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An Obstruction to Represent Abelian Lie Algebras by Unipotent Matrices

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Introduction.

The aim of this paper is the study of abelian Lie algebras as subalgebras of the nilpotent Lie algebra $\mathfrak{g}_n$ associated with Lie groups of upper-triangular square matrices whose main diagonal is formed by 1.

We also give an obstruction to obtain the abelian Lie algebra of dimension one unit less than the corresponding to $\mathfrak{g}_n$ as a Lie subalgebra of $\mathfrak{g}_n$. Moreover, we give a procedure to obtain abelian Lie subalgebras of $\mathfrak{g}_n$ up to the dimension which we think it is the maximum.

There are several reasons to study nilpotent Lie algebras. By one side, the problem of their classification is still unsolved, being only known up to dimension 7 (see [1, 2]). By the other side, we think that the information obtained about the simply connected Lie groups associated with them will translate in information about the algebras themselves, and finally it will mean a step forward in the above mentioned problem.

It is known that given a fixed Lie group, there exists a Lie algebra associated with it. The converse, that is, every Lie algebra is associated with some Lie group, was locally proved by Lie, in his Third Theorem, and globally by Ado. Consequently it can be proved that any finite-dimensional complex Lie algebra is isomorphic to some matrix Lie algebra (see [4]).

In this way, the study of Lie algebras reduces to the study of Lie algebras associated with matrix Lie groups. In fact, Proposition 3.6.6 of [4] states that every nilpotent Lie algebra is obtained as Lie subalgebra of the Lie algebra associated with the Lie group G_n that consists in upper-triangular square matrices with "1" in their main diagonal.

We have asked ourselves about the dimension of the abelian Lie algebras contained in the Lie algebra $\mathfrak{g}_n$ associated with G_n. This paper deals with the maximal dimension of abelian Lie algebras, considered as subalgebras of G_n, for a given $n \in \mathbb{N} \setminus \{1\}$.

We give a procedure to obtain abelian Lie algebras in $\mathfrak{g}_n$, considering the cases n odd or even. We formulate a conjecture about the maximal dimension of these algebras in $\mathfrak{g}_n$. Finally, the main result of the paper proves that the Lie algebra $\mathfrak{g}_n$, of dimension $\mathsf{d}_{\mathfrak{g}_n}$ cannot contain the abelian Lie algebra of dimension $\mathsf{d}_{\mathfrak{g}_n} - 1$ as a Lie subalgebra (see Theorem 3.1 and Corollary 3.2).

1. Preliminaries

We will remind some preliminary concepts on Lie groups and Lie algebras that will be used in the paper. For a general overview on Lie groups and Lie algebras, the reader can consult [4].

If a Lie group is denoted by G, we will denote its associated Lie algebra by $\mathfrak{g}$. Note that the dimensions of G and $\mathfrak{g}$ are the same.

A representation of a Lie group of dimension n is a homomorphism of Lie groups $\phi \colon G \to GL(n, \mathbb{C})$.

If $\mathcal{L}$ is a Lie algebra, its central series is given by:

$$\mathcal{C}^1(\mathcal{L}) = \mathcal{L},\ \mathcal{C}^2(\mathcal{L}) = [\mathcal{L}, \mathcal{L}],\ \mathcal{C}^3(\mathcal{L}) = [\mathcal{C}^2(\mathcal{L}), \mathcal{L}],\ \dots,\ \mathcal{C}^k(\mathcal{L}) = [\mathcal{C}^{k-1}(\mathcal{L}), \mathcal{L}],\ \dots$$

Then, $\mathcal{L}$ is called *nilpotent* if there exists a natural number m such that $\mathcal{C}^m(\mathcal{L}) \equiv 0$.

A Lie algebra $\mathcal{L}$ is called abelian if $[X, Y] = 0$, for all X, $Y \in \mathcal{L}$.

2. The Lie group G_n of unipotent matrices.

Since an abelian Lie algebra is nilpotent, its simply connected Lie group can be represented by unipotent matrices (that is, upper-triangular square matrices with "1" in the main diagonal). However, we do not know, a priori, the minimal order of matrices verifying such a condition.

