

Suppose that $\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ is another matrix that commutes with E. Then

$$b_1 c_2 N_1 - b_1 N_2 c_2 = b_1 c_2 N_1 - N_1 b_1 c_2 = 0$$

and hence

$$[b_1 c_2, N_1] = 0.$$

Since N_1 is standard non-derogatory $a_1, a_2, b_1 c_2$ and $b_2 c_1$ are polynomials in N_1. The $m \times m$ upper left block in the commutator of $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ is given by $[a_1, a_2] + b_1 c_2 - b_2 c_1$. Again, since N_1 is non-derogatory it follows that $[a_1, a_2] = 0$. Now the rank of $b_1 c_2$ is not more than n and the same is true of $b_2 c_1$. Since both are upper triangular and of the special form coming from the fact that they commute with the N_1, the same is true of $b_1 c_2 - b_2 c_1$. Hence $m - 1 \leq n$ and since $k = m + n$ we have $m \leq \frac{k+1}{2}$. ∎

LEMMA 2.4. *Consider in $\mathbf{R^k}$ the matrix equation*

$$[A, B] = E$$

where A and B commute with E which is nilpotent. Then if E has a decomposition of the following form

$$E = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix},$$

where N_1 and N_2 are standard non-derogatory nilpotent of size $m \times m$ and $n \times n$, respectively and $m + n = k$, then m=n.

Proof. According to Lemma 2.3 $m \leq \frac{k+1}{2}$. Since both N_1 and N_2 are both non-derogatory nilpotent, there are, up to change of basis, just two possibilities, namely, $m = n = \frac{k}{2}$ or $m = \frac{k+1}{2}$ and $n = \frac{k-1}{2}$. We suppose that we are in the second of these cases and we show that it is impossible. We suppose that $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ are as in Lemma 2.3 and commute with E. A straightforward calculation shows that the $(1,2)$ and $(k-1,k)$ entries are given by $b_{1(11)}c_{2(12)} - b_{2(11)}c_{1(12)}$ and $-b_{1(11)}c_{2(12)} + b_{2(11)}c_{1(12)}$, respectively, and hence contradicts the assumption that N_1 and N_2 are non-derogatory standard nilpotent. ∎

To illustrate these results consider a nilpotent 5×5 matrix E which is in the center of some 5×5 Lie algebra representation. There are, up to change of basis, five possible normal forms for a 5×5 nilpotent matrix. It follows from the remark in the first paragraph of this section that E cannot have rank 4. It follows from Lemma 2.4 that E cannot have rank 3. For rank two we have the following two canonical forms:

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$E = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Lemma 2.5. *The matrix E cannot be of the first form above.*

Proof. A matrix that commutes with E has the form

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} \\ 0 & Q_{11} & Q_{12} & 0 & 0 \\ 0 & 0 & Q_{11} & 0 & 0 \\ 0 & 0 & Q_{43} & Q_{44} & Q_{45} \\ 0 & 0 & Q_{53} & Q_{54} & Q_{55} \end{bmatrix}.$$

However, the commutator of two such matrices has its $(1,2)$ entry zero and hence cannot be E. ∎

With a lot more work it is possible to show that the second of the canonical forms for a rank two nilpotent matrix also leads to a contradiction. Many of the results above were inspired by algebra 5.39 which is one of the few five-dimensional algebra that is not nilpotent, for which we do not have a mechanism for obtaining a representation. Clearly the results will be applicable to higher dimensional algebras. The upshot of the preceding analysis, whose details we shall forgo, is that one cannot construct a representation for algebra 5.39 by lifting the adjoint representation of algebra 4.12, which is the quotient algebra obtained from 5.39 by dividing by its center.

3. Some representation results for Lie algebras

In this section we give some results which explain how to obtain faithful representations for n-dimensional Lie algebras as subalgebras of $gl(n, \mathbf{R})$. We begin with a simple result that gives a bound on the dimension of the center of an algebra.

PROPOSITION 3.1. *Suppose that g is an indecomposable Lie algebra of dimension n and that z is the dimension of its center $Z(g)$. Then* $z \leq \frac{2n+1-\sqrt{8n+1}}{2}$.

Proof. Since g is indecomposable we must have that $Z(g) \subset [g, g]$, where $[g, g]$ denotes the derived algebra of g. Furthermore the dimension of $[g, g]$ is equal at most to the number of non-zero brackets relative to some basis. However, the latter number is clearly bounded above by $\binom{n-z}{2}$. Hence $z \leq \binom{n-z}{2}$. If we solve this quadratic inequality we find that z satisfies the inequality as stated. ∎

As far as we aware from examples the bound given above is sharp.

Next we shall consider Lie algebras that have a codimension one abelian nilradical ideal and for such algebras we are able to obtain representations directly.

THEOREM 3.1. *Suppose that the n-dimensional Lie algebra g has a codimension one abelian ideal. Then g has a faithful representation as a subalgebra of $gl(n, \mathbf{R})$.*

Proof. If g is not itself nilpotent the abelian ideal in question will necessarily be the nilradical of g. On the other hand g itself may be nilpotent. If $\{e_1, e_2, \ldots, e_{n-1}\}$ is a basis for the codimension one abelian ideal we can extend it to a basis for g by means of the vector e_n. Define the endomorphism A to be $ad(e_n)$ and let its matrix be a^i_j. The non-zero brackets of g are given by $[e_n, e_i] = \sum_{k=1}^{n-1} a^k_i e_k$. To obtain the representation, map e_n to A; for each vector e_i $(i \leq n-1)$ map it to the $n \times n$ matrix E_i whose only non-zero entry is a 1 in the $(i, n)^{th}$ position. Clearly the $E_i's$ commute. Then note that the matrix product $E_i A$ is zero and so

$$[A, E_i] = \sum_{k=1}^{n-1} a^k_i E_k$$

and we have the required representation. ∎

COROLLARY 3.1. *An n-dimensional Lie algebra g that has a codimension one abelian ideal is isomorphic to the Lie algebra of a subgroup of $GL(n, \mathbf{R})$, that can be described explicitly.*

Proof. We resume from the previous Corollary. The subgroup of $GL(n, \mathbf{R})$ that we seek is given by

$$S = \begin{bmatrix} e^{(x_n A)} & x \\ 0 & 1 \end{bmatrix}$$

where x denotes the column $(n-1)$-vector, with entries $x_1, x_2, \ldots, x_{n-1}$. Clearly it is a group since the first $n-1$ entries in the last column are arbitrary and its Lie algebra is isomorphic to g as can be seen by differentiating with respect to each of the parameters and setting them equal to zero. ∎

We continue now in a different direction. The next result explains how under favorable circumstances the two-dimensional non-abelian Lie algebra can be used to obtain to a representation starting from a codimension one representation.

THEOREM 3.2. (i) *Suppose that h is an $(n+1)$-dimensional Lie algebra, that there exists an n-dimensional subalgebra g and that there exists a basis $\{e_1, \ldots, e_n, e_{n+1}\}$ of h such that $\{e_1, \ldots, e_{n-1}, e_n\}$ is a basis for g and that*

$$[e_{n+1}, e_n] = be_{n+1}$$

for some non-zero b and that

$$[e_{n+1}, e_i] = 0, \text{ for } 1 \leq i \leq n-1.$$

Then the subspaces spanned by $\{e_{n+1}\}$ and $\{e_1, e_2, \ldots e_{n-1}\}$ are ideals in h and g, respectively.

