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Boundedness for Multilinear Littlewood-Paley Operators on Hardy and Herz-Hardy Spaces

Lanzhe Liu

College of Mathematics and Computer, Changsha University of Science and Technology, Changsha 410077, *P.R. of China*
e-mail: lanzheliu@263.net

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1. Introduction

Let T be a Calderon-Zygmund operator, a classical result of Coifman, Rochberg and Weiss (see [7]) states that the commutator $[b, T] = T(bf) - bTf$ (where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$; Chanillo (see [2]) proves a similar result when T is replaced by the fractional integral operator. However, it was observed that $[b, T]$ is not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)$ for $p \leq 1$. But, the boundedness holds if b belongs to Lipschitz spaces $Lip_\beta(R^n)$ (see [3],[15]). This shows the difference of $b \in BMO(R^n)$ and $b \in Lip_\beta(R^n)$. The purpose of this paper is to prove the boundedness properties for some multilinear operators generated by Littlewood-Paley operators and Lipschitz functions on Hardy and Herz-Hardy spaces.

2. Preliminaries and results

In this paper, we will consider a class of multilinear operators related to Littlewood-Paley operators, whose definitions are following.

Let m be a positive integer and A be a function on R^n. We denote

$$R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha A(y)(x - y)^\alpha,$$

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$$Q_{m+1}(A;x,y)=R_m(A;x,y)-\sum_{|\alpha|=m}\frac{1}{\alpha!}D^\alpha A(x)(x-y)^\alpha$$

and $\Gamma(x)=\{(y,t)\in R_+^{n+1}:|x-y|<t\}$ as well as the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$.

Fix $\varepsilon>0$ and $\mu>1$. Let ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n}\psi(x)dx=0$,

(2) $|\psi(x)|\le C(1+|x|)^{-(n+1)}$,

(3) $|\psi(x+y)-\psi(x)|\le C|y|^\varepsilon(1+|x|)^{-(n+1+\varepsilon)}$ when $2|y|<|x|$;

The multilinear Littlewood-Paley operators are defined by

$$g_\psi^A(f)(x)=\left(\int_0^\infty|F_t^{\psi,A}(f)(x)|^2\frac{dt}{t}\right)^{1/2},$$

$$S_\psi^A(f)(x)=\left[\iint_{\Gamma(x)}|F_t^{\psi,A}(f)(x,y)|^2\frac{dydt}{t^{n+1}}\right]^{1/2}$$

and

$$g_\mu^A(f)(x)=\left[\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}|F_t^{\psi,A}(f)(x,y)|^2\frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^{\psi,A}(f)(x)=\int_{R^n}\frac{R_{m+1}(A;x,y)}{|x-y|^m}\psi_t(x-y)f(y)dy,$$

$$F_t^{\psi,A}(f)(x,y)=\int_{R^n}\frac{R_{m+1}(A;x,z)}{|x-z|^m}f(z)\psi_t(y-z)dz,$$

and $\psi_t(x)=t^{-n}\psi(x/t)$ for $t>0$. The variants of g_ψ^A, S_ψ^A and g_μ^A are defined by

$$\tilde g_\psi^A(f)(x)=\left(\int_0^\infty|\tilde F_t^{\psi,A}(f)(x)|^2\frac{dt}{t}\right)^{1/2},$$

$$\tilde S_\psi^A(f)(x)=\left[\iint_{\Gamma(x)}|\tilde F_t^{\psi,A}(f)(x,y)|^2\frac{dydt}{t^{n+1}}\right]^{1/2},$$

and

$$\tilde g_\mu^A(f)(x)=\left[\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}|\tilde F_t^{\psi,A}(f)(x,y)|^2\frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$\tilde{F}_t^{\psi,A}(f)(x) = \int_{R^n} \frac{Q_{m+1}(A;x,y)}{|x-y|^m}\psi_t(x-y)f(y)dy$$

and

$$\tilde{F}_t^{\psi,A}(f)(x,y) = \int_{R^n} \frac{Q_{m+1}(A;x,z)}{|x-z|^m}\psi_t(y-z)f(z)dz.$$

