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Non-Expansive Mappings in Spaces of Continuous Functions

Tomás Domínguez Benavides, María A. Japón Pineda

Departamento de Análisis Matemático, Universidad de Sevilla, 41071 *Sevilla, Spain*
e-mail: tomasd@us.es, japon@us.es

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1. Introduction

Let (X, d) be a metric space and $T : X \to X$ a k-Lipschitz mapping, i.e.

$$d(Tx, Ty) \leq kd(x, y).$$

If $k < 1$, it is well known (Contractive Mapping Principle) that T has a fixed point. However, it is clear (consider, for instance, a translation in $\mathbb{R}$) that the result is not longer true when $k = 1$. In this case, the mapping T is said to be non-expansive. In 1965 some fixed point theorems for non-expansive mappings appeared:

Theorem 1. (Browder's Theorem [5]) *Let C be a convex bounded closed subset of a uniformly convex Banach space (a preliminary version was given for Hilbert spaces) and $T : C \to C$a nonexpansive mapping. Then T has a fixed point.*

Theorem 2. (Kirk's Theorem [11]) *Let C be a convex bounded closed subset of a reflexive Banach space with normal structure. If $T : C \to C$ is nonexpansive, then T has a fixed point.*

These results were, in some sense, surprising. Indeed, the convexity assumption on C seems to be more suitable for fixed point theorems concerning compact operators (Schauder fixed point theorem). Furthermore, the assumptions on X (uniform convexity, reflexivity, normal structure, etc) had usually been considered in Linear Functional Analysis, and it is noteworthy that these assumptions can assure the existence of fixed point for nonlinear operators. From this starting point, a large number of fixed point results have been obtained for non-expansive mappings by using geometric properties of Banach spaces. In general, we say that a Banach space X has the fixed point property (FPP) if every non-expansive mappings defined from a convex weakly compact subset C of X into C has a fixed point. In this terminology, Kirk's theorem and Browder's theorem can be reformulated saying that either Banach spaces which are uniformly convex or reflexive Banach spaces with normal structure has the FPP. A long-standing open problem was the following: Does any Banach space X have the FPP? The answer was given in 1981 by Alspach [1] who proved that $L^1([0,1])$ fails to have this property. An interesting research subject appears in this moment: Which Banach spaces do have the FPP? No general answer is known to this question. In fact, many special cases of this question remain open. For instance: Does any reflexive space have the FPP? Does any Banach space which is isomorphic to a Hilbert space have the FPP? We are interested in these problems when X is a space of real continuous functions defined on a metric compact set K. On one hand, it is known that $C([0,1])$ is universal for separable Banach spaces. Thus $C([0,1])$ isometrically contains $L^1([0,1])$ and, a fortiori, it fails to have the FPP. On the other hand, if K is a finite set, we know that $C(K)$ has finite dimension and so it has the FPP. Furthermore, if K is the one point compactification of $\mathbb{N}$, $C(K)$ becomes the space c of all real convergent sequences and Borwein and Sims [4] have proved that c has the FPP. Thus $C(K)$ can either enjoy or fail the FPP depending on the topological structure of K. A natural problem in this subject is to give a complete characterization of those sets K such that $C(K)$ has the FPP. This problem is still open. In this survey we will show what is known and what is unknown with respect to this problem. The application of special techniques in metric fixed point theory will let us obtain positive results in this direction. Alspach's example will be the key to obtain Banach spaces of continuous functions which fail to have the FPP. Most results in section 2-4 and some of those in section 5 are "classical" and can be found in specialized books [2], [14], [9], [10] and [18]. We include some proofs by completeness.

2. Cardinal numbers, ordinal numbers and ordinal topological spaces

Definition 1. Let $\leq$ be a partial ordering on a set P. This order is said to be a well ordering if every nonempty subset A of P has a smallest element, i.e. there exists $a \in A$ such that $a \leq x$ for every $x \in A$.

The following Theorem is a consequence of (and, in fact, it is equivalent to) Zorn's Lemma (see, for instance, [10]).

Theorem 3. (Well-ordering Theorem) *Every set can be well ordered, i.e. if S is a set, then there exists some well-ordering on S.*

On well-ordered sets we can use induction arguments in a similar way as the induction principle is used for the set of natural numbers.

Definition 2. Let $(W, \leq)$ be a well-ordered set and let $a \in W$. The set $I(a) = \{x \in W : x \leq a,\ x \neq a\}$ is called the initial segment of W determined by a.

Theorem 4. (Principle of Transfinite Induction) *Let $(W, \leq)$ be a well-ordered set and let $A \subset W$ be such that $a \in A$ whenever $I(a) \subset A$. Then $A = W$.*

Proof. Assume that $W \setminus A \neq \emptyset$ and let a be the smallest member of $W \setminus A$. Then we have $I(a) \subset A$, so $a \in A$. But $a \in W \setminus A$. ∎

We shall recall now the definitions of ordinal and cardinal numbers. With an intuitive approach to set theory we can define these numbers by means of an equivalence relation. We can say that two linearly ordered sets A and B are order isomorphic if there exists an order isomorphism from A to B, i.e. a one-to-one function f from A onto B such that $x \leq y$ implies $f(x) \leq f(y)$. We can associate a symbol to any set which is order isomorphic to A. In particular, if A is well ordered we say that the order type of A is an ordinal, denoted $ordA$. However, we must be very careful because several well-known paradoxes can appear when we consider "too large" sets. For instance we cannot define the set of ordinal numbers as the quotient set of the "set of all well ordered set" under the above equivalence relation. To avoid this problem we can define directly ordinal and cardinal numbers, following von Neumann's definition, where an ordinal is the set of all preceding ordinals.

In this way, the first ordinal, zero, is the empty set 0. The second ordinal is the set $1 = \{0\}$ consisting of one element. The third ordinal is the set $2 = \{0, 1\} = \{0, \{0\}\}$, and so on. The first infinite ordinal is the set of all finite ordinals, i.e. $\omega = \{0, 1, 2, 3, ...\}$. The next is $\omega + 1 = \omega \cup \{\omega\} = \{1, 2, 3, ..., \omega\}$, and so on. An ordinal number α is called compact ordinal if $\alpha = \beta + 1$ for some ordinal number β. Otherwise α is said to be a limit ordinal.

Two sets A and B are said to have the same cardinality if there exists a one-to-one mapping from A onto B. It is clear that this is an equivalence relation which lets define card (A) in an intuitive approach. However we prefer to define cardinal numbers using ordinals: A cardinal number is an ordinal number which is the first ordinal between all ordinals with the same cardinality, i.e. an ordinal number α is cardinal if for every ordinal $\gamma \neq \alpha$ which has the same cardinality as α we have $\alpha \leq \gamma$ (equivalently: $\alpha \in \gamma$ or $\alpha \subset \gamma$).