(ii) *Given the data of* (i) *suppose that g has a faithful representation as an n-dimensional subalgebra of $gl(n, \mathbf{R})$ in which each matrix has a zero bottom row. Then h has a faithful representation as an $(n+1)$-dimensional subalgebra of $gl(n+1, \mathbf{R})$.*

Proof. (i) It is clear that the one-dimensional subspace spanned by e_{n+1} is an ideal in h. Let us write the nonzero Lie brackets as

$$[e_i, e_j] = \sum_{k=1}^{n-1} C_{ij}^k e_k + C_{ij} e_n$$

and

$$[e_i, e_n] = \sum_{k=1}^{n-1} \Gamma_i^k e_k + \Gamma_i e_n.$$

Then considering the Jacobi identity

$$\big[e_{n+1}, [e_i, e_j]\big] + \big[e_j, [e_{n+1}, e_i]\big] + \big[e_i, [e_j, e_{n+1}]\big] = 0.$$

where $1 \leq i, j \leq n-1$ we find that $bC_{ij} = 0$ and hence $C_{ij} = 0$ since $b \neq 0$. It follows that the subspace spanned by $e_1, e_2, \ldots, e_{n-1}$ is a subalgebra of g.

Next considering the Jacobi identity

$$\big[e_n, [e_i, e_{n+1}]\big] + \big[e_{n+1}, [e_n, e_i]\big] + \big[e_i, [e_{n+1}, e_n]\big] = 0,$$

where $1 \leq i \leq n-1$ we find that $b\Gamma_i = 0$ and hence $\Gamma_i = 0$. Thus the subspace spanned by $\{e_1, e_2, \ldots, e_{n-1}\}$ is actually an ideal in g.

(ii) Suppose that for $1 \leq i \leq n$ the basis vector e_i is represented by the matrix E_i which has a zero bottom row. For $1 \leq i \leq n-1$ map e_i to the $(n+1) \times (n+1)$ matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & E_i \end{bmatrix}$$

and map e_n to the matrix

$$\begin{bmatrix} b & 0 \\ 0 & E_n \end{bmatrix}.$$

Finally map e_{n+1} to the $(n+1) \times (n+1)$ matrix whose only non-zero entry is 1 in the $(1, n+1)^{th}$ position. We obtain thereby a representation of h in $gl(n+1, \mathbf{R})$. ∎

Let us take stock of the situation and see how our results can be applied to obtain representations for indecomposable Lie algebras of dimension five and less. For dimension two a representation of the non-abelian algebra is given by $\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$ for arbitrary real x and y, as follows from Corollary 2.1. In dimension three only one, the Heisenberg algebra has a non-trival center and its representation too can be obtained from Theorem 3.1. In dimension four only four, namely, 4.1, 4.3, 4.8 and 4.10 have non-trival centers. The representations for 4.1 and 4.3 can be obtained from Theorem 3.1. For dimension five the first six algebras are nilpotent and at the moment we are able to obtain representations only for 5.1 and 5.2 by invoking Theorem 3.1. The algebras $5.7-18$ have a four-dimensional nilradical and their representations follow from Theorem 3.1. The algebras $5.19-32$ have a four-dimensional non-abelian nilradical and their representations are obtained from Theorem 3.1. Of the remaining algebras, $5.33-39$ have three-dimensional nilradicals but only 5.38 and 5.39 have non-trivial centers. Furthermore Theorem 3.2 can be applied to 5.38. As for algebra 5.40 it is peculiar in being the only five-dimensional algebra that is not solvable; however it has a trivial center as well as being known to be the Lie algebra of the special affine group and so even has a three-dimensional representation. In conclusion only the cases 5.3, 5.4, 5.5, 5.6, $5.30(a=-1)$ and 5.39 are not yet amenable to some sort of theory that yields a representation. It should be mentioned too that representations for several of the algebras can be obtained by applying more than one of the representation results given above.

4. The representations

Dimension 2:

$$S = \begin{bmatrix} e^x & y \\ 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields $D_y, D_x + yD_y$.

Dimension 3:

Of the three-dimensional algebras only 3.8 has a representation as a subalgebra of $gl(2, \mathbf{R})$.

3.1 $[e_2, e_3] = e_1$: center $\langle e_1 \rangle$

$$S = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields $D_z, D_y, D_x + yD_z$.

3.2 $[e_1, e_3] = e_1$, $[e_2, e_3] = e_1 + e_2$: trivial center

$$S = \begin{bmatrix} e^z & ze^z & x \\ 0 & e^z & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields $D_x, D_y, D_z + (x + y)D_x + yD_y$.

3.3 $[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$: trivial center

$$S = \begin{bmatrix} e^z & 0 & x \\ 0 & e^z & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields $D_x, D_y, D_z + xD_x + yD_y$.

3.4 $[e_1, e_3] = e_1$, $[e_2, e_3] = -e_2$: trivial center

$$S = \begin{bmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields $D_x, D_y, D_z + xD_x - yD_y$.

3.5a $(a \neq \pm 1)$ $[e_1, e_3] = e_1$, $[e_2, e_3] = ae_2$: trivial center

$$S = \begin{bmatrix} e^z & 0 & x \\ 0 & e^{az} & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields $D_x, D_y, D_z + xD_x + ayD_y$.

3.6 $[e_1, e_3] = -e_2$, $[e_2, e_3] = e_1$: trivial center

$$S = \begin{bmatrix} \cos(z) & \sin(z) & x \\ -\sin(z) & \cos(z) & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields $D_x, D_y, D_z + yD_x - xD_y$.

3.7a $(a \neq 0)$ $[e_1, e_3] = ae_1 - e_2$, $[e_2, e_3] = e_1 + ae_2$: trivial center

$$S = \begin{bmatrix} e^{az}\cos(z) & e^{az}\sin(z) & x \\ -e^{az}\sin(z) & e^{az}\cos(z) & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields $D_x, D_y, D_z + (ax + y)D_x + (-x + ay)D_y$.

3.8 (semi-simple $sl(2,\mathbf{R})$) $[e_1,e_3]=-2e_2$, $[e_1,e_2]=e_1$, $[e_2,e_3]=e_3$

$$S=\begin{bmatrix} \cosh(x)+\sinh(x)\cosh(y) & -\sinh(x)\sinh(y)\,e^{-z} \\ \sinh(x)\sinh(y)\,e^{z} & \cosh(x)-\sinh(x)\cosh(y) \end{bmatrix}$$

Right-invariant vector fields:

$$\Big\{\tfrac{1}{2}\big(\sinh(y)e^z D_x - \tfrac{\cosh(x)\cosh(y)-\sinh(x)}{\sinh(x)}e^z D_y - \tfrac{\sinh(x)\cosh(y)-\cosh(x)}{\sinh(y)\sinh(x)}e^z D_z\big),$$
$$\tfrac{1}{2}\big(\cosh(y)D_x - \tfrac{\cosh(x)\sinh(y)}{\sinh(x)}D_y - D_z\big),$$
$$\tfrac{1}{2}\big(-\sinh(y)e^{-z}D_x + \tfrac{\cosh(x)\cosh(y)+\sinh(x)}{\sinh(x)}e^{-z}D_y$$
$$+\tfrac{\sinh(x)\cosh(y)+\cosh(x)}{\sinh(y)\sinh(x)}e^{-z}D_z\big)\Big\}.$$

3.9 (semi-simple $so(3)$) $[e_1,e_2]=e_3$, $[e_2,e_3]=e_1$, $[e_3,e_1]=e_2$

$$S=\begin{bmatrix} \cos(x)\cos(y)\cos(z) - \sin(x)\sin(z) & \sin(x)\cos(y)\cos(z) + \cos(x)\sin(z) & -\sin(y)\cos(z) \\ -\cos(x)\cos(y)\sin(z) - \sin(x)\cos(z) & -\sin(x)\sin(z)\cos(y) + \cos(x)\cos(z) & \sin(y)\sin(z) \\ \cos(x)\sin(y) & \sin(x)\sin(y) & \cos(y) \end{bmatrix}$$

Right-invariant vector fields:

$$\Big\{D_z,\ \tfrac{\sin(z)}{\sin(y)}D_x+\cos(z)D_y-\tfrac{\cos(y)\sin(z)}{\sin(y)}D_z,\ \tfrac{\cos(z)}{\sin(y)}D_x-\sin(z)D_y-\tfrac{\cos(y)\cos(z)}{\sin(y)}D_z\Big\}.$$

(We refer the reader to [1] for a complete discussion of $so(3)$, $su(2)$ and the Euler angles.)