We denote that $F_t^{\psi}(f)(y) = f * \psi_t(y)$. We also define that

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t^\psi(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$S_\psi(f)(x) = \left(\iint_{\Gamma(x)} |F_t^\psi(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}$$

and

$$g_\mu(f)(x) = \left(\iint_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t^\psi(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

which are the Littlewood-Paley operators (see [19]). For S_ψ^A, $\tilde{S}_\psi^A$ and g_μ^A, $\tilde{g}_\mu^A$, we have the following pointwise estimates (see [19, p.317]):

$$S_\psi^A(f)(x) \le C g_\mu^A(f)(x) \text{ and } \tilde{S}_\psi^A(f)(x) \le C\tilde{g}_\mu^A(f)(x).$$

Let $\psi = \varphi * \chi_B$, where B is a ball of R^n. It is easy to see that

$$F_t^{\psi,A}(f)(x) = \frac{1}{t^n}\int_{|x-y|\le t} F_t^{\varphi,A}(f)(x,y)dy,$$

thus

$$g_\psi^A(f)(x) \le C S_\varphi^A(f)(x) \text{ and } \tilde{g}_\psi^A(f)(x) \le C\tilde{S}_\varphi^A(f)(x).$$

Notice that if φ satisfies the properties (1),(2)and (3), then ψ also satisfies similar estimates.

Note that when $m = 0$, g_ψ^A, S_ψ^A and g_μ^A are just the commutator of Littlewood-Paley operators (see [1],[12],[13]), while when $m > 0$, they are non-trivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [3-6],[8],[9]). In [3] and [20], authors obtain the boundedness of multilinear singular integral operators generated by singular integrals and Lipschitz functions on $L^p(p > 1)$ and some Hardy spaces. The main purpose of this paper is to discuss the boundedness properties of the multilinear Littlewood-Paley operators on Hardy and Herz-Hardy spaces. Let us first introduce some definitions (see [10],[16],[17],[18]). Throughout this paper, $M(f)$ will denote the Hardy-Littlewood maximal function of f, Q will denote a cube of R^n with side parallel to the axes. Denote the Hardy spaces by $H^p(R^n)$. It is well known that $H^p(R^n)(0 < p \le 1)$ has the atomic decomposition characterization (see [19]). For $\beta > 0$, the Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$||f||_{Lip_\beta} = \sup_{\substack{x,h \in R^n \\ h \neq 0}} |f(x+h) - f(x)|/|h|^\beta < \infty.$$

Let $B_k = \{x \in R^n : |x| \le 2^k\}$ and $C_k = B_k \setminus B_{k-1}$ for $k \in Z$. Denote by χ_k the characteristic function of C_k and $\tilde{\chi}_k$ the characteristic function of C_k for $k \ge 1$ and $\tilde{\chi}_0$ the characteristic function of B_0.

Definition 1. Let $0 < p$, $q < \infty$, $\alpha \in R$.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(R^n) = \{f \in L_{loc}^q(R^n \setminus \{0\}) : ||f||_{\dot{K}_q^{\alpha,p}(R^n)} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p\right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \{f \in L_{loc}^q(R^n) : ||f||_{K_q^{\alpha,p}(R^n)} < \infty\},$$

where

$$||f||_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p + ||f\chi_{B_0}||_{L^q}^p\right]^{1/p}.$$

Definition 2. Let $\alpha \in R$, $0 < p, q < \infty$.

(1) The homogeneous Herz type Hardy space is defined by

$$H\dot{K}_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in \dot{K}_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{H\dot{K}_q^{\alpha,p}} = ||G(f)||_{\dot{K}_q^{\alpha,p}};$$

(2) The nonhomogeneous Herz type Hardy space is defined by

$$HK_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in K_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{HK_q^{\alpha,p}} = ||G(f)||_{K_q^{\alpha,p}};$$

where $G(f)$ is the grand maximal function of f.