As usual, if A is a linearly ordered set we denote $(a, b) = \{x \in A : a < x < b\}$, $(a, \infty) = \{x \in A : x > a\}$ and $(\infty, b) = \{x \in A : x < b\}$. It is easy to check that the open intervals $(a, b), (\infty, b), (a, \infty)$ form a base for a topology in A. This topology is called the interval topology of A. In particular, any ordinal number α is an ordered set. Thus we can consider α as a topological space. Note that a compact ordinal number $\beta \in \alpha$ is an isolated point and so a compact subset of α. However a limit ordinal $\beta \in \alpha$ is an accumulation point of α.

3. Scattered and perfect sets

Definition 3. Let M be a topological space and A a subset of M. The set A is said to be perfect if it is closed and has no isolated points, i.e. A is equal to the set of its own accumulation points. The space M is said to be scattered if it contains no perfect non-void subset.

If A is a subset of a topological space M, the derived set of A is the set $A^{(1)}$ of all accumulation points of A. If α is an ordinal number, we define the αth-derived set by transfinite induction:

$$A^{(0)} = A \qquad A^{(\alpha+1)} = (A^{(\alpha)})^{(1)} \qquad A^{(\lambda)} = \cap_{\alpha<\lambda} A^{(\alpha)}$$

if λ is a noncompact ordinal.

Theorem 5. (Cantor-Bendixson) *Let A be a topological space. Then there exists an ordinal number α such that $A^{(\alpha+1)} = A^{(\alpha)}$. Moreover $A^{(\alpha)} = \emptyset$ if and only if A is scattered.*

Proof. First, we will show that there exists an ordinal number α such that $A^{(\alpha)} = A^{(\alpha+1)}$. Indeed, let α be an ordinal number greater than card (A). Assume that $A^{(\beta)} \setminus A^{(\beta+1)} \neq \emptyset$ for every $\beta < \alpha$. Since the sets $E_\beta = A^{(\beta)} \setminus A^{(\beta+1)}$ are pairwise disjoint we can define an 1-1 mapping $f : \alpha \to \cup_{\beta<\alpha} E_\beta$ (choosing $f(\beta) \in E_\beta$ for every $\beta \in \alpha$). Hence, card $(A) \geq$ card $(\cup_{\beta<\alpha} E_\beta) \geq \alpha$, a contradiction. Since $A^{(\alpha)}$ is a perfect set, we have the second statement. ∎

PROPOSITION 1. *If M is a second-countable scattered topological space, then M is countable. In particular, every scattered compact metric space is countable.*

Proof. Assume that $\{G_n : n \in \omega\}$ is a countable basis for the topology of M. A point $x \in M$ will be called a condensation point of M if U is uncountable for each neighborhood U of x. Let $P = \{x \in M : x$ is a condensation point of $M\}$ and let $C = M \setminus P$. Let $x \in C$. Since x is not a condensation point of M there exists a neighborhood G_n such that $G_n \cap M$ is countable. Thus $C \subset \cup\{M \cap G_n : n \in \omega\}$ is countable. Furthermore, let x be a point in P and U a neighborhood of x. Since the set $U \cap M$ is uncountable and $U \cap C$ is countable we have that $U \cap P = (U \cap M) \setminus (U \cap C)$ is uncountable. Thus $P \subset P^{(1)}$. Moreover, P is a closed set. Indeed, assume that s belongs to $P^{(1)}$ and U is a neighborhood of x. Since $U \cap P \neq \emptyset$ we know that U is a neighborhood od a point in P. Hence U is uncountable. Thus $P = P^{(1)}$ which implies that P is a perfect set. Since M is scattered we know that $P = \emptyset$ and $M = C$ is countable.

The second assertion is now clear because every compact metric space is second countable. Indeed, for every $n \in \mathbb{N}$ there exist finite many balls $B_{i,n}, ..., B_{k(n),n}$ such that diam $B_{i,n} < 1/n$. Thus the collection of all $B_{i,n}$ is a countable base of open sets for the metric topology. ∎

THEOREM 6. (Mazurkiewicz-Sierpiński) *Every compact scattered first-countable space is homeomorphic to a countable compact ordinal.*

We do not give the proof of this classic theorem. We only recall that for any compact scattered set K we can define the characteristic system (α, m) of K, where α is the smallest ordinal such that $K^{(\alpha)} = \emptyset$ and m is the (finite) number of elements in $K^{(\alpha-1)}$. It can be proved that K is homeomorphic to $\omega^\alpha m + 1$ if (α, m) is the characteristic system of K.

Remark. It is necessary to be careful about the meaning of αm and ω^α for an ordinal number α. The ordinal αm must be understood as the smallest ordinal greater than $\{\alpha(m-1) + n : n < \omega\}$(where $\alpha 1$ is defined as α) , i.e.

$$\alpha m = \cup_{n<\omega}\{\alpha(m-1) + n\}.$$

Analogously we define $\omega^0 = 1$ and

$$\omega^\alpha = \cup_{\beta<\alpha}\omega^\beta$$

which is different of $\{f : \alpha \to \omega\}$. In particular, $\omega^\omega = \cup_{\beta<\omega}\omega^\beta$ is a countable union of countable sets and so it is a countable set.

The next results are concerned with existence of continuous functions whose rang is all of [0,1].

LEMMA 1. *Let S and T be compact Hausdorff spaces and f a continuous onto map from S to T. If P is a closed subset of T, there exists a minimal closed set $F \subset S$ such that $f(F) = P$.*

Proof. Let $\mathcal{K}$ be the class of all closed $K \subset S$ such that $f(K) = P$ (note that $f^{-1}(F) \in \mathcal{K}$). Then $\mathcal{K}$ is partially ordered under set inclusion. If $\mathcal{C}$ is a chain in $\mathcal{K}$, then $\mathcal{C}$ has the finite intersection property and hence $\emptyset \neq \cap\mathcal{C} = K_0$. Let $t \in P$ and for each $K \in \mathcal{C}$ let $s_K \in K$ with $f(s_K) = t$. Since $\mathcal{C}$ is ordered, $\{s_K\}$ is a net and has a subnet which converges to some $s \in K_0$. Clearly $f(s) = t$ and it follows that K_0 is a lower bound to $\mathcal{C}$. Thus by Zorn's lemma, $\mathcal{K}$ has a minimal element. ∎

LEMMA 2. *Let S and T be compact Hausdorff spaces and suppose T contains a perfect set. If $f : S \to T$ is a continuous onto map, then S has a perfect set.*

