EXTRACTA MATHEMATICAE Vol. **20**, Núm. 1, 43 – 50 (2005)

Conditions ensuring $T^{-1}(Y) \subset Y$

Dagmar Medková *

Mathematical Institute of the Academy of Sciences of the Czech Republic
Žitná 25, 115 67 *Praha* 1, *Czech Republic*
Czech Technical University, Faculty of Mechanical Engineering, Department of Technical Mathematics, Karlovo nám. 13, 121 35 *Praha* 2, *Czech Republic*
e-mail: medkova@math.cas.cz

(Presented by M. González)

AMS *Subject Class.* (2000): 47A15 *Received May* 13, 2004

Introduction

The following theorem is the main result of the paper: Let X be a complex Banach space and $T \in L(X)$. Suppose that 0 lies at the unbounded component of the set of those λ such that $\lambda I - T$ is a Fredholm operator. Let Y be a dense subspace of the dual space X' and S be a closed operator from Y to X such that $T'(Y) \subset Y$ and $TSy = ST'y$ for each $y \in Y$. Then for each vector $x \in X'$, $T'x \in Y$ if and only if $x \in Y$.

The paper is motivated by requirements of the integral equation method used in solving boundary value problems for partial differential equations. The solution is looked for in the form of a suitable potential. On the Banach space X of boundary conditions we get a bounded linear operator T. The potential corresponding to $x \in X$ is a solution of the boundary value problem with the boundary condition y if and only if $Tx = y$. In the classical situation, when we study the boundary value problem on a domain G with smooth boundary, the operator T has a form $\frac{1}{2}I + K$, where I is the identity operator and K is a compact operator on X. If we stop to suppose that G has smooth boundary the situation changes dramatically. We only know that T is Fredholm or that 0 lies in the unbounded component of $\Phi(T)$, the set of all $\lambda \in \mathbb{C}$ for which $\lambda I - T$ is a Fredholm operator. There are two moments when we need to know that the solution $x \in X$ of the problem $Tx = y$ is an element of some subspace Y of the Banach space X. First situation is when we prove the uniqueness of the problem. In the classical situation we use Green's formula

*The paper is supported by the grant no. KSK 1019101

for the proof of the uniqueness of the solution. If the boundary conditions are too general (for example real measures) the corresponding potentials are not smooth enough we can use Green's formula. So, we must show that the solution x of $Tx = 0$ lies in some subspace Y of X, what ensures that the potential corresponding to x is smooth enough Green's formula can be used. J. Král in [2] is in this situation when he studies the solution of the Neumann problem for the Laplace equation. I. Netuka was in the similar situation in [4], when he studied the third problem for the Laplace equation. In both articles Y is a dense subspace of X, $T(Y) \subset Y$ and there are a bounded linear operator $\tilde{T}$ on a Banach space $\tilde{X}$ such that T is the adjoint operator of $\tilde{T}$ and $\tilde{T}Sy = STy$ for each $y \in Y$. So, our result solves quickly the problem.

The second situation, when we need to know that the solution $x \in X$ of the problem $Tx = y$ is an element of some subspace Y of the Banach space X, is if we study the behaviour of solutions of boundary value problems. When I looked for necessary and sufficient conditions the weak solution of the Neumann problem for the Laplace equation to be continuous on the closure of the domain I knew that the solution has a form of a single layer potential where the corresponding real measure ν is a solution of the equation $T'\nu = \mu$ and μ is the boundary condition. Moreover, I knew that $\nu = \mu + \sum (I - 2T')^j (2I - T')\mu$. Using this fact I proved that the single layer potential corresponding to the real measure ν is continuous on the closure of the domain if and only if the single layer potential corresponding the real measure μ is continuous on the closure of the domain (see [3]). But in general situation we are not able to calculate the solution of the problem. We only know that the solution has a form of an appropriate potential where the corresponding real measure is a solution of the equation $T'\nu = \mu$ where μ is the boundary condition. Moreover, we know that T' is the adjoint operator of some operator T on a Banach space X, $\lambda I - T$ is a Fredholm operator for each nonnegative number λ and the space Y of all real measures, for which the corresponding single layer potential has the required property, is dense in the space of all boundary conditions. Moreover, the single layer operator is a closed operator from Y to X such that $TSy = ST'y$ for each $y \in Y$. Therefore our result solves the problem.

