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On the Moore-Penrose Inverse in C^*-algebras

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1. Introduction

Given an unitary ring A, an element $a \in A$ will be called *regular*, if it has a *generalized inverse*, also called *pseudo-inverse*, in A, that is if there exists $a' \in A$ for which

$$a = aa'a.$$

It is clear that in this case aa' and $a'a$ are idempotents of A.

In addition, a generalized inverse a' of a regular element $a \in A$ will be called *normalized*, if a' is regular and a is a pseudo-inverse of a', that is if

$$a = aa'a, \qquad a' = a'aa'.$$

In the presence of an involution $*\colon A \to A$, it is also possible to enquire if the idempotents aa' and $a'a$ are self-adjoint, equivalently whether or not

$$(aa')^* = aa', \qquad (a'a)^* = a'a.$$

In this case a' is called the *Moore-Penrose inverse* of a, and it is denoted by $a^\dagger$, see [16], where this concept was introduced for matrices, and the related works [10], [11], and [14].

In [10] it was proved that each regular element a in a C^*-algebra A has a Moore-Penrose inverse, which in addition is unique. Consequently, the Moore-Penrose inverse of a regular element $a \in A$ is the unique solution $x \in A$ to the following set of equations:

$$a = axa, \qquad x = xax, \qquad (ax)^* = ax, \qquad (xa)^* = xa.$$

According to the uniqueness of the Moore-Penrose inverse of a regular element a, a^* also has a Moore-Penrose inverse and

$$(a^*)^\dagger = (a^\dagger)^*.$$

Furthermore, according to the above equations, if a is a regular element, then $a^\dagger$ also is and

$$(a^\dagger)^\dagger = a.$$

The so-called reverse order law is one of the most important properties of the Moore-Penrose inverse that have been deeply studied, that is under what condition the equation

$$(ab)^\dagger = b^\dagger a^\dagger$$

holds.

In the well-known article [7], T.N.E. Greville proved that the following facts are equivalent:

(i) $(ab)^\dagger = b^\dagger a^\dagger$,
(ii) $a^\dagger abb^*a^* = bb^*a^*$ and $bb^\dagger a^*ab = a^*ab$,
(iii) $a^\dagger a$ commutes with bb^* and a^*a with $bb^\dagger$,
(iv) $a^\dagger abb^*a^*abb^\dagger = bb^*a^*a$,
(v) $a^\dagger ab = b(ab)^\dagger ab$ and $bb^\dagger a^* = a^*ab(ab)^\dagger$,

where a and b are two matrices. However, it is worth noticing that the proofs in [7] are also valid in the more general context of C^*-algebras.

The key results of [7] were extended in some works devoted to generalized inverses of matrices, see for example [2], [3], and [18]. As regard Hilbert space operators, in [4] R. Bouldin gave a characterization in terms of invariant subspaces, which was refined in [12] and [17]. Observe that the main result in [4], Theorem 3.1, is equivalent to the generalization of Theorem 2 of [7], the above mentioned condition (iii), to Hilbert space operators, see Remark 3.2 of [4].

On the other hand, in the work [14] M. Mbekhta studied the reverse order law for generalized inverses in the frame of C^*-algebras. In fact, given two regular elements a and b in a C^*-algebra A, it was proved that the following statements are equivalent:

(i) $b'a'$ is a generalized inverse of ab,
(ii) $a(pq - qp)b = 0$,
(iii) qp is an idempotent,

where a' and b' are generalized inverses of a and b respectively, $p = bb'$ and $q = a'a$, see Theorem 3.1 of [14]. Naturally, this characterization remains true in Banach algebras, in fact in a ring. Furthermore, in [5] R. Bouldin proved the same characterization for Banach space operators.

In addition, in [14] M. Mbekhta posed the problem of finding necessary and sufficient conditions, analogues to the ones of Theorem 3.1 of [14], which ensures that

$$(ab)^\dagger = b^\dagger a^\dagger,$$

for a and b in a C^*-algebra A.

In the work [13] it was claimed that the question of M. Mbekhta in [14] was solved. However, the answer to this problem, Theorem 5 of [13], not only does not provide conditions analogues to the one of Theorem 3.1 in [14], but also it consists in the formulation of the well-known Theorems 1 and 2 of [7] in C^*-algebras, the above reviewed conditions (i), (ii), and (iii), whose proofs are also valid in C^*-algebras.

The first and main objective of the present work consists in solving the problem posed by M. Mbekhta, that is to give a characterization of the reverse order law for the Moore-Penrose inverse in C^*-algebras which is analogue to the one of Theorem 3.1 of [14]. Due to the fact that the Moore-Penrose inverse is determined by four equations instead of one, and that it involves not only the product but also the involution, several modifications must be made, however the form of M. Mbekhta's characterization is preserved. What is more, in Section 3 four equivalent characterizations with this characteristic will be proved. To this end it will be necessary to reformulate the equations that define the Moore-Penrose inverse of a regular element, which will be done in Section 2 following an argument in [16].

On the other hand, given a regular element a in a C^*-algebra A, according to a general argument, or even as an application of the results of Section 3, it is easy to prove that $(aa^\dagger)^\dagger = aa^\dagger$ and $(a^\dagger a)^\dagger = a^\dagger a$. Now well, since the Moore-Penrose inverse is a particular generalized inverse, it can be thought of a sort of inverse, however, these two identities also suggest that the Moore-Penrose inverse has properties that are similar to the ones of the involution of the algebra. This observation has led to the second objective of this work, namely, the study of the regular elements $a \in A$ for which $a^\dagger = a$. These elements will be called Moore-Penrose hermitian, and its basic properties will be studied in Section 4. Furthermore, in Section 5 Moore-Penrose hermitian elements will be fully characterize both in the Hilbert space and in the C^*-algebra setting.

