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On BVPs in $l^\infty(A)$

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1. Introduction

Let $A \neq \emptyset$ be a set, and let $l^\infty(A)$ denote the real Banach space of all bounded functions $x = (x_\alpha)_{\alpha\in A} : A \to \mathbb{R}$, endowed with the supremum norm $||\cdot||$. Let $l^\infty(A)$ be ordered by the cone

$$K = \{x : x_\alpha \geq 0 \ (\alpha \in A)\},$$

that is $x \leq y :\Leftrightarrow y - x \in K$. Inequalities for functions with values in $l^\infty(A)$ are always intended pointwise.

For two functions $v, w : [0,1] \to l^\infty(A)$ with $v \leq w$ we consider

$$S(v,w) = \big\{(t,x) \in (0,1) \times l^\infty(A) : v(t) \leq x \leq w(t) \ (t \in (0,1))\big\},$$

and a function $f : S(v,w) \times l^\infty(A) \to l^\infty(A)$. We will assume that v, w is a pair of generalized upper and lower functions, that f is continuous and satisfies a Nagumo condition, that f is quasimonotone increasing in its second variable, and that f is diagonally depending on the third variable.

Under these conditions we will prove the existence of a maximal and a minimal solution of the boundary value problem

$$u''(t) + f(t, u(t), u'(t)) = 0, \quad u(0) = u(1) = 0.$$

2. Extremal solutions of scalar BVPs

For a function $u : [0,1] \to \mathbb{R}$ let

$$D_-u(t),\ D^-u(t) \ \ (t \in (0,1]), \quad D_+u(t),\ D^+u(t) \ \ (t \in [0,1))$$

denote the Dini derivatives of u, and for $t \in (0,1)$ let

$$D_2 u(t) := \liminf_{h\to 0} \frac{u(t+h) - 2u(t) + u(t-h)}{h^2},$$

$$D^2 u(t) := \limsup_{h\to 0} \frac{u(t+h) - 2u(t) + u(t-h)}{h^2}$$

denote the Schwarz derivatives of u.

Now, let $v, w : [0,1] \to \mathbb{R}$, $v \le w$,

$$S(v,w) = \big\{(t,x) \in (0,1) \times \mathbb{R} : v(t) \le x \le w(t)\ (t \in (0,1))\big\},$$

and $f : S(v,w) \times \mathbb{R} \to \mathbb{R}$ be given, and consider the scalar boundary value problem

$$u''(t) + f(t, u(t), u'(t)) = 0, \quad u(0) = u(1) = 0. \tag{1}$$

We employ the following notion for lower and upper functions to (1):

The function $v : [0,1] \to \mathbb{R}$ is called lower function for (1), if it is Lipschitz continuous, if we have $v(0) \le 0$, $v(1) \le 0$, $D^- v(t) \le D_+ v(t)$ $(t \in (0,1))$, and if for each $t \in (0,1)$ such that $v'(t)$ exists we have

$$D^2 v(t) + f(t, v(t), v'(t)) \ge 0.$$

Analogously $w : [0,1] \to \mathbb{R}$ is called upper function for (1), if it is Lipschitz continuous, if $w(0) \ge 0$, $w(1) \ge 0$, $D_- w(t) \ge D^+ w(t)$ $(t \in (0,1))$, and if for each $t \in (0,1)$ such that $w'(t)$ exists we have

$$D_2 w(t) + f(t, w(t), w'(t)) \le 0.$$

The function f satisfies a Nagumo condition with respect to v and w, if there exists a continuous function $q : [0,\infty) \to (0,\infty)$ with

$$\int_0^\infty \frac{s}{q(s)}\, ds = \infty,$$

such that

$$|f(t,x,p)| \le q(|p|) \quad ((t,x,p) \in S(v,w) \times \mathbb{R}).$$

The following Nagumo type theorem [10] is due to Akǒ [1] Theorem 1.1. Our concept of lower and upper functions is a simplification of the concept of lower and upper functions in the sense of Akǒ. We will give a proof of Theorem 1 for this reason.

THEOREM 1. *Let $v, w : [0,1] \to \mathbb{R}$ with $v \le w$ and $f : S(v,w) \times \mathbb{R} \to \mathbb{R}$ be such that f is continuous and satisfies a Nagumo condition with respect to v and w, and that v, w are lower and upper functions for* (1), *respectively. Then* (1) *has a minimal and a maximal solution in $C([0,1],\mathbb{R}) \cap C^2((0,1),\mathbb{R})$* (*whose graph is in $S(v,w)$*).