1. Conditions for $T^{-1}(Y) \subset Y$

Let X be a complex Banach space and $T \in L(X)$, the algebra of all bounded linear operators on X. We denote by $\operatorname{Ker} T$ the kernel of T, by $\sigma(T)$ the spectrum of T, by $\rho(T) = \mathbb{C} \setminus \sigma(T)$ the resolvent set of T, by X' the dual space of X and by T' the adjoint operator of T. Denote by I the identity operator. The operator T is called Fredholm if the dimension of $\operatorname{Ker} T$ and the codimension of $T(X)$ are both finite. Denote by $\Phi(T)$ the set of all $\lambda \in \mathbb{C}$ for which $\lambda I - T$ is a Fredholm operator. The ascent of T, $p(T)$, and the descent of T, $q(T)$ are given by

$$p(T) = \inf \{ n \in N_0 \,:\, \operatorname{Ker}(T^n) = \operatorname{Ker}(T^{n+1}) \},$$

$$q(T) = \inf \{ n \in N_0 \,:\, T^n(X) = T^{n+1}(X) \},$$

where $\inf \emptyset = +\infty$.

Inspired by [2] we introduce the following terminology: We say that T, $\tilde{T}$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y, if X and $\tilde{X}$ are Banach spaces, Y is a subspace of X, $T \in L(X)$, $\tilde{T} \in L(\tilde{X})$, S is a closed linear operator from Y to $\tilde{X}$, $T(Y) \subset Y$, and $\tilde{T}Sy = STy$ for each $y \in Y$.

Lemma 1. *Let X, $\tilde{X}$ be Banach spaces, $T \in L(X)$, $\tilde{T} \in L(\tilde{X})$, $T_n \in L(X)$, $\tilde{T}_n \in L(\tilde{X})$, $n \in N$, be such that $T_n \to T$, $\tilde{T}_n \to \tilde{T}$ as $n \to \infty$. If T_n, $\tilde{T}_n$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y for each n then T, $\tilde{T}$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y.*

Proof. Fix $y \in Y$. Then $Ty = \lim T_n y$, $T_n y \in Y$ and $\lim ST_n y = \lim \tilde{T}_n Sy = \tilde{T}Sy$. Since S is closed we have $Ty \in Y$, the domain of S, and $STy = \tilde{T}Sy$. ∎

Lemma 2. *Let X, $\tilde{X}$ be complex Banach spaces, T, $\tilde{T}$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y. Let Ω be a component of $\rho(T) \cap \rho(\tilde{T})$. If there is $\lambda \in \Omega$ such that $(\lambda I - T)^{-1}$, $(\lambda I - \tilde{T})^{-1}$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y, then $(\mu I - T)^{-1}$, $(\mu I - \tilde{T})^{-1}$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y for each $\mu \in \Omega$.*

Proof. Denote by U the set of all $\mu \in \Omega$ for which $(\mu I - T)^{-1}$, $(\mu I - \tilde{T})^{-1}$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y. We show that U is open. Suppose that $\mu \in U$, $\nu \in \Omega$, $|\nu - \mu| < \min(\|(\mu I - T)^{-1}\|^{-1}, \|(\mu I - \tilde{T})^{-1}\|^{-1})$. According to [5, Chapter VI, Theorem 3.9] we have

$$(\nu I - T)^{-1} = \sum_{k=1}^{\infty} (\mu - \nu)^{k-1} [(\mu I - T)^{-1}]^k ,$$

$$(\nu I - \tilde{T})^{-1} = \sum_{k=1}^{\infty} (\mu - \nu)^{k-1} [(\mu I - \tilde{T})^{-1}]^k \,.$$

Since $(\mu - \nu)^{k-1}[(\mu I - T)^{-1}]^k(Y) \subset Y$ and $S(\mu - \nu)^{k-1}[(\mu I - T)^{-1}]^k y = (\mu - \nu)^{k-1}[(\mu I - \tilde{T})^{-1}]^k S y$ for each $y \in Y$ and $k \in N$, Lemma 1 gives that $(\nu I - T)^{-1}$, $(\nu I - \tilde{T})^{-1}$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y. Thus $\nu \in U$.