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 3. Let $\alpha \in R$, $1 < q < \infty$. A function $a(x)$ on R^n is called a central (α, q)-atom (or a central (a, q)-atom of restrict type), if
1) $\operatorname{Supp} a \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$),
2) $||a||_{L^q} \leq |B(0, r)|^{-\alpha/n}$,
3) $\int a(x)x^\gamma dx = 0$ for $|\gamma| \leq [\alpha - n(1 - 1/q)]$.

Lemma 1. (See [17]) *Let $0 < p < \infty$, $1 < q < \infty$ and $\alpha \geq n(1 - 1/q)$. A temperate distribution f belongs to $H\dot{K}_q^{\alpha,p}(R^n)$(or $HK_q^{\alpha,p}(R^n)$) if and only if there exist central (α, q)-atoms (or central (α, q)-atoms of restrict type) a_j supported on $B_j = B(0, 2^j)$ and constants λ_j, $\sum_j |\lambda_j|^p < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$(or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(R^n)$ sense, and*

$$||f||_{H\dot{K}_q^{\alpha,p}} (\text{or } ||f||_{HK_q^{\alpha,p}}) \approx \left(\sum_j |\lambda_j|^p \right)^{1/p}.$$

We will prove the following theorems in Section 4.

Theorem 1. *Let $0 < \beta \leq 1$, $\max(n/(n + \beta), n/(n + \varepsilon)) < p \leq 1$ and $1/p - 1/q = \beta/n$. If $D^\alpha A \in Lip_\beta(R^n)$ for $|\alpha| = m$. Then g_μ^A is bounded from $H^p(R^n)$ to $L^q(R^n)$.*

Theorem 2. *Let $0 < \beta < \min(1, \varepsilon)$. If $D^\alpha A \in Lip_\beta(R^n)$ for $|\alpha| = m$. Then $\tilde{g}_\mu^A$ is bounded from $H^{n/(n+\beta)}(R^n)$ to $L^1(R^n)$.*

Theorem 3. *Let $0 < \beta < \min(1, \varepsilon)$. If $D^\alpha A \in Lip_\beta(R^n)$ for $|\alpha| = m$. Then g_μ^A is bounded from $H^{n/(n+\beta)}(R^n)$ to weak $L^1(R^n)$.*

THEOREM 4. *Let* $0<\beta\le 1$, $0<p<\infty$, $1<q_1,q_2<\infty$, $1/q_1-1/q_2=\beta/n$ *and* $n(1-1/q_1)\le\alpha<\min(n(1-1/q_1)+\beta, n(1-1/q_1)+\varepsilon)$. *If* $D^\alpha A\in Lip_\beta(R^n)$ *for* $|\alpha|=m$. *Then* g_μ^A *is bounded from* $H\dot{K}_{q_1}^{\alpha,p}(R^n)$ *to* $\dot{K}_{q_2}^{\alpha,p}(R^n)$.

Remark 1. By the pointwise estimates of g_ψ^A, S_ψ^A and g_μ^A(or $\tilde{g}_\psi^A$, $\tilde{S}_\psi^A$ and $\tilde{g}_\mu^A$), Theorem 1, 2, 3 and 4 also hold for g_ψ^A and S_ψ^A(or $\tilde{g}_\psi^A$ and $\tilde{S}_\psi^A$).

Remark 2. Theorem 4 also hold for the nonhomogeneous Herz type Hardy space.

3. Some lemmas

We begin with some preliminary lemmas.