Dimension 4:

We remark that the first six algebras have 3-dimensional abelian nilradical; 4.7 – 4.11 have the 3-dimensional Heisenberg algebra as their nilradical and 4.12 has a 2-dimensional abelian nilradical. Clearly the four-dimensional algebras cannot have representations as subalgebras of $gl(2,\mathbf{R})$. As far as we are aware only 4.12 has a representation as a subalgebra of $gl(3,\mathbf{R})$.

4.1 $[e_2, e_4] = e_1$, $[e_3, e_4] = e_2$: center $\langle e_1 \rangle$

$$S = \begin{bmatrix} 1 & w & \frac{w^2}{2} & x \\ 0 & 1 & w & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_x, D_y, D_z, D_w + yD_x + zD_y$.

4.2a $(a \neq 0)$ $[e_1, e_4] = ae_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_2 + e_3$: trivial center

$$S = \begin{bmatrix} e^{aw} & 0 & 0 & x \\ 0 & e^w & we^w & y \\ 0 & 0 & e^w & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_x, D_y, D_z, D_w + axD_x + (y+z)D_y + zD_z$.

4.3 $[e_1, e_4] = e_1$, $[e_3, e_4] = e_2$: center $\langle e_2 \rangle$

$$S = \begin{bmatrix} e^w & 0 & 0 & x \\ 0 & 1 & w & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_x, D_y, D_z, D_w + xD_x + zD_y$.

4.4 $[e_1, e_4] = e_1$, $[e_2, e_4] = e_1 + e_2$, $[e_3, e_4] = e_2 + e_3$: trivial center

$$S = \begin{bmatrix} e^w & we^w & \frac{w^2}{2}e^w & x \\ 0 & e^w & we^w & y \\ 0 & 0 & e^w & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_x, D_y, D_z, D_w + (x+y)D_x + (y+z)D_y + zD_z$.

4.5ab $(0 \leq ab,\ -1 \leq a \leq b \leq 1)$ $[e_1, e_4] = e_1$, $[e_2, e_4] = ae_2$, $[e_3, e_4] = be_3$: trivial center

$$S = \begin{bmatrix} e^w & 0 & 0 & x \\ 0 & e^{aw} & 0 & y \\ 0 & 0 & e^{bw} & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_x, D_y, D_z, D_w + xD_x + ayD_y + bzD_z$.

4.6ab $(a \neq 0,\ b \geq 0)$ $[e_1, e_4] = ae_1$, $[e_2, e_4] = be_2 - e_3$, $[e_3, e_4] = e_2 + be_3$: trivial center

$$S = \begin{bmatrix} e^{aw} & 0 & 0 & x \\ 0 & e^{bw}\cos(w) & e^{bw}\sin(w) & y \\ 0 & -e^{bw}\sin(w) & e^{bw}\cos(w) & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_x, D_y, D_z, D_w + axD_x + (by+z)D_y + (bz-y)D_z$.

4.7 $[e_2, e_3] = e_1$, $[e_1, e_4] = 2e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_2 + e_3$: trivial center

$$S = \begin{bmatrix} e^{2w} & -ze^{w} & ye^{w} & x \\ 0 & e^{w} & we^{w} & y + zw \\ 0 & 0 & e^{w} & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $-\frac{1}{2}D_x, zD_x + D_y, D_z - (y+zw)D_x - wD_y, D_w + 2xD_x + yD_y + zD_z$.

4.8 $[e_2, e_3] = e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = -e_3$: center $\langle e_1 \rangle$

$$S = \begin{bmatrix} 1 & 0 & xe^{w} & y \\ 0 & e^{-w} & 0 & x \\ 0 & 0 & e^{w} & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_y, D_z, D_x + zD_y, D_w - xD_x + zD_z$.

4.9b $(-1 < b \leq 1)$ $[e_2, e_3] = e_1$, $[e_1, e_4] = (b+1)e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = be_3$: trivial center

$$S = \begin{bmatrix} e^{(b+1)w} & -xe^{w} & ye^{bw} & z \\ 0 & e^{w} & 0 & y \\ 0 & 0 & e^{bw} & bx \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $(b+1)D_z, D_y - bxD_z, D_x + yD_z, (b+1)D_z, D_w + bxD_x + yD_y + (b+1)zD_z$.

4.10 $[e_2, e_3] = e_1$, $[e_2, e_4] = -e_3$, $[e_3, e_4] = e_2$: center $\langle e_1 \rangle$

$$S = \begin{bmatrix} 1 & -y\cos(w) + x\sin(w) & y\sin(w) + x\cos(w) & z \\ 0 & \cos(w) & -\sin(w) & x \\ 0 & \sin(w) & \cos(w) & y \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $\frac{1}{2}D_z, D_x + yD_z, -D_y + xD_z, D_w - yD_x + xD_y$.

4.11a ($a > 0$) $[e_2, e_3] = e_1$, $[e_1, e_4] = 2ae_1$, $[e_2, e_4] = ae_2 - e_3$, $[e_3, e_4] = e_2 + ae_3$: trivial center

$$S = \begin{bmatrix} e^{2aw} & -e^{aw}(x\sin(w) + y\cos(w)) & e^{aw}(x\cos(w) - y\sin(w)) & z \\ 0 & e^{aw}\cos(w) & e^{aw}\sin(w) & ax + y \\ 0 & -e^{aw}\sin(w) & e^{aw}\cos(w) & ay - x \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $-\frac{1}{a+1}D_z, D_x + (ay - x)D_z, D_y - (ax + y)D_z, D_w + (ax + y)D_x + (ay - x)D_y + 2azD_z$.

4.12 $[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$, $[e_1, e_4] = -e_2$, $[e_2, e_4] = e_1$: trivial center

$$S = \begin{bmatrix} e^z\cos(w) & e^z\sin(w) & x & y \\ -e^z\sin(w) & e^z\cos(w) & y & -x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_z + xD_x + yD_y, D_w + yD_x - xD_y, D_x, D_y$.

Alternative representation:

$$S = \begin{bmatrix} 1 & x & y \\ 0 & e^w\cos(z) & e^w\sin(z) \\ 0 & -e^w\sin(z) & e^w\cos(z) \end{bmatrix}.$$

Left-invariant vector fields: $D_x, D_y, -D_w + xD_x + yD_y, D_z + yD_x - xD_y$.

Dimension 5:

The first six algebras are nilpotent. We give in each case a nilpotent Lie group whose algebra is isomorphic to the given algebra. The six algebras are distinguished by their index of nilpotence and the dimension of the derived algebra except for the filiforms 5.2 and 5.6. However, 5.2 has a codimension one abelian ideal whereas 5.6 does not so the six algebras are mutually non-isomorphic. Remark that the algebras $5.7-18$ have 4-dimensional abelian nilradical; $5.19-32$ have 4-dimensional non-abelian nilradical and $5.33-39$ have a 3-dimensional nilradical. 5.40 is the only algebra that is not solvable and it has a 2-dimensional abelian nilradical. Clearly the five-dimensional algebras cannot have representations as subalgebras of $gl(2,\mathbf{R})$. Since all but algebra 5.40 is solvable, by Lie's theorem, if any of them have representations as subalgebras of $gl(3,\mathbf{R})$ they would have upper triangular representations. (Strictly speaking, Lie's theorem applies only over $\mathbf{C}$ rather than $\mathbf{R}$.) The algebra of 3×3 upper triangular matrices is a decomposable six-dimensional algebra: in fact it is isomorphic to the direct sum of algebra 5.36 and $\mathbf{R}$. Thus only 5.36 and 5.40 have representations as subalgebras of $gl(3,\mathbf{R})$. Algebra 5.4, the five-dimensional Heisenberg algebra has a representation as a subalgebra of $gl(4,\mathbf{R})$ and in fact the $2n+1$-dimensional Heisenberg algebra has a representation as a subalgebra of $gl(n+2,\mathbf{R})$: see Section 5. We cannot definitively exclude the possibility that some of the other algebras might have representations as subalgebras of $gl(4,\mathbf{R})$.