Since the mappings $\nu \mapsto (\nu I - T)^{-1}$, $\nu \mapsto (\nu I - \tilde{T})^{-1}$ are continuous in Ω (see [6, Chapter VIII, §1]), Lemma 1 yields that U is closed in Ω. Since U is a nonempty open and closed subset of the connected set Ω, we obtain that $U = \Omega$. ∎

Lemma 3. *Let X, $\tilde{X}$ be complex Banach spaces, T, $\tilde{T}$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y. Let Ω be an unbounded component of $\rho(T) \cap \rho(\tilde{T})$. Then $(\mu I - T)^{-1}$, $(\mu I - \tilde{T})^{-1}$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y for each $\mu \in \Omega$.*

Proof. Suppose that $|\lambda| > \max(\|T\|, \|\tilde{T}\|)$. According to [6, Chapter VIII, Theorem 3] we have

$$(\lambda I - T)^{-1} = \sum_{k=1}^{\infty} \lambda^{-k} (T)^{k-1} \,,$$

$$(\lambda I - \tilde{T})^{-1} = \sum_{k=1}^{\infty} \lambda^{-k} (\tilde{T})^{k-1} \,.$$

Since $\lambda^{-k} T^{k-1}(Y) \subset Y$ and $S \lambda^{-k} T^{k-1} y = \lambda^{-k} \tilde{T}^{k-1} S y$ for each $y \in Y$ and $k \in N$, Lemma 1 shows that $(\lambda I - T)^{-1}$, $(\lambda I - \tilde{T})^{-1}$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y. Lemma 2 yields that $(\mu I - T)^{-1}$, $(\mu I - \tilde{T})^{-1}$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y for each $\mu \in \Omega$. ∎

Lemma 4. *Let X, $\tilde{X}$ be complex Banach spaces, T, $\tilde{T}$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y. Let Ω be a component of $\Phi(T) \cap \Phi(\tilde{T})$ such that there is $\lambda \in \Omega \cap \rho(T) \cap \rho(\tilde{T})$ for which $(\lambda I - T)^{-1}$, $(\lambda I - \tilde{T})^{-1}$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y. Let Γ be a smooth Jordan curve in $\Omega \cap \rho(T) \cap \rho(\tilde{T})$. Denote*

$$P = \frac{1}{2\pi i} \int_{\Gamma} (tI - T)^{-1} \, \mathrm{d}t \,, \qquad \tilde{P} = \frac{1}{2\pi i} \int_{\Gamma} (tI - \tilde{T})^{-1} \, \mathrm{d}t \,.$$

Then P, $\tilde{P}$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y.

Proof. Let $\gamma(t)$, $0 \le t \le 1$, be a smooth parametrization of the curve Γ. Since the mapping $t \mapsto (tI - T)^{-1}$ is continuous by [6, Chapter VIII, §1], the integral P is the limit of the Riemann's sums

$$R_n = \frac{1}{2\pi i} \sum_{k=1}^{n} \frac{1}{n} \gamma'(k/n)(\gamma(k/n)I - T)^{-1}.$$

Similarly, the integral $\tilde{P}$ is the limit of the Riemann's sums

$$\tilde{R}_n = \frac{1}{2\pi i} \sum_{k=1}^{n} \frac{1}{n} \gamma'(k/n)(\gamma(k/n)I - \tilde{T})^{-1}.$$

According to Lemma 2 we see that R_n, $\tilde{R}_n$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y. Lemma 1 shows that P, $\tilde{P}$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y. ∎

THEOREM 5. *Let X, $\tilde{X}$ be complex Banach spaces, T, $\tilde{T}$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y. Denote by Ω the unbounded component of $\Phi(T) \cap \Phi(\tilde{T})$. Let $\mu \in \Omega$ be such that $\operatorname{Ker}(T - \mu I)^n \subset Y$ for each $n \in N$. If $x, y \in X$, $(T - \mu I)x = y$ then $x \in Y$ if and only if $y \in Y$.*

Proof. If $x \in Y$ then $y \in Y$. Suppose that $y \in Y$. Since $\Omega \cap (\sigma(T) \cup \sigma(\tilde{T}))$ is an isolated set in Ω (see [1, Satz 51.3 and Satz 51.2]), the set $\Omega \cap \rho(T) \cap \rho(\tilde{T})$ is an unbounded domain. If $\mu \in \Omega \cap \rho(T) \cap \rho(\tilde{T})$ then $(\mu I - T)^{-1}$, $(\mu I - \tilde{T})^{-1}$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y by Lemma 3 and thus $x \in Y$. Therefore we can suppose that $\mu \in \sigma(T) \cup \sigma(\tilde{T})$. Since μ is not an accumulated point of $\sigma(T) \cup \sigma(\tilde{T})$ there is $r > 0$ such that $\{\lambda; 0 < |\lambda - \mu| < r\} \subset \rho(T) \cap \rho(\tilde{T})$. Put $\Gamma = \{\lambda; |\lambda - \mu| = r/2\}$,

$$\tilde{P} = \frac{1}{2\pi i} \int_\Gamma (tI - \tilde{T})^{-1} \,\mathrm{d}t\,, \qquad P = \frac{1}{2\pi i} \int_\Gamma (tI - T)^{-1} \,\mathrm{d}t\,.$$

