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Equilateral, Diametral, Centered Sets and Subsets of Spheres

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1. Introduction and definitions

Let X be a Banach space. Set, for $x \in X$ and $r \geq 0$

$$U(x,r) = \{y \in X : ||x-y|| = r\}.$$

Given a nonempty, bounded set $A \subset X$, we set

$r(A,x) = \sup\{||x-a|| : a \in A\}$ $(x \in X)$ (radius of A with respect to x);

$r(A) = \inf\{r(A;x) : x \in X\}$ (radius of A);

$\delta(A) = \sup\{||a-b|| : a, b \in A\}$ (diameter of A).

Clearly, $\delta(A) \leq 2r(A)$ always.

Define, for A, the following properties:

A is *diametral*: $r(A,x) = \delta(A)$ for every $x \in A$;

A is *equilateral*: there exists $k \in R^+$ such that $||a-b|| = k$ for all $a, b \in A$;

A *lies on a sphere*: there exist $x \in X$ and $\alpha > 0$ such that for all $a \in A$, $||x-a|| = \alpha$ (that is, $A \subset U(x,\alpha)$). Clearly, $A \subset U(x,\alpha) \Leftrightarrow \frac{A-x}{\alpha} \in U(\theta,1)$.

A is *centered* (with respect to x): there exists an $x \in X$ such that $a - x \in A \Rightarrow x - a \in A$. In other terms, A is centered if it is the translate of a symmetric set ($A - x$ is symmetric with respect to the origin).

It is simple to see that if A is centered and bounded, then the point x appearing in the definition is unique.

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We use the term centered to distinguish from the term centred introduced in [3], to denote sets as above, but with the additional properties: A is closed and $x \in A$. Note that diametral sets can be centered (according to our definition), but not centred (according to the definition in [3]).

Also, note that centred sets have properties not shared by centered sets. For example, let X be one of the spaces ℓ_1 or ℓ_2; take $A_n = \{\pm e_n, \pm e_{n+1}, \ldots\}$ ($n \in \mathbb{N}$); then we have a decreasing sequence of bounded, closed, centered sets with respect to θ, whose intersection is empty (compare with Corollaries 1 and 2 in [3])

Note that all notions introduced, apart from that of centered set which can be given also for A unbounded, refer to bounded set.

In Section 2, we shall compare the above properties of sets. In Section 3 we shall give results concerning special classes of Banach spaces. Finally, in Section 4, we shall collect several examples related to these properties.

2. Equilateral, diametral, centered sets and sets lying on a sphere

Let A be a bounded and closed set. Consider now the following properties:

(P_1) A is an equilateral set;

(P_2) A is diametral;

(P_3) A lies on a sphere;

(P_4) A is centered;

(P_5) $\delta(A) = 2r(A)$.

Note that all these properties are preserved if we translate the set A, or we consider a homothetic copy of A.

Proposition 2.1. *Among the above properties, only the following implications hold:*

$$(P_1) \Rightarrow (P_2); \qquad (P_4) \Rightarrow (P_5).$$

Proof. The first implication is trivial. The second one can be proved in this way: we can assume, without restriction, that the (bounded) set A is centered with respect to θ, so $a \in A$ implies $-a \in A$. Let $r = r(A, \theta) = \sup\{||a|| : a \in A\}$; given $\varepsilon > 0$, there exists $a \in A$ such that $||a|| > r - \varepsilon$, so $\delta(A) \geq ||a - (-a)|| > 2(r - \varepsilon)$; this is true for any $\varepsilon > 0$, so $\delta(A) \geq 2r \geq 2r(A)$, and then

$$\delta(A) = 2r(A, \theta) = 2r(A), \tag{$*$}$$

so (P_5) is true.