Proof. Let $P \subset T$ be a perfect set. By Lemma 1 there is a minimal closed set $F \subset S$ such that $f(F) = P$. We claim that F is perfect. Indeed, assume that $s \in F$ is an isolated point and $f(s) = t \in P$. Since P is a perfect set there exists a net t_i convergent to s such that $t \neq t_i$. Choose $s_i \in F$ such that $f(s_i) = t_i$. Then $\{s_i\}$ has a convergent subnet to $s' \neq s$ and $f(s') = t$. Thus $F' = F \setminus \{s\}$ is closed set such that $f(F') = P$ which is a contradiction because F is minimal. ∎

LEMMA 3. *Let T be a perfect compact Hausdorff space. If F is a closed set in T and U is an open set in T with $F \subsetneq U$, then there is an open set V with $F \subsetneq V \subset \overline{V} \subsetneq U$.*

Proof. Let t_1, t_2 be two distinct points in $U \setminus F$ (note that $U \setminus F$ is not a singleton because T is perfect). For each $t \in F$, $s \in \partial U$ choose disjoint open sets $V_{t,s}$ containing t, t_2, $V_{t,s} \subset U$ and $W_{t,s}$ containing t_1, s. There are finitely many $W_{t,s_1}, ..., W_{t,s_k}$ which cover ∂U. Thus $V_t = \cap_{i=1}^k V_{t,s_i}$ and $W_t = \cup_{i=1}^k W_{t,s_i}$ are open sets such that $t, t_2 \in V_t$ and $\{t_1\} \cup \partial U \subset W_t$. Then $V_t \subset \overline{V_t} \subset U \setminus \{t_1\}$. There are finitely many $V_{t_3}, ..., V_{t_n}$ which cover F and $V = V_{t_3} \cup ... \cup V_{t_n}$ will suffice. ∎

Lemma 4. *Let T be a compact Hausdorff space, D a dense set in $[0,1]$, $\{A_r\}(r \in D)$ a family of nonempty open sets in T such that if r, s are in D with $r < s$ then $\overline{A_r} \subsetneqq A_s$. Then there is a continuous function f from T onto $[0,1]$.*

Proof. Define

$$f(t) = \sup\{r \in D : t \notin U_r\}$$

for $t \in T$ where $\sup \emptyset = 0$. We claim that f is continuous. Indeed, let $f(t) = c \in (0,1)$. For an arbitrary $\epsilon > 0$ choose $r_1, r_2 \in D$ such that $|r_1 - r_2| < \epsilon/2$ and $r_1 < c < r_2$. Then $t \in A_{r_2}$ and $t \notin A_{r_1}$. Take $r_3 \in D \cap (r_1 - \epsilon/2, r_1)$. Since $\overline{A_{r_3}} \subset A_{r_1}$ we have $t \notin \overline{A_{r_3}}$. Thus $t \in A_{r_2} \setminus \overline{A_{r_3}}$ which is an open set. If $s \in A_{r_2} \setminus \overline{A_{r_3}}$ we have $r_3 < f(s) < r_2$ which implies $|f(s) - f(t)| < \epsilon$. Minor modifications of these arguments let prove the continuity at 0 and 1.

We will prove now that f is onto. Choose $c \in (0,1)$ and r_1, r_2 in D such that $r_1 < c < r_2 < r_1 + 1/n$. If $t_n \in A_{r_2} \setminus A_{r_1}$ we have $r_1 \leq f(t_n) < r_2$ which implies $|f(t_n) - c| < 1/n$. Since $\{t_n\}$ has a convergent subsequence we obtain that $c = f(t)$ for some $t \in T$. Furthermore, the finite intersection property implies that $\cap_{t \in D} \overline{U_t} \neq \emptyset$ and $\cap_{t<1} {U_t}^c \neq \emptyset$ and we have $f(s) = 0$ is $s \in \cap_{t \in D} \overline{U_t}$ and $f(s) = 1$ if $s \in \cap_{t<1} {U_t}^c$. ∎

Theorem 7. *Let T be a compact Hausdorff space. Then T has a perfect set if and only if there is a continuous map of T onto $[0,1]$.*

Proof. The necessity is Lemma 2. By the Tietze Extension Theorem it suffices to assume that T is a perfect set for the sufficiency. Let D be the set of all dyadic rationals in [0,1]. Choose V_0 as any closed proper subset of T and $V_1 = T$. We can construct V_r $(r \in D)$ by induction. By Lemma 3 there is an open set $V_{1/2}$ such that $V_0 \subsetneqq V_{1/2} \subset \overline{V_{1/2}} \subsetneqq V_1$. Employing Lemma 3 again, we can construct open sets $V_{1/4}; V_{3/4}$ such that

$$V_0 \subsetneqq V_{1/4} \subset \overline{V_{1/4}} \subsetneqq V_{1/2} \subset \overline{V_{1/2}} \subsetneqq V_{3/4} \subset \overline{V_{3/4}} \subsetneqq V_1.$$

By induction, an open set V_r can be constructed for any dyadic rational in [0,1] satisfying the assumption in Lemma 4. The result is now a consequence of Lemma 4. ∎

4. Some properties of the continuous function spaces

The classical Banach-Stone Theorem states that the spaces $C(X)$ and $C(Y)$ are isometric if and only if X and Y are homeomorphic. We only need a much more elementary result:

Theorem 8. *Let M, N be topological spaces such that there exist a continuous mapping from M onto N. Then $C(N)$ can be isometrically embedded in $C(M)$.*

Proof. Assume that $\phi : M \to N$ is an onto continuous mapping. Define $T : C(N) \to C(M)$ by $T(f) = f \circ \phi$. It is easy to check that T is an isometry. ∎

Corollary 1. *Let K be a compact metric non-scattered space. Then $C(K)$ contains isometrically $C([0,1])$.*

Proof. Is a straightforward consequence of Theorems 7 and 8. ∎

From Corollary 1 we know that $C(K)$ contains isometrically $L^1([0,1])$ when K is not scattered. In fact, a stronger result can be proved:

Theorem 9. *Let K be a metric compact space. Then $C(K)$ contains isometrically $L_1([0,1])$ if and only if K is non-scattered.*

Proof. Assume that K is a scattered compact metric space. We claim that the dual space $C(K)^*$ can be identified with ℓ^1. If we assume that $L^1([0,1])$ is isometrically embedded in $C(K)$ we obtain that its dual space $L^\infty([0,1])$ is a quotient space of a separable space, which is a contradiction.