LEMMA 2. (See [6]) *Let* A *be a function on* R^n *and* $D^\alpha A\in L^q(R^n)$ *for* $|\alpha|=m$ *and some* $q>n$. *Then*

$$|R_m(A;x,y)|\le C|x-y|^m\sum_{|\alpha|=m}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}|D^\alpha A(z)|^q dz\right)^{1/q},$$

where $\tilde{Q}(x,y)$ *is the cube centered at* x *and having side length* $5\sqrt{n}|x-y|$.

LEMMA 3. (See [3, p.418, Theorem 2.3]) *Let* T^A *be the multilinear operators defined by*

$$T^A(f)(x)=\int_{R^n}\frac{R_{m+1}(A;x,y)}{|x-y|^{m+n}}f(y)dy.$$

If $0<\beta<1$, $1<p<n/\beta$, $1/q=1/p-\beta/n$ *and* $D^\alpha A\in Lip_\beta(R^n)$ *for* $|\alpha|=m$. *Then* T^A *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$, *that is*

$$||T^A(f)||_{L^q}\le C||f||_{L^p}.$$

LEMMA 4. *Let* $0<\beta\le 1$, $1<p<n/\beta$, $1/q=1/p-\beta/n$ *and* $D^\alpha A\in Lip_\beta(R^n)$ *for* $|\alpha|=m$. *Then* g_ψ^A, S_ψ^A *and* g_μ^A *are all bounded from* $L^p(R^n)$ *to* $L^q(R^n)$.

Proof. By the pointwise estimates of g_ψ^A, S_ψ^A and g_μ^A, we only need to give the proof of g_μ^A. Note that

$$\int_0^\infty\frac{tdt}{(t+|x-z|)^{2n+2}}=C|x-z|^{-2n}$$

and

$$t^{-n}\int_{R^n}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{dy}{(t+|y-z|)^{2n+2}}$$
$$\leq CM\left(\frac{1}{(t+|x-z|)^{2n+2}}\right)\leq C\frac{1}{(t+|x-z|)^{2n+2}},$$

by using Minkowski's inequality and the condition of ψ, we obtain

$$\begin{aligned} g_\mu^A(f)(x) &\leq C\int_{R^n}\frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \\ &\quad\cdot\left[\int_0^\infty\left(t^{-n}\int_{R^n}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{dy}{(t+|y-z|)^{2n+2}}\right)tdt\right]^{1/2}dz \\ &\leq C\int_{R^n}\frac{|f(z)|}{|x-z|^m}|R_{m+1}(A;x,z)|\left(\int_0^\infty\frac{tdt}{(t+|x-z|)^{2n+2}}\right)^{1/2}dz \\ &\leq C\int_{R^n}\frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^{m+n}}dz. \end{aligned}$$

Thus, the lemma follows from Lemma 3. ∎

4. Proofs of theorems

Proof of Theorem 1. It suffices to show that there exists a constant $C>0$ such that for every H^p-atom a,

$$||g_\mu^A(a)||_{L^q}\leq C.$$

Let a be a H^p-atom, that is that a supported on a cube $Q=Q(x_0,r)$, $||a||_{L^\infty}\leq |Q|^{-1/p}$ and $\int a(x)x^\gamma dx=0$ for $|\gamma|\leq[n(1/p-1)]$. We write

$$\int_{R^n}[g_\mu^A(a)(x)]^q dx=\left(\int_{2Q}+\int_{(2Q)^c}\right)[g_\mu^A(a)(x)]^q dx=I_1+I_2.$$

For I_1, taking $1<p_1<n/\beta$ and q_1 such that $1/p_1-1/q_1=\beta/n$, by Hölder's inequality and the (L^{p_1},L^{q_1})-boundedness of g_μ^A (see Lemma 4), we get

$$I_1\leq C||g_\mu^A(a)||_{L^{q_1}}^q|2Q|^{1-q/q_1}\leq C||a||_{L^{p_1}}^q|Q|^{1-q/q_1}\leq C.$$