Remark. Extremal solutions for boundary value problems have been studied by several authors for various equations, boundary conditions and generalizations of lower and upper functions, see for example [3] Chapter 5., [9], [11] and the references given there.

As an immediate consequence of Theorem 1 we will obtain monotone dependence of the extremal solutions on f. Consider a second boundary value problem

$$u''(t) + g(t, u(t), u'(t)) = 0, \quad u(0) = u(1) = 0. \tag{2}$$

THEOREM 2. *Under the assumptions of Theorem 1 let $g : S(v,w) \times \mathbb{R} \to \mathbb{R}$ be continuous, satisfy a Nagumo condition with respect to v and w, let $v, w : [0,1] \to \mathbb{R}$ be a lower and upper functions for* (2), *respectively, and let $f(t,x,p) \le g(t,x,p)$ on $S(v,w) \times \mathbb{R}$. Then the maximal* (*minimal*) *solution of* (1) *is $\le$ the maximal* (*minimal*) *solution of* (2).

3. The main result

Let $v, w : [0,1] \to l^\infty(A)$, $v \le w$ and for each $\alpha \in A$ let a function $f_\alpha : S(v,w) \times \mathbb{R} \to \mathbb{R}$ be given, such that

$$f(t,x,p) = \Big(f_\alpha(t,x,p_\alpha)\Big)_{\alpha \in A}$$

defines a function $f : S(v,w) \times l^\infty(A) \to l^\infty(A)$.

If $x \mapsto f(t,x,p)$ is continuous on $\{x : v(t) \le x \le w(t)\}$ for each $(t,p) \in (0,1) \times l^\infty(A)$, then the function f is quasimonotone increasing in its second variable, in the sense of Volkmann [13], if and only if

$$(t,x,p), (t,y,p) \in S(v,w) \times l^\infty(A),\ x \le y,\ \alpha \in A,\ x_\alpha = y_\alpha$$
$$\Rightarrow f_\alpha(t,x,p_\alpha) \le f_\alpha(t,y,p_\alpha),$$

compare [12].

We consider the boundary value problem

$$u''(t) + f(t, u(t), u'(t)) = 0, \quad u(0) = u(1) = 0 \tag{3}$$

in $l^\infty(A)$.

Now, $v : [0,1] \to l^\infty(A)$ is called lower function for (3), if it is Lipschitz continuous, if we have $v(0) \leq 0$, $v(1) \leq 0$, and if it has the following properties for each coordinate $\alpha \in A$: $D^- v_\alpha(t) \leq D_+ v_\alpha(t)$ $(t \in (0,1))$, and for each $t \in (0,1)$ such that $v'_\alpha(t)$ exists we have

$$D^2 v_\alpha(t) + f_\alpha(t, v(t), v'_\alpha(t)) \geq 0.$$

The definition of an upper function $w : [0,1] \to l^\infty(A)$ is now obvious.

We say that f satisfies a Nagumo condition with respect to v and w, if there exists a continuous function $q : [0,\infty) \to (0,\infty)$ with

$$\int_0^\infty \frac{s}{q(s)}\, ds = \infty,$$

such that for each $\alpha \in A$

$$|f_\alpha(t,x,r)| \leq q(|r|) \quad ((t,x,r) \in S(v,w) \times \mathbb{R}).$$

Remark. A Nagumo condition in particular implies that $f(S(v,w) \times B)$ is bounded for each bounded subset $B \subseteq l^\infty(A)$. It is a notable fact that in contrast to the finite dimensional case ($|A| < \infty$) and in contrast to the case of monotone functions, a continuous quasimonotone increasing function defined on an order interval may be unbounded. An example is $g : [0,1]^{\mathbb{N}} \to l^\infty(\mathbb{N})$ defined by

$$g(x) = \left(\frac{1 - x_n}{x_n + \sum_{k=1}^\infty (1 - x_k)/2^k} \right)_{n \in \mathbb{N}}.$$

We have

THEOREM 3. *Let $v, w : [0,1] \to l^\infty(A)$ with $v \leq w$ and $f_\alpha : S(v,w) \times \mathbb{R} \to \mathbb{R}$ $(\alpha \in A)$ be such that $f : S(v,w) \times l^\infty(A) \to l^\infty(A)$, $f(t,x,p) = (f_\alpha(t,x,p_\alpha))_{\alpha \in A}$ is continuous, quasimonotone increasing in its second variable, satisfies a Nagumo condition with respect to v and w, and that v, w are lower and upper functions for* (3)*, respectively. Then* (3) *has a minimal and a maximal solution in $C([0,1], l^\infty(A)) \cap C^2((0,1), l^\infty(A))$ (whose graph is in $S(v,w)$).*

