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A Note on the Range of Generalized Derivation

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1. Introduction

Let $\mathcal{L}(H)$ be the algebra of all bounded linear operators acting on a complex separable and infinite dimensional Hilbert space H. For operators $A, B \in \mathcal{L}(H)$ we define the generalized derivation $\delta_{A,B}$ associated with (A, B) by

$$\delta_{A,B}(X) = AX - XB \quad \text{for } X \in \mathcal{L}(H).$$

If $A = B$, then $\delta_{A,A} = \delta_A$ is called the inner derivation. The theory of derivations has been extensively dealt with in the literature (see for example [1, 2, 3, 5, 6, 7, 8, 9, 10, 16, 17, 18, 20] and [21]).

For a linear operator T acting on a Banach space X, we denote by T^*, $Ker\,T$ and $R(T)$ respectively the adjoint, the kernel and the range of T. Also we denote by $\overline{R(T)}$ and $\overline{R(T)}^{\omega}$ respectively the closure of the range of T respect to the norm topology and the weak operator topology.

In this work we give the extension of the results showed by Williams [21, p. 301] and Ho [13, p. 511] to $\delta_{A,B}$. We will give some conditions for $A,\ B \in \mathcal{L}(H)$ under which

$$\overline{R(\delta_{A,B})}^{\tau} \cap Ker\delta_{A^*,B^*} = \{0\},$$

where $\overline{R(\delta_{A,B})}^{\tau}$ denotes closure of $R(\delta_{A,B})$ respect to the norm topology or the weak operator topology.

In section 1, we prove that if A and B are isometries (resp. co-isometries) or if $P(A)$ and $P(B)$ are normal for some non-trivial polynomial P with degree ≤ 2, then

$$\overline{R(\delta_{A,B})} \cap Ker\,\delta_{A^*,B^*} = \{0\}.$$

Recall [12] that $A \in \mathcal{L}(H)$ is bloc-diagonal if there exists an increasing sequence $\{P_n\}_n$ of self-adjoint projectors of finite rank in $\mathcal{L}(H)$ such that $\lim_{sot} P_n = I$ and $P_nA = AP_n$ for all $n \in I\!N$, where $\lim_{sot}$ is the limit respect to the strong operator topology in $\mathcal{L}(H)$.

In section 2, we prove that if A is bloc-diagonal then every positive operator in $\overline{R(\delta_A)}^{\omega}$ vanishes. As a consequence of this we obtain that if A is bloc-diagonal then $\overline{R(\delta_{A,B})}^{\omega} \cap Ker\delta_{A^*,B^*} = \{0\}$ for every $B \in \mathcal{L}(H)$.

2. Conditions under which $\overline{R(\delta_{A,B})} \cap Ker\delta_{A^*,B^*} = \{0\}$

Let $\mathcal{C}_1(H)$ be the ideal of trace class operators, that is, the set of all compact operators $T \in \mathcal{L}(H)$ for which the eigenvalues of $(T^*T)^{\frac{1}{2}}$ counted according to their multiplicity are summable. The trace function is defined by $Tr(T) = \sum_n < Te_n, e_n >$, where (e_n) is any complete orthonormal sequence in H. Recall that the ultraweak continuous linear functionals on $\mathcal{L}(H)$ are those of the form f_T for some $T \in \mathcal{C}_1(H)$ and the weak continuous linear functionals on $\mathcal{L}(H)$ are those of the form f_T, where T is of finite rank.

Lemma 2.1. *Let* $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ *on* $H \oplus H$, *where* $A, B \in \mathcal{L}(H)$. *Then we have the following assertions :*

i) *If* $\overline{R(\delta_T)}^{\tau} \cap Ker\delta_{T^*} = \{0\}$, *then* $\overline{R(\delta_{A,B})}^{\tau} \cap Ker\delta_{A^*,B^*} = \{0\}$;

ii) *If* $R(\delta_T) \cap Ker\delta_{T^*} = \{0\}$, *then* $R(\delta_{A,B}) \cap Ker\delta_{A^*,B^*} = \{0\}$.

