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Homogeneous Polynomial Vector Fields of Degree 2 on the 2–Dimensional Sphere

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1. Introduction and statement of the main results

A *polynomial vector field X in* $\mathbb{R}^3$ is a vector field of the form

$$X = P(x,y,z)\frac{\partial}{\partial x} + Q(x,y,z)\frac{\partial}{\partial y} + R(x,y,z)\frac{\partial}{\partial z},$$

where P, Q, R are polynomials in the variables x, y and z with real coefficients. We denote $m = \max\{\deg P, \deg Q, \deg R\}$ the *degree* of the polynomial vector field X. In what follows X will denote the above polynomial vector field.

Let $\mathbb{S}^2$ be the 2–dimensional sphere $\{(x,y,z) \in \mathbb{R}^3 \,: x^2+y^2+z^2 = 1\}$. A *polynomial vector field X on* $\mathbb{S}^2$ is a polynomial vector field in $\mathbb{R}^3$ such that restricted to the sphere $\mathbb{S}^2$ defines a vector field on $\mathbb{S}^2$; i.e. it must satisfy the following equality

$$xP(x,y,z) + yQ(x,y,z) + zR(x,y,z) = 0, \tag{1}$$

for all points (x,y,z) of the sphere $\mathbb{S}^2$.

Let $f \in \mathbb{R}[x,y,z]$, where $\mathbb{R}[x,y,z]$ denotes the ring of all polynomials in the variables x, y and z with real coefficients. The algebraic surface $f = 0$ is an *invariant algebraic surface* of the polynomial vector field X if for some polynomial $K \in \mathbb{R}[x,y,z]$ we have

$$Xf = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} + R\frac{\partial f}{\partial z} = Kf.$$

The polynomial K is called the *cofactor* of the invariant algebraic surface $f = 0$. We note that since the polynomial system has degree m, then any cofactor has at most degree $m - 1$.

The algebraic surface $f = 0$ defines an *invariant algebraic curve* $\{f = 0\} \cap \mathbb{S}^2$ of the polynomial vector field X on the sphere $\mathbb{S}^2$ if

(i) for some polynomial $K \in \mathbb{R}[x, y, z]$ we have

$$Xf = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} + R\frac{\partial f}{\partial z} = Kf,$$

on all the points (x, y, z) of the sphere $\mathbb{S}^2$, and

(ii) the intersection of the two surfaces $f = 0$ and $\mathbb{S}^2$ is transversal; i.e. for all points $(x, y, z) \in \{f = 0\} \cap \mathbb{S}^2$ we have that

$$(x, y, z) \wedge \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \neq 0,$$

where $\wedge$ denotes the cross product in $\mathbb{R}^3$.

Again the polynomial K is called the *cofactor* of the invariant algebraic curve $\{f = 0\} \cap \mathbb{S}^2$.

Note that, if the curve $\{f = 0\} \cap \mathbb{S}^2$ satisfies the above definition, then it is formed by trajectories of the vector field X. This justifies to call $\{f = 0\} \cap \mathbb{S}^2$ an invariant algebraic curve, since in this case it is invariant under flow defined by X on $\mathbb{S}^2$.

If the invariant algebraic curve $\{f = 0\} \cap \mathbb{S}^2$ is contained in some plane, then we say that $\{f = 0\} \cap \mathbb{S}^2$ is an *invariant circle* of the polynomial vector field X on the sphere $\mathbb{S}^2$. Moreover, if the plane contains the origin, then $\{f = 0\} \cap \mathbb{S}^2$ is an *invariant great circle*.

Let U be an open subset of $\mathbb{R}^3$. Here a nonconstant analytic function $H : U \to \mathbb{R}$ is called a *first integral* of the system on U if it is constant on all solutions curves $(x(t), y(t), z(t))$ of the vector field X on U; i.e. $H(x(t), y(t), z(t)) =$ constant for all values of t for which the solution $(x(t), y(t), z(t))$ is defined in U. Clearly H is a first integral of the vector field X on U if and only if $XH \equiv 0$ on U. If X is a vector field on $\mathbb{S}^2$, the definition of *first integral* on $\mathbb{S}^2$ is the same substituting U by $U \cap \mathbb{S}^2$.

In what follows we say that two phase portraits of the vector fields X_1 and X_2 on $\mathbb{S}^2$ are (*topologically*) *equivalent*, if there exists a homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ such that h applies orbits of X_1 into orbits of X_2, preserving or reversing the orientation of all orbits.

In [10] the Darboux theory of integrability from polynomial vector fields on $\mathbb{R}^2$ (see [7]) has been extended to polynomial vector fields on $\mathbb{S}^2$. The Darboux theory of integrability analyze how to construct a first integral of a polynomial vector field by using a sufficient number of invariant algebraic curves. Therefore, to study the existence and number of invariant algebraic curves of a polynomial vector field X in dimension 2 (and in particular in $\mathbb{S}^2$), is an interesting subject of recent papers [1,4,5,6,7,9,10]. The first step in this direction is determine the maximum number of invariant circles for a polynomial vector fields X on $\mathbb{S}^2$, when X has finitely many invariant circles. A similar study but for invariant straight lines and polynomial vector fields in $\mathbb{R}^2$ was made in [1].

In this paper we consider homogeneous polynomial vector fields of degree two on $\mathbb{S}^2$, and we determine the number of invariant circles when it has finitely many invariant circles.

The main results of this paper are the following theorems.

THEOREM 1.1. *Let X be a homogeneous polynomial vector field of degree 2 on $\mathbb{S}^2$. If X has finitely many invariant circles, then every invariant circle is a great circle of $\mathbb{S}^2$.*

THEOREM 1.2. *Let X be a homogeneous polynomial vector field of degree 2 on $\mathbb{S}^2$. Suppose that X has invariant circles on $\mathbb{S}^2$, then it has either at most two invariant circles, or it has infinitely many invariant circles on $\mathbb{S}^2$. Moreover, the invariant circles are never limit cycles.*

Theorems 1.1 and 1 will be proved in Section 4.

THEOREM 1.3. *Let X be a homogeneous polynomial vector field of degree 2 on $\mathbb{S}^2$. Suppose that X has exactly two invariant circles on $\mathbb{S}^2$, then the phase portrait of X is equivalent to one of Figures* 1 *or* 2.

THEOREM 1.4. *Let X be a homogeneous polynomial vector field of degree 2 on $\mathbb{S}^2$. Suppose that X has exactly one invariant circle on $\mathbb{S}^2$, then the phase portrait of X is equivalent to one of the phase portraits of Figures* 3, 4, 5, 6 *or* 7.

COROLLARY 1.5. *Let X be a homogeneous polynomial vector field of degree 2 on $\mathbb{S}^2$. Suppose that X has at least one invariant circle on $\mathbb{S}^2$, then X does not have limit cycles.*

Theorem 1.3 will be proved in Section 6, Theorem 1.4 and Corollary 1.5 will be proved in Section 7.

2. Quadratic homogeneous polynomial vector fields on $\mathbb{S}^2$

In what follows we assume that $m = 2$ and P, Q, R are homogeneous polynomials, i.e. X is vector field associated to the differential system

$$\begin{aligned} \dot{x} &= P(x,y,z) = b_1x^2 + b_2y^2 + b_3z^2 + b_4xy + b_5xz + b_6yz, \\ \dot{y} &= Q(x,y,z) = b_7x^2 + b_8y^2 + b_9z^2 + b_{10}xy + b_{11}xz + b_{12}yz, \\ \dot{z} &= R(x,y,z) = b_{13}x^2 + b_{14}y^2 + b_{15}z^2 + b_{16}xy + b_{17}xz + b_{18}yz. \end{aligned} \tag{2}$$

Proposition 2.1. *Let X be the vector field associated to* (2). *Then X is a polynomial vector field on $\mathbb{S}^2$ if and only if system* (2) *can be written as*

$$\begin{aligned} \dot{x} &= P(x,y,z) = a_1xy + a_2y^2 + a_3z^2 + a_4xz + a_5yz, \\ \dot{y} &= Q(x,y,z) = -a_1x^2 - a_2xy + a_6z^2 + a_7xz + a_8yz, \\ \dot{z} &= R(x,y,z) = -a_4x^2 - a_8y^2 - (a_5 + a_7)xy - a_3xz - a_6yz. \end{aligned} \tag{3}$$

Before proving Proposition 2.1 we state some results that will be needed.