5.1 $[e_3,e_5]=e_1$, $[e_4,e_5]=e_2$: center $\langle e_1,e_2\rangle$, nilpotent of index 2

$$S=\begin{bmatrix}1&0&w&0&q\\0&1&0&w&x\\0&0&1&0&y\\0&0&0&1&z\\0&0&0&0&1\end{bmatrix}.$$

Right-invariant vector fields: $-D_q,-D_x,D_y,D_z,D_w-yD_q-zD_x$.

5.2 $[e_2,e_5]=e_1$, $[e_3,e_5]=e_2$, $[e_4,e_5]=e_3$: center $\langle e_1\rangle$, nilpotent of index 4 (filiform)

$$S=\begin{bmatrix}1&w&\frac{w^2}{2}&\frac{w^3}{6}&q\\0&1&w&\frac{w^2}{2}&x\\0&0&1&w&y\\0&0&0&1&z\\0&0&0&0&1\end{bmatrix}.$$

Right-invariant vector fields: $D_q,D_x,D_y,D_z,D_w+xD_q+yD_x+zD_y$.

5.3 $[e_3, e_4] = e_2$, $[e_3, e_5] = e_1$, $[e_4, e_5] = e_3$: center $\langle e_1, e_2\rangle$, nilpotent of index 3

$$S = \begin{bmatrix} 1 & 0 & -z & y - zw & q \\ 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 1 & w & 2y \\ 0 & 0 & 0 & 1 & 2z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $2D_x, -4D_q, D_y + 2zD_q, D_z - 2yD_q, D_w + 2yD_x + zD_y$.

5.4 $[e_2, e_4] = e_1$, $[e_3, e_5] = e_1$: center $\langle e_1\rangle$ nilpotent of index 2

$$S = \begin{bmatrix} 1 & z & w & q \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_q, D_z, D_w, D_x + zD_q, D_y + wD_q$.

5.5 $[e_3, e_4] = e_1$, $[e_2, e_5] = e_1$, $[e_3, e_5] = e_2$: center $\langle e_1\rangle$ nilpotent of index 3

$$S = \begin{bmatrix} 1 & w & \frac{w^2}{2} & q & x \\ 0 & 1 & w & 0 & y \\ 0 & 0 & 1 & 0 & q \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields are: $D_x, D_y, D_q + zD_x, -D_z, D_w + qD_y + yD_x$.

5.6 $[e_3, e_4] = e_1$, $[e_2, e_5] = e_1$, $[e_3, e_5] = e_2$, $[e_4, e_5] = e_3$: center $\langle e_1\rangle$ nilpotent of index 4 (filiform)

$$S = \begin{bmatrix} 1 & 2w & w^2 - z & y - zw + \frac{w^3}{3} & q \\ 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $-2D_q, -D_x, D_y + zD_q, D_z - yD_q, D_w + 2xD_q + yD_x + zD_y$.

5.7abc ($abc \neq 0,\ -1 \leq c \leq b \leq a \leq 1$) $[e_1, e_5] = e_1$, $[e_2, e_5] = ae_2$, $[e_3, e_5] = be_3$, $[e_4, e_5] = ce_4$: trivial center

$$S = \begin{bmatrix} e^w & 0 & 0 & 0 & q \\ 0 & e^{aw} & 0 & 0 & x \\ 0 & 0 & e^{bw} & 0 & y \\ 0 & 0 & 0 & e^{cw} & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_q, D_x, D_y, D_z, D_w + qD_q + axD_x + byD_y + czD_z$.

5.8c ($0 \leq c \leq 1$) $[e_2, e_5] = e_1$, $[e_3, e_5] = e_3$, $[e_4, e_5] = ce_4$: center $\langle e_1 \rangle$

$$S = \begin{bmatrix} e^{cw} & 0 & 0 & 0 & q \\ 0 & e^w & 0 & 0 & x \\ 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_x, D_y, D_z, D_q, D_w + cqD_q + yD_x + zD_z$.

5.9bc ($0 \neq c \leq b$) $[e_1, e_5] = e_1$, $[e_2, e_5] = e_1 + e_2$, $[e_3, e_5] = be_3$, $[e_4, e_5] = ce_4$: trivial center

$$S = \begin{bmatrix} e^{cw} & 0 & 0 & 0 & q \\ 0 & e^{bw} & 0 & 0 & x \\ 0 & 0 & e^w & we^w & y \\ 0 & 0 & 0 & e^w & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_y, D_z, D_x, D_q, D_w + bxD_x + cqD_q + (y + z)D_y + zD_z$.

5.10 $[e_2, e_5] = e_1$, $[e_3, e_5] = e_2$, $[e_4, e_5] = e_4$: center $\langle e_1 \rangle$

$$S = \begin{bmatrix} e^w & 0 & 0 & 0 & q \\ 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right invariant vector fields: $D_x, D_y, D_z, D_q, D_w + qD_q + yD_x + zD_y$.

5.11c ($c \neq 0$) $[e_1, e_5] = e_1$, $[e_2, e_5] = e_1 + e_2$, $[e_3, e_5] = e_2 + e_3$, $[e_4, e_5] = ce_4$: trivial center

$$S = \begin{bmatrix} e^{cw} & 0 & 0 & 0 & q \\ 0 & e^w & we^w & \frac{w^2e^w}{2} & x \\ 0 & 0 & e^w & we^w & y \\ 0 & 0 & 0 & e^w & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_x, D_y, D_z, D_q, D_w + (x+y)D_x + (y+z)D_y + zD_z + cqD_q$.

5.12 $[e_1, e_5] = e_1$, $[e_2, e_5] = e_1 + e_2$, $[e_3, e_5] = e_2 + e_3$, $[e_4, e_5] = e_3 + e_4$: trivial center

$$S = \begin{bmatrix} e^w & we^w & \frac{w^2e^w}{2} & \frac{w^3e^w}{6} & q \\ 0 & e^w & we^w & \frac{w^2e^w}{2} & x \\ 0 & 0 & e^w & we^w & y \\ 0 & 0 & 0 & e^w & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_q, D_x, D_y, D_z, D_w + (q+x)D_q + (x+y)D_x + (y+z)D_y + zD_z + qD_q$.

5.13apr ($ar \neq 0$, $|a| \leq 1$) $[e_1, e_5] = e_1$, $[e_2, e_5] = ae_2$, $[e_3, e_5] = pe_3 - re_4$, $[e_4, e_5] = re_3 + pe_4$: trivial center

$$S = \begin{bmatrix} e^w & 0 & 0 & 0 & x \\ 0 & e^{aw} & 0 & 0 & y \\ 0 & 0 & e^{pw}\cos(rw) & e^{pw}\sin(rw) & z \\ 0 & 0 & -e^{pw}\sin(rw) & e^{pw}\cos(rw) & q \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_q, D_x, D_y, -D_z, D_w + xD_x + yD_y + (pq - rz)D_q + (pz + rq)D_z$.