Example 4.1 will show that (P_4) does not imply (P_2) or (P_3); Example 4.2, bet also Example $4.2\frac{1}{2}$, will show that (P_1) or (P_3) does not imply (P_5); Example 4.3, that (P_3) does not imply (P_2) and that (P_5) does not imply (P_4); Example 4.4, that (P_2) does not imply (P_1); Example 4.5, or Example 4.6, that (P_1) does not imply (P_3); for another example see Example 6.1 in [1]; Example 4.7, that (P_5) does not imply (P_4). ∎

Remark. If A is equilateral, contains three points and $\dim(X) \geq 3$, then A always lies on a sphere; in fact, it is always possible to add a fourth point to A so that the enlarged set is still equilateral (see [7]).

We can ask the following: by using pairs of the above conditions, what implications can be proved? We have:

PROPOSITION 2.2. *The following implication holds:*

(j) $(P_2) \wedge (P_4) \Rightarrow (P_3)$;

Moreover, we have:

(i) $(P_2) \wedge (P_3) \wedge (P_4) \not\Rightarrow (P_1)$;

(ii) $(P_3) \wedge (P_5) \not\Rightarrow (P_4)$;

(iii) $(P_1) \wedge (P_5) \not\Rightarrow (P_4)$; also, $(P_1) \wedge (P_3) \wedge (P_5) \not\Rightarrow (P_4)$;

(iv) $(P_1) \wedge (P_3) \not\Rightarrow (P_5)$;

(v) $(P_3) \wedge (P_4) \not\Rightarrow (P_2)$.

Proof. (j): Let A be diametral and centered, say at θ. Let $r = r(A) = r(A, \theta) = \sup\{||a|| : a \in A\}$; according to Proposition 2.1 (see (∗)), we have $\delta(A) = 2r$. But $||\bar{a}|| = r - \varepsilon$ for some $\bar{a} \in A$ and $\varepsilon > 0$ would imply, for every $a \in A$: $r(A, \bar{a}) \leq ||\bar{a} - \theta|| + ||\theta - a|| \leq r - \varepsilon + r = \delta(A) - \varepsilon$, against the assumption that A is diametral. This proves (j).

(i) is proved by Example 4.4; (ii) by Example 4.3; (iii) by Example 4.7; (iv) by Example 4.2; (v) by Example $4.1\frac{1}{2}$. ∎

Our examples also indicate what implications are true, given a triple of conditions among (P_1) through (P_5).

We could raise the following question:

QUESTION. Do $(P_2) \wedge (P_5)$ imply (P_3)? Or at least, do $(P_1) \wedge (P_5)$ imply (P_3)?

A partial result will be given by Proposition 3.1.

It is trivial to see that a diametral set cannot have interior points. We recall another simple result concerning diametral sets.

Proposition 2.3. *If A is diametral and compact, then $r(A) < \delta(A)$.*

Proof. If A is diametral, then $\overline{co}(A)$ is convex and compact; also $\overline{co}(A)$ has the same radius and diameter of A; therefore (see e.g. [5], p. 39) there exists $x_o \in \overline{co}(A)$ non diametral, so $r(A) \leq r(A, x_o) < \delta(A)$. ∎

3. Results in some spaces

We recall that there are spaces where all finite sets are contained on a sphere: this topic has been studied in [2].

Of course there are spaces where other implications hold.

We consider now spaces where the following property holds:

(E) for every bounded set A, there exists $c \in X$ such that $r(A, c) = r(A)$.

For some general results concerning spaces with this property, see for example [6], § 33. Note that in general, also when A is an equilateral set, the existence of a point x as before is not assured (see [1], Example 5.2).

Next proposition should be compared with the question raised in Section 2.

Proposition 3.1. *Let X be a space with property* (E)*; then the following implication holds:*

$$(\mathrm{P}_2) \wedge (\mathrm{P}_5) \Rightarrow (\mathrm{P}_3).$$

Proof. Let X satisfy (E). Assume that A is a diametral set and that $\delta(A) = 2r(A) = 2r$; let $c \in X$ be a point such that $r(A, c) = r$; then $A \subset U(c, r)$. In fact, let $x \in A$ be such that $||x - c|| < r$; we would have, for any $a \in A$, $||x - a|| \leq ||x - c|| + ||c - a|| < r + r = \delta(A)$; a contradiction proving the proposition. ∎

Now we indicate a result, which should be compared with Proposition 2.2 (iii).

