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Non-trivial Derivations on Commutative Regular Algebras

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1. Introduction

The theory of derivations on operator algebras (in particular, C^*-algebras, AW^*-algebras and W^*-algebras) is an important and well developed integral part of the general theory of operator algebras and modern mathematical physics (see e.g. [13], [14], [4]). It is well-known that every derivation of a C^*-algebra is norm continuous and that every derivation of an AW^*-algebra, and in particular of a W^*-algebra is always inner. A very detailed study of derivations on C^*-algebras is given in [14] and a comprehensive account of such a theory in general Banach algebras is given in the recent monograph [7]. In particular, the latter book details the conditions guaranteeing the automatic continuity of derivations on various classes of Banach algebras.

The development of noncommutative integration theory initiated in [15], has brought about new classes of (not necessarily Banach) algebras of unbounded operators, which by their algebraic and order-topological structure are still somewhat similar to C^*, W^* and AW^*-algebras (see e.g. [8]). Special importance here is attached to the algebra $L(M)$ of all measurable operators affiliated with a von Neumann algebra M. In the classical case, when $M = L_\infty[0,1]$, the algebra $L(M)$ coincides with the familiar space $S(0,1)$ of all (classes of) measurable functions on $[0,1]$.

This development has naturally led to the question concerning the description of derivations on algebras $L(M)$ and their properties. Some partial results in this direction are contained in [9], [10], [2]. In particular, in [2] the following two problems are explicitly stated:

1) Is every derivation on $L(M)$ continuous with respect to the measure topology on $L(M)$?

2) Is every derivation on $L(M)$ necessarily inner?

The present paper answers both questions in the negative for the setting of commutative von Neumann algebras M. In fact, in our study of derivations we have been able to deal with a general class of commutative algebras which are regular in the sense of von Neumann, which contains as a special and important case, the subclass of algebras of measurable operators $L(M)$ affiliated with commutative von Neumann algebras M. Our key results concern the extension of a derivation δ living on a subalgebra $\mathcal{B}$ of a commutative algebra $\mathcal{A}$ which is regular in the sense of von Neumann to the algebra itself.

The paper consists of two sections and the present Introduction. In Section 2 following, we catalogue some necessary facts and tools from inverse semigroup theory and the theory of algebras which are regular in the sense of von Neumann. In particular, we describe various relevant properties of derivations on a such algebra $\mathcal{A}$ and those of a special metric ρ on $\mathcal{A}$ associated with a finite strictly positive measure living on the Boolean algebra of all idempotents in $\mathcal{A}$. In Section 3, we present results showing the existence of non-zero derivations on $\mathcal{A}$, provided the latter is complete with respect to the metric ρ. The main result in that section (and that of the paper) is given in Theorem 3.1, asserting that every derivation $\delta : \mathcal{B} \to \mathcal{A}$ extends from a subalgebra $\mathcal{B}$ to a derivation on $\mathcal{A}$. The proof of this result is partitioned into several steps, some of which correspond to various natural extensions of the subalgebra $\mathcal{B}$; in particular, the embedding of $\mathcal{B}$ into the least regular in the sense of von Neumann subalgebra in $\mathcal{A}$, the extension of $\mathcal{B}$ by an integral element with respect to $\mathcal{B}$ and the extension of $\mathcal{B}$ by a weakly transcendental element with respect to $\mathcal{B}$. The execution of the first step (as well as the proof of several other auxiliary results) is done in Section 2, while two latter steps are performed in Section 3. It is perhaps worth mentioning that in the previously considered cases, where $\mathcal{A}$ is a field, or a unital integral domain, these two steps are similar to the operations of algebraic and transcendental extensions of $\mathcal{B}$, respectively. In those cases, the procedures of extending a derivation δ are well-known, see e.g. [7, §1.8]. However, in our setting, when $\mathcal{A}$ is a commutative algebra regular in the sense of von Neumann, the extension of δ to an "integral extension of $\mathcal{B}$" encountered significant technical difficulties. It is of interest to observe, that although the idea of such an extension in the present paper remains essentially the same as for the "algebraic extension of $\mathcal{B}$" in the cases when $\mathcal{A}$ is a field, or a unital integral domain (see [7, Theorem 1.8.15]), its technical execution is based on an entirely different method. To explain this point in a little more detailed manner, we remark that the idea of the extension of δ in this instance is based on the equality

$$\delta(p(a)) = p^{\delta}(a) + p'(a)\delta(a) = 0\,,$$

where a is an algebraic element with respect to $\mathcal{B}$ and $p(x) = \sum\limits_{k=0}^{n} b_k x^k$ is a polynomial with coefficients from $\mathcal{B}$, such that $p(a) = 0$ and where $p^{\delta}(x) = \sum\limits_{k=0}^{n} \delta(b_k)x^k$ (these notations are further explained in Section 3). To use the equality above for the definition of an extension $\delta(a)$, it is necessary to justify the "division" by $p'(a)$. In the case, when $\mathcal{A}$ is a field, this justification is relatively simple [19, Ch. 10, §76], while for the case that $\mathcal{A}$ is a unital integral domains it is based on the introduction of a specially designed homomorphism (naturally linked with δ) [7, Theorem 1.8.15]. In our case of commutative algebra regular in the sense of von Neumann, none of the approaches above works (due to the existence of rich families of idempotents and zero divisors). Our approach, at this point, is entirely different. Using the properties of supports of elements from $\mathcal{A}$ (explained and introduced in Section 2) and assuming that a is an integral element with respect to $\mathcal{B}$ (in our setting there is a significant difference between the notion of an algebraic element and that of an integral element; see relevant definitions and examples in Section 3), we have managed to partition the support of a into a finite family of pairwise disjoint idempotents ℓ_k's, such that $p'(a)\ell_k$ is invertible in the reduced algebra $\mathcal{A}\ell_k$ (Proposition 3.4). This idea allows correct definition of the extension of δ to the "integral" extension $\mathcal{B}(a)$ of the subalgebra $\mathcal{B}$ (Proposition 3.5 and Proposition 3.6). The final step of our programme is the extension of δ to the subalgebra $\mathcal{B}(a)$, in the case when a is not necessarily an integral element with respect to $\mathcal{B}$. Again, if a were a transcendental element with respect to $\mathcal{B}$, then such an extension could have been easily realized through previously known schemes (see e.g. [7, §1.8]). However, in our case, there are examples of elements $a \in \mathcal{A}$, which are neither integral, nor transcendental with respect to $\mathcal{B}$ (Example 3.1) and this again renders all previously known approaches inapplicable. To deal with this obstacle, we introduce and utilize the notion of a weakly transcendental element with respect to $\mathcal{B}$. For such elements, it is still possible to define the extension of δ to the subalgebra $\mathcal{B}(a)$ (Proposition

3.7) and this makes it possible to complete the proof of Theorem 3.1 by a (standard) method involving an appeal to Zorn's lemma (cf. [19], [7]).

Having established this extension result for derivations, we apply it to obtain a criterion for existence of a non-zero derivation on an arbitrary ρ-complete commutative algebra $\mathcal{A}$ regular in the sense of von Neumann. With the help of this criterion, one can easily see that for a commutative von Neumann algebra M, the algebra $L(M)$ admits non-zero derivations, if and only if the Boolean algebra of all idempotents in M is not atomic. This further implies (see Remark 3.1) that there exists a non-zero derivation $\delta : L_\infty[a, b] \to S[a, b]$ (recall, that any derivation from $L_\infty[a, b]$ into itself is necessarily inner [13]).

We use terminology and notation from regular rings theory [16], [12], inverse semigroup theory [6], von Neumann algebra theory [17], [13] and theory of measurable operators from [15], [20], [8].

Some results from this paper have been announced in [3].

2. Preliminary information and results

The main objective of the present section is to describe the properties of an arbitrary commutative algebra that is regular in the sense of von Neumann, which will be utilized for building the extension of a differentiation living on a subalgebra of such an algebra.

Firstly, we need to review some notions and tools from the theory of inverse semigroups. Let S be an arbitrary semigroup. The elements a and b from S are called *inverses* to each other, if $aba = a$, $bab = b$ (see [6]). The semigroup S is called inverse, if for any $a \in S$ there exists a unique inverse element to a, which is denoted by $i(a)$. For any elements a, b from an inverse semigroup S it follows that: $i(ab) = i(b)i(a)$; $i(i(a)) = a$; in addition, $e = ai(a)$ is an idempotent in S, i.e., $e^2 = e$, and $ea = a$ (see, for example, [6, §1.9]). In particular, for any idempotent $f \in S$ we have $f = f \cdot f \cdot f$, i.e., $f = i(f)$, and therefore $i(af) = i(f)i(a) = fi(a)$. It is shown in [6, §1.9] that $e = ai(a)$ is the unique idempotent in S, for which $aS = eS$.

We need the following criterion for S to be inverse (see [6, §1.9]).

Proposition 2.1. *A semigroup S is inverse if and only if any two idempotents commute and for any $a \in S$ there exists an element $b \in S$ such that $a = aba$.*

The semigroup S is said to be regular if for every $a \in S$, there exists an element $b \in S$ such that $a = aba$. A regular subsemigroup of the inverse semigroup S is also an inverse semigroup and its inversion coincides with the inversion in S.

Let $\mathcal{A}$ be a commutative algebra with unit $\mathbf{1}$ over field K. Obviously, $\mathcal{A}$ is a commutative semigroup with respect to multiplication in $\mathcal{A}$. We denote by ∇ the set of all idempotents in $\mathcal{A}$.

For any $e, f \in \nabla$, we set $e \leq f$, if $ef = e$. It is well known (see, for example, [12, Prop. 1.6, p. 137]), that this relation is a partial order relation in ∇, with respect to which ∇ is a Boolean algebra. In addition, for the lattice operations and the operation of taking the complement in ∇, the following equalities are true: $e \vee f = e + f - ef$, $e \wedge f = ef$, $Ce = \mathbf{1} - e$. As usual, we denote by $e \triangle f = (e \vee f) \wedge C(e \wedge f)$ the symmetric difference of idempotents e and f.

We recall, that the nonzero element q from the Boolean algebra ∇ is called an *atom*, if from the conditions $0 \neq e \leq q$, $e \in \nabla$ it follows that $e = q$. The Boolean algebra ∇ is called *atomic*, if for any nonzero $e \in \nabla$ there exists an atom q, such that $q \leq e$.

A ring $\mathcal{A}$ is called *regular in the sense of von Neumann*, if for any $a \in \mathcal{A}$ there exists $b \in \mathcal{A}$ such that $a = aba$ (see [16]).

A commutative unital algebra, which is simultaneously a regular in the sense of von Neumann ring is called a *commutative regular in the sense of von Neumann algebra*. For brevity, everywhere below we refer to such algebras simply as to *regular algebras*.

Further, we shall always assume that $\mathcal{A}$ *is a unital commutative regular algebra over the field* K, and that ∇ is the Boolean algebra of all idempotents in $\mathcal{A}$. Clearly, with respect to the operation of multiplication in $\mathcal{A}$, the algebra $\mathcal{A}$ is a commutative regular semigroup, which is an inverse semigroup (see Proposition 2.1).

The idempotent $e \in \nabla$ is said to be the *support* of $a \in \mathcal{A}$, if $ea = a$ and from $ga = a$, $g \in \nabla$, it follows that $e \leq g$. As we have already mentioned above, for every $a \in \mathcal{A}$, we have $e := ai(a)$ belongs to ∇ and $ea = a$. If $g \in \nabla$ and $ga = a$, then $gi(a) = i(ga) = i(a)$, i.e., $ge = gi(a) \cdot a = i(a)a = e$, and therefore $e \leq g$. This means that the idempotent $e = ai(a)$ is the support of a.

We shall denote by $s(a)$ the support of the element $a \in \mathcal{A}$. If $e \in \nabla$, then $i(e) = e$, and therefore $s(e) = ei(e) = e$.

The following proposition lists some simple properties of supports.

PROPOSITION 2.2. *The following relations hold for any* $a, b \in \mathcal{A}$, $e, f \in \nabla$*:*

(i) $s(i(a)) = s(a)$, $s(a^n) = s(a)$, $s(\alpha a) = s(a)$, *for all* $\alpha \in K$, $\alpha \neq 0$, $n = 1, 2, \ldots$;

(ii) $s(ab) = s(a)s(b)$;

(iii) $ab = 0 \Leftrightarrow s(a)s(b) = 0$;

(iv) *if* $ab = 0$, *then* $i(a + b) = i(a) + i(b)$, $s(a + b) = s(a) + s(b)$;

(v) $s(b - a) \geq s(b) \vartriangle s(a)$, $s(e - f) = e \vartriangle f = (e - f)^2$.

Proof. The proofs of (i), (ii) and (iii) follow immediately from the definition of support and properties of inverse elements.

(iv) Let $ab = 0$. Since $s(a)s(b) = 0$, then $(a + b)(s(a) + s(b)) = as(a) + bs(b)s(a) + as(a)s(b) + bs(b) = a + b$, i.e., $s(a) + s(b) \geq s(a + b)$. On the other hand $s(a + b)s(a) = s((a + b)a) = s(a^2 + bs(b)s(a)a) = s(a^2) = s(a)$, and similarly $s(a + b)s(b) = s(b)$. This means that $s(a + b) \geq s(a) \vee s(b) = s(a) + s(b)$. Consequently, $s(a + b) = s(a) + s(b)$.

Using this equality together with (i), (iii), we get that $(a + b)(i(a) + i(b)) = ai(a) + ai(b) + bi(a) + bi(b) = s(a) + as(a)s(b)i(b) + bs(b)s(a)i(a) + s(b) = s(a) + s(b) = s(a + b)$. Consequently, $i(a + b) = i(a + b)s(a + b) = i(a + b)(a + b)(i(a) + i(b)) = s(a + b)(i(a) + i(b)) = (s(a) + s(b))(s(a)i(a) + s(b)i(b)) = i(a) + i(b)$.

(v) Let $e_1 = s(a) - s(a)s(b)$, $e_2 = s(b) - s(a)s(b)$, $e_3 = s(a)s(b)$. Then $e_i \in \nabla$, $e_i e_j = 0$, $i \neq j$, $i, j = 1, 2, 3$, and $s(a) \vee s(b) = e_1 \vee e_2 \vee e_3 = e_1 + e_2 + e_3$. Therefore, $b - a = (b - a)(s(a) \vee s(b)) = (b - a)e_1 + (b - a)e_2 + (b - a)e_3$. Since $s(b)e_1 = 0$, $s(a)e_2 = 0$, we have for $a_1 := (b - a)e_1 = -ae_1$, $b_1 := (b - a)e_2 = be_2$ and $d := (b - a)e_3$ that $a_1 b_1 = a_1 d = b_1 d = 0$. It now follows from (i) that $s(b - a) = s(a_1) + s(b_1) + s(d) \geq s(a_1) + s(b_1)$. However, $s(a_1) = s(-a)s(e_1) = s(a)e_1 = e_1$, $s(b_1) = s(b)s(e_2) = e_2$, and thus $s(b - a) \geq e_1 \vee e_2 = s(a) \vartriangle s(b)$. Finally, if $e, f \in \nabla$, then we get (see (iv)) that $s(e - f) = s((e - ef) + (ef - f)) = s(e - ef) + s(ef - f) = e - ef + f - ef = e \vartriangle f$. ∎

We list below a few important examples of commutative unital regular algebras.

EXAMPLE 2.1. Let Δ be an arbitrary set, and let $\mathcal{A} := K^\Delta = \big\{\{\alpha_q\}_{q \in \Delta} : \alpha_q \in K \text{ for all } q \in \Delta\big\}$ be the direct product of a Δ copies of the field K. With respect to the pointwise algebraic operations, the set $\mathcal{A}$ is a commutative algebra (over K) with unit $\mathbf{1} := \{\mathbf{1}_q\}_{q \in \Delta}$, where $\mathbf{1}_q = \mathbf{1}_K$ is the unit in K. Any idempotent in $\mathcal{A}$ has the form $e = e_B = \{\alpha_q\}_{q\in\Delta}$, where $B \subset \Delta$, $\alpha_q = \mathbf{1}_K$ for $q \in B$, $\alpha_q = 0$ for $q \in (\Delta \setminus B)$. Hence, the Boolean algebra ∇ of all idempotents in $\mathcal{A}$ coincides with the atomic Boolean algebra of all subsets of Δ. Let $a = \{\alpha_q\}_{q\in\Delta}$, $b = \{\beta_q\}_{q\in\Delta} \in \mathcal{A}$, $\beta_q := \alpha_q^{-1}$, if $\alpha_q \neq 0$, and $\beta_q := 0$, if $\alpha_q = 0$. Then $a = a^2 b$, i.e., $\mathcal{A}$ is a regular algebra, in particular, $i(a) = b$, $s(a) = e_B$, where $B = \{q \in \Delta : \alpha_q \neq 0\}$.