Proof. i) Let $C \in \overline{R(\delta_{A,B})}^{\tau} \cap Ker\delta_{A^*,B^*}$. Then there exists a sequence $\{X_\alpha\}_\alpha$ of elements of $\mathcal{L}(H)$ such that $\lim_{\tau} AX_\alpha - X_\alpha B = C$ and $A^*C = CB^*$. Let $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, $Y_\alpha = \begin{pmatrix} 0 & X_\alpha \\ 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}$ on $H \oplus H$. Then

$$\begin{aligned} \lim_{\tau} TY_\alpha - Y_\alpha T &= \lim_{\tau} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & X_\alpha \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & X_\alpha \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \\ &= \lim_{\tau} \begin{pmatrix} 0 & AX_\alpha - X_\alpha B \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

If $\lim_{\omega} \begin{pmatrix} 0 & AX_\alpha - X_\alpha B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$ on $H \oplus H$. Then

$$\left| \left\langle \begin{pmatrix} L_{11} & L_{12} - (AX_\alpha - X_\alpha B) \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} y \\ 0 \end{pmatrix} \right\rangle \right|$$

converges to 0, hence $| < [L_{12} - (AX_\alpha - X_\alpha B)]x, y > |$ converges to 0 for all $x, y \in H$, which implies that

$$\lim_\omega AX_\alpha - X_\alpha B = L_{12}.$$

As the same we prove that

$$L_{11} = L_{21} = L_{22} = 0.$$

This implies that

$$\begin{aligned}\lim_\omega \begin{pmatrix} 0 & AX_\alpha - X_\alpha B \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & \lim_\omega AX_\alpha - X_\alpha B \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

If $\lim \begin{pmatrix} 0 & AX_\alpha - X_\alpha B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$ on $H \oplus H$, then

$$\left\| \begin{pmatrix} L_{11} & L_{12} - [AX_\alpha - X_\alpha B] \\ L_{21} & L_{22} \end{pmatrix} \right\| \quad \text{converges to } 0,$$

hence

$$\|L_{12} - [AX_\alpha - X_\alpha B\| \text{ converges to } 0 \text{ and } L_{11} = L_{21} = L_{22} = 0.$$

This implies that

$$\begin{aligned}\lim \begin{pmatrix} 0 & AX_\alpha - X_\alpha B \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & \lim AX_\alpha - X_\alpha B \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

Hence, $\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} = S \in \overline{R(\delta_T)}^\tau$. Since $ST^* = T^*S$, then $S \in \overline{R(\delta_T)}^\tau \cap Ker\, \delta_{T^*} = \{0\}$. So $C = 0$. This completes the proof of i). To prove ii) it suffices to replace $\overline{R(\delta_T)}^\tau$ with $R(\delta_T)$.

In the following theorem we give an extension of the result of [21, p. 301] and [13, p. 511] to $\delta_{A,B}$.

Theorem 2.1. *Let $A, B \in \mathcal{L}(H)$. If A and B are isometries (resp. co-isometries) or $P(A)$ and $P(B)$ are normal for some non-trivial polynomial P with degree ≤ 2 then*

$$\overline{R(\delta_{A,B})} \cap Ker\delta_{A^*,B^*} = \{0\}.$$

Proof. i) If A and B are isometries (resp. co-isometries), then

$$T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

is also an isometry (resp. co-isometry) on $\mathcal{L}(H \oplus H)$. By [21, p. 301], we have $\overline{R(\delta_T)} \cap Ker\delta_{T^*} = \{0\}$. Hence from Lemma 2.1, we conclude that $\overline{R(\delta_{A,B})} \cap Ker\delta_{A^*,B^*} = \{0\}$.