Recall the following equivalent formulation of *local uniform rotundity*, (LUR) for short: X is locally uniformly rotund if

$$||x|| = ||x_n|| = 1 \;\forall n \in \mathbb{N} \text{ and } \lim_{n\to\infty} ||x - x_n|| = 2 \quad \Rightarrow \quad \lim_{n\to\infty} ||x + x_n|| = 0.$$

PROPOSITION 3.2. *Let X be a* (LUR) *space, satisfying* (E) *(for example, let X be uniformly convex). Assume that A is a diametral set; also, let $\delta(A) = 2r(A)$. Then A is centered; that is, the following implication is true:*

$$(\mathrm{P}_2) \wedge (\mathrm{P}_5) \Rightarrow (\mathrm{P}_4).$$

Proof. Let A be diametral, and $\delta(A) = 2r(A)$. If X satisfies (E), then according to Proposition 3.1, $(\mathrm{P}_2)\wedge(\mathrm{P}_5)$ imply that A is contained in a sphere of radius $r(A)$; we can assume, without restriction, that $A \subset U(\theta, 1)$ and that $\delta(A) = 2$. Let $x \in A$, then take a sequence (x_n) in A (so $||x_n|| = 1$) such that $\lim_{n\to\infty} ||x - x_n|| = 2$; since X is (LUR), this implies $\lim_{n\to\infty} ||x + x_n|| = 0$, so $-x = \lim_{n\to\infty} x_n \in A$ (since A is closed). This shows that A is centered. ∎

According to [1], Theorems 3.1 and 3.2, every finite equilateral set in a Hilbert space lies on a sphere. But the same is true for countable sets; in fact we have.

PROPOSITION 3.3. (see [4], Theorem 3.3) *Let X be a Hilbert space; if we have a sequence $(x_n)_{n\in\mathbb{N}}$ such that $||x_i - x_j||$ is constant $(i, j \in \mathbb{N})$, then $\{x_n\}_{n\in\mathbb{N}}$ is contained on a sphere.*

QUESTION. We do not know if the above result can be extended to uncountable, equilateral sets in (necessarily not separable) Hilbert spaces.

Remark. The implication $(\mathrm{P}_2) \Rightarrow (\mathrm{P}_3)$ is not true, neither in Hilbert spaces: see Example 4.8 in Section 4.

4. Some examples concerning sets which are diametral, centered, ...

In this section we collect several examples concerning the conditions and implications considered in Section 2. All sets A of these examples are closed and bounded.

EXAMPLE 4.1. Let $X = R$ (the Euclidean line); $A = [-1, -1/2] \cup \{\theta\} \cup [1/2, 1]$. The set A is centered (at θ); it does not lie on a sphere nor is diametral (so (P_4) does not imply (P_2) or (P_3)).

EXAMPLE $4.1\frac{1}{2}$. Let $X = R^2$ with the sup norm; consider $\mathrm{A}_1 = \{(x, 1)$ with $x \in A\}$, A as in Example 4.1. Then A_1 is centered with respect to $(0, 1)$;

it is contained in the sphere centered at the origin, with radius 1; but it is not diametral (so $(P_3) \wedge (P_4)$ do not imply (P_2)).

Example 4.2. Let X be the Euclidean plane and $A = \{(0,1); (\frac{\sqrt{3}}{2}, -\frac{1}{2}); (-\frac{\sqrt{3}}{2}, -\frac{1}{2})\}$; the set A is equilateral and lying on a sphere; it does not satisfy $\delta(A) = 2r(A)$ (so (P_1) or (P_3) does not imply (P_5)).

Example $4.2\frac{1}{2}$. Let $X = c_o$ and A the set containing the elements of the natural basis. Then A satisfies (P_1) and (P_3), but not (P_5) since $r(A) = \delta(A) = 1$.