Proposition 2.2. *Let $\gamma = \{\varphi(t) \ : t \in \mathbb{R}\} \subset \mathbb{S}^2$ an orbit of X, where X is a homogeneous polynomial vector field of degree 2 on $\mathbb{S}^2$. Then X is tangent to the surface $S(\gamma) = \{sp \ : s \in \mathbb{R},\ p \in \gamma\}$. Moreover, if γ is a curve formed by the union of orbits of X the statement also holds.*

The proof of Proposition 2.2 is contained in the proof of next result, see also [3].

Corollary 2.3. *Let $f(x,y,z) = x^2 + y^2 + z^2 - 1$ and let X be a homogeneous polynomial vector field of degree 2 on $\mathbb{S}^2$. Then, f is a first integral of X, i. e.*

$$P(x,y,z)x + Q(x,y,z)y + R(x,y,z)z = 0$$

for all $(x,y,z) \in \mathbb{R}^3$.

Proof. We denote by $\varphi(t,x)$ the flow of X on $\mathbb{S}^2$. For each positive $s \in \mathbb{R}$ we have that $s\varphi(t,x)$ define a flow on the sphere of radius s in $\mathbb{R}^3$. Now

$$X(s\varphi(t,x)) = s^2 X(\varphi(t,x)) = s^2 \frac{\partial\varphi}{\partial t}(t,x).$$

Therefore, X is tangent to sphere of of radius s in $\mathbb{R}^3$, i.e. f is a first integral of X. ∎

Proof of Proposition 2.1. Let $f \in \mathbb{R}[x,y,z]$ such that $f(x,y,z) = x^2 + y^2 + z^2 - 1$. By Corollary 2.3, we have that f is a first integral of X on $\mathbb{R}^3$. Therefore, solving the equation $Xf(x,y,z) = 0$ and renaming the coefficients of (2) the proposition follows. ∎

Proposition 2.2 and Corollary 2.3 also hold for any homogeneous polynomial vector field of degree m on $\mathbb{S}^2$. We call the surfaces $S(\gamma) = \{sp\ : s \in \mathbb{R},\ p \in \gamma\}$ *invariant cones* of X.

The equation of a plane in $\mathbb{R}^3$ is given for

$$ax + by + cz + d = 0 \tag{4}$$

and $|d|/\sqrt{a^2+b^2+c^2}$ measures the distance from the plane to the origin $(0,0,0)$. Any circle on the sphere lies in a plane $ax + by + cz + d = 0$, where we can assume that $a^2 + b^2 + c^2 = 1$ and $0 \leq -d < 1$. In that follows when we talk about a circle on the sphere we always will assume that it is contained in a plane (4) satisfying $a^2 + b^2 + c^2 = 1$ and $0 \leq -d < 1$. We say that a circle on $\mathbb{S}^2$ is a *great circle* if it has radius equal to 1.

3. Invariant circles

In this section we always assume that C is a circle on the sphere $\mathbb{S}^2$ and X is a homogeneous polynomial vector field on $\mathbb{S}^2$. We suppose that C is formed by trajectories of the vector field X. Then by Proposition 2.2 $\mathcal{C} = \{sp\ : s \in \mathbb{R},\ p \in C\}$ is an invariant cone of X.

The proof of the next proposition is elementary and we do not give it here.

Proposition 3.1. *Set a, b, c, $d \in \mathbb{R}$ satisfing $a^2 + b^2 + c^2 = 1$ and $0 \leq -d < 1$, such that C lies in the plane $ax + by + cz + d = 0$. Then, the equation of the invariant cone $\mathcal{C}$ is*

$$(a^2 - d^2)x^2 + (b^2 - d^2)y^2 + (c^2 - d^2)z^2 + 2abxy + 2acxz + 2bcyz = 0. \tag{5}$$

In the proof of the next proposition we will use the Hilbert's Nullstellensatz (see [8]).

Proposition 3.2. *Let*

$$f_{\mathcal{C}}(x,y,z) = (a^2-d^2)x^2+(b^2-d^2)y^2+(c^2-d^2)z^2+2abxy+2acxz+2bcyz \tag{6}$$

if $d \neq 0$, and let $f_{\mathcal{C}}(x,y,z) = ax + by + cz$ if $d = 0$. If $f_{\mathcal{C}} = 0$ is an invariant cone of X, i.e. $Xf_{\mathcal{C}}\ |_{f_{\mathcal{C}}=0} = 0$, then $f_{\mathcal{C}} = 0$ is an invariant algebraic surface of X.

Proof. Note that in both cases $f_{\mathcal{C}}$ is a irreducible polynomial. Therefore, this proposition is a straightforward consequence of Hilbert's Nullstellensatz. ∎

4. Number of invariant circles

If X is a homogeneous vector field of degree 2 on $\mathbb{S}^2$, then the differential system associated to it is invariant with respect to the change of variables $(x, y, z, t) \mapsto (-x, -y, -z, -t)$. Thus, in particular the phase portrait of X at the northern hemisphere of $\mathbb{S}^2$ is symmetric with respect to the origin to the phase portrait at the southern hemisphere with the time reverse. Hence, if C is a circle on $\mathbb{S}^2$ formed by trajectories of the vector field X, then $-C = \{-p : p \in C\}$ also is a circle on $\mathbb{S}^2$ formed by trajectories of X. Now, by Propositions 2.2 and 3.2, the invariant cone $\mathcal{C} = \{sp\ : s \in \mathbb{R},\ p \in C\}$ associated to C is a invariant algebraic surface of X and $\{C \cup -C\} = \{f_{\mathcal{C}} = 0\} \cap \mathbb{S}^2$ are invariants circles of X on $\mathbb{S}^2$, where $f_{\mathcal{C}} = 0$ is the equation of $\mathcal{C}$ given by (5). Note that if $d = 0$ the equation of cone $\mathcal{C}$ becomes

$$f_{\mathcal{C}}(x, y, z) = a^2x^2 + b^2y^2 + c^2z^2 + 2abxy + 2acxz + 2bcyz = (ax + by + cz)^2 = 0,$$

this mean that $f_{\mathcal{C}} = 0$ is the plane $ax + by + cz = 0$ and C is a great circle. Therefore, if $d \neq 0$ for each invariant algebraic cone of X of the form (5) we have two invariant circles of X on $\mathbb{S}^2$ and if $d = 0$ we have only one. Obviously, the converse also holds. Thus, the problem of determining the maximum number of invariant circles on $\mathbb{S}^2$ for a homogeneous polynomial vector fields of degree 2 is equivalent to determine the maximum number of invariant cones of the form (5), supposing that X has finitely many invariant cones. Therefore, we have to study the equation

$$P\frac{\partial f_{\mathcal{C}}}{\partial x} + Q\frac{\partial f_{\mathcal{C}}}{\partial y} + R\frac{\partial f_{\mathcal{C}}}{\partial z} = K_{f_{\mathcal{C}}} f_{\mathcal{C}}, \tag{7}$$

where $K_{f_{\mathcal{C}}} \in \mathbb{R}[x, y, z]$ is a homogeneous polynomial of degree 1, because X is a homogeneous polynomial vector field of degree 2.

Proof of Theorem 1.1. By Proposition 2.1, we have that X is a vector field of the form (3). Let C be an invariant circle of X on $\mathbb{S}^2$. Consider the plane $ax + by + cz + d = 0$ containing C and also the invariant cone $\mathcal{C}$ given by (5). Note that, as X is invariant by rotations of $SO(3)$, we can assume that $(a, b, c) = (0, 0, 1)$. Now, to prove this theorem it is sufficient to show that $d = 0$.