ii) The result of [13, Theorem 3 (1)] asserts that if $T \in \mathcal{L}(H)$ is such that $P(T)$ is normal for some non-trivial polynomial P with degree ≤ 2, then $\overline{R(\delta_T)} \cap Ker\delta_{T^*} = \{0\}$. Indeed, suppose that $T^2 - 2\alpha T - \beta = N$ is a normal operator. Let $\lim TX_n - X_nT = S^* \in \overline{R(\delta_T)} \cap Ker\delta_{T^*}$. Then

$$\lim(N + 2\alpha T)X_n - X_n(N + 2\alpha T) = \lim T^2X_n - X_nT^2 = TS^* + S^*T.$$

This implies that $TS^* + S^*T - 2\alpha S^* \in \overline{R(\delta_N)} \cap Ker\delta_{N^*}$ so that $TS^* + S^*T - 2\alpha S^* = 0$ by [4, Theorem 1.7]. Hence

$$(S + S^*)(T - \alpha) = (T - \alpha)(S - S^*) \text{ and } (T - \alpha)S^* = -S^*(T - \alpha).$$

The Putnam-Fuglede theorem then gives

$$(S^* + S)(T - \alpha) = (T - \alpha)(S^* - S) \text{ and } (T - \alpha)S = -S(T - \alpha).$$

Combining these equations we get

$$(T - \alpha)(S^* + S) = 0 \text{ and } (S^* + S)(T - \alpha) = 0.$$

Hence $S^*T = TS^*$. Therefore $S^*S \in \overline{R(\delta_T)} \cap Ker\delta_{T^*}$ so that $S = 0$ by [13, Lemma 3]. Now, if $P(A)$ and $P(B)$ are normal for some non-trivial polynomial P with degree ≤ 2, then $P(T)$ is also normal for $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, hence from the previous result $\overline{R(\delta_T)} \cap Ker\delta_{T^*} = \{0\}$. From Lemma 2.1, we conclude that

$$\overline{R(\delta_{A,B})} \cap Ker\delta_{A^*,B^*} = \{0\}.$$

LEMMA 2.2. *Let* $A, B \in \mathcal{L}(H)$. *If* $T \in \overline{R(\delta_{A,B})}^{\tau} \cap Ker\delta_{A^*,B^*}$, *then* $T^*T \in \overline{R(\delta_B)}^{\tau}$ *and* $TT^* \in \overline{R(\delta_A)}^{\tau}$.

Proof. If $T \in \overline{R(\delta_{A,B})}^{\tau} \cap Ker\delta_{A^*,B^*}$, then there exists a sequence $\{X_\alpha\}_\alpha$ of elements of $\mathcal{L}(H)$ such that $T = \lim_\tau AX_\alpha - X_\alpha A = 0$ and $A^*T - TB^* = 0$. Hence

$$T^*T = \lim_{\tau} T^*AX_\alpha - T^*X_\alpha B = \lim_{\tau} BT^*X_\alpha - T^*X_\alpha B,$$

and

$$TT^* = \lim_{\tau} AX_\alpha T^* - X_\alpha BT^* = \lim_{\tau} AX_\alpha T^* - X_\alpha T^*A,$$

since right multiplication and left multiplication are continuous with respect to the topology τ.

The following lemma is proved in [19], we need it to prove the next theorem.

LEMMA 2.3. *Let* $B \in \mathcal{L}(H)$ *be a normal operator and* $X \in \mathcal{C}_2(H)$ *such that* $BX - XB \in \mathcal{C}_1(H)$, *then* $Tr(BX - XB) = 0$.

For the unilateral right shift with a non null weight, we have the following result.

THEOREM 2.2. *Let* $S \in \mathcal{L}(H)$ *be the unilateral right shift with a non null weight* $(\alpha_n)_n$; $\alpha_n \neq 0$ *for all* $n \in I\!N$ *and let* $B \in \mathcal{L}(H)$ *be normal. Then*

$$R(\delta_{S,B}) \cap Ker\delta_{S^*,B^*} = \{0\}.$$

Proof. Let $T \in R(\delta_{S,B}) \cap Ker\delta_{S^*,B^*}$. By the same argument as in the proof of Lemma 2.2, we get that $TT^* \in R(\delta_S)$, hence from [13] $TT^* \in \mathcal{C}_1(H)$. Which is equivalent to $T \in \mathcal{C}_2(H)$. On the other hand $T^*T = BT^*X - T^*XB$ with $T^*T \in \mathcal{C}_1(H)$, $T^*X \in \mathcal{C}_2(H)$ and B is normal. Hence by Lemma 2.3, we conclude that $Tr(T^*T) = 0$. Since T^*T is positive, then $T = 0$.