To obtain the estimate of I_2, we need to estimate $g_\mu^A(a)(x)$ for $x\in(2Q)^c$. Let $\tilde{Q}=5\sqrt{n}Q$ and $\tilde{A}(x)=A(x)-\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}}x^\alpha$. Then $R_m(A;x,y)=$

$R_m(\tilde{A};x,y)$ and $D^\alpha\tilde{A}(y)=D^\alpha A(y)-(D^\alpha A)_Q$). We write, by the vanishing moment of a,

$$F_t^{\psi,A}(a)(x,y) = \int_{R^n}\left[\frac{\psi_t(y-z)R_m(\tilde{A};x,z)}{|x-z|^m}-\frac{\psi_t(y-x_0)R_m(\tilde{A};x,x_0)}{|x-x_0|^m}\right]dz$$
$$-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\frac{\psi_t(y-z)(x-z)^\alpha D^\alpha\tilde{A}(z)}{|x-z|^m}a(z)dz.$$

By Lemma 2 and the following inequality, for $b\in Lip_\beta(R^n)$,

$$|b(x)-b_Q|\le\frac{1}{|Q|}\int_Q||b||_{Lip_\beta}|x-y|^\beta dy\le||b||_{Lip_\beta}(|x-x_0|+r)^\beta,$$

we get

$$|R_m(\tilde{A};x,y)|\le\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}(|x-y|+r)^{m+\beta}.$$

On the other hand, by the formula (see [6]):

$$R_m(\tilde{A};x,y)-R_m(\tilde{A};x,x_0)=\sum_{|\eta|<m}\frac{1}{\eta!}R_{m-|\eta|}(D^\eta\tilde{A};x_0,y)(x-x_0)^\eta$$

and note that $|x-y|\sim|x-x_0|$ for $y\in Q$ and $x\in R^n\setminus Q$, we obtain, similar to the proof of Lemma 4,

$$g_\mu^A(a)(x)\le C\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}\int\left[\frac{|y-x_0|}{|x-x_0|^{n+1-\beta}}+\frac{|y-x_0|^\varepsilon}{|x-x_0|^{n+\varepsilon-\beta}}\right.$$
$$\left.+\sum_{|\eta|<m}\frac{|y-x_0|^{m+\beta-|\eta|}}{|x-x_0|^{n+m-|\eta|}}+\frac{|y-x_0|^\beta}{|x-x_0|^n}\right]|a(y)|dy$$
$$\le C\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}\left[\frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^n}+\frac{|Q|^{\varepsilon/n+1-1/p}}{|x-x_0|^{n+\varepsilon-\beta}}\right];$$

Thus,

$$\begin{aligned}I_2 &\leq \sum_{k=1}^{\infty} \int_{2^{k+1}Q\setminus 2^kQ} [T^A(a)(x)]^q dx \\ &\leq C \left(\sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta} \right)^q \sum_{k=1}^{\infty} \left[2^{kqn(1/p-(n+\beta)/n)} + 2^{kqn(1/p-(n+\varepsilon)/n)} \right] \\ &\leq C \left(\sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta} \right)^q ,\end{aligned}$$

which together with the estimate for I yields the desired result. This finishes the proof of Theorem 1. ∎

From Theorem 1, we get

Corollary. *Let $0 < \beta \leq 1$. If $D^\alpha A \in Lip_\beta(R^n)$ for $|\alpha| = m$. Then g_ψ^A, S_ψ^A and g_μ^A are all bounded from $L^{n/\beta}(R^n)$ to $BMO(R^n)$.*

Proof of Theorem 2. It suffices to show that there exists a constant $C > 0$ such that for every $H^{n/(n+\beta)}$-atom a supported on $Q = Q(x_0, r)$, we have

$$||\tilde{g}_\mu^A(a)||_{L^1} \leq C.$$

We write

$$\int_{R^n} \tilde{g}_\mu^A(a)(x)dx = \left[\int_{2Q} + \int_{(2Q)^c} \right] \tilde{g}_\mu^A(a)(x)dx := J_1 + J_2.$$