Let $f_{\mathcal{C}}$ be as in (6) and $K_{f_{\mathcal{C}}}(x,y,z) = c_1x + c_2y + c_3z$. We have that X, $f_{\mathcal{C}}$ and $K_{f_{\mathcal{C}}}$ satisfies (7). Thus, substituting $a = 0$, $b = 0$ and $c = 1$ in (7), we obtain the following system of equations

$$\begin{array}{lcllcl} d^2c_1 & = & 0, & c_1(d^2-1) - 2a_3 & = & 0,\\ d^2c_2 & = & 0, & -2a_8 + d^2c_3 & = & 0,\\ c_3(d^2-1) & = & 0, & c_2(d^2-1) - 2a_6 & = & 0,\\ -2a_4 + d^2c_3 & = & 0, & a_5 + a_7 & = & 0. \end{array} \tag{8}$$

It follows from (8) that (7) admits only 4 solutions, i.e.

$$S_1 = \{d,\ a_1,\ a_2,\ a_3 = 0,\ a_4 = 0,\ a_5 = -a_7,\ a_6 = 0,\ a_8 = 0,\\ c_1 = 0,\ c_2 = 0,\ c_3 = 0\},$$

$$S_2 = \{d = 1,\ a_1,\ a_2,\ a_3 = 0,\ a_5 = -a_7,\ a_6 = 0,\ a_8 = a_4,\\ c_1 = 0,\ c_2 = 0,\ c_3 = 2a_4\},$$

$$S_3 = \{d = -1,\ a_1,\ a_2,\ a_3 = 0,\ a_5 = -a_7,\ a_6 = 0,\ a_8 = a_4,\\ c_1 = 0,\ c_2 = 0,\ c_3 = 2a_4\},$$

$$S_4 = \{d = 0,\ a_1,\ a_2,\ a_4 = 0,\ a_5 = -a_7,\ a_8 = 0,\ c_1 = -2a_3,\\ c_2 = -2a_6,\ c_3 = 0 \text{ and } a_3 \neq 0 \text{ or } a_6 \neq 0\}.$$

The solution S_1 implies that there exist infinitely many invariant circles of X on $\mathbb{S}^2$ having center on the z–axis. The solutions S_2 and S_3 imply that C is one point. Therefore, the solution that satisfies the hypotheses of the theorem is S_4. Hence $d = 0$, and C is a great circle. ∎

Proof of Theorem 1.2. By Proposition 2.1, we have that X is a vector field associated to (3). Suppose that X has only invariant great circles on $\mathbb{S}^2$, otherwise, from Theorem 1.1, X has infinitely many invariant circles on $\mathbb{S}^2$. Let C be an invariant great circle of X on $\mathbb{S}^2$. Then the equation of the cone associated to C is

$$f_{\mathcal{C}}(x,y,z) = ax + by + cz = 0, \tag{9}$$

with $a^2 + b^2 + c^2 = 1$. Now, as X is invariant by rotations of $SO(3)$, we can assume that $(a,b,c) = (0,0,1)$, i.e. C is the equator of $\mathbb{S}^2$. Therefore, by proof of Theorem 1.1, it follows that the coefficients of the vector field X must satisfy

$$a_4 = 0,\ \ a_7 = -a_5,\ \ a_8 = 0 \ \text{ and } \ a_3 \neq 0 \ \text{ or } \ a_6 \neq 0. \tag{10}$$

Thus, (3) becomes

$$\begin{array}{rcl} \dot{x} & = & a_1xy + a_2y^2 + a_3z^2 + a_5yz, \\ \dot{y} & = & -a_1x^2 - a_2xy + a_6z^2 - a_5xz, \\ \dot{z} & = & -a_3xz - a_6yz, \end{array} \tag{11}$$

with $a_3 \neq 0$ or $a_6 \neq 0$.

We will study if there are additional invariant circles different from C. Therefore, substituting expressions (11), (9) respectively for X, $f_{\mathcal{C}}$ and $K_{f_{\mathcal{C}}} = c_1x + c_2y + c_3z$ in (7), we want to determine the other possible solutions (9) of (7) for $a, b, c \in \mathbb{R}$ with $(a, b, c) \neq (0, 0, 1)$. Thus, we obtain the following system of equations

$$\begin{gathered} -ac_1 - ba_1 = aa_2 - bc_2 = aa_3 + ba_6 - cc_3 = aa_1 - ac_2 - ba_2 - bc_1 = 0, \\ -ac_3 - ba_5 - ca_3 - cc_1 = aa_5 - bc_3 - ca_6 - cc_2 = 0. \end{gathered} \tag{12}$$

For solving this system we consider separately the case $b \neq 0$ and the case $b = 0$.

Suppose that $b \neq 0$. We have that

$$\begin{gathered} S_5 = \{a_1 = \frac{a}{b}a_2,\ a_2,\ a_3,\ a_5 = \frac{c}{b}(a_2 - a_3),\ a_6 = -\frac{a}{b}a_3, \\ c_1 = -a_2,\ c_2 = \frac{a}{b}a_2,\ c_3 = 0\} \end{gathered}$$

is the solution of system (12) in this case.

Now, assume that $b = 0$ and $a \neq 0$. In this case

$$S_6 = \{a_1,\ a_2 = 0,\ a_3 = 0,\ a_5 = \frac{c}{a}(a_6 + a_1),\ a_6,\ c_1 = 0,\ c_2 = a_1,\ c_3 = 0\}$$

is the solution of system (12).

The solutions S_5 and S_6 are the unique possible solutions of system (12) for fixed $(a, b, c) \neq (0, 0, 1)$. Note that, the coefficients of X cannot satisfy S_5 and S_6 simultaneously, otherwise we obtain $X \equiv 0$.

We will study the solutions S_5 and S_6. Firstly, we distinguish the following four cases.

Case 1: $a_1 = a_2 = 0$. In this case if $b \neq 0$, then, by S_5, $a_3 \neq 0$, otherwise $a_3 = a_6 = 0$ and this is not possible by (10). Hence, we have that

$$\begin{gathered} a = -\frac{a_6}{a_3}b,\ b \neq 0,\ c = -\frac{a_5}{a_3}b,\ a_1 = 0,\ a_2 = 0,\ a_3 \neq 0, \\ a_5,\ a_6,\ c_1 = c_2 = c_3 = 0. \end{gathered} \tag{13}$$

Since $a^2+b^2+c^2=1$, we obtain that $b=\pm\alpha$, where $\alpha=\sqrt{a_3^2/(a_5^2+a_6^2+a_3^2)}$. Note that the two vectors $(a,b,c)=\pm\left(-(a_6/a_3)\alpha,\ \alpha,\ -(a_5/a_3)\alpha\right)$, determine the same plane. Now, $f(x,y,z)=-(a_6/a_3)\alpha x+\alpha y-(a_5/a_3)\alpha z$ is a first integral of X, because in this case $c_1=c_2=c_3=0$. Thus, if the coefficients of X satisfy (10) and (13), then X has infinitely many invariant circles on $\mathbb{S}^2$ determined by $f=$ constant.

In a similar way the same conclusion is obtained, if we suppose that $b=0$.

Case 2: $a_1=a_6=0$, $a_2a_3\neq 0$ and $a_2\neq a_3$. In this case the coefficients of system (11) satisfy only the solution S_5. Therefore, we have that

$$
\begin{gathered}
a=0,\ b\neq 0,\ c=\frac{a_5}{(a_2-a_3)}b,\\
a_1=0,\ a_2\neq 0,\ a_3\neq a_2,\ a_5,\ a_6=0,\ c_1=-a_2,\ c_2=0,\ c_3=0.
\end{gathered}
\tag{14}
$$

As in Case 1, we obtain that $b=\pm\alpha$, where $\alpha=\sqrt{(a_2-a_3)^2/((a_2-a_3)^2+a_5^2)}$. Note that the two vectors $(a,b,c)=\pm\left(0,\ \alpha,\ (a_5/(a_2-a_3))\alpha\right)$, determine the same plane. Thus, if the coefficients of X satisfy (10) and (14), then additionally to the great circle C the vector field X has a unique invariant great circle on $\mathbb{S}^2$, because now the cofactor of $f(x,y,z)=\alpha y+(a_5/(a_2-a_3))\alpha z=0$ is non–zero.

Case 3: $a_1\neq 0$, $a_2=a_3=0$ and $a_6=-a_1$. If $a_5=0$, then in this case the coefficients of system (11) satisfy only the solution S_6. Therefore, we have that

$$
\begin{gathered}
a\neq 0,\ b=0,\ c,\\
a_1\neq 0,\ a_2=0,\ a_3=0,\ a_5=0,\ a_6=-a_1,\ c_1=0,\ c_2=a_1,\ c_3=0.
\end{gathered}
\tag{15}
$$

Note that in this case if the coefficients of X satisfy (10) and (15), then X has infinitely many invariant circles on $\mathbb{S}^2$, because the equation $a^2+c^2=1$ has infinitely many solutions for $c\in[-1,1]$.

Now, if $a_5\neq 0$, then in this case the coefficients of X do not satisfy any of the solutions S_5 and S_6. Thus, C is the unique invariant circle of X on $\mathbb{S}^2$.

Case 4: $a_1a_3\neq 0$ and $a_2=0$. In this case the coefficients of X do not satisfy any of the solutions S_5 and S_6. Thus, C is the unique invariant circle of X on $\mathbb{S}^2$.