It is easy to prove that $C(K)^*$ can be identified with ℓ^1. Indeed, Riesz's Theorem implies that $C(K)^*$ is the set of regular measures on K. Since K is countable, say $K = \{t_n : n \in \mathbb{N}\}$, (Proposition 1) μ is purely atomic. Denote $a_n = \mu(\{t_n\})$. It is easy to check that $\mu = \sum_{n=1}^{\infty} a_n \delta_{t_n}$ where $\delta_t(A) = 1$ if $t \in A$ and $\delta_t(A) = 0$ otherwise. Now, the identification $\mu \in C(K)^* \leftrightarrow (a_n) \in \ell_1$ proves our claim. ∎

Remark. The metrizability assumption in Theorem 9 is not necessary. Indeed, using more technical arguments Pelczynski and Semadeni [16] proved that the statement of this theorem still holds if K is a compact Hausdorff topological space.

The classical Tietze Extension Theorem Theorem states that every bounded continuous scalar-valued function on a closed subset L of a normal space K can be extended to a bounded continuous scalar-valued function on the whole K. This fact lets define an embedding from $C(L)$ into $C(K)$. However, this extension is not, in general, linear and we cannot consider $C(L)$ as a subspace of $C(K)$. When K is a metric space, we can improve the Tietze-Urysohn theorem, obtaining a linear embedding from $C(L)$ into $C(K)$.

THEOREM 10. (Borsuk-Dugundji) *Let L be a closed nonempty subset of a metric space K. Then there exists a linear extension $\Lambda : C(L) \to C(K)$ such that $\|\Lambda\| = 1$.*

Proof. For each $t \in K \setminus L$, let $V_t = B(t, \frac{1}{3}d(t, L))$. Since this family is an open cover of $K \setminus L$, by the Stone Theorem there exists a locally finite refinement $(W_i)_{i\in I}$ of $(V_t)_{t\in K\setminus L}$, i.e. for any $i \in I$ there exists $t \in K \setminus L$ such that $W_i \subset V_t$. Assume that $(p_i)_{i\in I}$ is a partition of unity subordinate to the refinement. For each $i \in I$, select a w_i in W_i and a v_i in L such that $d(v_i, w_i) < 2d(w_i, L)$. Let $f \in C(L)$. We define $(\Lambda f)(t) = f(t)$ if $t \in L$ and $(\Lambda f)(t) = \sum_{i\in I} f(v_i)p_i(t)$ if $t \in K \setminus L$. Since each p_i is a continuous function and the sum is locally finite, it is clear that Λf is separately continuous on L and $K \setminus L$. We have to prove that it is continuous at any point of ∂L. Let $t_0 \in \partial L$ and $\epsilon > 0$. Since f is continuous on L there exists $\delta > 0$ such that $|f(t_0) - f(t)| < \epsilon$ if $t \in L$, $d(t, t_0) < \delta$. Assume that $s \in K \setminus L$ and $d(s, t_0) < \delta/6$. We will prove that $|f(s) - f(t_0)| < \epsilon$. Since the set $I_0 = \{i \in I : s \in W_i\}$ is finite and $p_i(t) = 0$ if $i \notin I_0$ we have

$$(\Lambda f)(s) = \sum_{i\in I_0} f(v_i)p_i(s).$$

Let $i \in I_0$. There exists $t \in K \setminus L$ such that $W_i \subset V_t$. Therefore $s \in V_t$ and

$$d(t, L) \le d(t, t_0) \le d(t, s) + d(s, t_0) \le \frac{1}{3}d(t, L) + \frac{\delta}{6}.$$

These inequalities imply that $d(t, L) < \delta/4$ and $d(t, t_0) < \delta/4$. Hence

$$d(w_i, t_0) \le d(w_i, s) + d(s, t_0) < \frac{1}{3}d(t, L) + \frac{\delta}{4} < \frac{\delta}{3}$$

and

$$d(v_i, t_0) \le d(v_i, w_i) + d(w_i, t_0) \le 2d(w_i, L) + \frac{\delta}{3} \le 2d(w_i, t_0) + \frac{\delta}{3} < \delta.$$

Hence $|f(v_i) - f(t_0)| < \epsilon$ which implies

$$\left| \sum_{i \in I_0} f(v_i) p_i(s) - f(t_0) \right| < \epsilon.$$

Since p_i and v_i depend on $(W_i)_{i \in I}$ but do not depend on f, the operator Λ is linear. Furthermore, for any t we have that $(\Lambda f)(t)$ is in the convex hull of $f(L)$ which implies $\|\Lambda f\| \leq \|f\|$. ∎

5. Some techniques in fixed point theory

According to Alspach's example and Theorem 9 we know that $C(M)$ fails to have the FPP if M is not scattered. In this section we will prove that $C(M)$ enjoys the FPP when M is a scattered set such that $M^{(\omega)} = \emptyset$. We denote by $\ell_\infty(X)$ (respectively $c_0(X)$) the linear space of all bounded sequences (respectively all sequences convergent to zero) in the Banach space X. By $[X]$ we denote the quotient space $\ell_\infty(X)/c_0(X)$ endowed with the norm $\|[z^n]\| = \limsup_n \|z^n\|$ where $[z^n]$ is the equivalent class of $(z^n) \in \ell_\infty(X)$. By identifying $x \in X$ with the class $[(x, x, \cdots)]$ we can consider X as a subset of $[X]$. If C is a subset of X we can define the set $[C] = \{[z^n] \in [X] : z^n \in C \text{ for every } n \in \mathbb{N}\}$. If T is a mapping from C into C, then $[T] : [C] \to [C]$ given by $[T]([x^n]) = [Tx^n]$ is a well defined mapping. Notice that if (x^n) is an approximated fixed point sequence of T (that is: $\lim_{n\to\infty} \|x^n - Tx^n\| = 0$) its equivalent class $[x^n]$ is a fixed point for $[T]$. We first recall a "classical" result in Metric Fixed Point Theory: Goebel-Karlovitz' Lemma. We need some previous definitions and results.

Definition 4. Let X be a Banach space, A a bounded subset of X and B an arbitrary subset of X. The Chebyshev radius of A with respect to B is defined by

$$r(A, B) = \inf\{\sup\{\|x - y\| : x \in A\} : y \in B\}$$

where we write $r(A)$ instead of $r(A,\ \mathrm{co}(A))$. The Chebyshev center of A with respect to B is defined by

$$Z(A, B) = \{y \in B : \sup\{\|x - y\| : x \in A\} = r(A, B)\}$$

where we write $Z(A)$ instead of $Z(A,\ \mathrm{co}(A))$.