For J_1, by the following equality

$$Q_{m+1}(A; x, z) = R_{m+1}(A; x, z) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-z)^\alpha (D^\alpha A(x) - D^\alpha A(z)),$$

we have, similar to the proof of Lemma 4,

$$\tilde{g}_\mu^A(a)(x) \leq g_\mu^A(a)(x) + C \sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n} |a(y)|dy,$$

thus, $\tilde{g}_\mu^A$ is (L^p, L^q)-bounded by Lemma 4 and [1],[2], where $n/\beta > p > 1$ and $1/q = 1/p - \beta/n$. We see that

$$J_1 \leq C||\tilde{g}_\mu^A(a)||_{L^q}|2Q|^{1-1/q} \leq C||a||_{L^p}|Q|^{1-1/q} \leq C|Q|^{1+1/p-1/q-(n+\beta)/n} \leq C.$$

To obtain the estimate of J_2, we denote that $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A)_{2B} x^\alpha$. Then $Q_m(A;x,y) = Q_m(\tilde{A};x,y)$. We write, by the vanishing moment of a and $Q_{m+1}(A;x,y) = R_m(A;x,y) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha D^\alpha A(x)$, for $x \in (2Q)^c$,

$$\begin{aligned}
\tilde{F}_t^{\psi,A}(a)(x,y) &= \int_{R^n} \frac{\psi_t(y-z) R_m(\tilde{A};x,z)}{|x-z|^m} a(z)dz \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{\psi_t(y-z) D^\alpha \tilde{A}(z)(x-z)^\alpha}{|x-z|^m} a(z)dz \\
&= \int_{R^n} \left[\frac{\psi_t(y-z) R_m(\tilde{A};x,z)}{|x-z|^m} - \frac{\psi_t(x-x_0) R_m(\tilde{A};x,x_0)}{|x-x_0|^m} \right] a(z)dz \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \left[\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m} \right. \\
&\qquad \left. - \frac{\psi_t(x-x_0)(x-x_0)^\alpha}{|x-x_0|^m} \right] D^\alpha \tilde{A}(z) a(z)dz,
\end{aligned}$$

thus, similar to the proof of Theorem 1, we obtain, for $x \in (2Q)^c$

$$\begin{aligned}
|\tilde{g}_\mu^A(a)(x)| &\le C|Q|^{-\beta/n} \sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta} \left(\frac{|Q|^{1/n}}{|x-x_0|^{n+1-\beta}} + \frac{|Q|^{\varepsilon/n}}{|x-x_0|^{n+\varepsilon-\beta}} \right) \\
&\quad + C|Q|^{-\beta/n} \sum_{|\alpha|=m} |D^\alpha \tilde{A}(x)| \left(\frac{|Q|^{1/n}}{|x-x_0|^{n+1}} + \frac{|Q|^{\varepsilon/n}}{|x-x_0|^{n+\varepsilon}} \right),
\end{aligned}$$

so that,

$$J_2 \le C \sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta} \sum_{k=1}^\infty [2^{k(\beta-1)} + 2^{k(\beta-\varepsilon)}] \le C,$$

which together with the estimate for J_1 yields the desired result. This finishes the proof of Theorem 2. ∎

Proof of Theorem 3. By the following equality

$$R_{m+1}(A;x,z) = Q_{m+1}(A;x,z) + \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-z)^\alpha (D^\alpha A(x) - D^\alpha A(z))$$

and similar to the proof of Lemma 4, we get

$$g_\mu^A(f)(x) \le \tilde{g}_\mu^A(f)(x) + C \sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(z)|}{|x-z|^n} |f(z)|dz,$$

from Theorem 1, 2 and [15], we obtain

$$\begin{aligned}
&|\{x \in R^n : g_\mu^A(f)(x) > \lambda\}| \\
&\quad \le |\{x \in R^n : \tilde{g}_\mu^A(f)(x) > \lambda/2\}| \\
&\quad + \left|\left\{x \in R^n : \sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(z)|}{|x-z|^n} |f(z)| dz > C\lambda \right\}\right| \\
&\quad \le C\lambda^{-1} ||f||_{H^{n/(n+\beta)}}.
\end{aligned}$$