Since P, $\tilde{P}$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y by Lemma 4, the operators $I - P$, $I - \tilde{P}$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y. Therefore $\tilde{Y} = (I - P)Y \subset Y$. Put $Z = (I - P)X$, $\tilde{Z} = (I - \tilde{P})\tilde{X}$. Then $\tilde{Y} \subset Z$. If $\mu \in \rho(T)$ ($\mu \in \rho(\tilde{T})$) then $P = 0$ ($\tilde{P} = 0$), respectively. If $\mu \in \sigma(T)$ ($\mu \in \sigma(\tilde{T})$) then P ($\tilde{P}$) is the spectral projection corresponding to the spectral set $\{\mu\}$ and the operator T (operator $\tilde{T}$), respectively. So, Z is a closed subspace of X and $\tilde{Z}$ is a closed subspace of $\tilde{X}$ (see [5, p. 147]). Moreover, $TP = PT$, $\tilde{T}\tilde{P} = \tilde{P}\tilde{T}$ by [5, Chapter VI, Lemma 3.6], and $T(Z) \subset Z$, $\tilde{T}(\tilde{Z}) \subset \tilde{Z}$, $T(\tilde{Y}) \subset \tilde{Y}$. If $z \in \tilde{Y} \subset Z$ then $z = (I-P)z$ because $I-P$ is a projection with the range Z. Since $I-P$, $I-\tilde{P}$, S form Plemejl's triplet of operators with respect to X, $\tilde{X}$ and Y we have $Sz = S(I-P)z = (I-\tilde{P})Sz \in \tilde{Z}$ for $z \in \tilde{Y}$. We now show that S is a closed operator from $\tilde{Y}$ to $\tilde{Z}$. Let $z_n \in \tilde{Y}$, $z_n \to z$ and $Sz_n \to w$. Since S is a closed operator from Y to $\tilde{X}$ we obtain that $z \in Y$ and $Sz = w \in \tilde{X}$. Since $Sz_n \in \tilde{Z}$ and $\tilde{Z}$ is a closed subspace of $\tilde{X}$ we deduce that $Sz = w \in \tilde{Z}$. Since $z_n \in Z$ and Z is a closed subspace of X we have $z \in Z$. Since $I-P$ is a projection with the range Z we obtain $z = (I-P)z \in \tilde{Y}$. Thus S is a closed operator form $\tilde{Y}$ to $\tilde{Z}$. Hence T, $\tilde{T}$, S form Plemejl's triplet of operators with respect to Z, $\tilde{Z}$ and $\tilde{Y}$.

According to [5, Chapter VI, Theorem 4.1] we have $\Omega \cup \{\mu\} \subset \rho(T|Z) \cap \rho(\tilde{T}|\tilde{Z})$. (Here $I|Z$ denotes the restriction of the operator T onto Z.) Since T, $\tilde{T}$, S form Plemejl's triplet of operators with respect to Z, $\tilde{Z}$ and $\tilde{Y}$ Lemma 3 yields that $((\mu I - T)|Z)^{-1}(\tilde{Y}) \subset \tilde{Y}$. If $\mu \in \rho(T)$ then $P = 0$, $\tilde{Y} = Y$ and $x = (T-\mu I)^{-1}y \in Y$. Suppose now that $\mu \in \sigma(T)$. Then P is the spectral projection corresponding to the spectral set $\{\mu\}$ and the operator T. Since $p(\mu I - T) = q(\mu I - T) < \infty$ by [1, Theorem 51.1], there is $n \in N$ such that $P(X) = \operatorname{Ker}(T-\mu I)^n \subset Y$ by [1, Satz 50.2]. Then $Px \in P(X) \subset Y$ and $(T-\mu I)Px \in Y$. Thus $(T-\mu I)(I-P)x = y - (T-\mu I)Px \in Y$. Since $(I-P)x \in Z$ we have $(T-\mu I)(I-P)x \in Z$. Since $(I-P)$ is a projection with the range Z we have $(T-\mu I)(I-P)x = (I-P)(T-\mu I)(I-P)x \in (I-P)(Y) = \tilde{Y}$. Thus $(I-P)x = ((T-\mu I)|Z)^{-1}[(T-\mu I)(I-P)x] \in ((\mu I - T)|Z)^{-1}(\tilde{Y}) \subset \tilde{Y} \subset Y$. Since $Px \in Y$ we obtain $x = Px + (I-P)x \in Y$. ∎

Definition 6. Let X, Y be normed spaces, $b(x,y)$ be a bilinear form on $X \times Y$. We say that X, Y, b form a dual system if $b(x,y) = 0$ for all $y \in Y$ implies $x = 0$ and $b(x,y) = 0$ for all $x \in X$ implies $y = 0$.