5.14 $[e_2, e_5] = e_1$, $[e_3, e_5] = pe_3 - e_4$, $[e_4, e_5] = e_3 + pe_4$: center $\langle e_1 \rangle$

$$S = \begin{bmatrix} 1 & x & 0 & 0 & q \\ 0 & 1 & 0 & 0 & w \\ 0 & 0 & e^{pw}\cos(w) & e^{pw}\sin(w) & y \\ 0 & 0 & -e^{pw}\sin(w) & e^{pw}\cos(w) & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $-D_q, D_x + wD_q, D_y, D_z, D_w + (py + z)D_y + (pz - y)D_z$.

5.15a $[e_1, e_5] = e_1$, $[e_2, e_5] = e_1 + e_2$, $[e_3, e_5] = ae_3$, $[e_4, e_5] = e_3 + ae_4$: trivial center $a \neq 0$; center $\langle e_3 \rangle$ if $a = 0$:

$$S = \begin{bmatrix} e^w & we^w & 0 & 0 & q \\ 0 & e^w & 0 & 0 & x \\ 0 & 0 & e^{aw} & we^{aw} & y \\ 0 & 0 & 0 & e^{aw} & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_q, D_x, D_y, D_z, D_w + (q + x)D_q + xD_x + (ay + z)D_y + azD_z$.

5.16pr $(r \neq 0)$ $[e_1, e_5] = e_1$, $[e_2, e_5] = e_1 + e_2$, $[e_3, e_5] = pe_3 - re_4$, $[e_4, e_5] = re_3 + pe_4$: trivial center

$$S = \begin{bmatrix} e^w & we^w & 0 & 0 & q \\ 0 & e^w & 0 & 0 & x \\ 0 & 0 & e^{pw}\cos(rw) & e^{pw}\sin(rw) & y \\ 0 & 0 & -e^{pw}\sin(rw) & e^{pw}\cos(rw) & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_q, D_x, D_y, D_z, D_w + (q + x)D_q + xD_x + (py + rz)D_y + (-ry + pz)D_z$.

5.17prs $(s \neq 0)$ $[e_1, e_5] = pe_1 - e_2$, $[e_2, e_5] = e_1 + pe_2$, $[e_3, e_5] = re_3 - se_4$, $[e_4, e_5] = se_3 + qe_4$: trivial center

$$S = \begin{bmatrix} e^{pw}\cos(w) & e^{pw}\sin(w) & 0 & 0 & x \\ -e^{pw}\sin(w) & e^{pw}\cos(w) & 0 & 0 & y \\ 0 & 0 & e^{rw}\cos(sw) & e^{rw}\sin(sw) & z \\ 0 & 0 & -e^{rw}\sin(sw) & e^{rw}\cos(sw) & q \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_x, D_y, D_z, D_q, D_w + (px + y)D_x + (py - x)D_y + (rz + sq)D_z + (rq - sz)D_q$.

5.18p $(p \geq 0)$ $[e_1, e_5] = pe_1 - e_2$, $[e_2, e_5] = e_1 + pe_2$, $[e_3, e_5] = e_1 + pe_3 - e_4$, $[e_4, e_5] = e_2 + e_3 + pe_4$: trivial center

$$S = \begin{bmatrix} e^{pw}\cos(w) & e^{pw}\sin(w) & we^{pw}\cos(w) & we^{pw}\sin(w) & x \\ -e^{pw}\sin(w) & e^{pw}\cos(w) & -we^{pw}\sin(w) & we^{pw}\cos(w) & y \\ 0 & 0 & e^{pw}\cos(w) & e^{pw}\sin(w) & z \\ 0 & 0 & -e^{pw}\sin(w) & e^{pw}\cos(w) & q \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_x, D_y, D_z, D_q, D_w + (px + y + z)D_x + (py - x + q)D_y + (q + pz)D_z + (pq - z)D_q$.

5.19ab $(a \neq 0)$ $[e_2, e_3] = e_1$, $[e_1, e_5] = ae_1$, $[e_2, e_5] = e_2$, $[e_3, e_5] = (a-1)e_3$, $[e_4, e_5] = be_4$: trivial center

$$S = \begin{bmatrix} e^{bw} & 0 & 0 & 0 & q \\ 0 & e^{aw} & -xe^{w} & ye^{(a-1)w} & z \\ 0 & 0 & e^{w} & 0 & y \\ 0 & 0 & 0 & e^{(a-1)w} & (a-1)x \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $-aD_z, D_y - (a-1)xD_z, D_x + yD_z, D_q, D_w + (a-1)xD_x + yD_y + azD_z + bqD_q$.

5.19ab $(a=0)$ $[e_2, e_3]=e_1$, $[e_2, e_5]=e_2$, $[e_3, e_5]=-e_3$, $[e_4, e_5]=be_4$: center $\langle e_1 \rangle$

$$S = \begin{bmatrix} e^{bw} & 0 & 0 & 0 & q \\ 0 & 1 & 0 & ye^{w} & z \\ 0 & 0 & e^{-w} & 0 & y \\ 0 & 0 & 0 & e^{w} & x \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_z, D_x, D_y + xD_z, D_q, D_w - yD_y + zD_z + xD_x + bqD_q$.

5.20 $(a \neq 0)$ $[e_2, e_3] = e_1$, $[e_1, e_5] = ae_1$, $[e_2, e_5] = e_2$, $[e_3, e_5] = (a-1)e_3$, $[e_4, e_5] = e_1 + ae_4$: trivial center

$$S = \begin{bmatrix} e^{aw} & -e^{w}q & ze^{(a-1)w} & we^{aw} & x \\ 0 & e^{w} & 0 & 0 & z \\ 0 & 0 & e^{(a-1)w} & 0 & (a-1)q \\ 0 & 0 & 0 & e^{aw} & y \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_x, D_z - (a-1)qD_x, D_q + zD_x, D_y, D_z - (a-1)qD_x, (a-1)qD_q + (ax + y)D_x + ayD_y + zD_z + D_w$.

5.20 $(a = 0)$ $[e_2, e_3] = e_1$, $[e_2, e_5] = e_2$, $[e_3, e_5] = -e_3$, $[e_4, e_5] = e_1$: center $\langle e_1 \rangle$

$$S = \begin{bmatrix} 1 & q & 0 & xe^w & y \\ 0 & 1 & 0 & 0 & w \\ 0 & 0 & e^{-w} & 0 & x \\ 0 & 0 & 0 & e^w & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_y, D_z, D_x + zD_y, -D_q - wD_y, D_w - xD_x + zD_z$.

5.21 $[e_2, e_3] = e_1$, $[e_1, e_5] = 2e_1$, $[e_2, e_5] = e_2 + e_3$, $[e_3, e_5] = e_3 + e_4$, $[e_4, e_5] = e_4$: trivial center

$$S = \begin{bmatrix} e^{2w} & 0 & ze^w & (z - y + zw)e^w & q \\ 0 & e^w & we^w & \frac{w^2 e^w}{2} & x \\ 0 & 0 & e^w & we^w & y \\ 0 & 0 & 0 & e^w & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $-2D_q, (y+z)D_q + D_z, D_y - zD_q, D_x, D_w + 2qD_q + (x + y)D_x + (y + z)D_y + zD_z$.

5.22 $[e_2, e_3] = e_1$, $[e_2, e_5] = e_3$, $[e_4, e_5] = e_4$: center $\langle e_1 \rangle$

$$S = \begin{bmatrix} e^z & 0 & 0 & 0 & q \\ 0 & 1 & w & \frac{w^2}{2} & x \\ 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_x, D_w + yD_x + zD_y, -D_y, D_q, D_z + qD_q$.

5.23b $(b \neq 0)$ $[e_2, e_3] = e_1$, $[e_1, e_5] = 2e_1$, $[e_2, e_5] = e_2 + e_3$, $[e_3, e_5] = e_3$, $[e_4, e_5] = be_4$: trivial center

$$S = \begin{bmatrix} e^{bw} & 0 & 0 & 0 & q \\ 0 & e^{2w} & -ze^w & ye^w & x \\ 0 & 0 & e^w & we^w & y + zw \\ 0 & 0 & 0 & e^w & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $-\frac{1}{2}D_x, zD_x + D_y, D_z - (y + zw)D_x - wD_y, D_q, D_w + qD_q + 2xD_x + yD_y + zD_z$.