Example 2.2. Let (Ω, Σ, μ) be a localizable measure space (see [15]). We denote by $S = S(\Omega, \Sigma, \mu)$ the algebra of all (classes of) measurable functions on (Ω, Σ, μ) with values in the field K, where $K = \mathbb{R}$ is the field of all real numbers, or else $K = \mathbb{C}$ is the field of complex numbers. Clearly, S is a commutative regular algebra with unit $\mathbf{1}$ given by $\mathbf{1}(\omega) \equiv 1$, $\omega \in \Omega$. The Boolean algebra ∇ of all idempotents in S coincides with the Boolean algebra of classes of almost everywhere equal sets from Σ. For any $a \in S$, the support $s(a)$ is a class from ∇ with the representative $\{\omega \in \Omega : |a(\omega)| \neq 0\} \in \Sigma$.

Example 2.3. Let M be a commutative AW^* algebra, in particular, W^* algebra, and let $L(M)$ be the algebra of all measurable operators affiliated with M. It is well-known that $L(M)$ (respectively, its self-adjoint part $L_h(M) := \{a \in L(M) : a^* = a\}$) is a commutative regular algebra with a unit over the field $\mathbb{C}$ (respectively, over the field $\mathbb{R}$) (see, for example, [8], [15]). We note that the regular algebra from Example 2.1 is a special case of the regular algebra in this example.

Let $\mathcal{M}$ be an arbitrary algebra, and let $\mathcal{B}$ be a subalgebra of $\mathcal{M}$. A linear map $\delta : \mathcal{B} \to \mathcal{M}$ is called a *derivation*, if $\delta(ab) = \delta(a)b + a\delta(b)$ for any $a, b \in \mathcal{B}$. The derivation δ is called inner, if there exists an element $a \in \mathcal{M}$, such that $\delta(b) = [b, a] = ba - ab$ for all $b \in \mathcal{B}$. Clearly, every inner derivation on commutative algebras is identically equal to zero, that is in such algebras any nonzero derivation is necessarily not inner.

We list below a few simple properties of derivations, which will be frequently used in the sequel.

Proposition 2.3. *If $\mathcal{B}$ is a subalgebra in $\mathcal{A}$ and $\delta : \mathcal{B} \to \mathcal{A}$ is a derivation, then the following hold for all $b \in \mathcal{B}$ and $e \in \mathcal{B} \cap \nabla$:*

(i) $\delta(b^n) = nb^{n-1}\delta(b)$, $n = 1, 2, \ldots$;

(ii) $\delta(e) = 0$;

(iii) $\delta(be) = \delta(b)e$;

(iv) *if* $s(b) \in \mathcal{B}$, *then* $s(\delta(b)) \leq s(b)$;

(v) $\delta(i(b)) = -\delta(b)i(b^2)$ *if* $i(b) \in \mathcal{B}$.

Proof. The assertions (i), (ii) are established in [7, §1.8]. The assertion (iii) follows from (ii).

(iv) If $s(b) \in \mathcal{B}$, then appealing to (iii) we get, $\delta(b) = \delta(s(b)b) = s(b)\delta(b)$, whence $s(\delta(b)) \leq s(b)$.

(v) If $i(b) \in \mathcal{B}$, then $s(b) = b\,i(b) \in \mathcal{B}$, and therefore $0 = \delta(s(b)) = \delta(b \cdot i(b)) = \delta(b)i(b) + \delta(i(b))b$, i.e., $\delta(i(b))b = -\delta(b)i(b)$. Consequently, via Proposition 2.2 (i) and (iii) above, we conclude

$$\begin{aligned}\delta(i(b)) &= \delta(i(b)s(b)) = \delta(i(b))s(b)\\ &= \delta(i(b))b \cdot i(b) = -\delta(b)i(b)i(b) = -\delta(b)i(b^2)\,.\end{aligned}$$

∎

In what follows, we shall be examining derivations $\delta : \mathcal{B} \to \mathcal{A}$, such that $s(\delta(b)) \leq s(b)$ for each $b \in \mathcal{B}$. This condition holds automatically, if $\mathcal{B}$ is a regular subalgebra in $\mathcal{A}$, i.e., when $\mathcal{B}$ is a regular ring. Indeed, in this case, due to the uniqueness of the inverse element for every $b \in \mathcal{B}$ (taken in $\mathcal{B}$ and/or in $\mathcal{A}$), we see that the element $s(b) = bi(b)$ belongs to $\mathcal{B}$. Thus, according to Proposition 2.3 (iv), we have $s(\delta(b)) \leq s(b)$.

It is probably worthwhile to mention that there are no non-trivial derivations on the regular commutative algebra K^{Δ} from Example 2.1 (in the special case that $K = \mathbb{C}$ this fact is pointed out in [2]). Indeed, for any derivation $\delta : K^{\Delta} \to K^{\Delta}$ and $x = \{x_q\} \in K^{\Delta}$, we have

$$e_q\delta(x) = \delta(e_q x) = x_q\delta(e_q) = 0\,, \qquad \forall q \in \Delta\,,$$

where e_q is the idempotent in ∇ corresponding to the point $q \in \Delta$. This means that $\delta(x) = 0$. On the other hand, the situation with commutative regular algebras in Example 2.2 can be quite different and the demonstration of this phenomenon is the main objective of the present paper.

Generally speaking, our task in this paper consists in extending a derivation δ defined on a subalgebra $\mathcal{B} \subseteq \mathcal{A}$ to a derivation defined on the whole algebra $\mathcal{A}$. The first step in this programme is to show that δ extends to the subalgebra $\mathcal{A}(\mathcal{B}, \nabla)$, the smallest subalgebra in $\mathcal{A}$ containing $\mathcal{B}$ and the Boolean algebra ∇ of all idempotents from $\mathcal{A}$. Clearly,

$$\mathcal{A}(\mathcal{B}, \nabla) = \left\{ \sum_{i=1}^{n} b_i e_i + \sum_{j=1}^{m} \alpha_j f_j \,:\, b_i \in \mathcal{B},\ e_i, f_j \in \nabla,\ \alpha_j \in K,\ n, m \in \mathbb{N} \right\},$$

where $\mathbb{N}$ is the set of all natural numbers.

PROPOSITION 2.4. *If $\mathcal{B}$ is a subalgebra in $\mathcal{A}$ and $\delta : \mathcal{B} \to \mathcal{A}$ is a derivation such that $s(\delta(b)) \leq s(b)$ for all $b \in \mathcal{B}$, then there exists the unique derivation $\delta_1 : \mathcal{A}(\mathcal{B}, \nabla) \to \mathcal{A}$ such that $\delta_1(b) = \delta(b)$ for all $b \in \mathcal{B}$.*

Proof. For each $a = \sum\limits_{i=1}^{n} b_i e_i + \sum\limits_{j=1}^{m} \alpha_j f_j \in \mathcal{A}(\mathcal{B}, \nabla)$, we set $\delta_1(a) = \sum\limits_{i=1}^{n} \delta(b_i) e_i$. We shall show that this definition is well-defined. Let $a_k = \sum\limits_{i=1}^{n_k} b_i^{(k)} e_i^{(k)} + \sum\limits_{j=1}^{m_k} \alpha_j^{(k)} f_j^{(k)}$, $k = 1, 2$, and let $a_1 = a_2$. We denote by ∇_0 the finite Boolean subalgebra in ∇, generated by the idempotents $e_1^{(1)}, \ldots, e_{n_1}^{(1)}, e_1^{(2)}, \ldots, e_{n_2}^{(2)}$, $f_1^{(1)}, \ldots, f_{m_1}^{(1)}, f_1^{(2)}, \ldots, f_{m_2}^{(2)}$. Let $q_1, \ldots, q_l$ be all the atoms in ∇_0.

Since $a_1 = a_2$, then $a_1 q_k = a_2 q_k$ for all $k = 1, \ldots, l$, i.e.,

$$q_k \sum_{i:\, e_i^{(1)} \geq q_k} b_i^{(1)} + q_k \sum_{j:\, f_j^{(1)} \geq q_k} \alpha_j^{(1)} = q_k \sum_{i:\, e_i^{(2)} \geq q_k} b_i^{(2)} + q_k \sum_{j:\, f_j^{(2)} \geq q_k} \alpha_j^{(2)} .$$

Setting

$$c_k := \sum_{i:\, e_i^{(1)} \geq q_k} b_i^{(1)} - \sum_{i:\, e_i^{(2)} \geq q_k} b_i^{(2)} ,$$

$$\beta_k := \sum_{j:\, f_j^{(1)} \geq q_k} \alpha_j^{(1)} - \sum_{j:\, f_j^{(2)} \geq q_k} \alpha_j^{(2)} ,$$

we get $q_k(c_k + \beta_k) = 0$, $1 \leq k \leq l$. If $\beta_k = 0$, then $q_k c_k = 0$, and therefore, by Proposition 2.2 (iii), $q_k s(c_k) = 0$ and hence, by the assumption, $q_k s(\delta(c_k)) = 0$, in particular, $q_k \delta(c_k) = 0$. We shall now show that the latter equality continues to hold also for those indices $k = 1, 2, \ldots, l$ for which $\beta_k \neq 0$. Indeed, for such k's, we define $d_k = -\beta_k^{-1} c_k$ and note that it follows from the equality $q_k c_k = -\beta_k q_k$, that $q_k d_k = q_k$. Therefore, $q_k = q_k^2 = q_k d_k^2$, and hence $q_k(d_k^2 - d_k) = 0$. Again using Proposition 2.2 (iii), we obtain that $q_k s(d_k^2 - d_k) = 0$. Since, by the assumption, $s(\delta(b)) \leq s(b)$ for all $b \in \mathcal{B}$, we infer $q_k s(\delta(d_k^2 - d_k)) = 0$, in particular, $q_k \delta(d_k^2 - d_k) = 0$. Therefore, it follows from Proposition 2.3 (i) combined with the equality $q_k d_k = q_k$ that $q_k \delta(d_k) = q_k \delta(d_k^2) = 2 q_k \delta(d_k)$, i.e., $q_k \delta(d_k) = 0$, and $q_k \delta(c_k) = -\beta_k q_k \delta(d_k) = 0$. Hence, $q_k \delta(c_k) = 0$ for all $k = 1, \ldots, l$.

This implies that

$$\sum_{i=1}^{n_1} \delta(b_i^{(1)}) e_i^{(1)} = \sum_{k=1}^{l} q_k \sum_{i:\, e_i^{(1)} \geq q_k} \delta(b_i^{(1)})$$
$$= \sum_{k=1}^{l} q_k \sum_{i:\, e_i^{(2)} \geq q_k} \delta(b_i^{(2)}) = \sum_{i=1}^{n_2} \delta(b_i^{(2)}) e_i^{(2)} .$$

This means that the map δ_1 is well-defined on $\mathcal{A}(\mathcal{B}, \nabla)$. The linearity of δ_1 and the equality $\delta_1(b) = \delta(b)$ for all $b \in \mathcal{B}$ immediately follow from the definition. We shall now show that $\delta_1(ad) = \delta_1(a)d + a\delta_1(d)$ for all $a, d \in \mathcal{A}(\mathcal{B}, \nabla)$. Let $a, b \in \mathcal{B}$, $e, f \in \nabla$. Then

$$\delta_1((ae)(bf)) = \delta_1(abef) = \delta(ab)ef = (\delta(a)b + a\delta(b))ef = \delta_1(ae)bf + ae\delta_1(bf)$$

and $\delta_1(aef) = \delta(a)ef = \delta_1(ae)f$. Now, let

$$a = \sum_{i=1}^{k} b_i e_i + \sum_{j=1}^{m} \alpha_j f_j , \qquad d = \sum_{s=1}^{l} b'_s e'_s + \sum_{p=1}^{t} \alpha'_p f'_p$$

be arbitrary elements from $\mathcal{A}(\mathcal{B}, \nabla)$. Then

$$\delta_1(ad) = \delta_1 \left(\sum_{i,s} b_i e_i b'_s e'_s + \sum_{i,p} b_i e_i \alpha'_p f'_p + \sum_{j,s} b'_s e'_s \alpha_j f_j + \sum_{j,p} \alpha_j \alpha'_p f_j f_p \right)$$
$$= \sum_{i,s} \big(\delta_1(b_i e_i) b'_s e'_s + b_i e_i \delta_1(b'_s e'_s) \big) + \sum_{i,p} \delta_1(b_i e_i) \alpha'_p f'_p + \sum_{j,s} \delta_1(b'_s e'_s) \alpha_j f_j$$
$$= \delta_1(a)d + a\delta_1(d) .$$

This completes the proof that $\delta_1 : \mathcal{A}(\mathcal{B}, \nabla) \to \mathcal{A}$ is a derivation. Now, let $\delta_2 : \mathcal{A}(\mathcal{B}, \nabla) \to \mathcal{A}$ be a derivation for which, $\delta_2(b) = \delta(b)$ for all $b \in \mathcal{B}$. Then, it follows from Proposition 2.3 (ii), (iii) that for any $b_i \in \mathcal{B},\ e_i, f_j \in \nabla,\ \alpha_j \in K$ we have

$$\delta_2 \left(\sum_{i=1}^{n} b_i e_i + \sum_{j=1}^{m} \alpha_j f_j \right) = \sum_{i=1}^{n} \delta_2(b_i) e_i = \sum_{i=1}^{n} \delta(b_i) e_i$$
$$= \delta_1 \left(\sum_{i=1}^{n} b_i e_i + \sum_{j=1}^{m} \alpha_j f_j \right) ,$$

i.e., $\delta_1 = \delta_2$. ∎

The following proposition contains the second step of our programme; in particular, it shows that an arbitrary derivation $\delta : \mathcal{A}(\mathcal{B}, \nabla) \to \mathcal{A}$ extends to a derivation on the least regular subalgebra containing $\mathcal{B}$ and ∇.

PROPOSITION 2.5. *Let $\mathcal{B}$ be a subalgebra in $\mathcal{A}$ such that $\nabla \subset \mathcal{B}$ and let $\delta : \mathcal{B} \to \mathcal{A}$ be a derivation. If $\mathcal{B}(i) := \{a \cdot i(b) : a, b \in \mathcal{B}\}$, then $\mathcal{B}(i)$ is the least regular subalgebra in $\mathcal{A}$ containing $\mathcal{B}$, and there exists a unique derivation $\delta_1 : \mathcal{B}(i) \to \mathcal{A}$ such that $\delta_1(b) = \delta(b)$ for all $b \in \mathcal{B}$.*

Proof. Clearly, $K \cdot \mathcal{B}(i) \subseteq \mathcal{B}(i)$. Since $i(ab) = i(b)i(a)$ and $i(i(a)) = a$, we have $i(\mathcal{B}(i)) = \mathcal{B}(i)$.

Let $a_k, b_k \in \mathcal{B}$, $d_k = a_k i(b_k) \in \mathcal{B}(i)$, $k = 1, 2$. Then $a_1 i(b_1) \cdot a_2 i(b_2) = a_1 a_2 i(b_1 b_2)$, i.e., $\mathcal{B}(i) \cdot \mathcal{B}(i) \subseteq \mathcal{B}(i)$.