Remarks. 1. We will prove Theorem 3 by a variant of Tarski's fixed point Theorem. For existence results of solutions of boundary value problems in $\mathbb{R}^n$ involving quasimonotonicity and upper and lower functions see [6], [7] and the

references given there.
2. For existence results of extremal solutions for initial value problems of first order equations in $l^\infty(A)$ see [4], [8] and the references given there.

4. Proof of Theorem 1

We make use of Nagumo's Lemma [5, Chapter VII, Lemma 5.1]:

Proposition 1. *Let $q:[0,\infty)\to(0,\infty)$ be continuous, let $z\in C([0,1],\mathbb{R})$ $\cap C^2((0,1),\mathbb{R})$, $z(0)=z(1)=0$, and let*

$$\max_{t\in[0,1]} z(t) - \min_{t\in[0,1]} z(t) \le \int_0^L \frac{s}{q(s)}ds.$$

Then $|z''(t)|\le q(|z'(t)|)$ $(t\in(0,1))$ implies $|z'(t)|\le L$ $(t\in(0,1))$.

Extend f to $(0,1)\times\mathbb{R}^2$ by

$$\widetilde{f}(t,x,p)=\begin{cases} f(t,w(t),p)-\dfrac{x-w(t)}{1+x-w(t)} & (x>w(t)) \\ f(t,v(t),p)+\dfrac{v(t)-x}{1+v(t)-x} & (x<v(t)) \end{cases}$$

and choose $L\ge 0$ such that

$$\int_0^L \frac{s}{q(s)}ds \ge \max_{t\in[0,1]} w(t) - \min_{t\in[0,1]} v(t).$$

Without loss of generality L is a Lipschitz constant for both v and w. Next, let $S:\mathbb{R}\to\mathbb{R}$ be continuous such that $0\le S(p)\le 1$ $(p\in\mathbb{R})$, and

$$S(p)=1 \quad (|p|\le L), \qquad S(p)=0 \quad (|p|\ge L+1).$$

Set

$$F(t,x,p)=S(p)\widetilde{f}(t,x,p) \quad ((t,x,p)\in(0,1)\times\mathbb{R}^2).$$

Then

$$|F(t,x,p)|\le q(|p|)+1,$$

so

$$|F(t,x,p)|\le \max\{q(|p|)+1 : |p|\le L+1\}.$$

Thus, F is continuous and bounded on $(0,1) \times \mathbb{R}^2$. By Scorzà Dragoni's theorem there is a solution of

$$u''(t) + F(t, u(t), u'(t)) = 0, \quad u(0) = u(1) = 0,$$

which turns out to be in $S(v, w)$: If there was $t \in (0,1)$ such that $u(t) > w(t)$, there would exist an interval $[t_1, t_2] \subseteq [0,1]$ such that

$$u(t_1) = w(t_1), \quad u(t_2) = w(t_2), \quad u(t) > w(t) \ \ (t \in (t_1, t_2)).$$

The function $w - u$ would then have a negative minimum there, say for $t = t_0$, where evidently

$$\begin{aligned} D^-(w-u)(t_0) &= D^- w(t_0) - u'(t_0) \le 0, \\ D_+(w-u)(t_0) &= D_+ w(t_0) - u'(t_0) \ge 0, \\ D_2(w-u)(t_0) &= D_2 w(t_0) - u''(t_0) \ge 0. \end{aligned} \tag{4}$$

But then

$$D^+ w(t_0) \ge D_+ w(t_0) \ge u'(t_0) \ge D^- w(t_0) \ge D_- w(t_0) \ge D^+ w(t_0),$$

where the last inequality holds according to the definition of an upper function. So w is differentiable at t_0 with $w'(t_0) = u'(t_0)$. This implies $|u'(t_0)| \le L$, thus

$$\begin{aligned} u''(t_0) &= -F(t_0, u(t_0), u'(t_0)) = -\widetilde{f}(t_0, u(t_0), u'(t_0)) \\ &= -f(t_0, w(t_0), w'(t_0)) + \frac{u(t_0) - w(t_0)}{1 + u(t_0) - w(t_0)} \\ &> -f(t_0, w(t_0), w'(t_0)) \ge D_2 w(t_0), \end{aligned}$$

which contradicts (4).