If we denote by G_n the Lie group of unipotent matrices, elements in this group have the form:

$$g_n(x_{i,j}) = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n-1} & x_{1,n} \\ 0 & 1 & x_{2,3} & \cdots & x_{2,n-1} & x_{2,n} \\ 0 & 0 & 1 & \cdots & x_{3,n-1} & x_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & x_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \qquad (x_{i,j} \in \mathbb{C}).$$

As we proved in [3], the Lie algebra $\mathfrak{g}_n$ associated with G_n is nilpotent and the only nonzero brackets in $\mathfrak{g}_n$ are:

$$[X_{i,j}, X_{j,k}] = X_{i,k} \qquad i = 1, \ldots, n; \quad j = i+1, \ldots, n; \quad k = j+1, \ldots, n$$

That is, nonzero brackets are only obtained if we multiply a field of the j^{th} column times a field of the j^{th} row, for every $j \in \{2, \ldots, n\}$.

We will distinguish two cases, depending of the parity of the order of matrices in G_n.

2.1. Case 1: Matrices of even order. Let us consider before, as examples, the Lie groups G_2 and G_4, already studied in [3]:

$$G_2 = \begin{pmatrix} 1 & x_{1,2} \\ 0 & 1 \end{pmatrix} \qquad G_4 = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Clearly, only the 1-dimensional abelian Lie algebra can be obtained as subalgebra $\mathfrak{g}_2$ with G_2.

Let's consider $\mathfrak{g}_4$. We have three fields corresponding to the 4^{th} column; the 3^{rd} column adds two fields although the field corresponding to the 3^{rd} row has to be removed. So, we have four fields. If we now add the 2^{nd} column (which has a unique field), we would have to remove the two fields corresponding to the 2^{nd} row and then this last step does not improve the situation.

Inspired in these two examples, we will show a procedure to get abelian Lie algebras from $\mathfrak{g}_{2k}$ for any k. It consists of the following steps:

Column $2k$. Firstly, we consider the $2k-1$ fields corresponding to the $(2k)^{th}$ column.

Column $(2k-1)$. We add the $2k-2$ fields corresponding to the $(2k-1)^{th}$ column and we remove the field of the $(2k-1)^{th}$ row.

⋮ ⋮

Column i. We add the $i-1$ fields corresponding to the i^{th} column and we remove the $2k-i$ fields of the i^{th} row. Hence the number of added fields is the difference between both numbers, that is, $2i-2k-1$.

⋮ ⋮

Column $k+1$. We stop the procedure in the $(k+1)^{th}$ column, since the difference $2i-2k-1$ is positive if and only if $i > k+\frac{1}{2}$. Then we add the k fields of the $(k+1)^{th}$ column and we remove the $k-1$ fields of the $(k+1)^{th}$ row.

In this way, we can obtain abelian Lie algebras whose dimension is less or equal than k^2. Besides, fields obtained in the procedure are the following:

$$\begin{array}{ccccc} X_{1,k+1} & X_{1,k+2} & \cdots & X_{1,2k-1} & X_{1,2k} \\ X_{2,k+1} & X_{2,k+2} & \cdots & X_{2,2k-1} & X_{2,2k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_{k-1,k+1} & X_{k-1,k+2} & \cdots & X_{k-1,2k-1} & X_{k-1,2k} \\ X_{k,k+1} & X_{k,k+2} & \cdots & X_{k,2k-1} & X_{k,2k} \end{array}$$

2.2. Case 2: Matrices of odd order. By repeating the same scheme as before, we firstly consider two particular examples, already studied in [3]:

$$G_3 = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} \\ 0 & 1 & x_{2,3} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad G_5 = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} \\ 0 & 0 & 0 & 1 & x_{4,5} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We now use the same procedure as in the previous case. So, from $\mathfrak{g}_3$, we can obtain the abelian Lie algebra $< X_{1,2}, X_{1,3} >$ of dimension 2, but we cannot obtain the abelian algebra of dimension 3, because $\mathfrak{g}_3$ itself is a non-abelian Lie algebra.

In $\mathfrak{g}_5$, if we consider the 5^{th} column, we have four fields; the 4^{th} column adds three fields and we have to remove the field corresponding to the 4^{th} row. So, we have six fields. If we add the 3^{rd} column (which has two fields), we would have to remove the two fields corresponding with the 3^{rd} row. However, it does not improve the situation.