We shall now show that $d_1 + d_2 \in \mathcal{B}(i)$. Clearly, to this end, it is sufficient to establish that $d_1 + d_2 = u \cdot i(v)$, where

$$u := a_1(s(d_1) - s(d_1 d_2)) + a_2(s(d_2) - s(d_1 d_2)) + (a_1 b_2 + a_2 b_1) s(d_1 d_2)\,,$$
$$v := b_1(s(d_1) - s(d_1 d_2)) + b_2(s(d_2) - s(d_1 d_2)) + b_1 b_2 s(d_1 d_2)\,.$$

For brevity, we also set

$$e = s(d_1) - s(d_1 d_2)\,, \qquad f = s(d_2) - s(d_1 d_2)\,, \qquad g = s(d_1 d_2)\,.$$

Since $s(d_k) = s(a_k)s(b_k)$, $k = 1, 2$ (see Proposition 2.2 (ii)), we get (using Proposition 2.2 (iv))

$$\begin{aligned} d_1 + d_2 &= a_1 i(b_1) e + (a_1 i(b_1) + a_2 i(b_2)) g + a_2 i(b_2) f \\ &= a_1 i(b_1) e + a_2 i(b_2) f + (a_1 b_2 i(b_2) i(b_1) s(d_1) s(d_2) \\ &\quad + a_2 b_1 i(b_1) i(b_2) s(d_1) s(d_2)) \\ &= \big(a_1 e + a_2 f\big) \cdot i\big(b_1 e + b_2 f\big) + (a_1 b_2 + a_2 b_1)(i(b_1 b_2) s(d_1 d_2)) \\ &= u \cdot i(v)\,. \end{aligned}$$

Thus, $\mathcal{B}(i)$ is a subalgebra in $\mathcal{A}$. For any element $a\, i(b)$ from $\mathcal{B}(i)$, the element $b\, i(a)$ also belongs to $\mathcal{B}(i)$, and since

$$a\, i(b) = a\, i(b) s(b) s(a) = (a\, i(b))(b\, i(a))(a\, i(b))\,,$$

it follows that $\mathcal{B}(i)$ is a regular ring. Combining the latter with the definition of $\mathcal{B}(i)$, we deduce that $\mathcal{B}(i)$ is the least regular subalgebra in $\mathcal{A}$, containing $\mathcal{B}$.

We define $\delta_1 : \mathcal{B}(i) \to \mathcal{A}$, as follows

$$\delta_1(a\, i(b)) = \delta(a)i(b) - a\delta(b)i(b^2)\,.$$

We have to verify first that this map is well-defined. Let $a_1 i(b_1) = a_2 i(b_2)$, a_k,$b_k \in \mathcal{B}$, $k = 1, 2$. Then $s(a_1)s(b_1) = s(a_1 i(b_1)) = s(a_2 i(b_2)) = s(a_2)s(b_2)$. We have that $s(b_1)\cdot s(b_2)(a_1b_2 - a_2b_1) = a_1 i(b_1)b_1 \cdot b_2 s(b_2) - a_2 i(b_2)b_2 b_1 s(b_1) = a_2 i(b_2)b_1b_2 - a_2 i(b_2)b_1b_2 = 0$. Since $e = s(a_1)s(b_1) = s(a_2)s(b_2) \le s(b_1) \wedge s(b_2) = s(b_1)\cdot s(b_2)$, then $e(a_1b_2 - a_2b_1) = 0$. Since $s(\delta(b)) \le s(b)$ for all $b \in \mathcal{B}$ (see Proposition 2.2 (iv)), we get $0 = e\delta(a_1b_2 - a_2b_1) = e(\delta(a_1)b_2 + \delta(b_2)a_1 - \delta(a_2)b_1 - \delta(b_1)a_2)$. We remark also that from the equality $a_1 i(b_1) = a_2 i(b_2)$, it follows that $a_1 s(b_1) = a_2 b_1 i(b_2)$, $a_2 s(b_2) = a_1 b_2 i(b_1)$. Therefore, using the inclusion $\nabla \subset \mathcal{B}$ and the equalities $s(i({b_1}^2{b_2}^2)) = s({b_1}^2)s(b_2^2) = s(b_1)s(b_2)$, we infer that

$$\begin{aligned}
&\big[\delta(a_1)i(b_1) - a_1\delta(b_1)i(b_1^2)\big] - \big[\delta(a_2)i(b_2) - a_2\delta(b_2)i(b_2^2)\big]\\
&\qquad = e\, i(b_1^2b_2^2)\big(\delta(a_1)b_1b_2^2 - a_1\delta(b_1)b_2^2 - \delta(a_2)b_2b_1^2 + a_2\delta(b_2)b_1^2\big)\\
&\qquad = e\, i(b_1^2b_2^2)\big(\delta(a_1)b_1b_2^2 - a_2b_1b_2\delta(b_1) - \delta(a_2)b_2b_1^2 + a_1b_1b_2\delta(b_2)\big)\\
&\qquad = e\, i(b_1b_2)\big(\delta(a_1)b_2 - a_2\delta(b_1) - \delta(a_2)b_1 + a_1\delta(b_2)\big) = 0\,.
\end{aligned}$$

This means that the map δ_1 is well-defined.

Clearly, $\delta_1(\lambda a\, i(b)) = \lambda\, \delta_1(a\, i(b))$ for all $\lambda \in K$. Thus, to show that δ_1 is a linear map, we need only to establish that $\delta_1(d_1 + d_2) = \delta_1(d_1) + \delta_1(d_2)$ for all $d_k = a_k i(b_k)$, a_k, $b_k \in \mathcal{B}$, $k = 1, 2$. To this end, using results and notation from the first part of the proof, we note first that $d_1 + d_2 = u \cdot i(v)$, and so, by the definition of δ_1, we have

$$\delta_1(d_1 + d_2) = \delta(u)i(v) - u\delta(v)\cdot i(v^2)\,.$$

Further, we note that it is a straightforward verification that

$$\begin{aligned}
e(d_1 + d_2) &= e\, u \cdot i(e\, v) = a_1 i(b_1)e\,,\\
f(d_1 + d_2) &= a_2 i(b_2) f\,,\\
g(d_1 + d_2) &= (a_1b_2 + a_2b_1)i(b_1b_2)g\,,
\end{aligned}$$

and that for any $h \in \nabla$, $a, b \in \mathcal{B}$ the following holds

$$\delta_1(h\, a\, i(b)) = \delta(h\, a)i(b) - h\, a\delta(b)i(b^2) = h\delta_1(a\, i(b))\,.$$

Hence,

$$e\delta_1(d_1+d_2) = \delta_1(e(d_1+d_2)) = e(\delta(a_1)i(b_1) - a_1\delta(b_1)i({b_1}^2)) = e\delta_1(d_1)\,,$$

$$f\delta_1(d_1+d_2) = \delta_1(f(d_1+d_2)) = f(\delta(a_2)i(b_2) - a\delta(b_2)i({b_2}^2)) = f\delta_1(d_2)\,,$$

$$\begin{aligned} g\delta_1(d_1+d_2) &= \delta_1(g(d_1+d_2)) \\ &= g\big((\delta(a_1)b_2 + a_1\delta(b_2) + \delta(a_2)b_1 + a_2\delta(b_1)\big)i(b_1b_2) \\ &\quad - (a_1b_2 + a_2b_1)(\delta(b_1)b_2 + b_1\delta(b_2)) \cdot i({b_1}^2{b_2}^2)) \\ &= g(\delta(a_1)i(b_1) - a_1\delta(b_1)i({b_1}^2) + \delta(a_2)i(b_2) - a_2\delta(b_2)i({b_2}^2)) \\ &= g(\delta_1(d_1) + \delta_1(d_2))\,. \end{aligned}$$

Since $s(u) \le s(d_1) \vee s(d_2) = e + f + g$, $s(v) \le e + f + g$, we have

$$s(\delta_1(d_1+d_2)) \le e + f + g\,,$$

and therefore, using the inequalities

$$\begin{aligned} s(\delta_1(d_1)) &\le s(d_1) = e + g\,, \\ s(\delta_1(d_2)) &\le s(d_2) = f + g\,, \end{aligned}$$

we obtain

$$\begin{aligned} \delta_1(d_1+d_2) &= \delta_1(d_1+d_2)(e+f+g) = e\delta(d_1) + f\delta_1(d_2) + g(\delta_1(d_1) + \delta_1(d_2)) \\ &= \delta_1(d_1) + \delta_1(d_2)\,. \end{aligned}$$

Thus, δ_1 is a linear map. In order to verify that δ_1 is a derivation on $\mathcal{B}(i)$, we note that for any $d_k = a_k i(b_k)$, $a_k, b_k \in \mathcal{B}$, $k = 1, 2$, we have

$$\begin{aligned} \delta_1(d_1d_2) &= \delta_1(a_1a_2i(b_1b_2)) \\ &= (\delta(a_1)a_2 + a_1\delta(a_2))i(b_1b_2) - a_1a_2(\delta(b_1)b_2 + b_1\delta(b_2))i({b_1}^2{b_2}^2) \\ &= (\delta(a_1)i(b_1) - a_1\delta(b_1)i({b_1}^2))a_2i(b_2) \\ &\quad + (\delta(a_2)i(b_2) - a_2\delta(b_2)i({b_2}^2))a_1i(b_1) \\ &= \delta_1(d_1)d_2 + \delta_1(d_2)d_1\,. \end{aligned}$$

Thus, δ_1 is a derivation on $\mathcal{B}(i)$.

If $b \in \mathcal{B}$, then $\delta_1(b) = \delta_1(b\,i(\mathbf{1})) = \delta(b)i(\mathbf{1}) - b\delta(\mathbf{1})i(\mathbf{1}) = \delta(b)$.

Finally, let $\delta_2 : \mathcal{B}(i) \to \mathcal{A}$ be a derivation, such that $\delta_2(b) = \delta(b)$ for all $b \in \mathcal{B}$. By Proposition 2.3 (v), for $a, b \in \mathcal{B}$ we have

$$\delta_2(a\,i(b)) = \delta_2(a)i(b) + a\delta_2(i(b)) = \delta_2(a)i(b) - a\delta_2(b)i(b^2) = \delta_1(a\,i(b))\,,$$

i.e., $\delta_2 = \delta_1$. ∎

We now recall a few notions related to measure theory on boolean algebras. A non-negative function $\mu : \nabla \to [0,\infty]$ is said to be a measure on the Boolean algebra ∇, if $\mu(e \vee g) = \mu(e) + \mu(g)$ for all $e, g \in \nabla$ such that $e \wedge g = 0$ and $\mu(0) = 0$. The measure μ is said to have the direct sum property, if there exists a family $\{e_i\}_{i\in I} \subseteq \nabla$ of nonzero pairwise disjoint elements, such that $\sup\limits_{i\in I} e_i = \mathbf{1}$ and $\mu(e_i) < \infty$ for all $i \in I$. The measure μ is called finite, if $\mu(e) < \infty$ for all $e \in \nabla$. It is called strictly positive, if from the equality $\mu(e) = 0$, it follows that $e = 0$. The measure μ is called completely additive (respectively, countably additive), if for any (respectively, for any countable) family $\{e_i\}_{i\in I} \subseteq \nabla$ of nonzero pairwise disjoint elements such that $\sup\limits_{i\in I} e_i \in \nabla$, the equality $\mu\left(\sum\limits_{i\in I} e_i\right) = \sum\limits_{i\in I} \mu(e_i)$ holds.

We assume below that μ is a strictly positive finite measure on the Boolean algebra ∇ of all idempotents of a commutative regular algebra $\mathcal{A}$. We define the function $\rho : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$, by setting:

$$\rho(a,b) = \mu(s(a-b))\,.$$

The following proposition represent the third step in our programme: it shows that a derivation $\delta : \mathcal{B} \to \mathcal{A}$ extends from the subalgebra $\mathcal{B}$ to the closure of $\mathcal{B}$ in the topology defined by the metric ρ. The assertion below that ρ is a metric on $\mathcal{A}$ and that $(\mathcal{A}, \rho)$ is a topological ring is known for certain algebras of continuous admissible functions on a Stone compact equipped with a finite measure [1]; we supply full proofs here for the reader's convenience.

Proposition 2.6. (i) *ρ is a metric on $\mathcal{A}$;*

(ii) *$\rho(a,b) = \rho(a+c, b+c)$ for any $a, b, c \in \mathcal{A}$, $\rho(\lambda a, \lambda b) = \rho(a,b)$ for any $a, b \in \mathcal{A}$, $\lambda \in K$, $\lambda \neq 0$;*

(iii) *the operations $(a,b) \to a+b$, $(a,b) \to ab$ are bi-continuous and $a \to -a$ is continuous in the metric ρ, i.e., $\mathcal{A}$ is a topological ring in the metric topology given by ρ, moreover, $\rho(ac, bc) \le \rho(a,b)$;*

(iv) $\rho(i(a), i(b)) = \rho(a, b)$ *for all* $a, b \in \mathcal{A}$*;*

(v) *if* $\mathcal{B}$ *is a subalgebra in* $\mathcal{A}$*, such that* $\nabla \subset \mathcal{B}$ *and if* $\delta : \mathcal{B} \to \mathcal{A}$ *is a derivation, then* $\rho(\delta(a), \delta(b)) \leq \rho(a, b)$ *for all* $a, b \in \mathcal{B}$*, in particular,* δ *is uniformly continuous in the metric* ρ*;*

(vi) *if* $\mathcal{B}$ *is a subalgebra (respectively, a regular subalgebra) in* $\mathcal{A}$*, such that* $\nabla \subset \mathcal{B}$ *and if* $\delta : \mathcal{B} \to \mathcal{A}$ *is a derivation, then the closure* $\overline{\mathcal{B}}$ *of the algebra* $\mathcal{B}$ *in* $(\mathcal{A}, \rho)$ *is a subalgebra (respectively, a regular subalgebra) in* $\mathcal{A}$*, and there exists a unique derivation* $\delta_1 : \overline{\mathcal{B}} \to \mathcal{A}$*, such that* $\delta_1(b) = \delta(b)$ *for all* $b \in \mathcal{B}$*.*

Proof. The assertions (i), (ii), (iii) follow immediately from the definition of the metric ρ and properties of the support (see Proposition 2.2.).

(iv) Let a,$b \in \mathcal{A}$, $e = \mathbf{1} - s(a - b)$. Then $(a - b)e = 0$, i.e., $a\,e = b\,e$. Hence $i(a)e = i(a\,e) = i(b\,e) = i(b)e$, and therefore $(i(a) - i(b))e = 0$. Consequently, $e \leq \mathbf{1} - s(i(a) - i(b))$, i.e., $s(i(a) - i(b)) \leq s(a - b)$. Replacing in the argument above a by $i(a)$, and b by $i(b)$, we get that $s(a - b) \leq s(i(a) - i(b))$. Thus, $s(a - b) = s(i(a) - i(b))$ and $\rho(i(a), i(b)) = \rho(a, b)$.

(v) Since $\nabla \subset \mathcal{B}$, it follows from Proposition 2.3 (iv) that $s(\delta(a)) \leq s(a)$, and therefore for any $a, b \in \mathcal{B}$ we have that $\rho(\delta(a), \delta(b)) = \mu(s(\delta(a - b))) \leq \mu(s(a - b)) = \rho(a, b)$, which implies, in particular, that δ is uniformly continuous in the metric ρ.

(vi) It follows from (ii) and (iii) above that $\overline{\mathcal{B}}$ is a sub-ring in $\mathcal{A}$. Let $\lambda \in K$, $a \in \overline{\mathcal{B}}$, $a_n \in \mathcal{B}$, $\rho(a_n, a) \to 0$. If $\lambda \neq 0$, then $\rho(\lambda a_n, \lambda a) = \rho(a_n, a) \to 0$, i.e., $\lambda a \in \overline{\mathcal{B}}$. If $\lambda = 0$, then $\lambda a = 0 \in \mathcal{B} \subset \overline{\mathcal{B}}$. Consequently, $\overline{\mathcal{B}}$ is a subalgebra in $\mathcal{A}$. Furthermore, it follows from (iv) above, that $i(a) \in \overline{\mathcal{B}}$ for any $a \in \overline{\mathcal{B}}$, provided that $\mathcal{B}$ is a regular subalgebra. Thus, in this case, the subalgebra $\overline{\mathcal{B}}$ is also a regular subalgebra in $\mathcal{A}$. Now, using the assertions (iii) and (v), it is easily shown that the derivation δ extends uniquely to a derivation on $\overline{\mathcal{B}}$. ∎

We recall that the Boolean algebra ∇ is called *complete* (respectively, *σ-complete*), if for any subset (respectively, countable subset) $M \subset \nabla$ there exists a least upper bound $\sup M \in \nabla$. The Boolean algebra ∇ is said to have *a countable type*, if any family of nonzero pairwise disjoint elements from ∇ is no more than countable. Clearly, for Boolean algebras of countable type the notions of fully additive and countably additive measures coincide. It is shown in [18, chapter I, §6] that if ∇ admits a strictly positive measure, then it necessarily has a countable type. Every σ-complete Boolean algebra of a countable type is a complete Boolean algebra. If ∇ is a complete Boolean algebra of countable type, then for every subset $M \subset \nabla$, there exists a countable subset $M_1 \subset M$, such that $\sup M_1 = \sup M$ (see [18, chapter III, §2]). Let μ be a strictly positive countably additive measure on the Boolean algebra ∇, and let $d(e, f) := \mu(e \bigtriangleup f)$, $e, f \in \nabla$. It is shown in [18, chapter III, §5] that d is a metric on ∇, and if (∇, d) is a complete metric space, then ∇ is a complete Boolean algebra.