The inequality $v(t) \le w(t)$ is proven along the same lines.

Therefore

$$|u''(t)| = |S(u'(t)) f(t, u(t), u'(t))| \le q(|u'(t)|) \quad (t \in (0,1)),$$

and according to Proposition 1 $|u'(t)| \le L$, thus $S(u'(t)) = 1$ $(t \in (0,1))$.

To show that there is a maximal and a minimal solution, note that for each solution $u : [0,1] \to \mathbb{R}$ of (1), $u' : (0,1) \to \mathbb{R}$ can be extended to $[0,1]$ such that $u \in C^1([0,1], \mathbb{R})$, and that the set of all solutions to (1) is a compact subset of $C^1([0,1],\mathbb{R})$, as Proposition 1 implies $|u'(t)| \le L$ $(t \in (0,1))$ for each solution. Set

$$\overline{u}(t) = \max\{u(t) : u \text{ is a solution of } (1)\}.$$

Then $\overline{u}$ is Lipschitz continuous with constant L, and to each $t_0 \in (0,1)$ there is a solution u_0 of (1) satisfying $u_0(t_0) = \overline{u}(t_0)$. Because of $u_0 \le \overline{u}$ it follows

$$D_+\overline{u}(t_0) \ge u_0'(t_0) \ge D^-\overline{u}(t_0), \quad D_2\overline{u}(t_0) \ge u_0''(t_0),$$

and, in case $\overline{u}$ is differentiable at t_0,

$$\overline{u}'(t_0) = u_0'(t_0).$$

Therefore,

$$D^2\overline{u}(t_0) \ge D_2\overline{u}(t_0) \ge u_0''(t_0) = -f(t, u_0(t_0), u_0'(t_0)) = -f(t, \overline{u}(t_0), \overline{u}'(t_0)).$$

Summing up, $\overline{u}$ is a lower function for (1), and by the first part of the proof, there is a solution of (1) between $\overline{u}$ and w, which must be $\overline{u}$. So $\overline{u}$ is the maximal solution.

The existence of a minimal solution $\underline{u}$ follows by similar reasoning.

5. Proof of Theorem 2

Let $\overline{u}$ and $\overline{U}$ be the maximal solution of (1) and (2), respectively. Then, for $t \in (0,1)$ we get

$$\overline{u}''(t) + g(t, \overline{u}(t), \overline{u}'(t)) \ge \overline{u}''(t) + f(t, \overline{u}(t), \overline{u}'(t)) = 0,$$

and therefore $\overline{u}$ is a lower function of (2). Thus, (2) has a solution between $\overline{u}$ and w, in particular $\overline{u}(t) \le \overline{U}(t) \le w(t)$. Analogously, for the minimal solutions $\underline{u}$ and $\underline{U}$ we have

$$0 = \underline{U}''(t) + g(t, \underline{U}(t), \underline{U}'(t)) \ge \underline{U}''(t) + f(t, \underline{U}(t), \underline{U}'(t)),$$

thus $\underline{U}$ is an upper function of (1), and therefore $v(t) \le \underline{u}(t) \le \underline{U}(t)$.

6. Proof of Theorem 3

We make use of a fixed point Theorem of Bourbaki [2].

Proposition 2. *Let $\Omega \ne \emptyset$ be an ordered set, and let $T : \Omega \to \Omega$ be monotone increasing.*

1. *If* $\sup C$ *exists for each chain* $\emptyset \neq C \subseteq \Omega$, *and if there is* $\omega_0 \in \Omega$, $\omega_0 \leq T\omega_0$, *then* T *has a smallest fixed point in the set* $\{\omega \in \Omega : \omega_0 \leq \omega\}$.
2. *If* $\inf C$ *exists for each chain* $\emptyset \neq C \subseteq \Omega$, *and if there is* $\omega_1 \in \Omega$, $T\omega_1 \leq \omega_1$, *then* T *has a greatest fixed point in the set* $\{\omega \in \Omega : \omega \leq \omega_1\}$.