Remark. Roughly speaking, we can say that the Chebyshev radius $r(A,B)$ is the radius of the smallest ball centered at a point in B and covering the set A, the Chebyshev center $Z(A,B)$ being the set formed by all centers of these smallest balls. However, since the infimum appearing in the definition is not, necessarily attained, the set $Z(A,B)$ can be empty. In opposition, if for every $\varepsilon > 0$ we consider the set

$$Z_\varepsilon(A,B) = \{y \in B : r(A,y) \le r(A,B) + \varepsilon\},$$

then $Z_\varepsilon(A,B)$ is a nonempty, convex, bounded and closed set if B satisfies the same properties. Thus, $Z_\varepsilon(A,B)$ is convex, nonempty and weakly compact if so is B. Since

$$\bigcap_{\varepsilon>0} Z_\varepsilon(A,B) = Z(A,B),$$

the finite intersection property implies that $Z(A,B)$ is nonempty when B is a convex and weakly compact set.

Definition 5. A bounded convex closed subset A of a Banach space X is said to be diametral if $\operatorname{diam}(A) = r(A)$. Equivalently, if $Z(A) = A$.

Definition 6. The asymptotic radius and center of a sequence $\{x_n\}$ in a Banach space X are defined by:

$$r_a(\{x_n\}, B) = \inf\{\limsup_{n\to\infty} \|x_n - y\| : y \in B\},$$

$$Z_a(\{x_n\}, B) = \{y \in B : \limsup_{n\to\infty} \|x_n - y\| = r_a(\{x_n\}, B)\},$$

where B is an arbitrary subset of X. Whenever $B = \overline{\text{co}}(\{x_n\})$ we will write $r_a(\{x_n\})$ and $Z_a(\{x_n\})$ for $r_a(\{x_n\},\ \text{co}(\{x_n\}))$ and $Z_a(\{x_n\},\ \text{co}(\{x_n\}))$, resp.

If we assume that C is a convex bounded closed subset of a Banach space X and $T : C \to C$ a nonexpansive mapping, it is easy to prove that an approximated fixed point sequence exists in C.

Proposition 2. *Let K be a weakly compact convex subset of a Banach space X, and $T : K \to K$ be a nonexpansive mapping. Assume that K is minimal for T, that is, no closed convex bounded proper subset of K is invariant for T. If $\{x_n\}$ is an approximated fixed point sequence in K, then*

$$Z_a(\{x_n\}, K) = K.$$

Proof. Let

$$Z_{a,\varepsilon}(\{x_n\},K)=\{y\in K:\limsup_{n\to\infty}\|x_n-y\|\le r_a(\{x_n\},K)+\varepsilon\}.$$

It is easy to check that $Z_{a,\varepsilon}(\{x_n\},K)$ is nonempty, closed, convex and invariant for T. Thus $Z_{a,\varepsilon}(\{x_n\},K)=K$ and $Z_a(\{x_n\},K)=\bigcap_{\varepsilon>0}Z_{a,\varepsilon}(\{x_n\},K)=K$. ∎

LEMMA 5. (Goebel-Karlovitz) *Let K be a convex weakly compact subset of a Banach space X, and $T:K\to K$ a nonexpansive mapping. Assume that K is minimal with these properties and let $\{x_n\}$ be an approximated fixed point sequence for T in K, i.e. $\lim_n\|x_n-Tx_n\|=0$. Then*

$$\lim_{n\to\infty}\|y-x_n\|=\ \textit{diam}(K)$$

for every $y\in K$.

Proof. We claim that $Z(K)=K$ which implies that K is a diametral set. To prove that, it suffices to check that $Z(K)$ is a convex weakly compact subset of K which is invariant under T. Since $\overline{\text{co}}(T(K))\subset K$ and $T(\overline{\text{co}}(T(K)))\subset T(K)\subset\overline{\text{co}}(T(K))$ we have $K=\overline{\text{co}}(T(K))$. Let $x\in Z(K)$, that is, $r(K,x)=r(K)$. For every $y\in K$ we have $\|Ty-Tx\|\le\|y-x\|\le r(K)$. Thus $T(K)$ is contained in the closed ball $\overline{B}(Tx,r(K))$ which implies that $\overline{\text{co}}(T(K))=K\subset\overline{B}(Tx,r(K))$. Hence $r(K,Tx)\le r(K)$ which means $Tx\in Z(K)$. Thus $Z(K)$ is a convex weakly compact subset of K and is invariant under T. The minimality of K implies $Z(K)=K$.

We claim that $\limsup_{n\to\infty}\|y-x_n\|=\ \text{diam}(K)$ for every $y\in K$. Indeed, assume that there exists $y\in K$ such that $\limsup_{n\to\infty}\|y-x_n\|<\ \text{diam}(K)$. Denote $r=\limsup_{n\to\infty}\|y-x_n\|$, $d=\ \text{diam}(K)$ and consider the collection $\{\overline{B}(z,(r+d)/2)\cap K:z\in K\}$. Choose an arbitrary positive number ε such that $\varepsilon<(d-r)/2$. From Proposition 2 we know that $\limsup_{n\to\infty}\|x_n-z\|=r$ for every $z\in K$. Thus, for every finite subset $\{z_1,...,z_k\}$ of K there exists a nonnegative integer N such that $\|x_N-z_i\|\le r+\varepsilon<(r+d)/2$ for $i=1,...,k$. Hence x_N belongs to $\bigcap_{i=1}^k\overline{B}(z_i,(r+d)/2)$. The weak compactness of K implies the existence of $x_0\in\bigcap_{z\in K}\overline{B}(z,(r+d)/2)\cap K$ and this point is not diametral because

$$\sup_{z\in K}\|z-x_0\|<\frac{r+d}{2}<d=\ \text{diam}(K).$$

This contradiction proves the claim. If $\liminf_{n\to\infty}\|y-x_n\|<\ \text{diam}(K)$ for some $y\in K$ there exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $\limsup_{n\to\infty}\|y_n$

$-y\| = \liminf_{n\to\infty} \|x_n - y\| <$ diam(K), which is a contradiction according to the claim applied to the sequence $\{y_n\}$ which is again an approximated fixed point sequence. ∎

LEMMA 6. (Lin) *Let X be a Banach space and K be a weakly compact convex subset of X. Let $T : K \to K$ be a nonexpansive map and suppose K is a minimal invariant set for K. If $[W]$ is a nonempty closed convex subset of $[K]$ which is invariant under $[T]$ then*

$$\sup\{\|[w^n] - [x]\| : [w^n] \in [W]\} = \textit{diam}(K)$$

for every $x \in K$.