5.24ϵ ($\epsilon = \pm 1$) $[e_2, e_3] = e_1$, $[e_1, e_5] = 2e_1$, $[e_2, e_5] = e_2 + e_3$, $[e_3, e_5] = e_3$, $[e_4, e_5] = \epsilon e_1 + 2e_4$: trivial center

$$S = \begin{bmatrix} e^{2w} & xe^{w} & e^{w}(-y+xw) & -2\epsilon w e^{2w} & q \\ 0 & e^{w} & we^{w} & 0 & y \\ 0 & 0 & e^{w} & 0 & x \\ 0 & 0 & 0 & e^{2w} & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $-2D_q, D_x + yD_q, D_y - xD_q, D_z, D_w + 2(q - \epsilon z)D_q + xD_x + 2zD_z + (x+y)D_y$.

$5.25bp$ ($b \neq 0$, $p \neq 0$) $[e_2, e_3] = e_1$, $[e_1, e_5] = 2pe_1$, $[e_2, e_5] = pe_2 + e_3$, $[e_3, e_5] = pe_3 - e_2$, $[e_4, e_5] = be_4$: trivial center

$$S = \begin{bmatrix} e^{bw} & 0 & 0 & 0 & q \\ 0 & e^{2pw} & -e^{pw}\big(y\cos(w) + x\sin(w)\big) & e^{pw}\big(x\cos(w) - y\sin(w)\big) & z \\ 0 & 0 & e^{pw}\cos(w) & e^{pw}\sin(w) & px + y \\ 0 & 0 & -e^{pw}\sin(w) & e^{pw}\cos(w) & py - x \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_q, -D_x + (x - py)D_z, D_y - (px + y)D_z, -2pD_z, bqD_q + (px + y)D_x + (py - x)D_y + 2pzD_z + D_w$.

$5.25bp$ ($b \neq 0$, $p = 0$) $[e_2, e_3] = e_1$, $[e_2, e_5] = e_3$, $[e_3, e_5] = -e_2$, $[e_4, e_5] = be_4$: center $\langle e_1 \rangle$

$$S = \begin{bmatrix} e^{bw} & 0 & 0 & 0 & q \\ 0 & 1 & x\cos(w) + y\sin(w) & -x\sin(w) + y\cos(w) & z \\ 0 & 0 & \cos(w) & -\sin(w) & y \\ 0 & 0 & \sin(w) & \cos(w) & -x \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_y - xD_z, -(D_x + yD_z), -2D_z, D_q, bqD_q + xD_y - yD_x + D_w$.

5.26ϵ ($p \neq 0$, $\epsilon = \pm 1$) $[e_2, e_3] = e_1$, $[e_1, e_5] = 2pe_1$, $[e_2, e_5] = pe_2 + e_3$, $[e_3, e_5] = pe_3 - e_2$, $[e_4, e_5] = \epsilon e_1 + 2pe_4$: trivial center

$$S = \begin{bmatrix} e^{2pw} & -\big(y\cos(w) - x\sin(w)\big)e^{pw} & \big(\sin(w)y + x\cos(w)\big)e^{pw} & \epsilon w e^{2pw} & q \\ 0 & e^{pw}\cos(w) & -e^{pw}\sin(w) & 0 & -y+px \\ 0 & e^{pw}\sin(w) & e^{pw}\cos(w) & 0 & py+x \\ 0 & 0 & 0 & e^{2pw} & 2pz \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $-2pD_q, D_x + (py+x)D_q, D_y + (y-px)D_q, -D_z, D_w + 2p(q+\epsilon z)D_q + (px-y)D_x + (py+x)D_y + 2pzD_z$.

5.26 ($p = 0$, $\epsilon = \pm 1$) $[e_2, e_3] = e_1$, $[e_2, e_5] = e_3$, $[e_3, e_5] = -e_2$, $[e_4, e_5] = \epsilon e_1$: center $\langle e_1 \rangle$

$$S = \begin{bmatrix} 1 & x\sin(w) - y\cos(w) & x\cos(w) + y\sin(w) & \epsilon w & z \\ 0 & \cos(w) & -\sin(w) & 0 & x \\ 0 & \sin(w) & \cos(w) & 0 & y \\ 0 & 0 & 0 & 1 & q \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $-D_x - yD_z, -D_y + xD_z, -2D_z, -2\epsilon D_q, -yD_x + xD_y + D_w + q\epsilon D_z$.

5.27 $[e_2, e_3] = e_1$, $[e_1, e_5] = e_1$, $[e_3, e_5] = e_3 + e_4$, $[e_4, e_5] = e_1 + e_4$: trivial center

$$S = \begin{bmatrix} e^w & we^w & \frac{1}{2}(2x + w^2)e^w & -y & q \\ 0 & e^w & we^w & 0 & z \\ 0 & 0 & e^w & 0 & y \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_q, -D_x - yD_q, D_y, D_z, D_w + (q+z)D_q + (y+z)D_z + yD_y$.

5.28a ($a=0$) $[e_2,e_3]=e_1$, $[e_2,e_5]=-e_2$, $[e_3,e_5]=e_3+e_4$, $[e_4,e_5]=e_4$: center $\langle e_1\rangle$

$$S=\begin{bmatrix} e^w & 0 & we^w & 0 & q \\ 0 & 1 & 0 & xe^{-w} & y \\ 0 & 0 & e^w & 0 & x \\ 0 & 0 & 0 & e^{-w} & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_y, D_z, D_x+zD_y, D_q, D_w+(q+x)D_q+xD_x-zD_z$.

5.28a ($a\neq 0$) $[e_2,e_3]=e_1$, $[e_1,e_5]=ae_1$, $[e_2,e_5]=(a-1)e_2$, $[e_3,e_5]=e_3+e_4$, $[e_4,e_5]=e_4$: trivial center

$$S=\begin{bmatrix} e^{aw} & -ze^{(a-1)w} & 0 & xe^w & q \\ 0 & e^{(a-1)w} & 0 & 0 & (a-1)x \\ 0 & 0 & e^w & we^w & y \\ 0 & 0 & 0 & e^w & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $-aD_q, D_x+zD_q, D_z-(a-1)xD_q, D_y, D_w+aqD_q+(a-1)xD_x+zD_z+(y+z)D_y$.

5.29 $[e_2,e_4]=e_1$, $[e_1,e_5]=e_1$, $[e_2,e_5]=e_2$, $[e_4,e_5]=e_3$: center $\langle e_3\rangle$

$$S=\begin{bmatrix} e^w & 0 & 0 & x & q \\ 0 & e^w & 0 & 0 & x \\ 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $-D_q, D_x+zD_q, D_y, D_z, D_w+qD_q+xD_x+zD_y$.

5.30 ($a\neq -1$) $[e_2,e_4]=e_1$, $[e_1,e_5]=(a+1)e_1$, $[e_3,e_4]=e_2$, $[e_2,e_5]=ae_2$, $[e_3,e_5]=(a-1)e_3$, $[e_4,e_5]=e_4$ trivial center

$$S = \begin{bmatrix} 1 & xe^{w} & -4\,x^2e^{2\,w} & ze^{-w} & q \\ 0 & e^{w} & -8\,xe^{2\,w} & ye^{-w} & 4\,yx - 12\,z \\ 0 & 0 & e^{2\,w} & 0 & y \\ 0 & 0 & 0 & e^{-w} & 4\,x \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Right-invariant vector fields: $-16D_q, 4xD_q + D_z, D_y, -(12z - 4xy)D_q + D_x + y * D_z, -D_w + xD_x - 2yD_y - zD_z)$.