Let $L \geq 0$ be such that for each $\alpha \in A$

$$\int_0^L \frac{s}{q(s)} ds \geq \max_{t\in[0,1]} w_\alpha(t) - \min_{t\in[0,1]} v_\alpha(t),$$

and set

$$M = \sup \left\{ ||f(t,x,p)|| : (t,x,p) \in S(v,w) \times [-L,L]^{\mathbb{N}} \right\}.$$

Note that $M < \infty$ since $f(S(v,w) \times [-L,L]^{\mathbb{N}})$ is bounded, as a consequence of Nagumo's condition.

We consider the following subset Ω of $C^1([0,1], l^\infty(A))$:

$$\{\omega : \omega(0) = \omega(1) = 0,\ ||\omega'(t)|| \leq L,\ ||\omega'(t) - \omega'(s)|| \leq M||t-s||\ (t,s \in [0,1])\}$$

By standard reasoning $\sup C$ and $\inf C$ exist for each chain $\emptyset \neq C \subseteq \Omega$ (but Ω is not a lattice). First note that each solution of (3) is in Ω, by the choice of L and M, and by continuous extension of $u' : (0,1) \to l^\infty(A)$ to $[0,1]$.

We define a mapping T the following way:

Let $\omega : [0,1] \to l^\infty(A)$ be continuous with $v \leq \omega \leq w$ (not necessarily $\omega \in \Omega$), $\alpha \in A$,

$$S_\alpha(v,w) := \{(t,\xi) \in (0,1) \times \mathbb{R} : v_\alpha(t) \leq \xi \leq w_\alpha(t)\},$$

$$(Q_\alpha(x,\xi))_\beta = \begin{cases} x_\beta & \beta \neq \alpha \\ \xi & \beta = \alpha \end{cases} \qquad (x \in l^\infty(A), \xi \in \mathbb{R}),$$

and let $g_\alpha : S_\alpha(v,w) \times \mathbb{R} \to \mathbb{R}$ be defined by

$$g_\alpha(t,\xi,r) = f_\alpha(t, Q_\alpha(\omega(t),\xi), r).$$

Each function $Q_\alpha : l^\infty(A) \times \mathbb{R} \to l^\infty(A)$ is Lipschitz continuous, hence each g_α is continuous.

Consider the scalar boundary value problems

$$u_\alpha''(t) + g_\alpha(t, u_\alpha(t), u_\alpha'(t)) = 0, \quad u_\alpha(0) = u_\alpha(1) = 0. \tag{5}$$

Now, Theorem 1 applies to (5), since v_α, w_α are lower and upper functions for (5), respectively. For example if $t \in (0,1)$ and $v'_\alpha(t)$ exists, then the quasimonotonicity of f implies

$$D^2 v_\alpha(t) + f_\alpha(t, Q_\alpha(\omega(t), v_\alpha(t)), v'_\alpha(t)) \geq D^2 v_\alpha(t) + f_\alpha(t, v(t), v'_\alpha(t)) \geq 0.$$

Hence (5) has a maximal solution $\overline{u}_\alpha$ and according to Proposition 1 we have $|\overline{u}'_\alpha(t)| \leq L$, and therefore $|\overline{u}''_\alpha(t)| \leq M$ $(t \in (0,1))$. Thus, $T\omega := (\overline{u}_\alpha)_{\alpha \in A} \in \Omega$, and in particular $T(\Omega) \subseteq \Omega$.

Moreover $\omega \leq \widetilde{\omega}$ implies

$$f_\alpha(t, Q_\alpha(\omega(t), \xi), r) \leq f_\alpha(t, Q_\alpha(\widetilde{\omega}(t), \xi), r)$$

since f is quasimonotone increasing. Therefore $T\omega \leq T\widetilde{\omega}$ according to Theorem 2, and in particular T is monotone increasing on Ω.

To see that $\Omega \neq \emptyset$ note that $Tw \in \Omega$.

Next, consider $\omega_1 := Tw \leq w$. Then $T\omega_1 \leq \omega_1$ and Proposition 2 proves the existence of a greatest fixed point z of T in $\{\omega \in \Omega : \omega \leq \omega_1\}$, which is a solution of (3), since $Tz = z$ means that the maximal solution of

$$u''_\alpha(t) + f_\alpha(t, Q_\alpha(z(t), u_\alpha(t)), u'_\alpha(t)) = 0, \quad u_\alpha(0) = u_\alpha(1) = 0$$

is $u_\alpha = z_\alpha$, hence $Q_\alpha(z(t), u_\alpha(t)) = z(t)$.