Proposition 2.7. *Let $\mathcal{A}$ be a commutative unital regular algebra, and let μ be a strictly positive countably additive finite measure on the Boolean algebra ∇ of all idempotents from $\mathcal{A}$, $\rho(a, b) = \mu(s(a - b))$. If $(\mathcal{A}, \rho)$ is a complete metric space, then*

(i) *∇ is a complete Boolean algebra of the countable type;*

(ii) *if $a_i \in \mathcal{A}$, $a_i a_j = 0$ for $i \neq j$, $i, j = 1, 2, \ldots$, then the series $\sum_{i=1}^{\infty} a_i$ converges in $(\mathcal{A}, \rho)$ to some element a such that $a\, s(a_i) = a_i$, $i = 1, 2, \ldots$, and $s(a) = \sup_{i \geq 1} s(a_i)$.*

Proof. (i) Suppose $e_n \in \nabla$, $a \in \mathcal{A}$, $\rho(e_n, a) \to 0$ when $n \to \infty$. Then $e_n = {e_n}^2 \to a^2$, and therefore $a^2 = a$, i.e., $a \in \nabla$. Consequently, ∇ is a closed subset of $(\mathcal{A}, \rho)$, in particular, (∇, ρ) is complete metric space.

If $e, f \in \nabla$, then $s(e - f) = e \bigtriangleup f$ (see Proposition 2.2 (v)) and therefore $\mu(e \bigtriangleup f) = \rho(e, f)$. It follows now from [18, chapter III, §5] that ∇ is complete Boolean algebra.

(ii) Let $e_n = \sum_{k=1}^{n} s(a_k)$, $e = \sup_{k \geq 1} s(a_k)$. Since $s(a_i)s(a_j) = 0$ for $i \neq j$ (see Proposition 2.2 (iii)), we have $e_n \in \nabla$, $e_n \leq e_{n+1}$ and $e = \sup_{n \geq 1} e_n$, in particular, $\mu(e) = \lim_{n \to \infty} \mu(e_n)$ and $\lim_{\substack{m > n \\ m,n \to \infty}} \mu(e_m - e_n) = 0$. It follows, that for $m > n$, we have

$$\rho\left(\sum_{k=1}^{m} a_k, \sum_{k=1}^{n} a_k\right) = \rho\left(\sum_{k=n+1}^{m} a_k, 0\right) = \mu\left(s\left(\sum_{k=n+1}^{m} a_k\right)\right)$$
$$= \mu\left(\sum_{k=n+1}^{m} s(a_k)\right) = \mu(e_m - e_n) \quad \to 0 \quad \text{as} \quad m, n \to \infty\,.$$

Consequently, the sequence $\{a_n\}_{n \geq 1}$ is a Cauchy sequence in $(\mathcal{A}, \rho)$, and there exists an element $a \in \mathcal{A}$, such that $\rho\left(\sum_{k=1}^{n} a_k, a\right) \to 0$ as $n \to \infty$. For a fixed index $i \geq 1$, we have $\left(\sum\limits_{k=1}^{n} a_k\right) s(a_i) = a_i$ for every $n \geq i$. Therefore, by Proposition 2.6 (iii), we get $a\, s(a_i) = a_i$. It follows that $a_i s(a) = a\, s(a_i) s(a) = a\, s(a_i) = a_i$, and so $s(a_i) \leq s(a)$ for all $i = 1, 2, \ldots$. Consequently, $e \leq s(a)$. On the other hand, as $\left(\sum\limits_{k=1}^{n} a_k\right) e = \sum\limits_{k=1}^{n} a_k s(a_k) e = \sum\limits_{k=1}^{n} a_k$, we get by passing to the limit as $n \to \infty$, that $a\, e = a$. Consequently, $s(a) \leq e$, and therefore $s(a) = e$. ∎

In what follows, we shall be concerned with complete commutative regular algebras $(\mathcal{A}, \rho)$. The important examples of such algebras listed below are a specialization of the examples following Proposition 2.2.

Example 2.4. Let Δ be a countable set, let $\mathcal{A} = K^{\Delta}$ and let μ be a strictly positive countably additive measure on the Boolean algebra of all subsets in Δ (this Boolean algebra is naturally identified with the Boolean algebra of all idempotents in $\mathcal{A}$). For any $a = \{\alpha_q\}_{q \in \Delta}$, $b = \{\beta_q\}_{q \in \Delta} \in \mathcal{A}$, we have that $\rho(a, b) = \mu(s(a - b)) = \mu(\{q \in \Delta : \alpha_q \neq \beta_q\}$. Consequently, the topology generated by the metric ρ, coincides with the product topology in the product $\mathcal{A} = \prod_{q \in \Delta} K_q$ with $K_q = K$, $q \in \Delta$, and each K_q is equipped with the discrete topology. This implies that $(\mathcal{A}, \rho)$ is a complete metric space.

Example 2.5. Let $\mathcal{A} = S(\Omega, \Sigma, \mu)$, where μ is a finite countably additive measure on Σ, and $K = \mathbb{R}$, or else $K = \mathbb{C}$. The measure μ defines a strictly positive countably additive measure on the complete Boolean algebra ∇ of all idempotents in $\mathcal{A}$. For any $a, b \in \mathcal{A}$, we have $\rho(a, b) = \mu(s(a - b)) = \mu(\{\omega \in \Omega : a(\omega) \neq b(\omega)\})$. The metric space $(\mathcal{A}, \rho)$ is a complete. Due to the lack of a suitable reference, we supply below a short proof.

Let $\{a_n\}_{n \geq 1}$ be a Cauchy sequence in $(\mathcal{A}, \rho)$. Passing to a subsequence if necessary, we may assume that $\rho(a_{n+1}, a_n) < 1/2^n$ for all $n = 1, 2, \ldots$. For each fixed $k \geq 1$, we set $E_k := \bigcap\limits_{n=k}^{\infty} \{\omega \in \Omega : a_n(\omega) = a_k(\omega)\}$. Clearly, $E_k \in \Sigma$, $E_k \subset E_{k+1}$ and $\Omega \setminus E_k = \{\omega \in \Omega : a_n(\omega) \neq a_k(\omega)$ for some $n > k\}$. Let $m > k$ and $A_m := \{\omega \in \Omega : a_k(\omega) = a_{k+1}(\omega) = \cdots = a_{m-1}(\omega) \neq a_m(\omega)\}$. Then

$$A_m = \left(\bigcap_{n=k}^{m-1} \{\omega \in \Omega : a_n(\omega) = a_k(\omega)\}\right) \cap \{\omega \in \Omega : a_m(\omega) \neq a_k(\omega)\},$$

and so $A_m \in \Sigma$ and $\Omega \setminus E_k = \bigcup\limits_{m=k+1}^{\infty} A_m$. We denote by e_m the idempotent from ∇ corresponding to the set A_m. From the definition of the set A_m, we have that $e_m \leq s(a_m - a_{m-1})$. Therefore $\mu(e_m) \leq \mu(s(a_m - a_{m-1})) = \rho(a_m, a_{m-1}) < 1/2^{m-1}$. Let q_k be the idempotent in ∇ corresponding to the set E_k. Since $\mathbf{1} - q_k = \bigvee_{m=k+1}^{\infty} e_m$, we have $\mu(\mathbf{1} - q_k) \leq \sum_{m=k+1}^{\infty} \frac{1}{2^{m-1}} = \frac{1}{2^{k-1}}$. Consequently, $q_k \uparrow \mathbf{1}$, in particular, $\mu(\Omega \setminus (\bigcup_{k=1}^{\infty} E_k)) = 0$. We set

$$a(\omega) = \begin{cases} a_1(\omega)\,, & \omega \in E_1\,, \\ a_k(\omega)\,, & \omega \in E_k \setminus E_{k-1}\,,\ k \geq 2\,, \\ 0, & \omega \in \Omega \setminus (\bigcup_{k=1}^{\infty} E_k)\,. \end{cases}$$

Clearly, $a \in \mathcal{A}$ and $a(\omega) = a_n(\omega)$ for all $\omega \in E_k$ and $n \geq k$. Consequently, $s(a - a_k) \leq \mathbf{1} - q_k$, and so $\rho(a, a_k) \leq \frac{1}{2^{k-1}}$, i.e., the subsequence $\{a_k\}_{k\geq 1}$ converge to a in $(\mathcal{A}, \rho)$. Hence, $(\mathcal{A}, \rho)$ is a complete metric space.

Consider in $\mathcal{A}$ the metric $d(a,b) := \int_\Omega |a-b|(1+|a-b|)^{-1}\mathrm{d}\mu$, which generates in $\mathcal{A}$ the topology of a convergence in measure. It is well known that the set $\mathcal{B}$ of all function from $\mathcal{A}$ taking finitely many different values is dense in $(\mathcal{A}, d)$. Let δ be a derivation from $\mathcal{A}$ to $\mathcal{A}$. It follows now from Proposition 2.3 (ii) that $\delta(b) = 0$ for all $b \in \mathcal{B}$. Therefore, if δ is continuous with respect to the metric d, then $\delta(a) = 0$ for all $a \in \mathcal{A}$.

Example 2.6. Let M be a commutative AW^*-algebra and let μ be a strictly positive finite countably additive measure on the Boolean algebra of all projections in M. Then M is a W^*-algebra (see, for example, [8]) and so, the algebra of all measurable operators $L(M)$ may be identified with the algebra $S(\Omega, \Sigma, \mu)$ for the some measure space (Ω, Σ, μ) with a finite countably additive measure [15]. Therefore, by the preceding example, the commutative regular algebra $L(M)$ (respectively, $L_h(M)$) is a complete metric space with respect to the metric $\rho(a,b) = \mu(s(a-b))$. We also note, that it further follows from the preceding example that any continuous in topology of convergence in measure derivation on $L(M)$ vanishes.

Let $\mathcal{A}$ be a commutative unital regular algebra over the field K, and let μ be a strictly positive countably additive measure on the Boolean algebra ∇ of all idempotents from $\mathcal{A}$. Let us further assume that $\mathcal{A}$ is complete with respect to the metric $\rho(a,b) = \mu(s(a-b))$, $a, b \in \mathcal{A}$. An element $a \in \mathcal{A}$ is called finitely valued (respectively, countably valued), if $a = \sum_{k=1}^{n} \alpha_k e_k$, where $\alpha_k \in K$, $e_k \in \nabla$, $e_k\, e_j = 0$, $k \neq j$, $k, j = 1, \ldots, n$, $n \in \mathbb{N}$ (respectively, $a = \sum_{k=1}^{\omega} \alpha_k e_k$,

$\alpha_k \in K$, $e_k \in \nabla$, $e_k e_j = 0$, $k \neq j$, $k, j = 1, \ldots, \omega$, where ω is a natural number or $\omega = \infty$ (in the latter case the convergence of series is understood with respect to the metric ρ, see Proposition 2.7)). We denote by $K(\nabla)$ (respectively, $K_c(\nabla)$) the set of all finitely valued (respectively, countably valued) elements in $\mathcal{A}$. Clearly, $\nabla \subset K(\nabla) \subset K_c(\nabla)$ and $K(\nabla)$, $K_c(\nabla)$ are subalgebras in $\mathcal{A}$. If $a = \sum\limits_{k=1}^{n} \alpha_k e_k \in K(\nabla)$ (respectively $a = \sum\limits_{k=1}^{\omega} \alpha_k e_k \in K_c(\nabla)$), then $s(a) = sup\{e_k : \alpha_k \neq 0\}$ and $i(a) = \sum\limits_{k:\alpha_k \neq 0} \alpha_k^{-1} e_k \in K(\nabla)$ (respectively, $i(a) \in K_c(\nabla)$). Consequently, $K(\nabla)$ and $K_c(\nabla)$ are regular subalgebras in $\mathcal{A}$. Clearly, if the field K has characteristic zero, then $K(\nabla) = K_c(\nabla)$ if and only if ∇ is a finite Boolean algebra (in that case $K(\nabla) = K_c(\nabla) = \mathcal{A}$).

The following proposition shows, in particular, that the algebra $K_c(\nabla)$ is always closed in $(\mathcal{A}, \rho)$.

Proposition 2.8. *The closure $\overline{K(\nabla)}$ of the algebra $K(\nabla)$ in $(\mathcal{A}, \rho)$ coincides with $K_c(\nabla)$.*

Proof. Let $a \in K_c(\nabla) \setminus K(\nabla)$ be a countably valued but not finitely valued element in $\mathcal{A}$, i.e., $a = \sum\limits_{i=1}^{\infty} \alpha_i e_i$, where $\alpha_i \in K$, $\alpha_i \neq \alpha_j$, $e_i \in \nabla$, $e_i \neq 0$, $e_i e_j = 0$, $i \neq j$, $i, j \geq 1$. Let us set $a_n = \sum\limits_{i=1}^{n} \alpha_i e_i$. Then $a_n \in K(\nabla)$, $\rho(a_n, a) \leq \mu(\bigvee\limits_{i=n+1}^{\infty} e_i) = \sum\limits_{i=n+1}^{\infty} \mu(e_i) \to 0$ as $n \to \infty$. Consequently, $a \in \overline{K(\nabla)}$, and therefore $K_c(\nabla) \subset \overline{K(\nabla)}$.

Suppose now that $b \in \overline{K(\nabla)}$. For any $\alpha \in K$, we set

$$J(\alpha) := \{e \in \nabla : e \leq s(b),\ be = \alpha e\}.$$

Clearly, $J(\alpha)$ is an ideal in the complete Boolean algebra ∇. We set further $f_\alpha = \vee J(\alpha)$, and choose an increasing sequence $\{e_n\}_{n \geq 1} \in J(\alpha)$, so that $f_\alpha = \sup\limits_{n \geq 1} e_n$, in particular, $\rho(e_n, f_\alpha) \to 0$ as $n \to \infty$. Clearly, $f_\alpha \leq s(b)$. Since $\rho(\alpha e_n, \alpha f_\alpha) \to 0$ and $\rho(b e_n, b f_\alpha) \to 0$ as $n \to \infty$ (see Proposition 2.6), and since $\alpha e_n = b e_n$, $n = 1, 2, \ldots$, we have $b f_\alpha = \alpha f_\alpha$. Consequently, $f_\alpha \in J(\alpha)$, i.e., $J(\alpha) = f_\alpha \nabla$. If $\alpha \neq \beta$, $\alpha, \beta \in K$, $f_\alpha f_\beta = e \neq 0$, then $\alpha e = eb = \beta e$, which is impossible. Consequently, $f_\alpha f_\beta = 0$ for any $\alpha, \beta \in K$, $\alpha \neq \beta$. Since the Boolean algebra ∇ has a countable type, the set $\{f_\alpha : \alpha \in K,\ \mu(f_\alpha) > 0\}$ is at most countable. We denote this set as $\{f_1, f_2, \ldots, f_n, \ldots\}$, where $b f_n = \alpha_n f_n$,

$\alpha_n \in K$, $n = 1, 2, \dots$. Let us further set

$$f = \sum_{n\geq 1} f_n\,, \qquad b_1 = \sum_{n\geq 1} \alpha_n f_n\,, \qquad b_2 = b - b_1\,.$$

Since

$$bf = \rho - \lim_{k\to\infty} b \sum_{n=1}^{k} f_n = \rho - \lim_{k\to\infty} \sum_{n=1}^{k} \alpha_n f_n = b_1\,,$$

we have $b_2 = (s(b) - f)b$, $s(b_2) = s(b) - f$.