We will show now that system (11) is always equivalent to one of Cases 1, 2, 3 and 4, by a orthogonal linear change of variables. In Case 1 we have that if $a_1=a_2=0$ then system (11) has infinitely many invariant circles on

$\mathbb{S}^2$. Therefore, we can suppose that $a_1 \neq 0$ or $a_2 \neq 0$. Doing the change of variables

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \dfrac{a_1}{\sqrt{a_1^2+a_2^2}} & -\dfrac{a_2}{\sqrt{a_1^2+a_2^2}} & 0 \\ \dfrac{a_2}{\sqrt{a_1^2+a_2^2}} & \dfrac{a_1}{\sqrt{a_1^2+a_2^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix},$$

we have that system (11) becomes

$$\begin{aligned} \dot{\tilde{x}} &= \sqrt{a_1^2+a_2^2}\tilde{x}\tilde{y} + \frac{a_1a_3+a_6a_2}{\sqrt{a_1^2+a_2^2}}\tilde{z}^2 + a_5\tilde{y}\tilde{z}, \\ \dot{\tilde{y}} &= -\sqrt{a_1^2+a_2^2}\tilde{x}^2 + \frac{a_1a_6-a_2a_3}{\sqrt{a_1^2+a_2^2}}\tilde{z}^2 - a_5\tilde{x}\tilde{z}, \\ \dot{\tilde{z}} &= -\frac{a_1a_3+a_6a_2}{\sqrt{a_1^2+a_2^2}}\tilde{x}\tilde{z} - \frac{a_1a_6-a_2a_3}{\sqrt{a_1^2+a_2^2}}\tilde{y}\tilde{z}. \end{aligned} \tag{16}$$

If $a_1a_3 + a_2a_6 \neq 0$, then system (16) is a particular case of Case 4.

If $a_1a_3 + a_2a_6 = 0$ and $a_1 \neq 0$, then $a_3 = -(a_6a_2)/a_1$ and system (16) becomes

$$\begin{aligned} \dot{\tilde{x}} &= \sqrt{a_1^2+a_2^2}\tilde{x}\tilde{y} + a_5\tilde{y}\tilde{z}, \\ \dot{\tilde{y}} &= -\sqrt{a_1^2+a_2^2}\tilde{x}^2 + \frac{a_6}{a_1}\sqrt{a_1^2+a_2^2}\tilde{z}^2 - a_5\tilde{x}\tilde{z}, \\ \dot{\tilde{z}} &= -\frac{a_6}{a_1}\sqrt{a_1^2+a_2^2}\tilde{y}\tilde{z}. \end{aligned} \tag{17}$$

Note that $a_6 \neq 0$, because $a_6 = 0$ implies that $a_3 = 0$, and this is not possible by (10). Hence, if $a_6 \neq -a_1$ through the change of variables $(\tilde{x}, \tilde{y}, \tilde{z}) \mapsto (\bar{y}, \bar{x}, \bar{z})$ system (17) becomes a particular system of Case 2. Thus, in this case system (11) has exactly two invariant circles on $\mathbb{S}^2$. Now, if $a_6 = -a_1$, then system (17) is contained in Case 3.

If $a_1a_3 + a_2a_6 = 0$ and $a_2 \neq 0$, then $a_6 = -(a_1a_3)/a_2$ and system (16) becomes

$$\begin{aligned} \dot{\tilde{x}} &= \sqrt{a_1^2+a_2^2}\tilde{x}\tilde{y} + a_5\tilde{y}\tilde{z}, \\ \dot{\tilde{y}} &= -\sqrt{a_1^2+a_2^2}\tilde{x}^2 - \frac{a_3}{a_2}\sqrt{a_1^2+a_2^2}\tilde{z}^2 - a_5\tilde{x}\tilde{z}, \\ \dot{\tilde{z}} &= \frac{a_3}{a_2}\sqrt{a_1^2+a_2^2}\tilde{y}\tilde{z}. \end{aligned} \tag{18}$$

Note that $a_3 \neq 0$, because $a_3 = 0$ implies that $a_6 = 0$, and this is not possible by (10). Therefore, if $a_3 \neq a_2$ through the change of variables $(\tilde{x}, \tilde{y}, \tilde{z}) \mapsto (\bar{y}, \bar{x}, \bar{z})$ system (18) becomes a particular system of Case 2. Thus, in this case system (11) has exactly two invariant circles on $\mathbb{S}^2$. Now, if $a_3 = a_2$, then system (18) also is contained in Case 3. Therefore, if X has invariant circles on $\mathbb{S}^2$, then it has either at most two invariant circles, or it has infinitely many invariant circles on $\mathbb{S}^2$. ∎

5. Poincaré disc

Let X is be homogeneous vector field on $\mathbb{S}^2$, then the differential system associated to it is invariant with respect to the change of variables $(x, y, z, t) \mapsto (-x, -y, -z, -t)$ if its degree is even, or with respect to $(x, y, z, t) \mapsto (-x, -y, -z, t)$ if its degree is odd. Thus, in particular the phase portrait of X at the northern hemisphere of $\mathbb{S}^2$ is symmetric with respect to the origin to the phase portrait at the southern hemisphere with the time reverse if degree of X is even, or with the same time if the degree of X is odd. We now project the northern hemisphere of $\mathbb{S}^2$ orthogonally onto the plane Π containing the equator of $\mathbb{S}^2$, i.e. $\mathbb{S}^1$. The orbits of X on the northern hemisphere of $\mathbb{S}^2$ are mapped onto certain curves of the unit disc on Π. We called the unit disc, together with the corresponding induced phase portrait, the *Poincaré disc*.

Now, consider the homogeneous polynomial vector field X of degree 2 associated to (3). We identify $\mathbb{R}^2$ as the tangent plane to the sphere $\mathbb{S}^2$ at the point $p = (0, 0, -1)$, and we denote the points of $\mathbb{R}^2$ as $(u, v, -1)$. Let $\pi : \mathbb{R}^2 \to \mathbb{S}^2 \cap \{z < 0\}$ be the diffeomorphism given by $\pi(u, v) = 1/\lambda(x = u,\ y = v,\ z = -1)$, where $\lambda = \sqrt{1 + u^2 + v^2}$. That is, π is the inverse map of the central projection $\pi^{-1} : \mathbb{S}^2 \cap \{z < 0\} \to \mathbb{R}^2$ defined by

$$\pi^{-1}(x, y, z) = \left(u = -\frac{x}{z},\ v = -\frac{y}{z}, -1\right). \tag{19}$$

The homogeneous polynomial system (3) on $\mathbb{S}^2$ becomes, through the central projection π^{-1} and introducing the new independent variable s through $ds = (\sqrt{1 + u^2 + v^2})^{-1} dt$, the polynomial differential system

$$\begin{aligned} \dot{u} &= P(u, v) = -a_4 u^3 - (a_5 + a_7) u^2 v + a_3 u^2 - a_8 u v^2 + a_2 v^2 + \\ &\quad (a_1 + a_6) uv - a_4 u - a_5 v + a_3, \\ \dot{v} &= Q(u, v) = -a_8 v^3 - (a_5 + a_7) u v^2 - a_4 u^2 v - a_1 u^2 + a_6 v^2 + \\ &\quad (a_3 - a_2) uv - a_8 v - a_7 u + a_6. \end{aligned} \tag{20}$$

The next two results jointly with its respective proofs can be find in Camacho [3].

LEMMA 5.1. *Any straight line in* $\mathbb{R}^2$ *has at most two tangencies with the solutions of* (20) *or it is formed by solutions of* (20).

PROPOSITION 5.2. *Let* X *be the vector field associated to* (3), *and let* $s_1, -s_1 \in \mathbb{S}^2$ *be saddle points of* X *with a common separatrix* l, *then* l *is contained in a great circle through* s_1 *and* $-s_1$.

Proposition 5.2 is close to results proved by Sotomayor and Paterlini [13].

6. PHASE PORTRAITS FOR QUADRATIC HOMOGENEOUS POLYNOMIAL VECTOR FIELDS ON $\mathbb{S}^2$ WITH TWO INVARIANT CIRCLES

In the proof of Theorem 1.2 we saw that Case 2 has exactly two invariant great circles on $\mathbb{S}^2$ and that the other cases having also only two invariant great circles are reduced to Case 2. Now, we will go to study the phase portraits of Case 2.