This completes the proof of Theorem 3. ∎

Proof of Theorem 4. Let $f \in H\dot{K}_{q_1}^{\alpha,p}(R^n)$ and $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x)$ be the atomic decomposition for f as in Lemma 1. We write

$$\begin{aligned}
||g_\mu^A(f)||^p_{\dot{K}_{q_2}^{\alpha,p}} &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| ||g_\mu^A(a_j)\chi_k||_{L^{q_2}} \right)^p \\
&\quad + C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| ||g_\mu^A(a_j)\chi_k||_{L^{q_2}} \right)^p \\
&= L_1 + L_2.
\end{aligned}$$

For L_2, by the (L^{q_1}, L^{q_2}) boundedness of g_μ^A (see Lemma 4), we get

$$\begin{aligned}
L_2 &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| ||a_j||_{L^{q_1}} \right)^p \\
&\le \begin{cases} C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p} \right), \ 0 < p \le 1 \\ C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p/2} \right) \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p'/2} \right)^{p/p'}, \ p > 1 \end{cases} \\
&\le C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \le C||f||^p_{H\dot{K}_{q_1}^{\alpha,p}}.
\end{aligned}$$

For L_1, similar to the proof of Theorem 1, we have, for $x \in C_k$, $j \le k-3$,

$$\begin{aligned}
g_\mu^A(a_j)(x) &\le C \left(\frac{|B_j|^{\beta/n}}{|x|^n} + \frac{|B_j|^{\varepsilon/n}}{|x|^{n+\varepsilon-\beta}} \right) \int |a_j(y)| dy \\
&\le C(2^{j(\beta+n(1-1/q_1)-\alpha)}|x|^{-n} + 2^{j(\varepsilon+n(1-1/q_1)-\alpha)}|x|^{\beta-n-\varepsilon}),
\end{aligned}$$

thus

$$||g_\mu^A(a_j)\chi_k||_{L^{q_2}} \le C2^{-k\alpha}(2^{(j-k)(\beta+n(1-1/q_1)-\alpha)} + 2^{(j-k)(\varepsilon+n(1-1/q_1)-\alpha)}),$$

and

$$\begin{aligned} L_1 &\le C\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-3}|\lambda_j|(2^{(j-k)(\beta+n(1-1/q_1)-\alpha)} + 2^{(j-k)(1/2+n(1-1/q_1)-\alpha)} \\ &\qquad\qquad + 2^{(j-k)(\gamma+n(1-1/q_1)-\alpha)}\Big)^p \\ &\le \begin{cases} C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}\big(2^{(j-k)(\beta+n(1-1/q_1)-\alpha)} \\ \quad +2^{(j-k)(1/2+n(1-1/q_1)-\alpha)} + 2^{(j-k)(\gamma+n(1-1/q_1)-\alpha)}\big)^p, \quad 0<p\le 1 \\ C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\Big[\sum_{k=j+3}^{\infty}\big(2^{(j-k)p(\beta+n(1-1/q_1)-\alpha)/2} \\ \quad +2^{(j-k)p(1/2+n(1-1/q_1)-\alpha)/2} + 2^{(j-k)(\gamma+n(1-1/q_1)-\alpha)/2)}\big)\Big], \quad p>1 \end{cases} \\ &\le C\sum_{j=-\infty}^{\infty}|\lambda_j|^p \le C||f||^p_{H\dot K_{q_1}^{\alpha,p}}. \end{aligned}$$

This completes the proof of Theorem 4. ∎

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