Proof. We claim that $\limsup_{m\to\infty} \|[w^n]_m - [x]\| =$ diam(K) for every $x \in K$, $\{[w^n]_m\}$ being an approximated fixed point sequence for $[T]$ in $[W]$, and this claim clearly proves the lemma, because $[T]$ is also nonexpansive and we can find a sequence with this property in $[W]$. We denote a representative of the n-th element of the sequence $\{[w^n]_m\}$ as w^n_m and we write $d = \limsup_{m\to\infty} \|[w^n]_m - [x]\|$ and $\delta_m = \|[w^n]_m - [T][w^n]_m\|$. Thus $\lim_{m\to\infty} \delta_m = 0$. We shall prove $d =$ diam(K). Since we obviously have $d \le$ diam(K) we only need to prove the inequality $d \ge$ diam(K). To this end we construct a point $[w^k] \in [K]$ such that

(a) $[T][w^k] = [w^k]$.

(b) $\|[w^k] - [x]\| \le d$.

Thus Lemma 5 will imply

$$\text{diam}(K) = \lim_{k\to\infty} \|w^k - x\| = \|[w^k] - x\| \le d.$$

Choose a sequence $\{\varepsilon_k\} \to 0$. For a fixed $k \in \mathbb{N}$ a positive integer m_k exists such that $\|[w^n]_m - x\| \le d + \varepsilon_k$ if $m \ge m_k$. Since

$$\limsup_{n\to\infty} \|w^n_{m_k} - x\| \le d + \varepsilon_k \text{ and } \limsup_{n\to\infty} \|w^n_{m_k} - Tw^n_{m_k}\| = \delta_{m_k}$$

we can choose a large enough n_k, such that

$$\|w^{n_k}_{m_k} - x\| \le d + 2\varepsilon_k \text{ and } \|w^{n_k}_{m_k} - Tw^{n_k}_{m_k}\| \le \delta_{m_k} + \varepsilon_k.$$

Now consider the sequence $[w^k] = [w^{n_k}_{m_k}] \in [K]$. It is clear that $[w^k]$ satisfies (a) and (b). ∎

Definition 7. Let X be a Banach space and p a positive integer. We say that X is p-weakly orthogonal if for every weakly null sequence $(x_n) \subset X$ we have

$$\liminf_{n_p} ... \liminf_{n_1} \| |x_{n_p}| \wedge ... \wedge |x_{n_1}| \| = 0.$$

We say that X is ω-weakly orthogonal if for every weakly null sequence $(x_n) \subset X$ there exists some $p \in \mathbb{N}$ such that

$$\liminf_{n_p} ... \liminf_{n_1} \| |x_{n_p}| \wedge ... \wedge |x_{n_1}| \| = 0.$$

Theorem 11. *Assume that $C(M)$ is w-weakly orthogonal, then $C(M)$ has the FPP.*

Proof. By contradiction we assume that $C(M)$ fails to have the FPP. Thus we can find a convex weakly compact set C of $C(M)$ with $\operatorname{diam}(C) = 1$ and such that C is minimal invariant for a nonexpansive mapping T. Let (x_n) be an approximated fixed point sequence that, by translation, we can consider that is weakly null. Since X is ω-weakly orthogonal there exists some $p \in \mathbb{N}$ (depending on (x_n)) such that

$$\liminf_{n_p} ... \liminf_{n_1} \| |x_{n_p}| \wedge ... \wedge |x_{n_1}| \| = 0. \tag{1}$$

Next, we are going to construct $(x_{n_s(1)})_{s\in\mathbb{N}}$, $(x_{n_s(2)})_{s\in\mathbb{N}}$, ..., $(x_{n_s(p)})_{s\in\mathbb{N}}$ subsequences of (x_n) satisfying the following properties:

$$\lim_s \|x_{n_s(i)} - x_{n_s(j)}\| = 1; \qquad \text{for every } i, j \in \{1, ...p\},\ i \neq j \tag{2}$$

$$\lim_s \| |x_{n_s(1)}| \wedge |x_{n_s(2)}| \wedge ... \wedge |x_{n_s(p)}| \| = 0. \tag{3}$$

Indeed, fix $s \in \mathbb{N}$. From (1) we can find $n_s(1) \in \mathbb{N}$ such that

$$\liminf_{n_2} \dots \liminf_{n_p} \| |x_{n_s(1)}| \wedge |x_{n_2}| \wedge \dots \wedge |x_{n_p}| \| < \frac{1}{s}$$

¿From Goebel-Karlovitz's Lemma we know that $\lim_n \|x_n - x_{n_s(1)}\| = 1$ so we can find $n_s(2)$ large enough such that

$$\|x_{n_s(2)} - x_{n_s(1)}\| \geq 1 - \frac{1}{s}.$$

Given $k \in \{1,...,p-1\}$ suppose that we have found $n_s(1), n_s(2), \cdots, n_s(k)$ positive integers such that

$$\|x_{n_s(i)} - x_{n_s(j)}\| \geq 1 - \frac{1}{s}; \qquad i,j \in \{1,...,k\},\ i \neq j$$

$$\liminf_{n_{k+1}} ... \liminf_{n_p} \||x_{n_s(1)}| \wedge ... \wedge |x_{n_s(k)}| \wedge |x_{n_{k+1}}| \wedge ... \wedge |x_{n_p}|\| < \frac{1}{s}$$

From Goebel-Karlovitz's Lemma we know that $\lim_n \|x_n - x_{n_s(i)}\| = 1$ for every $i \in \{1,...,k\}$ so we can find $n_s(k+1)$ such that $\|x_{n_s(k+1)} - x_{n_s(i)}\| \geq 1 - \frac{1}{s}$ for every $i \in \{1,...,k\}$ and

$$\liminf_{n_{k+2}} ... \liminf_{n_p} \||x_{n_s(1)}| \wedge ... \wedge |x_{n_s(k+1)}| \wedge |x_{n_{k+2}}| \wedge ... \wedge |x_{n_p}|\| < \frac{1}{s}$$

Thus, by induction, we can construct $n_s(1), n_s(2), ...n_s(p)$ positive integers such that

$$\|x_{n_s(i)} - x_{n_s(j)}\| \geq 1 - \frac{1}{s}, \quad i,j \in \{1,...,p\}, i \neq j \tag{4}$$

and

$$\||x_{n_s(1)}| \wedge |x_{n_s(2)}| \wedge ... \wedge |x_{n_s(p)}|\| < \frac{1}{s}. \tag{5}$$

Using inductively the above argument for $s = 1, 2, ...$, we construct subsequences $(x_{n_s(1)})_{s\in\mathbb{N}}$, $(x_{n_s(2)})_{s\in\mathbb{N}}$, ..., $(x_{n_s(p)})_{s\in\mathbb{N}}$ of (x_n). It is clear that these subsequences are approximated fixed point sequences and from (4) and (5) we deduce properties (2) and (3).