We note then that this procedure is valid for obtaining abelian Lie algebras in $\mathfrak{g}_{2k+1}$ up to dimension $\frac{(2k+1)^2-1}{4}$. This conclusion is similar as the given in the previous case. Concretely:

Column $2k+1$.	Firstly, we consider the $2k$ fields corresponding to the $(2k+1)^{th}$ column.
Column $2k$.	We add the $2k-1$ fields corresponding to the $(2k)^{th}$ column and we remove the field corresponding to the $(2k)^{th}$ row.
$\vdots$	$\vdots$
Column j.	When dealing with the j^{th} column, we add $j-1$ fields and we remove the $2k+1-j$ fields corresponding with the j^{th} row.
$\vdots$	$\vdots$
Column $k+1$.	When dealing with the $(k+1)^{th}$ column, we add k fields and we remove the k fields of the $(k+1)^{th}$ row. We stop in this step, because, by considering the k^{th} column, we would add $k-1$ fields and we would remove the $k+1$ fields in the k^{th} row and, hence, the dimension would decrease.

So, the fields of $\mathfrak{g}_{2k+1}$ in this abelian Lie algebra are:

$$\begin{array}{ccccc}
X_{1,k} & X_{1,k+1} & \cdots & X_{1,2k} & X_{1,2k+1} \\
X_{2,k} & X_{2,k+1} & \cdots & X_{2,2k} & X_{2,2k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
X_{k-2,k} & X_{k-2,k+1} & \cdots & X_{k-2,2k} & X_{k-2,2k+1} \\
X_{k-1,k} & X_{k-1,k+1} & \cdots & X_{k-1,2k} & X_{k-1,2k+1}
\end{array}$$

Now, by taking into consideration both cases, a natural question appears: is it possible to obtain an abelian Lie algebra of higher dimension? By denoting the dimension of the algebra $\mathfrak{g}_n$ by $d_{\mathfrak{g}_n}$, we will see in the next section that it is not possible to obtain the abelian Lie algebra of dimension $d_{\mathfrak{g}_n}-1$. It is a first step in the attempt of proving the following:

CONJECTURE. The maximal dimension of an abelian Lie algebra h in g_n is given by:

$$\dim \mathsf{h} = \begin{cases} k^2, & \text{if } n = 2k, \quad \text{with } k \in \mathbb{N}, \\ \frac{(2k+1)^2-1}{4}, & \text{if } n = 2k+1, \quad \text{with } k \in \mathbb{N}. \end{cases}$$

We have already proved this result for $n \in \{2,3,4\}$ in [3].

3. ABELIAN LIE ALGEBRA OF DIMENSION $d_{\mathsf{g}_n} - 1$.

Now, coming back to the question of what abelian Lie algebras can be contained in a given Lie algebra g_n, we will prove that the abelian Lie algebra of dimension $d_{\mathsf{g}_n} - 1$ is not a subalgebra of g_n.

THEOREM 3.1. *If $n \in \mathbb{N}$, with $n \geq 4$, then the abelian Lie algebra of dimension $d_{\mathsf{g}_n} - 1$ is not a Lie subalgebra of g_n.*

If we use the relation between the Lie subgroups of a given Lie group and the Lie subalgebras of its associated Lie algebra, Theorem 3.1 immediately implies the following result:

COROLLARY 3.2. *If $n \in \mathbb{N}$, $n \geq 4$, then the simply connected Lie group associated with the abelian Lie algebra of dimension $d_{\mathsf{g}_n} - 1$ cannot be represented as a Lie subgroup of G_n.*

Proof. To prove Theorem 3.1 we will proceed by induction on n.

Let us suppose, in the first place, $n = 4$: as the dimension of G_4 is 6, the considered abelian Lie algebra has dimension 5. Then every basis of Lie subalgebras of g_4 can be expressed by $\{Y_i\}_{i=1}^{5}$, where:

$$Y_i = \sum_{\substack{k=j+1 \\ j=1}}^{\substack{j=3 \\ k=4}} a_{i,j,k} X_{j,k} \ , \quad (a_{i,j,k} \in \mathbb{C}) \ , \ (i = 1, \ldots, 5).$$

As the corresponding matrix of coefficients has rank 5, it is equivalent to the following matrix:

$$\begin{pmatrix} b_{1,1} & 0 & 0 & 0 & 0 & b_{1,6} \\ 0 & b_{2,2} & 0 & 0 & 0 & b_{2,6} \\ 0 & 0 & b_{3,3} & 0 & 0 & b_{3,6} \\ 0 & 0 & 0 & b_{4,4} & 0 & b_{4,6} \\ 0 & 0 & 0 & 0 & b_{5,5} & b_{5,6} \end{pmatrix},$$

where $b_{i,i} \neq 0$, for $1 \leq i \leq 5$. Therefore, to give a basis of every 5-dimensional subalgebra of $\mathfrak{g}_4$, we have to distinguish the following six possibilities, where $\lambda_i \in \mathbb{C}$:

$$\langle \lambda_1 X_{1,3} + \mu_1 X_{1,2}, \lambda_2 X_{1,4} + \mu_2 X_{1,2}, \lambda_3 X_{2,3} + \mu_3 X_{1,2}, \lambda_4 X_{2,4} + \mu_4 X_{1,2}, \lambda_5 X_{3,4} + \mu_5 X_{1,2} \rangle.$$
$$\langle \lambda_1 X_{1,2} + \mu_1 X_{1,3}, \lambda_2 X_{1,4} + \mu_2 X_{1,3}, \lambda_3 X_{2,3} + \mu_3 X_{1,3}, \lambda_4 X_{2,4} + \mu_4 X_{1,3}, \lambda_5 X_{3,4} + \mu_5 X_{1,3} \rangle.$$
$$\langle \lambda_1 X_{1,2} + \mu_1 X_{1,4}, \lambda_2 X_{1,3} + \mu_2 X_{1,4}, \lambda_3 X_{2,3} + \mu_3 X_{1,4}, \lambda_4 X_{2,4} + \mu_4 X_{1,4}, \lambda_5 X_{3,4} + \mu_5 X_{1,4} \rangle.$$
$$\langle \lambda_1 X_{1,2} + \mu_1 X_{2,3}, \lambda_2 X_{1,3} + \mu_2 X_{2,3}, \lambda_3 X_{1,4} + \mu_3 X_{2,3}, \lambda_4 X_{2,4} + \mu_4 X_{2,3}, \lambda_5 X_{3,4} + \mu_5 X_{2,3} \rangle.$$
$$\langle \lambda_1 X_{1,2} + \mu_1 X_{2,4}, \lambda_2 X_{1,3} + \mu_2 X_{2,4}, \lambda_3 X_{1,4} + \mu_3 X_{2,4}, \lambda_4 X_{2,3} + \mu_4 X_{2,4}, \lambda_5 X_{3,4} + \mu_5 X_{2,4} \rangle.$$
$$\langle \lambda_1 X_{1,2} + \mu_1 X_{3,4}, \lambda_2 X_{1,3} + \mu_2 X_{3,4}, \lambda_3 X_{1,4} + \mu_3 X_{3,4}, \lambda_4 X_{2,3} + \mu_4 X_{3,4}, \lambda_5 X_{2,4} + \mu_5 X_{3,4} \rangle.$$

We deal next with the first of the possibilities (the rest of them can be seen in [3]).

Making equal to zero the brackets between basic elements, we obtain a system which contains the following equations: $\lambda_3\mu_2 = 0$, $\lambda_3\mu_1 = 0$, $\lambda_3\mu_4 = 0$, $\mu_3\lambda_4 = 0$, $\lambda_3\lambda_5 = 0$, $\lambda_3\mu_5 = 0$, which gives a contradiction.

Let us suppose that $n > 4$ and, by the induction assumption, the result is true for $n-1$, that is, we cannot obtain the abelian Lie algebra of dimension $D(n-1) = d_{\mathfrak{g}_{n-1}} - 1$ in $\mathfrak{g}_{n-1}$.

Let us prove the result for n. The dimension of the abelian Lie algebra to study is $D(n) = d_{\mathfrak{g}_n} - 1 = \binom{n}{2} - 1$.

We will argue as in the case $n = 4$. Let us consider the elements $X_{i,j}$ (with $i = 1, \ldots, n-1$ and $j = i+1, \ldots, n-1$) in $\mathfrak{g}_n$ as coming from $\mathfrak{g}_{n-1}$ (considered as subalgebra of $\mathfrak{g}_n$). If $X_{h,k}$ is one of those elements and the basis $B_{h,k}$ of the $(d_{\mathfrak{g}_n} - 1)$ dimensional abelian subalgebra consists of elements of the form: $Y_{i,j} = \lambda_{i,j}\, X_{i,j} + \mu_{i,j}\, X_{h,k}$, with $(i,j) \neq (h,k)$, then the abelian Lie subalgebra $B =< Y_{i,j} >$, with $1 \leq i < j \leq n-1$, is an abelian Lie subalgebra of $\mathfrak{g}_{n-1}$ with dimension $D(n-1)$, against the induction assumption.