We claim that $f = s(b)$. If this is the case, then $b = b_1 \in K_c(\nabla)$ and Proposition 2.8 is proved. Let us suppose that $f < s(b)$ and work towards a contradiction. For any $\alpha \in K$, $e \in \nabla$, $0 < e \leq s(b) - f$ we then have that $b_2 e \neq \alpha e$. This implies that $s(b_2 e - \alpha e) = e$. Indeed, $s(b_2 e - \alpha e) \leq e$, and if it were that $q := e - s(b_2 e - \alpha e) \neq 0$, then $q(b_2 e - \alpha e) = 0$, and therefore $b_2 q = \alpha q$, but this is not the case. Let us now consider an arbitrary element

$$a = \sum_{n=1}^{k} \beta_n p_n \in K(\nabla)$$

such that $s(a) \leq s(b) - f$, $\beta_n \in K$, $\beta_n \neq 0$, $p_n\, p_j = 0$, $n \neq j$, $n, j = 1, \dots, k$. We shall show that for every such element a, the following estimate $\rho(b_2, a) = \mu(s(b) - f) > 0$ holds. Indeed, from one hand, we have

$$s(b) - f \geq s(b_2 - a) \geq s(b_2 p_n - a p_n) = s(b_2 p_n - \beta_n p_n) = p_n\,, \qquad n = 1, \dots, k\,,$$

i.e., $s(b_2 - a) \geq \sum\limits_{n=1}^{k} p_n = s(a)$. On the other hand, from the equality $(b_2 - a)(s(b) - f - s(a)) = b_2(s(b) - f - s(a))$, it follows that

$$s(b_2 - a)(s(b) - f - s(a)) = s(b_2)(s(b) - f) - s(b_2)s(a) = s(b) - f - s(a)\,,$$

i.e., $s(b_2 - a) \geq s(b) - f - s(a)$. Thus, $s(b_2 - a) = s(b) - f$, and this establishes our claim. Now, we note that since $b, b_1 \in \overline{K(\nabla)}$, we have $b_2 \in \overline{K(\nabla)}$, and therefore there exists a sequence $a_n \in K(\nabla)$, which converges to b_2 in $(\mathcal{A}, \rho)$. Setting, $c_n = (s(b) - f)a_n$, we see that $c_n \in K(\nabla)$, $s(c_n) \leq s(b) - f$, $\rho(b_2, c_n) \to 0$. However, for every element c_n, $n = 1, 2, \dots$, it follows from the claim established above that $\rho(b_2, c_n) = \mu(s(b) - f) > 0$. This contradiction shows that $f = s(b)$ and $b = b_1 \in K_c(\nabla)$. ∎

Suppose that $K_c(\nabla) = \mathcal{A}$ and let $\delta : \mathcal{A} \to \mathcal{A}$ be an arbitrary derivation. By Proposition 2.3 (ii), we have $\delta(a) = 0$ for all $a \in K(\nabla)$. Therefore, from Proposition 2.6 (v) and Proposition 2.8, it follows that $\delta(a) = 0$ for all $a \in \overline{K(\nabla)} = K_c(\nabla) = \mathcal{A}$, i.e., δ vanishes on $\mathcal{A}$. Thus, non-trivial derivations may exist only on those commutative regular algebras $\mathcal{A}$, for which $\mathcal{A} \neq K_c(\nabla)$. Such algebras are exemplified by the algebras $L(M)$ and $L_h(M)$, where M is a commutative von Neumann algebra whose Boolean algebra of all projections is not atomic and has a countable type. The latter assertion follows from the following well-known result.

Proposition 2.9. *If $\mathcal{A} = S(\Omega, \Sigma, \mu)$, where (Ω, Σ, μ) is a finite measure space, then $K_c(\nabla) = \mathcal{A}$ if and only if ∇ is an atomic Boolean algebra.*

We conclude from Proposition 2.8 and Proposition 2.9 that the closure $\overline{K(\nabla)}$ of the subalgebra $K(\nabla)$ in $(S(\Omega, \Sigma, \mu), \rho)$ coincides with $S(\Omega, \Sigma, \mu)$ if and only if the Boolean algebra ∇ is atomic. In that case, as we noted above, any derivation on $S(\Omega, \Sigma, \mu)$ vanishes.

3. Construction of a non-zero derivation on a commutative regular algebra

Everywhere in this section $\mathcal{A}$ is a commutative unital regular algebra over the field K. We shall always assume below that the Boolean algebra ∇ of all idempotents in $\mathcal{A}$ admits a finite strictly positive countably additive measure μ such that $\mathcal{A}$ is complete with respect to the metric ρ defined by $\rho(a, b) := \mu(s(a - b))$, $a, b \in \mathcal{A}$.

In this section we complete the presentation of our main results; in particular, the construction of an extension of a derivation $\delta : \mathcal{B} \to \mathcal{A}$ from a subalgebra $\mathcal{B} \subseteq \mathcal{A}$ to a derivation on $\mathcal{A}$. This construction allows us to present examples of non-zero derivations on the algebra $S = S(\Omega, \Sigma, \mu)$ of all measurable functions on a measure space (Ω, Σ, μ) such that the Boolean algebra ∇ of all idempotents in S is not atomic.

In this section, the field K is assumed to have the characteristic zero (this assumption is used in a crucial way in the proof of Proposition 3.4 below).

For a subalgebra $\mathcal{B}$ in $\mathcal{A}$, we denote by $\mathcal{B}[x]$ the algebra of all polynomials with coefficients from $\mathcal{B}$, i.e., $p \in \mathcal{B}[x]$, if $p(x) = \sum\limits_{k=0}^{n} a_k x^{n-k}$, where $a_k \in \mathcal{B}$, $k = 0, 1, \ldots, n$, $n \in \mathbb{N}$, $a_n x^0 = a_n$. The largest integer $n - k$, for which $a_k \neq 0$ is called the *degree* of the polynomial $p(x)$ (notation: $\deg p$). If $\mathbf{1} \in \mathcal{B}$

and the leading coefficient of a polynomial $p(x)$ is equal to $\mathbf{1}$, then $p(x)$ is called a *unitary* (*monic*) polynomial. The following result is established in [19, ch. 3, §14].

Proposition 3.1. (The division algorithm) *If $\mathbf{1} \in \mathcal{B}$ and $p \in \mathcal{B}[x]$ is a unitary polynomial, if $f \in \mathcal{B}[x]$ and $\deg f \geq \deg p$, then $f = gp + r$, where $g, r \in \mathcal{B}[x]$, $\deg r < \deg p$.*

Clearly, the algebra $\mathcal{B}[x]$ is naturally identified with a subalgebra in $\mathcal{A}[x]$, and we shall use in the sequel the embedding $\mathcal{B}[x] \subset \mathcal{A}[x]$ without any further comment. We note that for any element $a \in A$, the set $\mathcal{B}(a) = \{p(a) : p \in \mathcal{B}[x]\}$ is a subalgebra in $\mathcal{A}$, which is generated by $\mathcal{B}$ and the element a. Clearly, $\mathcal{B} \subset \mathcal{B}(a)$, and if $a \in \mathcal{B}$, then $\mathcal{B} = \mathcal{B}(a)$.

We shall first work with derivations $\ell : \mathcal{B}[x] \to \mathcal{A}[x]$, i.e., with linear maps from $\mathcal{B}[x]$ to $\mathcal{A}[x]$ such that $\ell(pg) = \ell(p)g + p\ell(g)$ for any $p, g \in \mathcal{B}[x]$.

Let $\delta : \mathcal{B} \to \mathcal{A}$ be a derivation. For any $p(x) = \left(\sum\limits_{k=0}^{n} a_k x^{n-k}\right) \in \mathcal{B}[x]$, we set

$$p'(x) = \sum_{k=0}^{n-1} (n-k) a_k x^{n-k-1}, \qquad p^{\delta}(x) = \sum_{k=0}^{n} \delta(a_k) x^{n-k}.$$

It is easy to verify the following assertion. We omit the details.

Proposition 3.2. *The maps $p \to p'$, $p \to p^{\delta}$ are derivations from $\mathcal{B}[x]$ to $\mathcal{A}[x]$.*

Let $\mathcal{B}$ be a unital subalgebra in $\mathcal{A}$. An element $a \in \mathcal{A}$ is said to be
– *algebraic with respect to* $\mathcal{B}$, if there exists a non-zero polynomial $p \in \mathcal{B}[x]$, such that $p(a) = 0$;
– *integral with respect to* $\mathcal{B}$, if there exists a unitary polynomial $p \in \mathcal{B}[x]$, such that $p(a) = 0$;
– *transcendental with respect to* $\mathcal{B}$, if a is not algebraic over $\mathcal{B}$;
– *weakly transcendental with respect to* $\mathcal{B}$, if $a \neq 0$, and for any non-zero idempotent $e \leq s(a)$ the element ea is not integral with respect to $\mathcal{B}$.

It is clear that an integral element with respect to $\mathcal{B}$ is algebraic with respect to $\mathcal{B}$. Moreover, if $\nabla \subset \mathcal{B}$, then any non-zero element $a \in \mathcal{A}$, such that $s(a) < \mathbf{1}$ is also algebraic with respect to $\mathcal{B}$, since for the polynomial $q(x) = (\mathbf{1} - s(a))x \in \mathcal{B}[x]$ we have $q(a) = 0$. In particular, this implies that any transcendental element c with respect to $\mathcal{B} \supset \nabla$ has the support $s(c) = \mathbf{1}$.

The example that follows shows that that there may be algebraic elements (with respect to $\mathcal{B}$), which are not integral.

EXAMPLE 3.1. Let $\mathcal{A} = S([0,1],\lambda)$, where as before λ is Lebesgue measure on the segment $[0,1]$, and let $\mathcal{B}$ be the subalgebra in $\mathcal{A}$, generated by the Boolean algebra of all idempotents in $\mathcal{A}$ and by the subalgebra of all polynomials from $\mathcal{A}$. Consider the idempotent e in $\mathcal{A}$, corresponding to the set $[0,1/2]$, and set $a := ec + (\mathbf{1} - e)d$, where $c = c(t) = t^{1/2}$, $d = d(t) = exp(t)$, $t \in [0,1]$. Then, $q(a) = 0$ for the polynomial $q(x) = (ex^2 - eb) \in \mathcal{B}[x]$, where $b \in \mathcal{B}$, $b(t) \equiv t$, i.e., a is an algebraic element with respect to $\mathcal{B}$. However, a is not an integral element with respect to $\mathcal{B}$, since for every unitary polynomial $p(x) \in \mathcal{B}[x]$, we have $p(a) \neq 0$.

We note further, that the element $a_1 := (\mathbf{1} - e)d$ has the support $s(a_1) = (\mathbf{1} - e) < \mathbf{1}$ and this easily implies that a_1 is an algebraic element with respect to $\mathcal{B}$. At the same time, it is easy to see that for every non-zero idempotent $g \leq s(a_1)$, the element $ga_1 = gd$ is not an integral element with respect to $\mathcal{B}$. Thus, the element a_1 is simultaneously an algebraic and weakly transcendental element with respect to $\mathcal{B}$.

It is important to point out, that if a is a transcendental element with respect to $\mathcal{B}$ and $\nabla \subset \mathcal{B}$, then a is necessarily also weakly transcendental with respect to $\mathcal{B}$. Indeed, if it were not so, then there exists a non-zero idempotent e, such that $e \leq s(a)$ and such that ea is an integral element with respect to $\mathcal{B}$, i.e., there exists a unitary polynomial $p(x) = (x^n + a_1 x^{n-1} + \cdots + a_n) \in \mathcal{B}[x]$ such that $0 = p(ea) = ea^n + ea_1 a^{n-1} + \cdots + a_n$. Since $\nabla \subset \mathcal{B}$, this means that a is an algebraic with respect to $\mathcal{B}$ and this is not the case.

Assume that $\mathcal{B} \subseteq \mathcal{A}$ is a regular subalgebra such that $\nabla \subset \mathcal{B}$. Our first objective in this section is to build an extension of a derivation $\delta : \mathcal{B} \to \mathcal{A}$ to a derivation $\delta_1 : \mathcal{B}(a) \to \mathcal{A}$, where a is an integral element with respect to $\mathcal{B}$. To this end we shall require several technical results whose necessity may be explained as follows.

Assume for a moment, that we have already built a derivation $\delta_1 : \mathcal{B}(a) \to \mathcal{A}$ extending $\delta : \mathcal{B} \to \mathcal{A}$. Then, for every polynomial $g(x) = \sum_{k=0}^{n} a_k x^{n-k} \in \mathcal{B}[x]$ we have

$$\begin{aligned}\delta_1(g(a)) &= \delta_1\left(\sum_{k=0}^{n} a_k a^{n-k}\right)\\ &= \sum_{k=0}^{n}(\delta(a_k)a^{n-k} + (n-k)a_k a^{n-k-1}\delta_1(a)) = g^{\delta}(a) + g'(a)\delta_1(a)\,.\end{aligned}$$

Let p be a unitary polynomial in $\mathcal{B}[x]$ for which $p(a) = 0$. Then,

$$0 = \delta_1(p(a)) = p^\delta(a) + p'(a)\delta_1(a)\,.$$

Suppose, we are given that

$$p'(a)i(p'(a)) = s(p'(a)) \geq s(\delta_1(a))\,.$$

Then,

$$\delta_1(a) = s(p'(a))\delta_1(a) = p'(a)i(p'(a))\delta_1(a) = -p^\delta(a)i(p'(a))\,,$$

and so the derivation δ_1 has to be of the form

$$\delta_1(g(a)) = g^\delta(a) - g'(a)p^\delta(a)i(p'(a))\,.$$

In order to (correctly) define such a mapping from $\mathcal{B}(a)$ into $\mathcal{A}$ some extra technical work is required. It should be noted, that in special cases when $\mathcal{A}$ is a field (respectively, a unital integral domain) and the element a is an algebraic with respect to $\mathcal{B}$, a similar approach to the extension of δ onto the subalgebra $\mathcal{B}(a)$ has been given in [19, Ch. 10, §76] (respectively, [7, §1.8]). However, in our situation, when $\mathcal{A}$ is a commutative regular algebra the proof of the inequality $s(p'(a)) \geq s(\delta_1(a))$ and hence that of the existence of $\delta_1(g(a)) = g^\delta(a) - g'(a)p^\delta(a)i(p'(a))$ requires extra efforts, even in the (seemingly) simpler case when a is an integral element with respect to $\mathcal{B}$. The technical details of our approach are entirely different from those of [19] and [7] and based on a crucial use of the internal algebraic structure of a commutative regular algebra.

Let μ be a finite strictly positive countably additive measure on the Boolean algebra ∇ of all idempotents in $\mathcal{A}$, let $\rho(a, b) = \mu(s(a - b))$, and suppose that $(\mathcal{A}, \rho)$ be a complete metric space.

Proposition 3.3. *Suppose that $\mathcal{B}$ is a regular subalgebra in $\mathcal{A}$, closed in $(\mathcal{A}, \rho)$ such that $\nabla \subset \mathcal{B}$. If $0 \neq a \in \mathcal{A}$ is an integral element with respect to $\mathcal{B}$, then there exist natural numbers $n_1 < n_2 < \cdots < n_m$ and pairwise disjoint idempotents $\ell_1, \ldots, \ell_m \in \nabla$, such that*

(i) $s(a) = \sum_{k=1}^{m} \ell_k$;

(ii) *for every non-zero idempotent $e \leq \ell_k$ the minimal degree of a unitary polynomial $g \in \mathcal{B}[x]$, for which $g(ae) = 0$ is n_k, $k = 1, \ldots, m$.*

Proof. Let $p(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ be a unitary polynomial from $\mathcal{B}[x]$, for which $p(a) = 0$. Then $q_e(ae) = ep(a) = 0$ for every $e \in \nabla$, where $q_e(x) = x^n + ea_1x^{n-1} + \cdots + ea_{n-1}x + ea_n$ is a unitary polynomial from $\mathcal{B}[x]$. Consequently, for $e \in \nabla$, $e \neq 0$, $e \leq s(a)$ we may define the number

$$d(e) := \min \{\deg(g) \,:\, g \in \mathcal{B}[x] \text{ is a unitary polynomial such that } g(ae) = 0\},$$

which clearly does not exceed n. Let $0 < e_1 \leq e_2 \leq s(a)$, $e_1, e_2 \in \nabla$, and let $g(x) = x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m$ be a unitary polynomial from $\mathcal{B}[x]$, for which $g(ae_2) = 0$. Then $f(x) = x^m + e_1g(x) - e_1x^m$ is also a unitary polynomial from $\mathcal{B}[x]$, $\deg g = \deg f$ such that $f(ae_1) = e_1g(ae_1) = e_1e_2g(ae_1) = e_1g(ae_2) = 0$. This shows that $d(e_1) \leq d(e_2)$.