3. POSITIVE OPERATORS IN $\overline{R(\delta_A)}^{\omega}$

DEFINITION 3.1. [12] An operator $A \in \mathcal{L}(H)$ is bloc-diagonal if there exists an increasing sequence $\{P_n\}_n$ of self-adjoint projectors of finite rank in $\mathcal{L}(H)$ such that $\lim\limits_{sot} P_n = I$ and $P_nA = AP_n$ for all $n \in I\!N$, where $\lim\limits_{sot}$ is the limit with respect to the strong operator topology in $\mathcal{L}(H)$.

Example 1. [12] Let $H = \bigoplus_{n=0}^{\infty} H_n$. If $A = \bigoplus_{n=0}^{n} A_n$ where $A_n = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ on $\mathbb{C}^2$, then A is bloc-diagonal.

For bloc-diagonal operators we have the following result.

Theorem 3.1. *Let $A \in \mathcal{L}(H)$. If A is bloc-diagonal then every positive operator in $\overline{R(\delta_A)}^{\omega}$ vanishes.*

Proof. Suppose that A is bloc-diagonal. Then there exists an increasing sequence $\{P_n\}_n$ of self-adjoint projectors of finite rank in $\mathcal{L}(H)$ such that

$$\lim_{sot} P_n = I \text{ and } P_n A = A P_n \text{ for all } n \in I\!N.$$

Let T a positive operator in $\overline{R(\delta_A)}^{\omega}$, then there exists a sequence $\{X_\alpha\}_\alpha$ in $\mathcal{L}(H)$ such that $T = \lim_{\omega} A X_\alpha - X_\alpha A$. By multiplication right and left by P_n, we obtain

$$P_n T P_n = \lim_{\omega} P_n A X_\alpha P_n - P_n X_\alpha A P_n,$$

since $AP_n = P_n A$, then

$$(*) \quad P_n T P_n = \lim_{\omega} P_n A P_n P_n X_\alpha P_n - P_n X_\alpha P_n P_n A P_n.$$

Since $AP_n = P_n A$ and $A^* P_n = P_n A^*$, then $R(P_n) = H_n$ reduces A. Hence A has the decomposition

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \quad \text{on } H = H_n \oplus H_n^{\perp}.$$

Let $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$, $X_\alpha = \begin{pmatrix} X_\alpha^{11} & X_\alpha^{12} \\ X_\alpha^{21} & X_\alpha^{22} \end{pmatrix}$ and $P_n = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ on $H = H_n \oplus H_n^{\perp}$. It follow from (*) that

$$\begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix} = \lim_{\omega} \begin{pmatrix} A_{11} X_\alpha^{11} - X_\alpha^{11} A_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence for all $x, y \in H_n$,

$$\left| \left\langle \begin{pmatrix} T_{11} - A_{11} X_\alpha^{11} - X_\alpha^{11} A_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ 0 \end{pmatrix} \right\rangle \right|$$

converges to 0. This implies that $\lim_{\omega} A_{11}X_{\alpha}^{11} - X_{\alpha}^{11}A_{11} = T_{11}$, that is $T_{11} \in \overline{R(\delta_{A_{11}})}^{\omega}$. Since dimension of H_n is finite, then $T_{11} \in R(\delta_{A_{11}})$, hence there exists $Y \in \mathcal{L}(H_n)$ such that $T_{11} = A_{11}Y - YA_{11}$, which implies that

$$Tr(T_{11}) = Tr(A_{11}Y) - Tr(YA_{11}) = 0.$$

Since P_n is auto-adjoint, then $P_nTP_n = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix}$ is positive, and hence T_{11} is positive. Since $Tr(T_{11}) = 0$, then $T_{11} = 0$, and hence $P_nTP_n = 0$ for all $n \in I\!N$. On the other hand, since