Now let us suppose that the basis of the abelian subalgebra, $B_{i,n}$, consists of elements that involve, all of them, the element $X_{i,n}$ and consider the basic elements:

$$\begin{array}{rclrcl} Y_{1,2,i} & = & \lambda_{1,2,i}X_{1,2}+\mu_{1,2,i}X_{i,n}, & Y_{1,3,i} & = & \lambda_{1,3,i}X_{1,3}+\mu_{1,3,i}X_{i,n}, \\ Y_{2,3,i} & = & \lambda_{2,3,i}X_{2,3}+\mu_{2,3,i}X_{i,n}, & Y_{3,4,i} & = & \lambda_{3,4,i}X_{3,4}+\mu_{3,4,i}X_{i,n}. \end{array}$$

The brackets $[Y_{1,2,i},Y_{2,3,i}]$ and $[Y_{1,3,i},Y_{3,4,i}]$ are given by:

$$[Y_{1,2,i},Y_{2,3,i}]=\lambda_{1,2,i}\lambda_{2,3,i}X_{1,3}+\lambda_{1,2,i}\mu_{2,3,i}[X_{1,2},X_{i,n}]+\mu_{1,2,i}\lambda_{2,3,i}[X_{i,n},X_{2,3}],$$
$$[Y_{1,3,i},Y_{3,4,i}]=\lambda_{1,3,i}\lambda_{3,4,i}X_{1,4}+\lambda_{1,3,i}\mu_{3,4,i}[X_{1,3},X_{i,n}]+\mu_{1,3,i}\lambda_{3,4,i}[X_{i,n},X_{3,4}].$$

According to the law of $\mathfrak{g}_n$, we have the brackets:

$$[X_{1,2},X_{i,n}]=\begin{cases}0, & \text{if } i\neq 2,\\ X_{1,n}, & \text{if } i=2.\end{cases} \qquad [X_{i,n},X_{2,3}]=\begin{cases}0, & \text{if } i\neq 3,\\ -X_{2,n}, & \text{if } i=3.\end{cases}$$

$$[X_{1,3},X_{i,n}]=\begin{cases}0, & \text{if } i\neq 3,\\ X_{1,n}, & \text{if } i=3.\end{cases} \qquad [X_{i,n},X_{3,4}]=\begin{cases}0, & \text{if } i\neq 4,\\ -X_{3,n}, & \text{if } i=4.\end{cases}$$

and, as a consequence, possible cases are:

a) If $i\neq 2,3,4$, we have:

$$\left\{\begin{array}{rcl}[Y_{1,2,i},Y_{2,3,i}] & = & \lambda_{1,2,i}\lambda_{2,3,i}X_{1,3},\\ [Y_{1,3,i},Y_{3,4,i}] & = & \lambda_{1,3,i}\lambda_{3,4,i}X_{1,4}.\end{array}\right.$$

b) If $i=2$, we have:

$$\left\{\begin{array}{rcl}[Y_{1,2,i},Y_{2,3,i}] & = & \lambda_{1,2,i}\lambda_{2,3,i}X_{1,3}+\lambda_{1,2,i}\mu_{2,3,i}X_{1,n},\\ [Y_{1,3,i},Y_{3,4,i}] & = & \lambda_{1,3,i}\lambda_{3,4,i}X_{1,4}.\end{array}\right.$$

c) If $i=3$, we have:

$$\left\{\begin{array}{rcl}[Y_{1,2,i},Y_{2,3,i}] & = & \lambda_{1,2,i}\lambda_{2,3,i}X_{1,3}-\mu_{1,2,i}\lambda_{2,3,i}X_{2,n},\\ [Y_{1,3,i},Y_{3,4,i}] & = & \lambda_{1,3,i}\lambda_{3,4,i}X_{1,4}+\lambda_{1,3,i}\mu_{3,4,i}X_{1,n}.\end{array}\right.$$

d) If $i=4$, we have:

$$\left\{\begin{array}{rcl}[Y_{1,2,i},Y_{2,3,i}] & = & \lambda_{1,2,i}\lambda_{2,3,i}X_{1,3},\\ [Y_{1,3,i},Y_{3,4,i}] & = & \lambda_{1,3,i}\lambda_{3,4,i}X_{1,4}-\mu_{1,3,i}\lambda_{3,4,i}X_{3,n}.\end{array}\right.$$

In every case, the equations $\lambda_{1,2,i}\lambda_{2,3,i}=0$ and $\lambda_{1,3,i}\lambda_{3,4,i}=0$ are obtained. Hence, two elements of the basis of the abelian subalgebra are linearly dependent, what gives a contradiction. This proves Theorem 3.1 and, consequently, Corollary 3.2. ∎

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