Proof of Theorem 1.3. By Proposition 2.1, we have that X is a vector field associated to (3). If the coefficients of the vector field associated to (3) satisfy Case 2 given in the proof of Theorem 1.2, then (3) becomes

$$\begin{aligned} \dot{x} &= P(x,y,z) = a_2y^2 + a_3z^2 + a_5yz, \\ \dot{y} &= Q(x,y,z) = -a_2xy - a_5xz, \\ \dot{z} &= R(x,y,z) = -a_3xz, \end{aligned} \tag{21}$$

with $a_2 \neq a_3$, and $a_2a_3 \neq 0$. The vector field given by (21) has exactly two invariant great circles, namely $\mathbb{S}^1$ and C on $\mathbb{S}^2$, determined respectively by the invariant planes $z = 0$ and $\sqrt{(a_2-a_3)^2/((a_2-a_3)^2+a_5^2)}(y + (a_5/(a_2 - a_3))z) = 0$. Note that $(1,0,0)$ and $(-1,0,0)$ are the unique singularities of this vector field on $\mathbb{S}^1$. We have that

$$\{0, -a_2, -a_3\} \tag{22}$$

are the eigenvalues of the linear part of (21) at $(1,0,0)$ with respective eigenvectors $(1,0,0)$, $(0,1,0)$ and $(0,1,(a_3-a_2)/a_5)$ if $a_5 \neq 0$, or $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ if $a_5 = 0$. Now,

$$\{0, a_2, a_3\} \tag{23}$$

are the eigenvalues of the linear part of (21) at $(-1,0,0)$ with the same eigenvectors.

We have by (20) that system (21) becomes

$$\dot{u} = a_3u^2 + a_2v^2 - a_5v + a_3, \quad \dot{v} = (a_3 - a_2)uv + a_5u. \tag{24}$$

Hence, for determining the phase portrait of (21) on $\mathbb{S}^2$ it is sufficient to study the phase portrait of (24) on $\mathbb{R}^2$.

We note that

$$v = -\frac{a_5}{a_3 - a_2} \tag{25}$$

is the unique invariant straight line of (24) which corresponds to the cental projection of an invariant great circle C of (21). If $a_5^2 - 4a_3a_2 > 0$, then system (24) has two singularities

$$\left(0, \frac{a_5 + \sqrt{a_5^2 - 4a_3a_2}}{2a_2}\right) \quad \text{and} \quad \left(0, \frac{a_5 - \sqrt{a_5^2 - 4a_3a_2}}{2a_2}\right), \tag{26}$$

with respective eigenvalues

$$\pm\frac{1}{2|a_2|}\sqrt{2a_2\sqrt{a_5^2 - 4a_3a_2}\left(a_5(a_3 + a_2) + \sqrt{a_5^2 - 4a_3a_2}(a_3 - a_2)\right)} \tag{27}$$

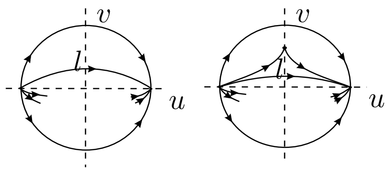
and

$$\pm\frac{1}{2|a_2|}\sqrt{-2a_2\sqrt{a_5^2 - 4a_3a_2}\left(a_5(a_3 + a_2) - \sqrt{a_5^2 - 4a_3a_2}(a_3 - a_2)\right)}. \tag{28}$$

For determining the phase portrait of (24), or equivalently the phase portrait of (21), we distinguish three cases.

Case 1: $a_5^2 - 4a_3a_2 < 0$. In this case $a_3a_2 > 0$. Therefore a_2 and a_3 have the same sign. Moreover, the unique singularities of (21) are on $\mathbb{S}^1$. Hence, by (22) and (23) those singularities on $\mathbb{S}^2$ are nodes. Thus, the phase portrait of (21) on the Poincaré disc is equivalent to that of Figure 1(a), where l is the projection of invariant circle C.

Case 2: $a_5^2 - 4a_3a_2 = 0$. Since $a_2a_3 \neq 0$, we have that $a_5 \neq 0$ and $a_3a_2 > 0$. Therefore, a_2 and a_3 have the same sign and the singularities of (21) on $\mathbb{S}^1$, as in Case 1, are nodes.



(a) $a_5^2 - 4a_3a_2 < 0$ (b) $a_5^2 - 4a_3a_2 = 0$

Figure 1: Phase portraits of Cases 1 and 2.

In this case system (24) has the singularity $(0, a_5/(2a_2))$, with eigenvalues $\{0,0\}$. Note that $a_5/(2a_2) = -a_5/(a_3 - a_2)$ implies that $a_3a_2 < 0$. Hence $(0, a_5/(2a_2))$ does not belong to the invariant straight line (25).

We have that $a_3 = a_5^2/(4a_2)$, so doing the change of variables $u = r$, $v = s + a_5/(2a_2)$, and introducing the variable τ through $d\tau = (a_5(4a_2^2 + a_5^2)/(8a_2^2))dt$, (24) becomes

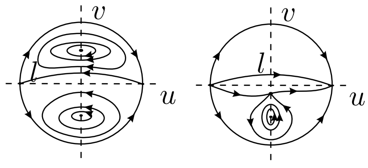
$$\dot{r} = \frac{8a_2^3}{a_5(4a_2^2 + a_5^2)}s^2 + \frac{2a_2a_5}{4a_2^2 + a_5^2}r^2, \quad \dot{s} = \frac{2a_2a_5^2 - 8a_2^3}{a_5(4a_2^2 + a_5^2)}rs + r. \tag{29}$$

By a result that can be find in [2] page 362, the singularity $(0,0)$ of (29) is equivalent to the cusp. Hence, we can conclude that the phase portrait of (21) on the Poincaré disc is equivalent to the one of Figure 1(b), where l is the projection of invariant circle C.

Case 3: $a_5^2 - 4a_3a_2 > 0$. In this case system (24) has two singularities given by (26). Substituting (26) into (25), we obtain the following equations

$$\frac{a_5(a_3 + a_2) \pm (a_3 - a_2)\sqrt{a_5^2 - 4a_3a_2}}{2a_2(a_3 - a_2)} = 0. \tag{30}$$

We denote, respectively, by α_+ and α_- the left hand side of equations (30). Therefore, $\alpha_+\alpha_- = 0$ if and only if $4a_2a_3(a_5^2 + (a_3 - a_2)^2) = 0$. Since $a_2 \neq a_3$ and $a_2a_3 \neq 0$, it follows that equations (30) do not have solution, i.e singularities (26) do not belong to the invariant straight line (25). Now, singularities (26) are at opposite sides with respect to the invariant straight line (25) if and only if $\alpha_+\alpha_- < 0$; i.e., if and only if a_2 and a_3 have opposite signs. Using the same kind of argument the singularities (26) belong to same half–plane determined by the invariant straight line (25) if and only if $\alpha_+\alpha_- > 0$; i.e., if and only if a_2 and a_3 have same signs.



(a) $a_2a_3 < 0$ (b) $a_2a_3 > 0$

Figure 2: Phase portraits of Case 3: $a_5^2 - 4a_3a_2 > 0$.

We denote, respectively, by

$$\left\{\frac{1}{2|a_2|}\sqrt{\tilde{\alpha}_+}, -\frac{1}{2|a_2|}\sqrt{\tilde{\alpha}_+}\right\} \quad \text{and} \quad \left\{\frac{1}{2|a_2|}\sqrt{\tilde{\alpha}_-}, -\frac{1}{2|a_2|}\sqrt{\tilde{\alpha}_-}\right\}$$

the eigenvalues given by (27) and (28), where

$$\tilde{\alpha}_\pm = \pm 2a_2\sqrt{a_5^2 - 4a_3a_2}\left(a_5(a_3 + a_2) \pm \sqrt{a_5^2 - 4a_3a_2}(a_3 - a_2)\right).$$

Note that $\tilde{\alpha}_+\tilde{\alpha}_- = -4a_2a_3(4a_2^2(a_5^2 - 4a_3a_2)(a_5^2 + (a_3 - a_2)^2))$. Therefore, in this case $\tilde{\alpha}_+\tilde{\alpha}_- \neq 0$. If $a_2a_3 > 0$, then one of the singularities (26) has real eigenvalues, i.e. it is a saddle and the other has complex eigenvalues, i.e. it is weak focus.

If $a_2a_3 < 0$, we have that $\tilde{\alpha}_+$ and $\tilde{\alpha}_-$ have the same signs, therefore the singularities (26) have either real eigenvalues, or complex eigenvalues and they are in opposite sides with respect to the invariant straight line (25). Now, we know that the Poincaré index of a singularity is -1 in the case of a saddle, 1 in the case of a node or focus, and by the Poincaré–Hopf Index Theorem (see, for instance [12]) we also know that the sum of the indices of all singularities of a vector field on $\mathbb{S}^2$ is 2. Thus, in this case by (22) and (23) we have that the singularities of (21) in $\mathbb{S}^1$ are saddles. Hence, in fact the singularities (26) in this case have complex eigenvalues, i.e. they are weak foci.