5.30 $(a = -1)$ $[e_2, e_4] = e_1$, $[e_3, e_4] = e_2$, $[e_2, e_5] = -e_2$, $[e_3, e_5] = -2e_3$, $[e_4, e_5] = e_4$: center $\langle e_1 \rangle$

$$\begin{bmatrix} 1 & xe^{w} & -4\,x^2e^{2\,w} & ze^{-w} & q \\ 0 & e^{w} & -8\,xe^{2\,w} & ye^{-w} & 4\,yx - 12\,z \\ 0 & 0 & e^{2\,w} & 0 & y \\ 0 & 0 & 0 & e^{-w} & 4\,x \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Right-invariant vector fields: $-16D_q, 4xD_q + D_z, D_y, -(12z - 4xy)D_q + D_x + yD_z, -D_w + xD_x - 2yD_y - zD_z$.

5.31 $[e_2, e_4] = e_1$, $[e_3, e_4] = e_2$, $[e_1, e_5] = 3e_1$, $[e_2, e_5] = 2e_2$, $[e_3, e_5] = e_3$, $[e_4, e_5] = e_3 + e_4$: trivial center

$$S = \begin{bmatrix} e^{3w} & -ze^{2w} & \frac{1}{2}z^2e^{w} & \frac{1}{2}e^{w}(z^2w + x - yz + \frac{3z^2}{2}) & q \\ 0 & e^{2w} & -ze^{w} & e^{w}(y - z - zw) & x \\ 0 & 0 & e^{w} & we^{w} & y \\ 0 & 0 & 0 & e^{w} & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $3D_q, -(2D_x + zD_q), D_y + zD_x, D_z - xD_q - (y+z)D_x, D_w + 3qD_q + 2xD_x + (y+z)D_y + +zD_z$.

5.32a $[e_2, e_4] = e_1$, $[e_3, e_4] = e_2$, $[e_1, e_5] = e_1$, $[e_2, e_5] = e_2$, $[e_3, e_5] = ae_1 + e_3$: trivial center

$$S = \begin{bmatrix} e^w & ze^w & \frac{1}{2}(2aw + z^2)e^w & x & q \\ 0 & e^w & ze^w & y & x \\ 0 & 0 & e^w & 0 & y \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_q, D_x, D_y, D_z + xD_q + yD_x, D_w + (ay + q)D_q + xD_x + yD_y$.

5.33ab, $(a^2 + b^2) \neq 0$ $[e_1, e_4] = e_1$, $[e_3, e_4] = be_3$, $[e_2, e_5] = e_2$, $[e_3, e_5] = ae_3$: trivial center

$$S = \begin{bmatrix} e^w & 0 & 0 & x & 0 \\ 0 & e^z & 0 & 0 & y \\ 0 & 0 & e^{(aw+bz)} & aq & bq \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_q, D_x, D_y, D_z + yD_y + aqD_q, D_w + xD_x + bqD_q$.

5.34a $[e_1, e_4] = ae_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_3$, $[e_1, e_5] = e_1$, $[e_3, e_5] = e_2$: trivial center

$$S = \begin{bmatrix} e^{az+w} & 0 & 0 & z & q \\ 0 & e^z & we^z & x & y \\ 0 & 0 & e^z & y & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_q, D_x, D_y, D_z + aqD_q + xD_x + yD_y, D_w + qD_q + yD_x$.

5.35ab $(a^2 + b^2 \neq 0)$ $[e_1, e_4] = be_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_3$, $[e_1, e_5] = ae_1$, $[e_2, e_5] = -e_3$, $[e_3, e_5] = e_2$: trivial center

$$S = \begin{bmatrix} e^{aw+bz} & 0 & 0 & bq & aq \\ 0 & e^z \cos(w) & e^z \sin(w) & -x & y \\ 0 & -e^z \sin(w) & e^z \cos(w) & y & x \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_q, -D_x, D_y, D_z + bqD_q + xD_x + yD_y, D_w + aqD_q - yD_x + xD_y$.

5.36 $[e_2, e_3] = e_1$, $[e_1, e_4] = e_1$, $[e_2, e_4] = e_2$, $[e_2, e_5] = -e_2$, $[e_3, e_5] = e_3$: trivial center

$$S = \begin{bmatrix} e^x & ze^x & qe^{-(x+y)} \\ 0 & e^y & we^{-(x+y)} \\ 0 & 0 & e^{-(x+y)} \end{bmatrix}.$$

Right-invariant vector fields: $-D_q, (D_z + wD_q), \frac{1}{3}(2D_x - D_y + 3qD_q + 3zD_z)$, $\frac{1}{3}(2D_y - D_x - 3zD_z + 3wD_w)$.

5.37 $[e_2, e_3] = e_1$, $[e_1, e_4] = 2e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_3$, $[e_2, e_5] = -e_3$, $[e_3, e_5] = e_2$: trivial center

$$S = \begin{bmatrix} e^{2z} & -e^z(\cos(w)x + \sin(w)y) & e^z(-\sin(w)x + \cos(w)y) & -\frac{(x^2+y^2)}{2} & q \\ 0 & \cos(w)e^z & \sin(w)e^z & x & y \\ 0 & -\sin(w)e^z & \cos(w)e^z & -y & x \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $-2D_q, D_y + xD_q, D_x - yD_q, D_w + 2qD_q + xD_x + yD_y, D_z - yD_x + xD_y$.

Alternative parametrization:

$$S = \begin{bmatrix} z^2 + w^2 & -zx - wy & zy - wx & -\frac{x^2+y^2}{2} & q \\ 0 & z & w & x & y \\ 0 & -w & z & -y & x \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $-2D_q, D_y + xD_q, D_x - yD_q, 2qD_q + zD_z + wD_w + xD_x + yD_y, wD_z - zD_w + yD_x - xD_y$.

5.38 $[e_1, e_4] = e_1$, $[e_2, e_5] = e_2$, $[e_4, e_5] = e_3$: center $\langle e_3 \rangle$

$$S = \begin{bmatrix} e^z & 0 & 0 & 0 & q \\ 0 & e^w & 0 & 0 & x \\ 0 & 0 & 1 & w & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_q, D_x, D_y, qD_q + D_z, D_w + xD_x + zD_y$.

5.39 $[e_1, e_4] = e_1$, $[e_2, e_4] = e_2$, $[e_1, e_5] = -e_2$, $[e_2, e_5] = e_1$, $[e_4, e_5] = e_3$: center $\langle e_3 \rangle$

$$S = \begin{bmatrix} 1 & x & y & -w & q \\ 0 & e^w \cos(z) & e^w \sin(z) & 0 & 0 \\ 0 & -e^w \sin(z) & e^w \cos(z) & 0 & 0 \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Left-invariant vector fields: $D_x, D_y, -D_q, -D_w + xD_x + yD_y, D_z + wD_q + yD_x - xD_y$.

5.40 $[e_1, e_2] = 2e_1$, $[e_1, e_3] = -e_2$, $[e_2, e_3] = 2e_3$, $[e_1, e_4] = e_5$, $[e_2, e_4] = e_4$, $[e_2, e_5] = -e_5$, $[e_3, e_5] = e_4$: trivial center

$$S = \begin{bmatrix} e^x & y & w \\ z & (1 + yz)e^{-x} & q \\ 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $ze^{-x}D_x + (yz + 1)e^{-x}D_y + qD_w, -qD_q + D_x + yD_y - zD_z + wD_w, wD_q + e^x D_z, D_q, D_w$.