Let $n_1 := \min\{d(e) \,:\, e \in \nabla,\ e \neq 0,\ e \leq s(a)\}$, $J_1 := \{e \in \nabla \,:\, d(e) = n_1,\ e \leq s(a)\} \cup \{0\}$. We shall show that J_1 is an ideal in ∇. If $0 < e_1 \leq e_2 \in J_1$, then $n_1 \leq d(e_1) \leq d(e_2) = n_1$, i.e., $d(e_1) = n_1$ and $e_1 \in J_1$. Let $e_1, e_2 \in J_1$, $e_1e_2 = 0$. Choose unitary polynomials $g_1, g_2 \in \mathcal{B}[x]$ of degree n_1, for which $g_1(ae_1) = 0 = g_2(ae_2)$. Set

$$f_1(x) = g_1(x) - x^{n_1}, \qquad f_2(x) = g_2(x) - x^{n_1},$$

and

$$q_1(x) = e_1f_1(x), \qquad q_2(x) = e_2f_2(x), \qquad r(x) = x^{n_1} + q_1(x) + q_2(x).$$

It follows from the definition that $r(x)$ is a unitary polynomial from $\mathcal{B}[x]$, $\deg r = n_1$ and that $r(a(e_1 + e_2)) = e_1a^{n_1} + e_2a^{n_1} + q_1(ae_1) + q_2(ae_2) = e_1a^{n_1} + e_2a^{n_1} + e_1f_1(ae_1) + e_2f_2(ae_2) = e_1g_1(ae_1) + e_2g_2(ae_2) = 0$. This implies that $(e_1 + e_2) \in J_1$. If $e_1, e_2 \in J_1$ and $e_1e_2 \neq 0$, then $(e_1 - e_1e_2) \in J_1$, and $e_1 \vee e_2 = (e_1 - e_1e_2) + e_2 \in J_1$. Thus, J_1 is an ideal in ∇.

We set $\ell_1 := \sup \{e \,:\, e \in J_1\}$ and claim that $\ell_1 \in J_1$. Since ∇ is a complete Boolean algebra of a countable type (see Proposition 2.7), there exists an increasing sequence $e_n \in J_1$, such that $e_n \uparrow \ell_1$. Setting $t_1 = e_1$, $t_n = e_n - e_{n-1}$, $n > 1$, we see that $t_n \in J_1$, $t_nt_m = 0$, $n \neq m$, $n, m = 1, 2, \ldots$, $\ell_1 = \bigvee_{n=1}^{\infty} t_n$. Next, pick unitary polynomials $g_n \in \mathcal{B}[x]$ of degree n_1, for which $g_n(at_n) = 0$, $n = 1, 2, \ldots$, and set

$$f_n(x) = g_n(x) - x^{n_1}, \qquad q_n(x) = t_nf_n(x).$$

It follows from the assumptions above that $\deg q_n \leq n_1 - 1$, and that the supports of all coefficients of the polynomial $q_n(x)$ belong to the Boolean algebra $t_n\nabla$, i.e., $q_n(x) = \sum\limits_{k=0}^{n_1-1} b_k^{(n)} x^{n_1-k-1}$, where $b_k^{(n)} \in \mathcal{B}$, $s(b_k^{(n)}) \le t_n$, for all $k = 0, 1, \dots, n_1 - 1$. Consequently, by Proposition 2.7 and since $\mathcal{B}$ is closed, the series $\sum\limits_{n=1}^{\infty} b_k^{(n)}$ composed from the coefficients of polynomials q_n accompanying k'th powers converges in $\mathcal{B}$ in the metric ρ for every $k = 1, 2, \dots, n_1 - 1$. Set

$$b_k := \sum_{n=1}^{\infty} b_k^{(n)}, \qquad q(x) = \sum_{k=0}^{n_1-1} b_k x^{n_1-k-1} \in \mathcal{B}[x]\,.$$

Clearly, the polynomial $r(x) = x^n + q(x)$ is a unitary polynomial from $\mathcal{B}[x]$ with $\deg r = n_1$, and such that $r(a\ell_1) = \ell_1 a^{n_1} + q(a\ell_1) = \sum\limits_{n=1}^{\infty} t_n a^{n_1} + \sum\limits_{n=1}^{\infty} q_n(at_n) = \sum\limits_{n=1}^{\infty} (t_n a^{n_1} + q_n(at_n)) = \sum\limits_{n=1}^{\infty} t_n g_n(at_n) = 0$. Thus, $\ell_1 \in J_1$ and so $J_1 = \ell_1\nabla$. If $\ell_1 = s(a)$, then the proof of Proposition 3.3 is completed.

Let us assume now that $\ell_1 \ne s(a)$. It follows from the definition of J_1, that for any non-zero idempotent $\nabla \ni e \le s(a)$, $e\ell_1 = 0$, we have $d(e) > n_1$. We set, $n_2 = \min\{d(e) : e \in \nabla,\ e \ne 0,\ e \le s(a) - \ell_1\}$, $J_2 = \{e \in \nabla : d(e) = n_2,\ e \le s(a) - \ell_1\} \cup \{0\}$. Clearly, $n_2 > n_1$. Repeating the previous argument, we obtain that J_2 is an ideal in ∇, $\ell_2 = \vee\{e : e \in J_2\} \in J_2$ and $J_2 = \ell_2\nabla$. If $\ell_1 + \ell_2 = s(a)$, then the construction of the numbers $\{n_i\}$ and that of the idempotents $\{e_i\}$ is completed. Otherwise, we continue this process until we have $\ell_1 + \dots + \ell_m = s(a)$. Since $n_1 < n_2 < \dots < n_m \le n$, it is guaranteed that this process of defining of $\{n_i\}$ and $\{\ell_i\}$ necessarily terminates for some natural number m. ∎

PROPOSITION 3.4. *Let a, $\{n_k\}_{k=1}^m$, $\{\ell_k\}_{k=1}^m$ be the same as in Proposition 3.3, and in addition $a \notin \mathcal{B}$. Let $g_k \in \mathcal{B}[x]$ be a unitary polynomial with $\deg g_k = n_k$, $g_k(a\ell_k) = 0$, $k = 1, 2, \dots, m$. Setting $p_a(x) := x^{n_m} + \sum\limits_{k=1}^{m} (g_k(x) - x^{n_k}) x^{n_m - n_k} \ell_k$, we have*

$$s(g_k'(a\ell_k)) \ge \ell_k\,, \qquad p_a(a) = 0\,, \qquad s(p_a'(a)) \ge s(a)\,.$$

Proof. Note, that it follows from the definition that the polynomial p_a is unitary. Since $g_k(x) = x^{n_k} + a_1 x^{n_k-1} + \dots + a_{n_k}$, we have

$$g_k'(x) = n_k x^{n_k-1} + (n_k - 1) a_1 x^{n_k-2} + \dots + a_{n_k-1}\,.$$

If $n_1 = 1$, then $g_1'(x) = \mathbf{1}$ and $s(g_1'(a\ell_1)) = \mathbf{1} \geq \ell_1$. Suppose that $n_1 > 1$. It follows that $n_k > 1$ for all $k = 1, \ldots, m$ and, since the field K has characteristic zero, the polynomial

$$q_k(x) = n_k^{-1} g_k'(x)$$

is a unitary polynomial with $\deg q_k = n_k - 1$, for which $s(q_k(a\ell_k)) = s(g_k'(a\ell_k))$. If

$$(\mathbf{1} - s(q_k(a\ell_k)))\ell_k := e \neq 0\,,$$

then

$$0 = q_k(a\ell_k)e = (ae)^{n_k-1} + (n_k - 1)n_k^{-1} a_1 (ae)^{n_k-2} + \cdots + n_k^{-1} a_{n_k-1} e\,,$$

and therefore $d(e) \leq n_k - 1$ (see the definition of the function d in the proof of Proposition 3.3), which contradicts Proposition 3.3 (ii). Consequently, $e = 0$, and therefore

$$s(g_k'(a\ell_k)) \geq \ell_k\,, \qquad k = 1, \ldots, m\,.$$

Further, since $s(a) = \sum_{k=1}^{m} \ell_k$ (Proposition 3.3 (i)), we have

$$\begin{aligned} p_a(x) &= x^{n_m} + \sum_{k=1}^{m} \ell_k g_k(x) x^{n_m - n_k} - \sum_{k=1}^{m} \ell_k x^{n_m} \\ &= x^{n_m} + \sum_{k=1}^{m} \ell_k g_k(x\ell_k) x^{n_m - n_k} - s(a) x^{n_m} \\ &= x^{n_m}(1 - s(a)) + \sum_{k=1}^{m} \ell_k g_k(x\ell_k) x^{n_m - n_k} \end{aligned}$$

and

$$\begin{aligned} p_a'(x) = n_m(1 - s(a))x^{n_m-1} &+ \sum_{k=1}^{m-1} (n_m - n_k) x^{n_m - n_k - 1} \ell_k g_k(x) \\ &+ \sum_{k=1}^{m-1} \ell_k g_k'(x) x^{n_m - n_k} + \ell_m g_m'(x)\,. \end{aligned}$$

Substituting a in the formulae above, we obtain $p_a(a) = 0$ and

$$p_a'(a) = \sum_{k=1}^{m} g_k'(a\ell_k)(a\ell_k)^{n_m - n_k} \ell_k\,.$$

It follows that $s(p'_a(a)) \le \sum_{k=1}^{m} \ell_k = s(a)$ and also that

$$p'_a(a)\ell_k = g'_k(a\ell_k)(a\ell_k)^{n_m - n_k}\ell_k\,, \qquad \forall\ k = 1, \dots, m\,.$$

Combining this fact with the inequality (see above) $s(g'_k(a\ell_k)) \ge \ell_k$, we get

$$s(p'_a(a)\ell_k) = \ell_k\,, \qquad \forall\ k = 1, \dots, m\,,$$

whence $s(p'_a(a)) \ge \ell_k$ for all $k = 1, \dots, m$, which further implies $s(p'_a(a)) \ge \sum_{k=1}^{m} \ell_k = s(a)$. Thus, $s(p'_a(a)) \ge s(a)$. ∎

Proposition 3.5. *Let $a, p_a(x)$ be the same as in Proposition 3.4, let $g \in \mathcal{B}[x]$, $g(a) = 0$, and let $\delta : \mathcal{B} \to \mathcal{A}$ be a derivation. Then $g^\delta(a) - g'(a)p_a^\delta(a)i(p'_a(a)) = 0$.*

Proof. Let $g(x) = \sum_{k=0}^{n} a_k x^{n-k}$, $a_k \in \mathcal{B}$ and set

$$u(x) = s(a)g(x)\,, \qquad v(x) = (\mathbf{1} - s(a))g(x)\,.$$

Clearly, $g(x) = u(x) + v(x)$. Since

$$v'(x) = (\mathbf{1} - s(a))\sum_{k=0}^{n-1}(n-k)a_k x^{n-k-1}\,, \qquad v^\delta(x) = (\mathbf{1} - s(a))\sum_{k=0}^{n}\delta(a_k)x^{n-k}\,,$$

we have $v'(a) = a_{n-1}(\mathbf{1} - s(a))$, $v^\delta(a) = \delta(a_n)(\mathbf{1} - s(a))$. Appealing now to the equalities $s(i(p'_a(a))) = s(p'_a(a)) = s(a)$ (see Proposition 3.4), we obtain

$$\begin{aligned} v^\delta(a) - v'(a)p_a^\delta(a)i(p'_a(a)) &= \delta(a_n)(\mathbf{1} - s(a)) \\ &\quad - a_{n-1}(\mathbf{1} - s(a))p_a^\delta(a)i(p'_a(a))s(a) \\ &= \delta(a_n(\mathbf{1} - s(a)))\,. \end{aligned}$$

Since $g(a) = 0$, we have $a_n(\mathbf{1} - s(a)) = g(a)(\mathbf{1} - s(a)) = 0$ and we further obtain

$$v^\delta(a) - v'(a)p_a^\delta(a)i(p'(a)) = 0\,.$$

Therefore

$$g^\delta - g'(a)p_a^\delta(a)i(p'_a(a)) = u^\delta(a) - u'(a)p_a^\delta(a)i(p'_a(a))\,. \tag{1}$$

Let $\{\ell_k\}_{k=1}^m$, $\{n_k\}_{k=1}^m$, $\{g_k\}_{k=1}^m$ be the same as in the assertion of Proposition 3.4. Set

$$u_k(x) = \ell_k u(x)\,, \qquad k = 1, \dots, m\,.$$

Since $s(a) = \sum\limits_{k=1}^{m} \ell_k$, we have $u(x) = \sum\limits_{k=1}^{m} u_k(x)$. By Proposition 3.1, there exist polynomials $f_k(x), r_k(x) \in \mathcal{B}[x]$, such that

$$u_k(x) = f_k(x) g_k(x) + r_k(x)\,, \qquad \deg r_k < \deg g_k = n_k\,, \qquad k = 1, \dots, m\,.$$

Since $u_k(x) = \ell_k u_k(x)$, we may (and shall) assume that the supports of all coefficients of the polynomials $f_k(x)$ and $r_k(x)$ belong to $\ell_k \nabla$. Further, from the equalities

$$g_k(a\ell_k) = 0\,, \qquad u_k(a) = \ell_k u(a) = \ell_k s(a) g(a) = 0\,,$$

and

$$u_k(a\ell_k) = \ell_k g(a\ell_k) = \ell_k (g(a)\ell_k + g(0)(\mathbf{1} - \ell_k)) = \ell_k g(a) = 0\,,$$

it follows that

$$r_k(a\ell_k) = u_k(a\ell_k) - f_k(a\ell_k) g_k(a\ell_k) = 0\,, \qquad k = 1, \dots, m\,.$$

We claim that, in fact, $r_k(x) \equiv 0$, i.e., all coefficients of the polynomial $r_k(x)$ vanish, $k = 1, \dots, m$. Let $r_k(x) = \sum\limits_{j=0}^{t} b_j x^{t-j}$, $b_0 \neq 0$, $t \in \mathbb{N}$. Consider the unitary polynomial $f(x) = x^t + i(b_0)(b_1 x^{t-1} + \cdots + b_{t-1} x + b_t)$. Since $s(b_0) = b_0 i(b_0)$, $s(b_j) \leq \ell_k$, $j = 0, 1, \dots, t$, we have

$$\begin{aligned} f(as(b_0)) &= a^t s(b_0) + i(b_0)(b_1 (as(b_0))^{t-1} + \cdots + b_t(as(b_0)) + b_t) \\ &= i(b_0) \sum_{j=0}^{t} b_j (as(b_0))^{t-j} = i(b_0) \sum_{j=0}^{t} \ell_k b_j (as(b_0))^{t-j} \\ &= i(b_0) \sum_{j=0}^{t} b_j (a\ell_k)^{t-j} = i(b_0) r_k(a\ell_k) = 0\,. \end{aligned}$$

Since $\deg f = \deg r_k = t < n_k$, this means that $d(s(b_0)) < n_k$, which (together with the inequality $s(b_0) \leq \ell_k$) contradicts Proposition 3.3 (ii). Consequently,

$r_k(x) \equiv 0$, and therefore $u_k(x) = f_k(x)g_k(x)$, $k = 1, \ldots, m$. Hence (see Proposition 3.2)

$$u_k'(x) = f_k'(x)g_k(x) + f_k(x)g_k'(x)\,,$$
$$u_k^\delta(x) = f_k^\delta(x)g_k(x) + f_k(x)g_k^\delta(x)\,.$$

Since, by the assumption, $g(a) = 0$, we deduce from the above that

$$u_k'(a\ell_k) = f_k(a\ell_k)g_k'(a\ell_k)\,, \qquad u_k^\delta(a\ell_k) = f_k(a\ell_k)g_k^\delta(a\ell_k)\,,$$

and further, appealing to the inequalities (which follow from Proposition 3.4 and the assumptions on f_k's and r_k's)

$$s(g_k'(a\ell_k)) \geq \ell_k\,, \qquad s(f_k(a\ell_k)) \leq \ell_k\,,$$

that

$$s(u_k'(a\ell_k)) \leq \ell_k\,, \qquad s(u_k^\delta(a\ell_k)) \leq \ell_k\,. \tag{2}$$

We use the relations above to show that

$$u_k^\delta(a\ell_k) - u_k'(a\ell_k)g_k^\delta(a\ell_k)i(g_k'(a\ell_k)) = 0\,. \tag{3}$$