Now, system (24) is invariant with respect to the change of variables $(u, v, t) \to (-u, v, -t)$, i.e system (24) is symmetric with respect to straight line $u = 0$. Therefore, when a singularity of (26) is a weak focus, then it is a center.

We conclude that if $a_2a_3 < 0$ the phase portrait of (21) on the Poincaré disc is equivalent to the one of Figure 2(a) and if $a_2a_3 > 0$ the phase portrait of (21) on the Poincaré disc is equivalent to the one of Figure 2(b), where l in both figures is the projection of the invariant circle C. ∎

7. Phase portraits for quadratic homogeneous polynomial vector fields on $\mathbb{S}^2$ with one invariant circle

Let X be a homogeneous polynomial vector field on $\mathbb{S}^2$ of degree 2. If X has only one invariant circle C on $\mathbb{S}^2$, then by the proof of Theorem 1.1 we can suppose that $C = \mathbb{S}^1$. Now, to prove Theorem 1.4, we have to consider only the Cases 3 and 4 of the proof of Theorem 1.2, i.e. we need to study only the following two systems

$$\begin{aligned} \dot{x} &= b_1xy + b_3z^2 + b_5yz, \\ \dot{y} &= -b_1x^2 + b_6z^2 - b_5xz, \\ \dot{z} &= -b_3xz - b_6yz, \end{aligned} \tag{31}$$

with $b_1b_3 \neq 0$, and

$$\begin{aligned} \dot{x} &= c_1xy + c_5yz, \\ \dot{y} &= -c_1x^2 - c_1z^2 - c_5xz, \\ \dot{z} &= c_1yz, \end{aligned} \tag{32}$$

with $c_1c_5 \neq 0$.

Before proving Theorem 1.4 we do a remark and we state a theorem that will be needed in its proof.

Remark 7.1. Let $q(x,y) = ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$ a conic,

$$\tilde{A} = \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

$$D_3 = \det \tilde{A}, \quad D_2 = \det A \text{ and } D_1 = \mathrm{trA}.$$

If $D_2 > 0$ and $D_3D_1 > 0$, then q is a complex ellipse. This implies that q does not change of sign.

For more details about classification of conics see [14].

Theorem 7.2. (**Dulac's criterion**) *Let $X = (P,Q)$ be a vector field on the plane, where P and Q are analytic functions. If there exists a C^1 function $D(x,y)$ on a simply connected region G such that* $\mathrm{div}(DX) = \dfrac{\partial DP}{\partial x} + \dfrac{\partial DQ}{\partial y}$ *has constant sign and is not identically zero on any open subset of G, then the vector field (P,Q) has neither periodic orbits, nor closed curves which are union of alternating orbits and singularities lying entirely in G.*

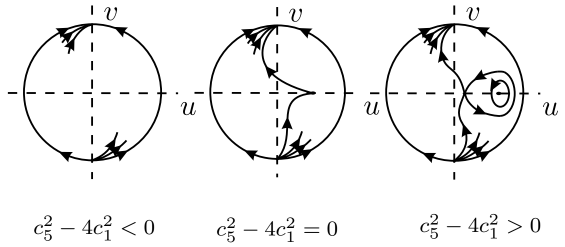


Figure 3: Phase portrait of system (32).

The proof of this theorem can be find in [2].

Proof of Theorem 1.4. First, we study the phase portraits of system (32) with $c_1 \neq 0$ and $c_5 \neq 0$. The vector field X associated to system (32) has two singularities $(0,1,0)$ and $(0,-1,0)$ on $\mathbb{S}^1$. We have that $\{0, c_1, c_1\}$ are the eigenvalues of the linear part of (32) at $(0,1,0)$ with eigenvector $(0,1,0)$ associated to 0, and $(1,0,0)$ associated to c_1. Now, $\{0, -c_1, -c_1\}$ are the eigenvalues of the linear part of (32) at $(0,-1,0)$ with the same eigenvectors. Since at Section 6, we considered the planar vector field induced by (32) through the central projection (19), i.e. by (20) we have that

$$\dot{u} = -c_5 v, \quad \dot{v} = -c_1 u^2 - c_1 v^2 + c_5 u - c_1. \tag{33}$$

If $c_5^2 - 4a_1^2 > 0$, the vector field associated to (33) has two singularities $((c_5 + \sqrt{c_5^2 - 4c_1^2})/(2c_1), 0)$ and $((c_5 - \sqrt{c_5^2 - 4c_1^2})/(2c_1), 0)$, with respective eigenvalues $\pm\sqrt{c_5\sqrt{c_5^2 - 4c_1^2}}$, and $\pm\sqrt{-c_5\sqrt{c_5^2 - 4c_1^2}}$. Using the same arguments than in the study of system (24) we prove that the phase portraits of system (33) or equivalently (32) on the Poincaré disc are equivalent to the ones of Figure 3.

Now, we study the phase portraits of (31). Since $b_1 \neq 0$, introducing the change of time $d\tau = b_1 dt$, we have that (31) becomes

$$\begin{array}{rcl} \dot{x} &=& xy + d_1 z^2 + d_2 yz, \\ \dot{y} &=& -x^2 + d_3 z^2 - d_2 xz, \\ \dot{z} &=& -d_1 xz - d_3 yz, \end{array} \tag{34}$$

with $d_1 \neq 0$. The vector field X associated to system (34) has two singularities $(0,1,0)$ and $(0,-1,0)$ on $\mathbb{S}^1$. We have that $\{0, 1, -d_3\}$ are the eigenvalues of the linear part of (34) at $(0,1,0)$ with respective eigenvectors $(0,1,0)$, $(1,0,0)$

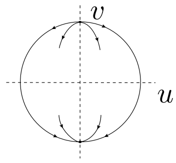


Figure 4: Phase portrait of Case 4: $d_2^2 + 4d_3 < 0$.

and $(1, 0, -(d_3 + 1)/d_2)$ if $d_2 \neq 0$, or $(0, 1, 0)$, $(1, 0, 0)$ and $(0, 0, 1)$ if $d_2 = 0$. Now, $\{0, -1, d_3\}$ are the eigenvalues of the linear part of (34) at $(0, -1, 0)$ with the same eigenvectors. As in Section 6, we consider the planar vector field induced by (34) through the central projection (19), i.e.

$$\begin{aligned} \dot{u} &= d_1 u^2 + (d_3 + 1)uv - d_2 v + d_1, \\ \dot{v} &= -u^2 + d_1 uv + d_3 v^2 + d_2 u + d_3. \end{aligned} \tag{35}$$

If $d_2^2 + 4d_3 > 0$ and $d_3 \neq 0$, the vector field associated to (35) has two singularities

$$\left(\frac{d_2 \pm \sqrt{d_2^2 + 4d_3}}{2}, -\frac{d_1 \left(d_2 \pm \sqrt{d_2^2 + 4d_3} \right)}{2d_3} \right). \tag{36}$$

For determining the phase portrait of (35), or equivalently the phase portrait of (34), we distinguish four cases.

Case 4: $d_2^2 + 4d_3 < 0$. In this case we have that $d_3 < 0$. Therefore, the unique singularities of (34) are on $\mathbb{S}^1$. Hence, since $\{0, 1, -d_3\}$ and $\{0, -1, d_3\}$ are its respective eigenvalues those singularities are nodes. Thus, the phase portrait of (34) or equivalently (31) is equivalent to the one of Figure 4.

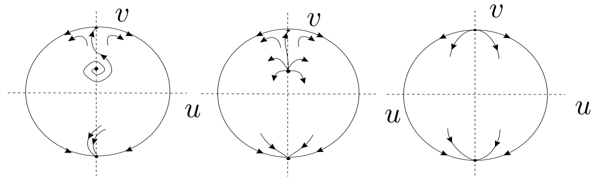
Case 5: $d_3 = 0$. We have that (35) becomes

$$\dot{u} = d_1 u^2 + uv - d_2 v + d_1, \quad \dot{v} = -u^2 + d_1 uv + d_2 u. \tag{37}$$

Doing the change of variables $u = r/s$, $v = 1/s$ and introducing the change of time $d\tau = s^{-1} dt$, system (37) becomes

$$\dot{r} = r^3 - d_2 r^2 s + d_1 s^2 + r - d_2 s, \quad \dot{s} = r^2 s - d_2 r s^2 - d_1 r s. \tag{38}$$

System (38) is induced by (34) through the central projection of $\mathbb{S}^2$ on the plane tangent to the point $(0, 1, 0)$, this fact can be find in [2] page 221. System



$(a): d_2 \neq 0$ $(b): d_2 \neq 0$ $(c): d_2 = 0$

Figure 5: Phase portrait of Case 5: $d_3 = 0$.