Finally z is the greatest solution of (3) between v and w: Let y be any solution of (3). In particular $y \in \Omega$ and $y \leq w$. We have

$$y''_\alpha(t) + f_\alpha(t, y(t), y'_\alpha(t)) = 0, \quad y_\alpha(0) = y_\alpha(1) = 0,$$

thus, y_α is a solution of

$$u''_\alpha(t) + f_\alpha(t, Q_\alpha(y(t), u_\alpha(t)), u'_\alpha(t)) = 0, \quad u_\alpha(0) = u_\alpha(1) = 0,$$

whereas $(Ty)_\alpha$ is the greatest solution of this boundary value problem. This proves $y \leq Ty$. Set $\omega_0 := y$. Again, by means of Proposition 2 there is a (smallest) fixed point $\widetilde{z}$ of T in $\{\omega \in \Omega : \omega_0 \leq \omega\}$. From $\widetilde{z} \leq w$ we obtain $\widetilde{z} \leq Tw = \omega_1$. Therefore, $\widetilde{z}$ is a fixed point of T in $\{\omega \in \Omega : \omega \leq \omega_1\}$. In particular $y \leq \widetilde{z} \leq z$.

Analogously one can prove the existence of a minimal solution of (3) between v and w.

7. A uniqueness condition

Let Ψ be a continuous positive linear functional on $l^\infty(A)$ with the following property:

$$x \geq 0,\ \Psi(x) = 0\ \Rightarrow\ x = 0.$$

Assume that v, w and f are as in Theorem 3, and that there are continuous functions $k, l : (0,1) \to \mathbb{R}$ such that:

1. The differential inequality $z''(t) + k(t)|z'(t)| + l(t)z(t) < 0$ has a positive solution $z \in C([0,1],\mathbb{R}) \cap C^2((0,1),\mathbb{R})$;
2. For $(t,x,p), (t,\widetilde{x},\widetilde{p}) \in S(v,w) \times l^\infty(A)$ with $x \leq \widetilde{x}$

$$\Psi(f(t,\widetilde{x},\widetilde{p}) - f(t,x,p)) \leq k(t)|\Psi(\widetilde{p}-p)| + l(t)\Psi(\widetilde{x}-x).$$

Let $\underline{u}, \overline{u}$ be the minimal and maximal solution of (3) according to Theorem 3, and set $h = \varphi(\overline{u} - \underline{u})$. Then, by means of 2., we have

$$h''(t) + k(t)|h'(t)| + l(t)h(t) \geq 0, \quad h(0) = h(1) = 0$$

By means of 1., standard reasoning proves $h(t) \leq 0$, hence $h(t) = 0$ $(t \in [0,1])$, and therefore $\overline{u} = \underline{u}$. In particular (3) is uniquely solvable between v and w.

8. An example

First note that our results hold for $[a,b] \subseteq \mathbb{R}$ instead of $[0,1]$ and for general boundary values instead of 0, as usual, by application of an affine transformation.

Let $A = \mathbb{Z}$. Let $g : \mathbb{R} \to \mathbb{R}$ be continuous, monotone increasing, $g(0) \geq 0$, and let $h = (h_n) : (-1,1) \to K$ be continuous and bounded: $||h(t)|| \leq c$ $(t \in (-1,1))$.

Consider the boundary value problem

$$u_n''(t) + h_n(t)g(u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)) - (u_n'(t))^2 = 0, \tag{6}$$

$$u_n(a) = \xi_n, \quad u_n(b) = \eta_n \tag{7}$$

in $l^\infty(\mathbb{Z})$. Let $e = (1)_{n\in\mathbb{Z}}$, and let $v, w : [-1,1] \to l^\infty(\mathbb{Z})$ and $f : S(v,w) \times l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$ be defined by $v(t) = -\max\{||\xi||, ||\eta||\}e$ $(t \in [-1,1])$,

$$w(t) = \begin{cases} (\max\{||\xi||,||\eta||\} + \sqrt{cg(0)}(1+t))e & (t \in [-1,0]), \\ (\max\{||\xi||,||\eta||\} + \sqrt{cg(0)}(1-t))e & (t \in [0,1]), \end{cases}$$

and

$$f_n(t,x,p_n) = h_n(t)g(x_{n+1} - 2x_n + x_{n-1}) - (p_n)^2.$$

Then, the transformed functions satisfy the assumptions of Theorem 3, in particular (6), (7) is solvable in $l^\infty(\mathbb{Z})$ for each choice of $\xi, \eta \in l^\infty(\mathbb{Z})$.

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