$$\lim_{sot} P_n = I, \ \lim_{sot} \|P_nTP_nx - TP_nx\| = 0$$

and

$$\lim_{sot} \|TP_nx - Tx\| \leq \lim_{sot} \|T\|\|P_nx - x\| = 0,$$

then $\lim_{sot} P_nTP_n = \lim_{sot} TP_n$ and $\lim_{sot} TP_n = T$. This implies that $\lim_{sot} P_nTP_n = T$ for all $n \in I\!N$. Finally, $T = 0$.

As an immediate consequence we have the following corollary:

COROLLARY 3.1. *Let $A \in \mathcal{L}(H)$. If A is bloc-diagonal, then*

$$\overline{R(\delta_{A,B})}^{\omega} \cap Ker\delta_{A^*,B^*} = \{0\}$$

for every $B \in \mathcal{L}(H)$.

Proof. If $A \in \mathcal{L}(H)$ is bloc-diagonal and $T \in \overline{R(\delta_{A,B})}^{\omega} \cap Ker\delta_{A^*,B^*}$, then by Lemma 2.2, $TT^* \in \overline{R(\delta_A)}^{\omega}$. By Theorem 3.1, we conclude that $TT^* = 0$, and hence $T = 0$.

Recall [12] that $A \in \mathcal{L}(H)$ is quasi-diagonal if there exists an increasing sequence $\{P_n\}_n$ of self-adjoint projectors of finite rank in $\mathcal{L}(H)$ such that $\lim_{sot} P_n = I$ and $\lim \|P_nA - AP_n\| = 0$ for all $n \in I\!N$. Every bloc-diagonal operator is quasi-diagonal and the converse is false, see [12]. The following example show that in general Theorem 3.1 does not hold for quasi-diagonal operators.

EXAMPLE 2. Let $A = S + S^*$ where S is the unilateral shift defined by $Se_n = e_{n+1}$ where $\{e_n\}_n$ is any complete orthonormal sequence in H. Since A is self-adjoint, then A is quasi diagonal [12]. Let $T = I - SS^*$, then

$T = (S+S^*)S - S(S+S^*) = AS - SA$. Hence $T \in R(\delta_A)$. On the other hand, we have

$$< Tx, x >=< (I - SS^*)x, x >= \|x\|^2 - \|S^*x\|^2, \quad \text{for all } x \in H.$$

Since $\|S^*\| \leq 1$, then $< Tx, x > \geq 0$ for all $x \in H$. Thus T is positive. Finally, T is a non null positive operator in $R(\delta_A)$.

4. A comment

In [1] (see also [15]) it is shown that *every finite rank operator in* $\overline{R(\delta_{A,B})}^{\omega} \cap Ker\delta_{A^*,B^*}$ *vanishes* and *every trace class operator in* $\overline{R(\delta_{A,B})}^{\omega^*} \cap Ker\delta_{A^*,B^*}$ *vanishes,* where $\overline{R(\delta_{A,B})}^{\omega^*}$ is the closure of $R(\delta_{A,B})$ with respect to the ultra-weak topology ω^*.

However in [11](see also [14]) the author ask; if every compact operator in $\overline{R(\delta_A)}^{\omega} \cap \{A^*\}'$ is quasinilpotent? A partial answer is given in [1] (see also [14]) if A or A^* is dominant and in [10] if A or A^* lies in $\mathcal{U}_0$.

Recall that $A \in \mathcal{L}(H)$ is dominant if for all $\lambda \in \mathbb{C}$, there exists a real number $M_\lambda \geq 1$ such that $\|(A-\lambda)^*x\| \leq M_\lambda \|(A-\lambda)x\|$ and A lie in $\mathcal{U}_0$ if A satisfies the absolute value condition $|A|^2 \leq |A^2|$ and every normal subspaces of A are reducing (An invariant subspace M of A is said to be a normal subspace of A if $A\mid_M$ is normal).

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