(38) has the singularities $(0,0)$ and $(0, d_2/d_1)$. Note that if $d_2 = 0$, then $(0,0)$ is the unique singularity of (38).

The linear part of (38) at $(0,0)$ has eigenvalues 0 and 1 with respective eigenvectors $(d_2, 1)$ and $(1,0)$. Now, we do the change of variables $r = d_2p+q$, $s = p$, and (38) becomes

$$\begin{aligned}
\dot{p} &= -d_1d_2p^2 - d_1pq + d_2p^2q + pq^2, \\
\dot{q} &= q + (d_1d_2^2 + d_1)p^2 + d_1d_2pq + d_2pq^2 + q^3.
\end{aligned}$$

Thus, applying Theorem 65 from [2] is easy to see that if $d_2 \neq 0$ then $(0,0)$ is a saddle–node of (38) and if $d_2 = 0$ then $(0,0)$ is a topological node of (38).

Now, if $d_2 \neq 0$ system (37) has the unique singularity $(0, d_1/d_2)$. The eigenvalues of the linear part of (37) at the singularity $(0, d_1/d_2)$ are

$$\frac{d_1}{2d_2} \pm \frac{\sqrt{d_1^2(1-4d_2^2) - 4d_2^4}}{2|d_2|}, \tag{39}$$

and its determinant is $d_1^2 + d_2^2$. Hence, $(0, d_1/d_2)$ is a node or a focus of (38).

Doing the change of variables $u = \tilde{r}$, $v = \tilde{s} + d_1/d_2$, system (37) can be written into the form

$$\begin{pmatrix} \dot{\tilde{r}} \\ \dot{\tilde{s}} \end{pmatrix} = \begin{pmatrix} \dfrac{d_1}{d_2} & -d_2 \\ \dfrac{d_1^2 + d_2^2}{d_2} & 0 \end{pmatrix} \begin{pmatrix} \tilde{r} \\ \tilde{s} \end{pmatrix} + \tilde{r} \begin{pmatrix} d_1 & 1 \\ -1 & d_1 \end{pmatrix} \begin{pmatrix} \tilde{r} \\ \tilde{s} \end{pmatrix}. \tag{40}$$

We will suppose that the singularity $(0, d_1/d_2)$ is a focus of system (37). If $d_1d_2 < 0$, by (39) $(0, d_1/d_2)$ is a stable focus and by Theorem B$(ii)'$ of [9], it follows that there does not exist limit cycles surrounding $(0, d_1/d_2)$.

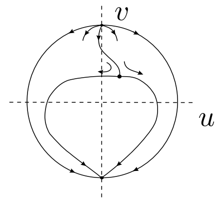


Figure 6: Phase portrait of Case 6: $d_2^2 + 4d_3 = 0$, $d_3 \neq 0$.

Now, if $d_1 d_2 > 0$, introducing the change of time $d\tau = -dt$ on system (40), we obtain that $(0, d_1/d_2)$ is a stable focus. Hence, again we applied Theorem B$(ii)'$ of [9] and obtain that there does not exist limit cycles surrounding $(0, d_1/d_2)$. Therefore, we can conclude that the phase portrait of (34) on the Poincaré disc is equivalent to ones of Figure 5. Note that, since (37) is a quadratic system, there does not exist limit cycles surrounding the node of Figure 5 (b), see for more details [15].

Case 6: $d_2^2 + 4d_3 = 0$, $d_3 \neq 0$. In this case we have that $d_3 < 0$ and $d_2 \neq 0$. Since the eigenvalues of two singularities of (34) on $\mathbb{S}^1$ are $\{0, 1, -d_3\}$ and $\{0, -1, d_3\}$ we have that they are nodes. We have that $d_3 = -d_2^2/4$. Note that by (36), $(d_2/2, 2d_1/d_2)$ is the unique singularity of system (35) and 0, $2d_1/d_2$ are the corresponding eigenvalues. Now, doing the change of variables $u = (d_2^2/(4d_1)r + (d_2^2+4)/(4d_1)s + d_2/2$, $v = r + s + 2d_1/d_2$, and introducing the new independent variable τ through $d\tau = 2d_1/d_2 dt$, we use Theorem 65 from [2] to see that $(d_2/2, 2d_1/d_2)$ is a saddle–node of (35). Thus, we can conclude that the phase portrait of (34) on the Poincaré disc is equivalent to the one of Figure 6.

Case 7: $d_2^2 + 4d_3 > 0$, $d_3 \neq 0$. In this case system (35) has two singularities $A_\pm = \Big((d_2 \pm \sqrt{d_2^2+4d_3})/2, -d_1(d_2 \pm \sqrt{d_2^2+4d_3})/(2d_3)\Big)$. We have that the traces of the linear part of (35) at the singularities $A_\pm$ are $\mathrm{tr}(DX(A_\pm)) = -(d_1(d_2 \pm \sqrt{d_2^2+4d_3}))/(2d_3)$, respectively, where X denotes the vector field determined by (35) and DX its Jacobian matrix. Note that, $\mathrm{tr}(DX(A_+)) \cdot \mathrm{tr}(DX(A_-)) = -d_1^2/d_3 \neq 0$. Denoting by $\det_{A_\pm} = \det(DX(A_\pm))$ the determinant of the Jacobian matrix of (35) at the singularities $A_\pm$, we have

$$\det\nolimits_{A_\pm} = \frac{\sqrt{d_2^2+4d_3}}{2d_3}\left((d_3 + d_3^2 + d_1^2)\sqrt{d_2^2+4d_3} \pm d_2(d_3^2 - d_3 + d_1^2)\right).$$

Since $d_3 \neq 0$ and d_1 is real, is easy to see that $\det_{A_+} \det_{A_-} \neq 0$. Therefore, the singularities A_+ and A_- are nodes, saddles or foci.

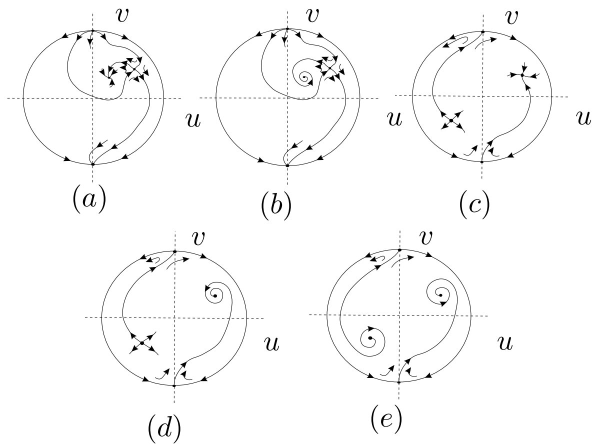


Figure 7: Phase portrait of Case 7: $d_2^2 + 4d_3 > 0$, $d_3 \neq 0$.

In this case the vector field (34) has six singularities on $\mathbb{S}^2$, i.e. two singularities on $\mathbb{S}^1$, two singularities on the northern hemisphere and two singularities on the southern hemisphere. The singularities on the northern hemisphere and on the southern hemisphere correspond to the singularities $A_{\pm}$. These singularities are saddles, nodes or foci. Now, we know that the Poincaré index of a singularity is -1 in the case of a saddle, 1 in the case of a node or focus, and by the Poincaré–Hopf Index Theorem (see, for instance [12]) we also know that the sum of the indices of all singularities of a vector field on $\mathbb{S}^2$ is 2. Since the eigenvalues of the singularities of (34) on $\mathbb{S}^1$ are $\{0, 1, -d_3\}$ and $\{0, -1, d_3\}$, if $d_3 < 0$ these singularities are nodes and their indices are 1. Therefore, the singularities of (34) on the northern hemisphere, respectively southern hemisphere, are a saddle and either a node or a focus. Now, if $d_3 > 0$ the singularities of (34) on $\mathbb{S}^1$ are saddles and their indices are -1. Thus, the singularities of (34) on the northern (respectively southern) hemisphere, are nodes or foci. Note that, by Proposition 5.2, in this case the separatrices of the saddles on $\mathbb{S}^1$, which are not contained in $\mathbb{S}^1$ do not connect, otherwise we would have two invariant circles on $\mathbb{S}^2$. Then, we can conclude that the phase portrait of (34) on the Poincaré disc is equivalent to ones of Figure 7. Note that on Figures 7 (a) and (c) do not exist limit cycles, because system (35) is quadratic and there are no limit cycles surrounding a node, see [15]. By the same argument there are no limit cycles surrounding the node of Figure 7 (d). Moreover, on Figures 7 (c), (d) and (e)