Consider now the space $[X]$ and define $[T] : [C] \to [C]$. We denote by $[x_1]$,..., $[x_p]$ the equivalence class of $(x_{n_s(1)})_{s\in\mathbb{N}}$, $(x_{n_s(2)})_{s\in\mathbb{N}}$, ..., $(x_{n_s(p)})_{s\in\mathbb{N}}$ respectively. Then $[x_1]$,..., $[x_p]$ are fixed points for $[T]$ and $\|[x_i] - [x_j]\| = 1$ for every $i,j \in \{1,...,p\}$, $i \neq j$. Define the following subset of $[X]$

$$[W] := \left\{ [t^n] \in [C] : \|[t^n] - [x_i]\| \leq \frac{p-1}{p} \text{ for every } i \in \{1,...,p\} \right\}$$

The set $[W]$ is nonempty since $\frac{1}{p}\sum_{i=1}^{p}[x_i] \in [W]$. It is also clear that $[W]$ is convex, closed and $[T]$-invariant.

Fix $[w^s] \in [W]$. Using the triangular inequality, it is not difficult to check that for every $s \in \mathbb{N}$ we have

$$\|w^s\| \leq \left(\|w^s - x_{n_s(1)}\| \vee ... \vee \|w^s - x_{n_s(p)}\|\right) + \||x_{n_s(1)}| \wedge ... \wedge |x_{n_s(p)}|\|$$

Taking limsup as $s \to \infty$ we obtain

$$\|[w^s]\| = \limsup_{s\to\infty} \|w^s\| \le \|[w^s]-[x_1]\| \vee \|[w^s]-[x_2]\| \vee ... \vee \|[w^s]-[x_p]\| \le \frac{p-1}{p}.$$

Thus $\sup\{\|[w^s]\| : [w^s] \in [W]\} \le \frac{p-1}{p} < 1$, which contradicts Lin's Lemma and X has the FPP. ∎

LEMMA 7. *Let K be a compact set and assume that there is some $p \in \mathbb{N}$ with $K^{(p)} = \emptyset$. Then $C(K)$ is p-weakly orthogonal.*

Proof. We use an induction argument on p. It is clear that the result holds if $p = 1$. Assume that the statement in Lemma 7 holds for any compact set L such that $L^{(p-1)} = \emptyset$ and let K be a compact set with $K^{(p)} = \emptyset$. Let (f_n) be a weakly null sequence in $C(K)$.

Since $K^{(p-1)}$ is a finite set, we can write $K^{(p-1)} = \{t_1, ..., t_m\}$. Fix a positive integer n_p and choose open neighborhoods V_i of t_i, $i = 1, ..., m$, such that $|f_{n_p}(t) - f_{n_p}(t_i)| < \frac{1}{n_p}$ if $t \in V_i$. Set $L = K \setminus \cup_{i=1}^m V_i$, which is a compact set with $L^{(p-1)} \subset K^{(p-1)} \cap L = \emptyset$. Consider the weakly null sequence $(g_n) \subset C(L)$ defined by $g_n(t) = f_n(t)$ for every $t \in L$. Therefore, according to the induction hypotheses we know that

$$\liminf_{n_{p-1}} ... \liminf_{n_1} \||g_{n_{p-1}}| \wedge ... \wedge |g_{n_1}|\| = 0.$$

Let $t \in K$. If $t \in L$ we have

$$|f_{n_p}| \wedge |f_{n_{p-1}}| \wedge ... \wedge |f_{n_1}|(t) \le |f_{n_{p-1}}| \wedge ... \wedge |f_{n_1}|(t) \le \||g_{n_{p-1}}| \wedge ... \wedge |g_{n_1}|\|.$$

If $t \in K \setminus L = \cup_{i=1}^m V_i$ we also have

$$|f_{n_p}| \wedge |f_{n_{p-1}}| \wedge ... \wedge |f_{n_1}|(t) \le |f_{n_p}|(t) \le \max_{i=1,...,m} |f_{n_p}(t_i)| + \frac{1}{n_p}.$$

Taking supremum we have

$$\||f_{n_p}| \wedge ... \wedge |f_{n_1}|\| \le \max\left\{\||g_{n_{p-1}}| \wedge ... \wedge |g_{n_1}|\|, \max_{i=1,...,m} |f_{n_p}(t_i)| + \frac{1}{n_p}\right\}$$

Finally, taking limits we obtain

$$\liminf_{n_p} ... \liminf_{n_1} \||f_{n_p}| \wedge ... \wedge |f_{n_1}|\| \le \lim_{n_p} \left(\max_{i=1,...,m} |f_{n_p}(t_i)| + \frac{1}{n_p}\right) = 0$$

which implies that $C(K)$ is p-weakly orthogonal. ∎

Corollary 2. *Let K be a compact metrizable set such that $K^{(\omega)} = \emptyset$. Then $C(K)$ has the FPP.*

Remark. When K is a metric compact space with infinitely many points, it is known (see [3]) that $K^{(w)} = \emptyset$ if and only if $C(K)$ is isomorphic to c_0. Thus, we can state the above theorem in the equivalent form:

Corollary 3. *Assume that $C(K)$ is isomorphic to c_0. Then $C(K)$ has the w-FPP.*

This result is, in some sense, surprising, because an isomorphic property implies the existence of fixed points for nonexpansive mappings which is, clearly, an isometric property. (Recall [7] that $L_1[0,1]$, which fails to have the w-FPP, can be renormed in such a way that the new space has normal structure (which implies the w-FPP) and this new norm is as close (in the Banach-Mazur distance) to the original norm as wanted). Moreover, it was known [4] that any Banach space X isomorphic to c_0 such that the Banach-Mazur distance between X and c_0 is less than 2, has the w-FPP. However, in Corollary 3 we prove the w-FPP for a class of spaces which are isomorphic to c_0 and the Banach-Mazur distance is arbitrarily large. Indeed, if $K^{(p)} \neq \emptyset$ and $K^{(p+1)} = \emptyset$, then the Banach-Mazur distance $d(c_0, C(K))$ is greater than p (see [3, Remark 1]).