Example 4.3. Let X be the Euclidean plane and A a semicircle; the set A lies on a sphere. Also, we have $\delta(A) = 2r(A)$, A is not centered (nor diametral), so this example proves (ii) in Proposition 2.2 (A satisfies (P_3), (P_5); not (P_2), (P_4)).

Example 4.4. Let X be the plane with the max norm and $A = \{(0,1); (1,1); (0,-1); (-1,-1)\}$. The set is diametral but not equilateral (so (P_2) does not imply (P_1)). Also, A satisfies (P_3) and (P_4).

Example 4.5. Let $X = R^3$ with the norm $||(x_1, x_2, x_3)|| = \sqrt{x_1^2 + x_2^2} + |x_3|$. Then take $A = \{x = (\frac{\sqrt{3}}{2}, \frac{1}{2}, 0); y = (-\frac{\sqrt{3}}{2}, \frac{1}{2}, 0); u = (0, -1, 0); v = (0, 0, \sqrt{3} - 1)\}$. The set A is equilateral (in fact the distance between any pair of its points is $\sqrt{3}$). Assume that for $z = (a, b, c)$ we have: $||z - x|| = ||z - y|| = ||z - u|| = ||z - v||$. From $||z - x|| = ||z - y||$ we obtain $a = 0$. Thus $||z - x|| = ||z - u||$ implies $\sqrt{\frac{3}{4} + (\frac{1}{2} - b)^2} + |c| = |1 + b| + |c|$, so $b = 0$. Then, since $z = (0, 0, c)$, $||z - u|| = ||z - v||$ implies $1 + |c| = |\sqrt{3} - 1 - c|$, which is impossible. This proves that A does not lie on a sphere, and so that (P_1) does not imply (P_3).

Example 4.6. Let $X = c_o$ and $A = \{$sequences with a finite or null number of components equal to 1 and the others equal to 0$\}$. A is equilateral (so also diametral) with $\delta(A) = 1$. Let $A \subset U(x, r)$ for some $x = (x_1, x_2, \ldots, x_n, \ldots) \in X$ and $r > 0$ ($r = ||x||$ since $\theta \in A$). Clearly $r(A, y) \geq 1$ for any $y \in X$, so $r \geq 1 = r(A)$. Let $||x|| = r = |x_p|$ for some $p \in \mathbb{N}$: since $||x - e_p|| = r$, we have $x_p \geq 0$ for such p, so $x_p = r$. Also, $x_p = r$ for finitely many indices, say $x_p = r$ for $p = n_1, n_2, \ldots, n_k$. Take $a = e_{n_1} + e_{n_2} + \cdots + e_{n_k}$; since $r \geq 1$, we have: $||x - a|| = \sup\{r - 1, \sup\{x_n \colon n \notin \{n_1, n_2, \ldots, n_k\}\}\} < r$:

a contradiction since $a \in A$. Therefore there exists no $x \in X$ such that $A \subset U(x, r)$. This shows that (P_1) does not imply (P_3) and that (P_1) does not imply (P_5).

Note that if we consider $A^- = A - \{\theta\}$, then $A^- \subset U(\theta, 1)$. Also, if we embed c_o in l_∞ and we take z=$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots)$, then $A \subset U(z, \frac{1}{2})$.

Example 4.7. Let $X = R^2$ with the max norm. Let $A = \{(0,1); (-1,-1); (1,-1)\}$. The set is equilateral, lies on a sphere and $\delta(A) = 2r(A) = 2$, but it is not centered (so (P_1)$\wedge$(P_3)$\wedge$(P_5) do not imply (P_4)).

Example $4.7\frac{1}{2}$. Another example similar to the previous one is the following: take $X = l^1$ and let A be the set of all elements of the natural basis.

Example 4.8. Let $X = R^2$ (the Euclidean plane) and $A = \{(\frac{\sqrt{3}}{2}, \frac{1}{2}); (-\frac{\sqrt{3}}{2}, \frac{1}{2}); (0,-1); (0, \sqrt{3}-1)\}$. The set A is diametral ($\delta(A) = \sqrt{3}$) but it does not lie on a sphere.

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