We now discuss briefly the fundamental groups of the matrix groups we have exhibited above. At the outset it is apparent that all with the possible exception of the following are simply-connected: in dimension three, 6, 7, 8, 9; in dimension four, 6, 10, 11, 12; in dimension five, 13, 14, 16, 17, 18, 25, 26, 35, 39, 40. With the exception of the two simple algebras 3.8 and 3.9, all the groups corresponding to algebras not in this list are upper triangular over $\mathbf{R}$ and they are contractible as can be seen by multiplying each of the coordinates by t and deforming from $t = 1$ to $t = 0$. As for algebra 3.8 the standard representation of the group $SL(2, \mathbf{R})$ as 2×2 matrices has fundamental group isomorphic to $\mathbf{Z}$. The coordinate representation given above is in terms of a chart in the neighborhood of the identity. More charts are required for the whole group. It is well known that there is no linear representation of the simply-connected cover of $SL(2, \mathbf{R})$. For algebra 3.9 the standard representation of the group $SO(3, \mathbf{R})$ as 3×3 matrices has fundamental group isomorphic to $\mathbf{Z}_2$. The simply-connected cover of $SO(3, \mathbf{R})$ can be realized as the unit quarternions. The coordinate representation given above is in terms of a chart in the neighborhood of the identity corresponding to the Euler angles.

As regards the Lie group that corresponds to the Lie algebra 3.6 above, by multiplying the coordinates x and y by t we may retract to the group $SO(2, \mathbf{R})$ whose fundamental group is isomorphic to $\mathbf{Z}$. As for the Lie group that corresponds to the Lie algebra 3.7, again we may retract away the x and y coordinates: since $a \neq 0$ what remains is a curve diffeomorphic to $\mathbf{R}$ and hence the group is simply-connected in this case.

Turning now to dimension four, for algebra 4.6 the x, y and z-coordinates in the group representation may be retracted away and since $a \neq 0$ the result is simply-connected. Similarly 4.11 can be shown to be simply-connected. On the other hand in cases 4.10 and 4.12 by retracting the x,y and z coordinates the fundamental group is seen to be isomorphic to $\mathbf{Z}$ in each case.

Let us now consider dimension five. For algebra 5.13 after retraction since $a \neq 0$ we obtain a curve diffeomorphic to $\mathbf{R}$ and hence the corresponding group is simply-connected. A similar analysis applies to algebras 5.14 and 5.16, the latter no matter what value p takes. As regards algebra 5.17 the fundamental group is isomorphic to $\mathbf{Z}$ only if and only if $p = q = 0$ and s is a rational number. On the other hand for the algebra 5.18 the fundamental group is trivial whatever the value of p.

For algebra 5.25($p = 0$) and 5.25($p \neq 0$) q, x, y, z may be retracted away and the result is a curve diffeomorphic to $\mathbf{R}$ and hence the group is simply-connected in this case. A similar remark applies also to algebra 5.26 regardless of the value of p.

For algebra 5.35 we retract away q, x, y, z and then the fundamental group will be either trivial or isomorphic to $\mathbf{Z}$ case according as $a \neq 0$ or $a = 0$. For algebra 5.39 the fundamental group is trivial. Finally for the algebra 5.40 there is a retraction to the group $SL(2, \mathbf{R})$ and so the fundamental group is isomorphic to $\mathbf{Z}$ in this case.

5. Some n-dimensional groups

In this short section we give a few particularly nice examples of groups that are valid in arbitrary dimensions.

1. The standard filiform Lie algebra: $[e_2, e_{n+1}] = e_1$, $[e_3, e_{n+1}] = e_2$, $[e_4, e_{n+1}] = e_3$, $[e_5, e_{n+1}] = e_4$, …, $[e_n, e_{n+1}] = e_{n-1}$: center $\langle e_n \rangle$

$$S = \begin{bmatrix} 1 & w & \frac{w^2}{2} & \frac{w^3}{6} & \cdots & \frac{w^{n-1}}{(n-1)!} & \frac{w^n}{n!} & x_n \\ 0 & 1 & w & \frac{w^2}{2} & \cdots & \frac{w^{n-2}}{(n-2)!} & \frac{w^{n-1}}{(n-1)!} & x_{n-1} \\ & \cdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & w & \frac{w^2}{2} & x_3 \\ 0 & 0 & 0 & 0 & \cdots & 1 & w & x_2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & x_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_{x_1}, D_{x_2}, D_{x_3}, \ldots, D_{x_n}, D_w + \sum_{i=1}^{n-1} x_{i+1} D_{x_i}$.

2. $[e_1, e_{n+1}] = e_1$, $[e_2, e_{n+1}] = e_1 + e_2$, $[e_3, e_{n+1}] = e_2 + e_3$, $[e_4, e_{n+1}] = e_3$, $[e_5, e_{n+1}] = e_4$, …, $[e_n, e_{n+1}] = e_{n-1} + e_n$: trivial center

$$S = \begin{bmatrix} e^w & we^w & \frac{w^2}{2}e^w & \frac{w^3}{6}e^w & \cdots & \frac{w^{n-1}}{(n-1)!}e^w & \frac{w^n}{n!}e^w & x_n \\ 0 & e^w & we^w & \frac{w^2}{2}e^w & \cdots & \frac{w^{n-2}}{(n-2)!}e^w & \frac{w^{n-1}}{(n-1)!}e^w & x_{n-1} \\ & \cdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & we^w & \frac{w^2}{2}e^w & x_3 \\ 0 & 0 & 0 & 0 & \cdots & e^w & we^w & x_2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & e^w & x_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_{x_1}, D_{x_2}, D_{x_3}, \ldots, D_{x_n}, D_w + \sum_{i=1}^{n-1} (x_i + x_{i+1}) D_{x_i}$.

3. The Heisenberg Lie algebra: $[e_1, e_2] = e_{2n+1}$, $[e_3, e_4] = e_{2n+1}$, $[e_5, e_6] = e_{2n+1}$, …, $[e_{2n-1}, e_{2n}] = e_{2n+1}$: center $\langle e_{2n+1} \rangle$

$$S = \begin{bmatrix} 1 & x_1 & x_2 & \cdots & x_n & z \\ 0 & 1 & 0 & \cdots & 0 & y_1 \\ 0 & 0 & 1 & \cdots & 0 & y_2 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & y_n \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_{y_1}, D_{x_1} + y_1 D_z, D_{y_2}, D_{x_2} + y_2 D_z, \ldots, D_{y_n}, D_{x_n} + y_n D_z, D_z$.

4. $[e_1, e_{n+1}] = a_1 e_1$, $[e_2, e_{n+1}] = a_2 e_2$, ..., $[e_n, e_{n+1}] = a_n e_{n+1}$: trivial center

$$S = \begin{bmatrix} e^{a_1 w} & 0 & 0 & \cdots & 0 & x_1 \\ 0 & e^{a_2 w} & 0 & \cdots & 0 & x_2 \\ 0 & 0 & e^{a_3 w} & \cdots & 0 & x_3 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & e^{a_n w} & x_n \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_{x_1}, D_{x_2}, D_{x_3}, \ldots, D_{x_n}, D_w + \sum_{i=1}^{n} a_i x_i D_{x_i}$.

5. $[e_{n+1},\ e_{2n+1}] = e_1$, $[e_{n+2},\ e_{2n+1}] = e_2$, ..., $[e_{2n}, e_{2n+1}] = e_n$: center $\langle e_1, e_2, \ldots, e_n \rangle$

$$S = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & w & 0 & 0 & \cdots & 0 & x_1 \\ 0 & 1 & 0 & \cdots & 0 & 0 & w & 0 & \cdots & 0 & x_2 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & w & \cdots & 0 & x_3 \\ \vdots & & & \ddots & & \ddots & & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & & \cdots & w & x_n \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & x_{n+1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & x_{n+2} \\ \vdots & & & \ddots & & & \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & x_{2n} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Right-invariant vector fields: $D_{x_1}, D_{x_2}, D_{x_3}, \ldots, D_{x_{2n}}, D_w + \sum_{i=1}^{n} x_{n+i} D_{x_i}$.

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