Indeed, the equality $u_k^\delta(a\ell_k) = f_k(a\ell_k)g_k^\delta(a\ell_k)$ together with

$$\begin{aligned} u_k'(a\ell_k)g_k^\delta(a\ell_k)i(g_k'(a\ell_k)) &= f_k(a\ell_k)g_k'(a\ell_k)g_k^\delta(a\ell_k)i(g_k'(a\ell_k)) \\ &= f_k(a\ell_k)s(f_k(a\ell_k))\ell_k g_k^\delta(a\ell_k)s(g_k'(a\ell_k) \\ &= f_k(a\ell_k)g_k^\delta(a\ell_k)\ell_k \end{aligned}$$

show that (3) holds. Since $u_k(x) = \ell_k u(x) = \ell_k(u(x) + v(x)) = \ell_k g(x)$ and $s(a) = \sum_{k=1}^m \ell_k$, we get

$$\begin{aligned} \sum_{k=1}^m u_k(a\ell_k) = \left(\sum_{k=1}^m a_0\ell_k\right) a^n + \cdots + \left(\sum_{k=1}^m a_{n-1}\ell_k\right) a \\ + \sum_{k=1}^m a_n\ell_k \;=\; s(a)g(a) \;=\; u(a)\,; \end{aligned} \tag{4}$$

$$\begin{aligned} \sum_{k=1}^m u_k^\delta(a\ell_k) = \left(\sum_{k=1}^m \delta(a_0)\ell_k\right) a^n + \ldots \\ + \left(\sum_{k=1}^m \delta(a_{n-1})\ell_k\right) a + \sum_{k=1}^m \delta(a_n)\ell_k \;=\; u^\delta(a)\,; \end{aligned} \tag{5}$$

$$\sum_{i=k}^{m} u'(a\ell_k) = n\left(\sum_{k=1}^{m} a_0\ell_k\right) a^{n-1} + \dots$$
$$+ \sum_{k=1}^{m} a_{n-1}\ell_k \;=\; s(a)g'(a) \;=\; u'(a)\,. \qquad (6)$$

Further, for

$$\begin{aligned} p_a(x) &= x^{n_m} + \sum_{k=1}^{m}(g_k(x) - x^{n_k})x^{n_m-n_k}\ell_k \\ &= x^{n_m} + \sum_{k=1}^{m}(g_k(x)x^{n_m-n_k}\ell_k - x^{n_m}\ell_k)\,, \end{aligned}$$

we have $p_a^\delta(x) = \sum_{i=k}^{m} g_k^\delta(x)x^{n_m-n_k}\ell_k$ and obviously, $p_a^\delta(a) = \sum_{k=1}^{n} g_k^\delta(a)a^{n_m-n_k}\ell_k$, i.e.,

$$p_a^\delta(a)\ell_k = g_k^\delta(a)a^{n_m-n_k}\ell_k\,.$$

Since $g_k(a\ell_k) = 0$, we have $0 = g_k(a\ell_k)\ell_k + g_k(0)(\mathbf{1}-\ell_k)$, i.e., $g_k(0)(\mathbf{1}-\ell_k) = 0$, and therefore $g_k(x)\ell_k = g_k(\ell_k x)$. This implies that $g_k^\delta(a)\ell_k = g_k^\delta(a\ell_k)$ and thus

$$p_a^\delta(a)\ell_k = g_k^\delta(a\ell_k)a^{n_m-n_k}\,.$$

It is shown, in the proof of Proposition 3.4 that

$$p_a'(a)\ell_k = g_k'(a\ell_k)(a\ell_k)^{n_m-n_k}\ell_k\,.$$

Consequently,

$$\begin{aligned} p_a^\delta(a)i(p_a'(a))\ell_k &= g_k^\delta(a\ell_k)a^{n_m-n_k}i(g_k'(a\ell_k))i(a)^{n_m-n_k}\ell_k \\ &= g_k^\delta(a\ell_k)i(g'(a\ell_k))s(a)\ell_k = g_k^\delta(a\ell_k)i(g_k'(a\ell_k))\ell_k\,. \qquad (7) \end{aligned}$$

Now, taking into account that $s(p_a'(a)) = s(a) = \sum_{k=1}^{m} \ell_k$, $s(u(a)) \le s(a)$, we deduce from (2)–(7) that

$$\begin{aligned} u^\delta(a) - u'(a)p_a^\delta(a)i(p_a'(a)) &= \sum_{k=1}^{m}(u^\delta(a) - u'(a)p_a^\delta(a)i(p_a'(a)))\ell_k \\ &= \sum_{k=1}^{m}(u_k^\delta(a\ell_k) - u_k'(a\ell_k)g_k^\delta(a\ell_k)i(g_k'(a\ell_k))) = 0\,, \end{aligned}$$

which together with (1) completes the proof of Proposition 3.5. ∎

We have now prepared all necessary technical tools to accomplish the fourth step in our extension programme. In the next proposition we build the extension of a derivation $\delta : \mathcal{B} \to \mathcal{A}$ onto the subalgebra $\mathcal{B}(a)$ for an integral element a with respect to $\mathcal{B}$.

Proposition 3.6. *Let $\mathcal{B} \subseteq \mathcal{A}$ be a regular ρ-closed subalgebra, such that $\mathcal{B} \supset \nabla$, and let $\delta : \mathcal{B} \to \mathcal{A}$ be a derivation. If a is an integral element with respect to $\mathcal{B}$, then there exists a unique derivation $\delta_1 : \mathcal{B}(a) \to \mathcal{A}$, such that $\delta_1(b) = \delta(b)$ for all $b \in \mathcal{B}$.*

Proof. Let $f, p \in \mathcal{B}[x]$ be such that $f(a) = p(a)$ and set $g(x) := f(x) - p(x)$. Then $g \in \mathcal{B}[x]$, $g(a) = 0$, and therefore by Proposition 3.5, we have

$$g^\delta(a) - g'(a)p_a^\delta(a)i(p_a'(a)) = 0\,.$$

Since $g^\delta(a) = f^\delta(a) - p^\delta(a)$, $g'(a) = f'(a) - p'(a)$, we get

$$f^\delta(a) - f'(a)p_a^\delta(a)i(p_a'(a)) = p^\delta(a) - p'(a)p_a^\delta(a)i(p_a'(a))\,.$$

Consequently, for any $f(a) \in \mathcal{B}(a)$, $f \in \mathcal{B}[x]$, the map

$$\delta_1(f(a)) = f^\delta(a) - f'(a)p_a^\delta(a)i(p_a'(a))$$

is well-defined. Since the maps $f \to f'$, $f \to f^\delta$ are derivations (see Proposition 3.2), the map $\delta_1 : \mathcal{B}(a) \to \mathcal{A}$ is also a derivation. If $b \in \mathcal{B}$, $f(x) \equiv b$, then $f'(x) \equiv 0$, $f^\delta(x) = \delta(b)$, $f(a) = b$, and therefore $f'(a) = 0$, $f^\delta(a) = \delta(b)$, i.e., $\delta_1(b) = \delta_1(f(a)) = \delta(b)$.

Suppose now that $\delta_2 : \mathcal{B}(a) \to \mathcal{A}$ is a derivation, such that $\delta_2(b) = \delta(b)$ for all $b \in \mathcal{B}$. For any $f(x) = \sum\limits_{k=0}^{n} a_k x^{n-k} \in \mathcal{B}[x]$, we have that

$$\begin{aligned}\delta_2(f(a)) = \delta_2\left(\sum_{k=0}^{n} a_k a^{n-k}\right) &= \sum_{k=0}^{n} \delta_2(a_k)a^{n-k} + \sum_{k=0}^{n-1}(n-k)a_k a^{n-k-1}\delta_2(a) \\ &= \sum_{k=0}^{n} \delta(a_k)a^{n-k} + f'(a)\delta_2(a) = f^\delta(a) + f'(a)\delta_2(a)\,.\end{aligned}$$

Since $p_a(a) = 0$, it follows $0 = \delta_2(p_a(a)) = p_a^\delta(a) + p_a'(a)\delta_2(a)$. Multiplying the latter equality by $i(p_a'(a))$ and using the equality $s(p_a'(a)) = s(a)$ established in Proposition 3.4, we obtain

$$-p_a^\delta(a)i(p_a'(a)) = p_a'(a)i(p_a'(a))\delta_2(a) = s(a)\delta_2(a) = \delta_2(s(a)a) = \delta_2(a)\,.$$

On the other hand, for $f(x) = x \in \mathcal{B}[x]$ we have $f(a) = a$, $f^\delta(a) = 0$, $f'(a) = \mathbf{1}$, and therefore

$$\delta_1(a) = \delta_1(f(a)) = -p_a^\delta(a)i(p_a'(a)) = \delta_2(a)\,.$$

Consequently, for any $f(x) \in \mathcal{B}[x]$, we have

$$\delta_2(f(a)) = f^\delta(a) + f'(a)\delta_2(a) = f^\delta(a) + f'(a)\delta_1(a) = \delta_1(f(a))\,,$$

i.e., $\delta_2 = \delta_1$. ∎

We have arrived at the penultimate step in our programme of extending a derivation $\delta : \mathcal{B} \to \mathcal{A}$ onto $\mathcal{A}$: the extension of δ onto a subalgebra of $\mathcal{A}$ which itself is an extension of $\mathcal{B}$ by a non-integral element with respect to $\mathcal{B}$. It is important to observe that a non-integral element a with respect to $\mathcal{B}$ is not necessarily transcendental (see Example 3.1), so we are not in a position to apply Theorem 1.8.16 from [7], in which the extension of δ on $\mathcal{B}(a)$ is presented for the case when a is transcendental with respect to $\mathcal{B}$. Our approach here is based on the (new) notion of a weakly transcendental element introduced at the beginning of the present section.

PROPOSITION 3.7. *Let $\mathcal{B}$ be a regular subalgebra in $\mathcal{A}$ such that $\nabla \subset \mathcal{B}$ and let $\delta : \mathcal{B} \to \mathcal{A}$ be a derivation. If $a \in \mathcal{A}$ is a weakly transcendental element with respect to $\mathcal{B}$, then for every $c \in \mathcal{A}$, such that $s(c) \le s(a)$, there exist a unique derivation $\delta_1 : \mathcal{B}(a) \to \mathcal{A}$, such that $\delta_1(b) = \delta(b)$ for all $b \in \mathcal{B}$ and $\delta_1(a) = c$.*

Proof. Let $\sum_{j=0}^{n} a_j x^{n-j} := g(x) \in \mathcal{B}[x]$ be such that $g(a) = 0$. We shall show that $g^\delta(a) = 0$ and $g'(a)s(a) = 0$. Since $s(a_0) = a_0 i(a_0)$, we have

$$\begin{aligned}0 = i(a_0)g(a) = i(a_0)\sum_{j=0}^{n} a_j a^{n-i} &= s(a_0)a^n + a_1 i(a_0)a^{n-1} + \cdots + i(a_0)a_n \\ &= (s(a_0)a)^n + a_1 i(a_0)(s(a_0)a)^{n-1} + \cdots + i(a_0)a_n\,.\end{aligned}$$

Since a is a weakly transcendental element with respect to $\mathcal{B}$, $i(a_0)a_j \in \mathcal{B}$, $j = 1, \ldots, n$, and since $s(a_0)s(a) \le s(a)$, we derive that $s(a_0)a = 0$. It follows, that $a_0 a^n = 0$ and so $g_1(a) = 0$, where

$$g_1(x) := \sum_{j=1}^{n} a_j x^{n-j} \in \mathcal{B}[x]\,.$$

Repeating the preceding argument n times, we see that

$$s(a_j)a = 0\,, \quad j = 0, 1, \ldots, n-1\,, \qquad a_n = 0\,.$$

Consequently, $\delta(a_j)a^{n-j} = \delta(a_j)s(a_j)a^{n-j} = 0$, $j = 0, \ldots, n-1$, $\delta(a_n) = 0$, hence

$$g^\delta(a) = 0\,, \qquad g'(a)s(a) = \sum_{j=0}^{n-1}(n-j)a_j a^{n-j-1}s(a) = 0\,.$$

We now define the map $\delta_1 : \mathcal{B}(a) \to \mathcal{A}$, by setting

$$\delta_1(f(a)) := f^\delta(a) + f'(a)c\,, \qquad \forall\ f \in \mathcal{B}[x]\,.$$

If $f, p \in \mathcal{B}[x]$ are such that $f(a) = p(a)$, then setting $g = f - p$, we have $g \in \mathcal{B}[x]$, $g(a) = 0$, and, by the previous argument,

$$f^\delta(a) = p^\delta(a)\,, \qquad f'(a)s(a) = p'(a)s(a)\,.$$

Since $s(c) \leq s(a)$, it follows $f'(a)c = p'(a)c$. This shows that the map δ_1 is defined correctly. Since the maps $f \to f'$, $f \to f^\delta$ are derivations (see Proposition 3.2), the map $\delta_1 : \mathcal{B}(a) \to \mathcal{A}$ is also a derivation such that $\delta_1(b) = \delta(b)$ for all $b \in \mathcal{B}$ (see the proof of Proposition 3.6). If $f(x) = x$, then $f(a) = a$, $f'(a) = \mathbf{1}$, $f^\delta(a) = 0$, i.e., $\delta_1(a) = c$.

Let $\delta_2 : \mathcal{B}(a) \to \mathcal{A}$ be a derivation, for which $\delta_2(b) = \delta(b)$ for all $b \in \mathcal{B}$ and $\delta_2(a) = c$. Then, for any $f \in \mathcal{B}[x]$ we have (see the proof of Proposition 3.6)

$$\delta_2(f(a)) = f^\delta(a) + f'(a)\delta_2(a) = f^\delta(a) + f'(a)c = \delta_1(f(a))\,,$$

i.e., $\delta_1 = \delta_2$. ∎

The following theorem is our first main result. It provides sufficient conditions for a derivation initially defined on a subalgebra of a commutative regular algebra to have an extension to the algebra.

Theorem 3.1. *Let $\mathcal{B}$ be a subalgebra of a commutative unital regular algebra $\mathcal{A}$ over a field K of characteristic zero with finite strictly positive countable-additive measure μ on the Boolean algebra ∇ of all idempotents in $\mathcal{A}$, and let $\rho(a, b) = \mu(s(a - b))$, $a, b \in \mathcal{A}$. If the metric space $(\mathcal{A}, \rho)$ is complete, then for any derivation $\delta : \mathcal{B} \to \mathcal{A}$ for which $s(\delta(b)) \leq s(b)$ for all $b \in \mathcal{B}$ there exists a derivation $\delta_0 : \mathcal{A} \to \mathcal{A}$, such that $\delta_0(b) = \delta(b)$ for all $b \in \mathcal{B}$.*

Proof. Thanks to Proposition 3.6 and Proposition 3.7, the proof of the theorem may be realized via a standard scheme involving the Zorn's lemma (see, for example, [7, §1.8]). Indeed, consider the set X of all pairs $(\mathcal{A}_\alpha, \delta_\alpha)$, where $\mathcal{A}_\alpha$ is a subalgebra in $(\mathcal{A}, \rho)$, $\mathcal{B} \subset \mathcal{A}_\alpha$ and $\delta_\alpha : \mathcal{A}_\alpha \to \mathcal{A}$ is a derivation, satisfying $\delta_\alpha(b) = \delta(b)$ for all $b \in \mathcal{B}$ and $s(\delta_\alpha(a)) \leq s(a)$ for all $a \in \mathcal{A}_\alpha$. The set X is not empty, since $(\mathcal{B}, \delta) \in X$. We define the partial order relation on X, by setting $(\mathcal{A}_{\alpha_1}, \delta_{\alpha_1}) \leq (\mathcal{A}_{\alpha_2}, \delta_{\alpha_2})$, if and only if $\mathcal{A}_{\alpha_1} \subset \mathcal{A}_{\alpha_2}$, $\delta_{\alpha_2}(a) = \delta_{\alpha_1}(a)$ for all $a \in \mathcal{A}_{\alpha_1}$. Let $Y := \{(\mathcal{A}_i, \delta_i)\}_{i\in I}$ be an arbitrary linearly ordered subset from X. Set $\mathcal{A}_I = \bigcup\limits_{i\in I} \mathcal{A}_i$. Clearly, $\mathcal{A}_I$ is a subalgebra in $\mathcal{A}$, $\mathcal{B} \subset \mathcal{A}_I$, and the derivation $\delta_I : \mathcal{A}_I \to \mathcal{A}$, given by $\delta_I(a) = \delta_i(a)$ for all $a \in \mathcal{A}_i$, $i \in I$ is well-defined and is such that $s(\delta_I(a) \leq s(a)$ for all $a \in \mathcal{A}_I$. This implies that $(\mathcal{A}_I, \delta_I) \in X$ and $(\mathcal{A}_i, \delta_i) \leq (\mathcal{A}_I, \delta_I)$ for any $i \in I$.