Theorem 7. *Let X be a complex Banach space, T', T, S form Plemejl's triple of operators with respect to X', X and Y. Suppose that X, Y and the bilinear form $b(x,y) = y(x)$ form a dual system. If μ lies in the unbounded component of $\Phi(T)$ then* $\operatorname{Ker}(\mu I - T')^n \subset Y$ *for each $n \in N$.*

Proof. Denote by Ω the unbounded component of $\Phi(T)$. Then $\Omega \setminus \sigma(T)$ is the unbounded component of $\rho(T)$, because $\Omega \cap \sigma(T)$ is an isolated set in Ω by

[1, Satz 51.3 and Satz 51.2]. According to Lemma 3 the operators $(\lambda I - T')^{-1}$, $(\lambda I - T)^{-1}$, S form Plemejl's triple of operators with respect to X', X and Y for each $\lambda \in \Omega \setminus \sigma(T)$. Let now $\mu \in \Omega \cap \sigma(T)$. Since $\Omega \cap \sigma(T)$ is an isolated set in Ω, there is a smooth Jordan curve Γ in $\Omega \setminus \sigma(T)$ so that μ is the only point of $\sigma(T)$ in the interior of Γ. Denote

$$P = \frac{1}{2\pi i} \int_\Gamma (tI - T)^{-1} \, \mathrm{d}t$$

the spectral projection corresponding to the spectral set $\{\mu\}$ and the operator T. Since $p(\mu I - T) = q(\mu I - T) < \infty$ by [1, Theorem 51.1],

$$P(X) = \operatorname{Ker}(\mu I - T)^k$$

for some k by [1, Satz 50.2]. Since $\mu I - T$ is Fredholm, the operator $(\mu I - T)^k$ is Fredholm too (see [5, Chapter V, Theorem 2.3]) and $m = \dim \operatorname{Ker}(\mu I - T)^k < \infty$. Choose $x_1, \dots, x_m \in X$ the base of $P(X) = \operatorname{Ker}(\mu I - T)^k$. Then there are $y_1, \dots, y_m \in X'$ so that

$$Px = \sum_{j=1}^{m} y_j(x) x_j$$

for each $x \in X$. This gives

$$P'z = \sum_{j=1}^{m} z(x_j) y_j \tag{1}$$

for each $z \in X'$.

Since X, Y, b form a dual system, [1, Satz 15.1] implies that there are $\tilde{y}_1, \dots, \tilde{y}_m \in Y$ so that $\tilde{y}_j(x_l) = \delta_{jl}$ for $j, l = 1, \dots, m$. (Here δ_{lj} means the Kronecker's delta.) We conclude from (1) that $y_j = P'(\tilde{y}_j) \subset P'(Y)$ for $j = 1, \dots, m$ and hence $P'(X') \subset P'(Y)$. Since

$$P' = \frac{1}{2\pi i} \int_\Gamma (\lambda I - T')^{-1} \, \mathrm{d}\lambda$$

is the spectral projection corresponding to the spectral set $\{\mu\}$ and the operator T' (see [5, Chapter VI]), $p(\mu I - T') = q(\mu I - T') < \infty$ by [1, Theorem 51.1],

$$P'(X') = \cup_{n=1}^{\infty} \operatorname{Ker}(\mu I - T')^n$$

by [1, Satz 50.2] and so $\operatorname{Ker}(\mu I - T')^n \subset P'(X') \subset P'(Y)$. Since $P'(Y) \subset Y$ by Lemma 4 we have $\operatorname{Ker}(\mu - T')^n \subset Y$. ∎

COROLLARY 8. *Let X be a complex Banach space, Y be a dense subspace of X' such that T', T, S form Plemejl's triple of operators with respect to X', X, Y. Let μ lies in the unbounded component of $\Phi(T)$. If $x, y \in X'$ such that $(T' - \mu I)x = y$, then $x \in Y$ if and only if $y \in Y$.*

Proof. X, Y and the bilinear form $b(x, y) = y(x)$ form a dual system because Y is a dense subspace of X'. Theorem 7 shows that $\mathrm{Ker}(\mu I - T')^n \subset Y$ for each $n \in N$. Therefore $x \in Y$ if and only if $y \in Y$ by Theorem 5. ∎

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