6. Characterization of the ω-weak orthogonality in $C(K)$

We have proved that $C(K)$ has the w-FPP if $K^{(\omega)} = \emptyset$. Since $C(K)$ fails to have the w-FPP if K is not scattered, a natural question is the following: Does $C(K)$ have the w-FPP if K is a scattered set and $K^{(\omega)} \neq \emptyset$? (Recall that, in this case, according to the Mazurkiewicz-Sierpiński Theorem the first ordinal α such that $K^{(\alpha)} = K^{(\alpha+1)}$ satisfies $\omega \leq \alpha < \omega_1$). We do not know the answer, but we can prove that our methods to prove the w-FPP do not work in this setting. Nominally:

Theorem 12. *Let K be a compact metrizable space. Then,*

(i) *$C(K)$ is p-weakly orthogonal if and only if $K^{(p)} = \emptyset$.*
(ii) *$C(K)$ is ω-weakly orthogonal if and only if $K^{(\omega)} = \emptyset$.*

Proof. To prove (i) we need to check that $C(K)$ is not p-weakly orthogonal if $K^{(p)} \neq \emptyset$. Assume that K satisfies this condition. So the characteristic system of K is (α, m) with $\alpha \geq p$. Thus K has a subset which is homeomorphic to $A_p = [\omega^{(p-1)}+1, \omega^{(p)}]$.We will define a weakly null sequence of $\{0,1\}$-valued functions in $C(A_p)$ such that for any ordered set $\{m_1 < ... < m_p\}$ of p positive integers, we have

$$\||f_{m_1}| \wedge ... \wedge |f_{m_p}|\| = 1$$

which implies that $C(A_p)$ is not p-weakly orthogonal. To simplify the notation, we shall write $\langle m_1, ..., m_k\rangle$ to denote the ordinal $\omega^{p-1}m_1 + ... + \omega^{p-k}m_k$. Consider the subset B_p of A_p defined by

$$B_p = \{\alpha = \langle m_1, ..., m_k\rangle : k = 1, 2, ..., p\, ,\, 1 < m_1 < ... < m_k\} \cup \{\omega^p\}.$$

We claim that B_p is a closed subset of A_p. Indeed, assume that $t = \lim_{s\to\infty} t_s$ where $t_s = \langle m_1(s), ..., m_{k(s)}(s)\rangle \in B_p$. There is a subsequence, denoted again t_s such that for any $i = 1, ..., p$ we have either $\lim_s m_i(s) = \infty$ or $m_i(s)$ is a constant, say m_i. If for every $i = 1, ..., p$ we have the second alternative, the result is clear. Otherwise, assume that $j = \min\{i : \lim_s m_i(s) = \infty\}$. Thus, $t = \langle m_1, ..., m_{j-1}, m_j + 1\rangle \in B_p$ if $j > 1$, or $t = \omega^p$ if $j = 1$.

We define a sequence $\{h_n\}$, $n > 1$ in $C(B_p)$ in the following way: $h_n(\langle m_1, ..., m_k\rangle) = 1$ if $n \in \{m_1, ..., m_{k-1}, m_k - 1\}$ and $h_n(t) = 0$ otherwise. We claim that h_n is a continuous function. It suffices to prove that $B_{n,p} = \{\langle m_1, ..., m_k\rangle) \in B_p : n \in \{m_1, ..., m_{k-1}, m_k - 1\}\}$ is an open and closed subset of B_p. To prove that $B_{n,p}$ is a closed subset of B_p, assume that $t_s = \langle m_1(s), ..., m_{k(s)}(s)\rangle$ is a sequence in $B_{n,p}$ (i.e. $m_1(s) < m_2(s) < ... < m_{k(s)}(s)$ and $n \in \{m_1(s), ..., m_{k(s)-1}, m_{k(s)}(s) - 1\}$) convergent to $t = \langle m_1, ..., m_k\rangle \in B_p$. Without loss of generality, we can assume that $m_i(s) = m_i$ for any s, $i = 1, ...j-1$ and $m_j(s) \to_s \infty$. Thus $n < m_i(s)$ for any $i \geq j$ and s large enough which implies that n belongs to $\{m_1, ..., m_{j-1}\}$ and $t = \langle m_1, ..., m_{j-1} + 1\rangle$ belongs to $B_{n,p}$. On the other hand, to prove that $B_{n,p}$ is an open subset of B_p, assume that $t_s \to_s t = \langle m_1, ..., m_k\rangle \in B_{n,p}$, where $t_s \in B_p$. For s large enough we have $t_s = t$ or $t_s = \langle m_1, ..., m_{k-1}, m_k - 1, m_{k+1}(s), ...\rangle \in B_{n,p}$.

It is easy to check that the sequence $\{h_n\}$ is weakly null. Furthermore $\||h_{n_1}| \wedge ... \wedge |h_{n_p}|\| = 1$ for any choice of distinct positive integers $n_1, ..., n_p$ which implies that $C(B_p)$ is not p-weakly orthogonal. By Borsuk-Dugundji theorem there is a linear extension $U_p : C(B_p) \to C(A_p)$. The sequence $\{g_n^{(p)}\}$, where $g_n^{(p)}$ is $U_p(h_n)$, is also weakly null and satisfies $g_n^{(p)}(\langle m_1, ..., m_j\rangle) = 1$ if $n \in \{m_1, ..., m_{j-1}, m_j - 1\}$ and $\langle m_1, ..., m_j\rangle \in B_p$.

To prove (ii) assume $K^{(w)} \neq \emptyset$. Then K contains homeomorphically the set

$$L = \omega^\omega + 1 = \bigcup_{p=1}^{\infty} A_p.$$

Fix $p \in \mathbb{N}$ and construct a weakly null sequence $(g_n^{(p)})_{n\in\mathbb{N}} \subset C(A_p)$ as in the proof of (i). Since every A_p is closed and open in L and the sets A_p are pairwise disjoint we can extend $g_n^{(p)}$ to the set L in the way $g_n^p(x) = 0$ if $x \in K \setminus A_p$ and we still have continuous functions in $C(L)$. Now define

$$f_n := \sum_{p=1}^{n} g_n^{(p)}$$

The sequence $(f_n)_{n\in\mathbb{N}}$ is a normalized weakly null sequence in $C(L)$. Furthermore, for every $p \in \mathbb{N}$ we have

$$\liminf_{n_1} ... \liminf_{n_p} \||f_{n_1}| \wedge ... \wedge |f_{n_p}|\|_{C(L)}$$

$$\geq \liminf_{n_1} ... \liminf_{n_p} \||g_{n_1}^{(p)}| \wedge ... \wedge |g_{n_p}^{(p)}\|_{C(A_p)} = 1$$

which implies that $C(L)$ is not ω-weakly orthogonal. Using again the Borsuk-Dugundji Theorem we obtain that $C(K)$ is not ω-weakly orthogonal either. ∎

Remark. It can be proved that $C(K)$ is not ω-weakly orthogonal if $K^{(\omega)} \neq \emptyset$ using the sequence of functions constructed in [17] where K is a compact subset of $\omega^\omega + 1$. (See also [6]).

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