According to the Zorn's lemma, there exists a maximal element $(\mathcal{A}_0, \delta_0) \in X$. By the propositions 2.4, 2.5 and 2.6, we have that $\nabla \subset \mathcal{A}_0$, that the algebra $\mathcal{A}_0$ is regular and closed in $(\mathcal{A}, \rho)$, and it follows from the assumption on $\mathcal{A}_0$ and Proposition 3.6, that every integral element with respect to $\mathcal{A}_0$ in fact belongs to $\mathcal{A}_0$. Suppose that $\mathcal{A}_0 \neq \mathcal{A}$ and select $a \in \mathcal{A}$, $a \notin \mathcal{A}_0$. The set $J = \{e \in \nabla : e \leq s(a),\, e\, a \in \mathcal{A}_0\}$ is an ideal in the complete Boolean algebra ∇. Set $e_0 = \vee J$ and select an increasing sequence $\{e_n\}_{n\geq 1} \subseteq J$ so that $e_0 = \bigvee\limits_{n=1}^{\infty} e_n$, in particular, $\rho(e_n, e_0) \to 0$ for $n \to \infty$. Then $e_0 \leq s(a)$, $e_n a \in \mathcal{A}_0$ and $\rho(e_n a, e_0 a) \to 0$ (Proposition 2.6 (iii)). Since $\mathcal{A}_0 = \overline{\mathcal{A}_0}$, we have $e_0 a \in \mathcal{A}_0$. This implies that $e_0 \in J$. Consider the element $c := (\mathbf{1} - e_0)a$. Since $a = c + e_0 a \notin \mathcal{A}_0$, we have $c \notin \mathcal{A}_0$. We shall show that c is a weakly transcendental element with respect to $\mathcal{A}_0$. If this is not the case, then there exists a non-zero idempotent $e \leq s(c) = (\mathbf{1} - e_0)s(a)$, such that ec is an integral element with respect to $\mathcal{A}_0$. It follows, $ea = ec \in \mathcal{A}_0$, and therefore $e \in J$, i.e., $e \leq (\mathbf{1} - e_0)e_0 = 0$. This contradiction shows that c is a weakly transcendental element with respect to $\mathcal{A}_0$. According to Proposition 3.7, the derivation δ_0 can be extended to the derivation δ_3 living on the subalgebra $\mathcal{A}_3$ in $(\mathcal{A}, \rho)$ such that $\mathcal{A}_0 \subset \mathcal{A}_3$, $c \in \mathcal{A}_3$. This implies that $(\mathcal{A}_3, \delta_3) \in X$ and $(\mathcal{A}_0, \delta_0) \leq (\mathcal{A}_3, \delta_3)$, $\mathcal{A}_0 \neq \mathcal{A}_3$, which contradicts to the maximality of the element $(\mathcal{A}_0, \delta_0)$ in X. Consequently, $\mathcal{A}_0 = \mathcal{A}$, and this completes the proof of Theorem 3.1. ∎

Remark 3.1. Let S be an algebra of all (classes of) measurable functions on the interval $([a, b], \lambda)$ (as before, λ is the linear Lebesgue measure), let $P[a, b]$ be the subalgebra of all polynomials in S, and let $\delta(p) = p'$ be the standard derivation on $P[a, b]$. Since $s(p) = \mathbf{1}$ for any $p \in P[a, b]$, $p \neq 0$, it follows from Theorem 3.1 that there exists a derivation $\delta : S \to S$, such that $\delta(p) = p'$ for all $p \in P[a,b]$. In particular, this implies that there exists a non-zero derivation from $L_\infty([a,b],\lambda)$ to S. In this connection, we recall that are no non-zero derivations from $L_\infty(\Omega,\Sigma,\mu) \to L_\infty(\Omega,\Sigma,\mu)$ (see, for example, [13]).

We shall now describe the class of commutative regular algebras which admit a non-zero derivation. Let $\mathcal{A}, K, \nabla, \mu, \rho$ be the same as in the assumptions of Theorem 3.1. Denote by $K_c(\nabla)$ the subalgebra of all countably-valued elements in $\mathcal{A}$.

Proposition 3.8. *If K is an algebraically closed field of characteristic zero, and $a \in \mathcal{A}$ is an integral element with respect to $K_c(\nabla)$, then $a \in K_c(\nabla)$.*

Proof. Since a is an integral element with respect to $K_c(\nabla)$, we have $p(a) = 0$ for the some polynomial $p(x) = x^n + a_1 x^{n-1} + \cdots + a_n \in K_c(\nabla)[x]$. Every coefficient a_k has a representation as a convergent series $a_k = \sum_{j=1}^{\infty} \alpha_j^{(k)} e_j^{(k)}$, where $\alpha_j^{(k)} \in K$, $e_j^{(k)} \in \nabla$, $e_i^{(k)} e_j^{(k)} = 0$, $i \neq j$, $i,j = 1,2,\ldots,$ $k = 1,\ldots,n$. Without loss of generality, we may (and shall) assume that $\sum_{j=1}^{\infty} e_j^{(k)} = \mathbf{1}$ for all $k = 1,\ldots,n$. Consider the atomic Boolean subalgebra ∇_0 of ∇, generated by the elements $e_j^{(k)}$, $k = 1,\ldots,n$, $j = 1,2,\ldots$. Atoms in ∇_0 are the elements of the form

$$e = e_{j_1}^{(1)} e_{j_2}^{(2)} \ldots e_{j_n}^{(n)}, \qquad e \neq 0, \quad j_k \geq 1, \quad k = 1,2,\ldots,n.$$

For any such atom, we have that

$$(ea)^n + \alpha_{j_1}^{(1)} (ea)^{n-1} + \alpha_{j_2}^{(2)} (ea)^{n-2} + \cdots + \alpha_{j_n}^{(n)} e = 0.$$

Since K is an algebraically closed field, we have (see, for example, [19, ch. 10, §72]) that

$$q(x) = x^n + \sum_{k=1}^{n} \alpha_{j_k}^{(k)} x^{n-k} = (x - \beta_1) \ldots (x - \beta_n)$$

for some $\beta_k \in K$, $k = 1,\ldots,n$. It follows,

$$0 = eq(ea) = (ea - \beta_1 e) \ldots (ea - \beta_n e).$$

Hence $s(ea - \beta_1 e) \ldots s(ea - \beta_n e) = 0$, and therefore $\mathbf{1} = \bigvee_{k=1}^{n} (\mathbf{1} - s(ea - \beta_k e))$.

Since $s(ea-\beta_k e) \le e$, $k=1,\ldots,n$, it follows $e = \bigvee_{k=1}^{n}(e - s(ea-\beta_k e))$. Pick pairwise disjoint elements $f_1, f_2, \ldots, f_n \in \nabla$ so that $e = \bigvee_{k=1}^{n} f_k$ and $f_k \le e - s(ea-\beta_k e)$ for all $k=1,\ldots,n$. Then,

$$\begin{aligned} af_k &= af_k(e - s(ea-\beta_k e)) = f_k(ae - as(ea-\beta_k e)) \\ &= f_k(ae - (ae-\beta_k e)s(ae-\beta_k e) - \beta_k s(ae-\beta_k e)) \\ &= f_k(ae - ae + \beta_k e - \beta_k s(ae-\beta_k e)) = \beta_k f_k - \beta_k f_k s(ae-\beta_k e)\,. \end{aligned}$$

This implies that $ae = \sum_{k=1}^{n} af_k \in K_c(\nabla)$ for any atom $e \in \nabla_0$. Since $\mathbf{1} = \rho - \sum_{i=1}^{\infty} q_i$ (the series converges in $(\mathcal{A}, \rho)$), where $\{q_i\}$ are atoms from ∇_0, we obtain $a = \rho - \sum_{i=1}^{\infty} aq_i$. From here and from Proposition 2.8, it follows that $a \in K_c(\nabla)$. ∎

The following theorem is our second main result. It provides necessary and sufficient conditions for a wide class of commutative regular algebras to have a non-zero derivation.

Theorem 3.2. *Let $\mathcal{A}$ be a commutative unital regular algebra over an algebraically closed field K of characteristic zero, let μ be a finite strictly positive countably additive measure on the Boolean algebra ∇ of all idempotents in $\mathcal{A}$ and let $\mathcal{A}$ be complete in the metric $\rho(a,b) = \mu(s(a-b))$, $a, b \in \mathcal{A}$. There are the non-zero derivations on $\mathcal{A}$ if and only if $K_c(\nabla) \neq \mathcal{A}$.*

Proof. If $K_c(\nabla) = A$, then the fact that every derivation on $\mathcal{A}$ vanishes is already established in Section 2.

Let $K_c(\nabla) \neq \mathcal{A}$. Since $K_c(\nabla)$ contains all integral elements with respect to $K_c(\nabla)$ (see Proposition 3.8), we may repeat the proof of Theorem 3.1 replacing $\mathcal{A}_0$ with $K_c(\nabla)$, obtaining that there exists a weakly transcendental element $c \in \mathcal{A}$ over $K_c(\nabla)$. By Proposition 3.7, there exists a non-zero derivation $\delta_1 : K_c(\nabla)(c) \to \mathcal{A}$, such that $\delta_1(c) = c \neq 0$, $\delta_1(a) = 0$ for all $a \in K_c(\nabla)$. Since $\nabla \subset K_c(\nabla) \subset K_c(\nabla)(c)$, we have $\delta_1(b) = \delta_1(s(b)b) = s(b)\delta_1(b)$, i.e., $s(\delta_1(b)) \le s(b)$ for all $b \in K_c(\nabla)(c)$. Therefore, it follows from Theorem 3.1 that there exists a derivation $\delta_0 : \mathcal{A} \to \mathcal{A}$, such that $\delta_0(c) = \delta_1(c) = c \neq 0$, i.e., δ_0 is a non-zero derivation on $\mathcal{A}$. ∎

The following theorem is an important specialization of the preceding result to the commutative regular algebra $S(\Omega, \Sigma, \mu)$.

THEOREM 3.3. *Let (Ω, Σ, μ) be a finite measure space and let $S = S(\Omega, \Sigma, \mu)$ be the algebra of all measurable function on (Ω, Σ, μ) with values in the field K, where $K = \mathbb{R}$, or else $K = \mathbb{C}$. The following conditions are equivalent:*

(i) *There exists a non-zero derivation $\delta : S \to S$;*

(ii) *The Boolean algebra ∇ of all idempotents in S is not an atomic Boolean algebra.*

Proof. Since $K = \mathbb{C}$ is an algebraically closed field of characteristic zero, the proof of the theorem for this case follows from Proposition 2.9 and Theorem 3.2.

Let $K = \mathbb{R}$, and denote by $S_{\mathbb{R}}, S_{\mathbb{C}}$ the algebras of all measurable functions on (Ω, Σ, μ) with values in $\mathbb{R}$ and $\mathbb{C}$ respectively. Set $S_h = \{a \in S_{\mathbb{C}} : a = \overline{a}\}$, where $\overline{a}$ is the complex conjugate for the function $a \in S_{\mathbb{C}}$. Clearly, S_h may be identified with $S_{\mathbb{R}}$, and that $S_{\mathbb{C}} = S_h + iS_h$, where $i \in \mathbb{C}$, $i^2 = -1$. Fix a derivation $\delta : S_{\mathbb{C}} \to S_{\mathbb{C}}$, and define the maps $\delta_k : S_h \to S_h$, $k = 1, 2$, by setting $\delta_1(a) := \operatorname{Re}\delta(a)$, $\delta_2(a) := \operatorname{Im}\delta(a)$, where, as usual, for $b \in S_{\mathbb{C}}$, we denote $\operatorname{Re} b = (b + \overline{b})/2$, $\operatorname{Im} b = (b - \overline{b})/2i$. Clearly, δ_k is a $\mathbb{R}$-linear map from S_h into S_h, $k = 1, 2$. For any $a, b \in S_h$, we have

$$\begin{aligned}\delta_1(ab) &= \operatorname{Re}\delta(ab) = \operatorname{Re}(\delta(a)b + a\delta(b))\\ &= (\operatorname{Re}\delta(a))b + a(\operatorname{Re}\delta(b)) = \delta_1(a)b + a\delta_1(b)\,,\end{aligned}$$

that is δ_1 is a derivation on S_h. Analogously, δ_2 is a derivation on S_h. For any function $a \in S_{\mathbb{C}}$, we have

$$\begin{aligned}\delta(a) &= \delta(\operatorname{Re} a) + i\delta(\operatorname{Im} a) = \operatorname{Re}\delta(\operatorname{Re} a) + i\operatorname{Im}\delta(\operatorname{Re} a)\\ &\quad + i\operatorname{Re}\delta(\operatorname{Im} a) - \operatorname{Im}\delta(\operatorname{Im} a)\\ &= \delta_1(\operatorname{Re} a) + i\delta_1(\operatorname{Im} a) - \delta_2(\operatorname{Im} a) + i\delta_2(\operatorname{Re} a)\,.\end{aligned}$$

Suppose that ∇ is a Boolean algebra which is not atomic and so, by the first part of the proof, there exists a non-zero derivation $\delta : S_{\mathbb{C}} \to S_{\mathbb{C}}$. In particular, $\delta(a) \neq 0$ for some $0 \neq a \in S_{\mathbb{C}}$, for which we also have (see the preceding equality) that at least one of the functions $\delta_1(\operatorname{Re} a)$, $\delta_1(\operatorname{Im} a)$, $\delta_2(\operatorname{Re} a)$, $\delta_2(\operatorname{Im} a)$ is non-zero. This implies that at least one of the derivations δ_1, δ_2 on $S_{\mathbb{R}}$ does not vanish.

To complete the proof, it remains to observe that if ∇ is an atomic Boolean algebra, then by Proposition 2.9, the algebra of all countably-valued elements in $S_{\mathbb{R}}$ coincides with $S_{\mathbb{R}}$, and therefore, according to the propositions 2.3, 2.6 and 2.8, any derivation from $S_{\mathbb{R}}$ to $S_{\mathbb{R}}$ vanishes. ∎

The following corollary follows from Theorem 3.3 and the definition of the first Hochschild cohomology group $H^1(\mathcal{A},\mathcal{A})$ as the factor-group of all derivations of the algebra $\mathcal{A}$ by the subgroup of all inner derivations of $\mathcal{A}$ (see [5, Ch. IX, §4] and [7, Section 1.9])

COROLLARY 3.1. *The cohomology group* $H^1\big(S(\Omega,\Sigma,\mu),S(\Omega,\Sigma,\mu)\big)$ *is non-trivial if and only if the Boolean algebra* ∇ *of all idempotents in* $S(\Omega,\Sigma,\mu)$ *is not atomic.*

Remark 3.2. The assertion of Theorem 3.3 holds also for localizable measure spaces (Ω,Σ,μ). To see that, it is sufficient to represent Ω as a union of pairwise disjoint measurable sets with a finite measure and to each of these sets apply Theorem 3.3.

Finally, we present a variant of Theorem 3.3 for commutative algebras of measurable operators.

THEOREM 3.4. *Let* M *be a commutative von Neumann algebra, let* $L(M)$ *be the algebra of all measurable operators, affiliated with* M. *The following conditions are equivalent:*

(i) *There exists a not-zero derivation* $\delta : L(M) \to L(M)$.

(ii) *The Boolean algebra* ∇ *of all projections in* M *is not an atomic Boolean algebra.*

Proof. It is well-known that the algebra $L(M)$ may be identified with the algebra $S_{\mathbb{C}}(\Omega,\Sigma,\mu)$ for some localizable measure space (Ω,Σ,μ) (see [15]). Therefore, the assertion of Theorem 3.4 follows from that of Theorem 3.3 and Remark 3.2. ∎

Thus, if the Boolean algebra ∇ of all projections in commutative von Neumann algebra M is not atomic, then there are non-zero derivations in $L(M)$, which are not continuous with respect to the measure topology.

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