the singularities corresponding to $A_\pm$ have always opposite stability, because $\operatorname{tr}(DX(A_+))\cdot\operatorname{tr}(DX(A_-)) = -d_1^2/d_3$ and in this cases $d_3 > 0$. We claim that do not exist limit cycles on the phase portraits of Figures 7 (b), (d), (e) and there is not homoclinic orbit on (a) and (b). In order to prove the claim we consider the function $D(x,y) = (ax+by+c)^r$. We have that

$$\operatorname{div}(DX)(x,y) = \frac{\partial DP}{\partial x}(x,y) + \frac{\partial DQ}{\partial y}(x,y) = (ax+by+c)^{r-1}q(x,y), \quad (41)$$

where X is the vector field associated to system (35), and

$$\begin{aligned} q(x,y) = {} & \big(ad_1(r+3) - br\big)x^2 \\ & + \big(ar(d_3+1) + bd_1(r+3) + a(3d3+1)\big)xy \\ & + \big(d_3(r+3)+1\big)by^2 + (brd_2 + 3d_1c)x \\ & + \big(c(3d_3+1) - ard_2\big)y + (ad_1 + bd_3)r \end{aligned} \quad (42)$$

is a conic.

We will prove that D is a *Dulac function*, i.e. we will show that (41) does not change of sign on the positive half plane determine by the straight line $ax+by+c=0$. For this we will show that it is possible to choose the parameters a, b, c and r of D such that q is a complex ellipse. Using the notation of Remark 7.1 we must show that $D_2 > 0$ and $D_3D_1 > 0$.

We have that $D_2 = e_2b^2 + e_1b + e_0$, where $e_2 = -\frac{1}{4}(d_1^2+4d_3)r^2 - \frac{1}{2}(6d_3+3d_1^2+2)r - \frac{9}{4}d_1^2$, and the other coefficients e_1, e_0 are functions of the parameters of X and D. Note that e_2 is a quadratic function in r. First we suppose that $d_1^2 + 4d_3 \neq 0$. It follows $r_\pm = -(6d_1^2+4(1+3d_3)\pm 4\sqrt{3d_1^2+(3d_3+1)^2})/(2(d_1^2+4d_3))$ are the roots of $e_2(r) = 0$. Similarly, we obtain that $D_3D_1 = \tilde{e}_2c^2 + \tilde{e}_1c + \tilde{e}_0$, where $\tilde{e}_2(r) = ((2ad_1(3d_3+1) + 3bd_1^2 + b(3d_3+1)^2)(ad_1 + b(d_3-1)/4)r^2 + ((2ad_1(3d_3+1) + 3bd_1^2 + b(3d_3+1)^2)(3ad_1 + b(3d_3+1))/4)r$. We have that

$$\begin{aligned} \tilde{e}_2(r_+)\tilde{e}_2(r_-) = {} & -\frac{9d_1^2\big((3d_1^2+(3d_3+1)^2)b + 2ad_1(3d_3+1)\big)^2}{4(d_1^2+4d_3)^2}\cdot \\ & \cdot\big(3d_1^2a^2 - (6bd_3d_1 + 2bd_1)a - b^2(3d_3+1)^2 - 4b^2d_1^2\big). \end{aligned} \quad (43)$$

Since by hypotheses $d_3d_1 \neq 0$, it follows that for b sufficiently large $(3d_1^2 + (3d_3+1)^2)b + 2ad_1(3d_3+1) \neq 0$, and $3d_1^2a^2 - (6bd_3d_1 + 2bd_1)a - b^2(3d_3+1)^2 - 4b^2d_1^2 = 0$ has two real roots on a given by $a_\pm = \big(3d_3+1\pm 2\sqrt{3d_1^2+(3d_3+1)^2}\big)/(3d_1)b$. Hence, by (43) we can choose $b, a \in \mathbb{R}$ such that $\tilde{e}_2(r_+)\tilde{e}_2(r_-) < 0$. This implies that $\tilde{e}_2(r_+)$ and $\tilde{e}_2(r_-)$ have opposite signs. Therefore, in a neighborhood of r_+ or r_- we can choose r such that $e_2(r) > 0$ and $\tilde{e}_2(r) > 0$. Thus, for b and c sufficiently large we have $D_2 > 0$ and $D_3D_1 > 0$ and so, by Remark 7.1, the conic (42) is a complex ellipse. Therefore, the function $D(x, y) = (ax + by + c)^r$ is a Dulac function for $ax + by + c > 0$, i.e. (41) does not change of sign on the positive half–plane determined by $ax + by + c = 0$. Note that $|c|/\sqrt{b^2 + a^2}$ is the distance of the straight line $ax + by + c = 0$ to the origin $(0, 0)$. So for c big enough we have that if system (35) have limit cycles or a homoclinic orbit, then they must belong to the half–plane determined by $ax + by + c = 0$ that contains the singularities of (35). If the singularities of (35) belongs to the positive half–plane then, by Theorem 7.2, system (35) has neither limit cycles, nor homoclinic orbits. Otherwise, the singularities of system (35) belongs to the positive half–plane determined by the straight line $-ax - by - c = 0$. Then, considering the function $D(x, y) = (-ax - by - c)^r$ we obtain the same result, since $\text{div}(DX)(x, y) = \dfrac{\partial DP}{\partial x}(x, y) + \dfrac{\partial DQ}{\partial y}(x, y) = -(-ax - by - c)^{r-1}q(x, y)$, where q is given by (42).

Now, we consider the case $d_3 = -d_1^2/4$. In this case $D_2 = e_2b^2 + e_1b + b_0$, where $e_2 = -(1/4)(4 + 3d_1^2)r - (9/4)d_1^2$. Note que e_2 is a straight line in r and $r_0 = -9d_1^2/(4 + 3d_1^2)$ is the solution of $e_2(r) = 0$. We also have that $D_3D_1 = \tilde{e}_2c^2 + \tilde{e}_1c + \tilde{e}_0$, where $\tilde{e}_2(r) = (-r(8ad_1(3d_1^2 - 4) - b(3d_1^2 + 4)^2)/256)(r(d_1(4a - bd_1) - 4b) + 3d_1(bd_1 + 4a) + 4b)$. Note that $\tilde{e}_2(r_0) = -9d_1^2(8d_1(3d_1^2 - 4)a - b(3d_1^2 + 4)^2)(b(9d_1^2 + 4) + 12d_1a)/256$. If $3d_1^2 - 4 \neq 0$, solving $\tilde{e}_2(r_0) = 0$ on a we obtain $a = -b(9d_1^2 + 4)/(12d_1)$ and $a = b(3d_1^2 + 4)^2/(8d_1(3d_1^2 - 4))$. Therefore, we can choose a such that $\tilde{e}_2(r_0) > 0$. Hence, there exists r in a neighborhood of r_0 such that $e_2(r) > 0$ and $\tilde{e}_2(r) > 0$. Thus, as in the previous case, it follows that system (35) has neither limit cycles, nor homoclinic orbits. Now if $3d_1^2 - 4 = 0$, we have that $a = -4b/(3d_1)$ is the solution of $\tilde{e}_2(r_0) = 0$ on a. As $\tilde{e}_2(r_0)$ is a straight line on a, we can choose a such that $\tilde{e}_2(r_0) > 0$. Then, using the same argument as in the previous case it follows that system (35) has neither limit cycles, nor homoclinic orbits. ∎

We can realize all the phase portraits of Figure 7. For example, system (34) has phase portrait equivalent to Figures 7 (a), if $d_1 = d_2 = 1$ and $d_3 = -6/25$, Figures 7 (b), if $d_1 = d_2 = 1$ and $d_3 = -1/5$, Figures 7 (c), if $d_1 = 3$, $d_2 = 0$ and $d_3 = 1/9$, Figures 7 (d), if $d_1 = d_2 = 1$ and $d_3 = 1/10$, Figures 7 (e), if $d_1 = 1$, $d_2 = 0$ and $d_3 = 1$.

Proof of Corollary 1.5. It is a straightforward consequence of Theorem 1.2 and of proofs of Theorems 1.3 and 1.4. ∎

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