In addition, it will be also proved that $a \in A$ is a normal Moore-Penrose hermitian element if and only if it is a hermitian partial isometry.

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2. Equivalent formulations of the Moore-Penrose inverse

Consider a C^*-algebra A, and $a \in A$ a regular element. In this section several equivalent formulations of the equations defining the Moore-Penrose inverse of a will be considered. These formulations will be central in the proof of the characterizations of the next section. In addition, the argument in Proposition 2.1 will follow ideas of Theorem 1 of [16].

Proposition 2.1. *Consider a C^*-algebra A, and two elements in A, a and x. Then,*

(i) *the equations $a = axa$ and $(ax)^* = ax$ are equivalent to $a = x^*a^*a$,*
(ii) *the equations $a = axa$ and $(xa)^* = xa$ are equivalent to $a = aa^*x^*$,*
(iii) *the equations $x = xax$ and $(ax)^* = ax$ are equivalent to $x = xx^*a^*$,*
(iv) *the equations $x = xax$ and $(xa)^* = xa$ are equivalent to $x = a^*x^*x$.*

Proof. The third equivalence was proved in Theorem 1 of [16]. The other three statements can be proved in a similar way. ∎

As a consequence, the following equivalent conditions are obtained.

Proposition 2.2. *Consider a C^*-algebra A and $a \in A$. Then the following statements are equivalent:*

(i) *$x \in A$ is the Moore-Penrose inverse of a,*
(ii) *$a = x^*a^*a$ and $x = a^*x^*x$,*
(iii) *$a = aa^*x^*$ and $x = xx^*a^*$.*

Proof. Is is a consequence of Proposition 2.1 and the equations defining the Moore-Penrose inverse. ∎

Remark 2.3. Consider a C^*-algebra A, $a \in A$ a regular element of A, and $x = a^\dagger$. Then, according to Proposition 2.2 and to the fact that a^* is also regular and $(a^*)^\dagger = (a^\dagger)^*$, the following statements are equivalent:

(i) $x \in A$ is the Moore-Penrose inverse of a,

(ii) $a^* = a^*ax$ and $x^* = x^*xa$,

(iii) $a^* = xaa^*$ and $x^* = axx^*$.

Next follows the equivalent formulations of the Moore-Penrose inverse that will be central in the next section.

PROPOSITION 2.4. *Consider a C^*-algebra A and $a \in A$. Then the following statements are equivalent:*

(i) *$x \in A$ is the Moore-Penrose inverse of a,*

(ii) *$a^* = xaa^*$ and $x = xx^*a^*$,*

(iii) *$a = aa^*x^*$ and $x^* = axx^*$,*

(iv) *$a^* = a^*ax$ and $x = a^*x^*x$,*

(v) *$a = x^*a^*a$ and $x^* = x^*xa$.*

Proof. It is a consequence of Proposition 2.2 and Remark 2.3. ∎

3. The reverse order law for the Moore-Penrose inverse

In this section the relationship between the product and the Moore-Penrose inverse will be studied. In fact, four equivalent characterizations of the so-called reverse order law for the Moore-Penrose inverse will be proved. These characterizations are analogue to the one given in Theorem 3.1 of [14] for the generalized inverse of the product of two C^*-algebra elements. The results of this section provide an answer to a question posed by M. Mbekhta in [14].

THEOREM 3.1. *Consider a C^*-algebra A, and two regular elements of A, a and b, such that ab is also regular. Define $p = bb^\dagger$, $q = a^\dagger a^{\dagger^*}$, $r = bb^*$ and $s = a^\dagger a$. Then, the following statements are equivalent:*

(i) $(ab)^\dagger = b^\dagger a^\dagger$,

(ii) *$a(pq - qp)b^{\dagger^*} = 0$, and $a(rs - sr)b^{\dagger^*} = 0$,*

(iii) *$spqp = qp$, and $srsp = sr$.*

Proof. First of all, observe that p, q, r and s are hermitian elements of A.

Consider $a^\dagger$, $b^\dagger$ and $(ab)^\dagger$, the Moore-Penrose inverses of a, b and ab respectively. According to the third statement of Proposition 2.4, the following equations hold:

$$a = aa^* a^{\dagger^*}, \qquad b = bb^* b^{\dagger^*}, \qquad ab = ab(ab)^*(ab)^{\dagger^*},$$
$$a^{\dagger^*} = aa^\dagger a^{\dagger^*}, \qquad b^{\dagger^*} = bb^\dagger b^{\dagger^*}, \qquad (ab)^{\dagger^*} = ab(ab)^\dagger(ab)^{\dagger^*}.$$

Furthermore, note that according again to the third statement of Proposition 2.4,

$$a = as, \qquad a^{\dagger^*} = aq, \qquad b = rb^{\dagger^*}, \qquad b^{\dagger^*} = pb^{\dagger^*}.$$

Now suppose that $(ab)^\dagger = b^\dagger a^\dagger$. Then, since $(ab)^* = b^* a^*$ and $(ab)^{\dagger^*} = (b^\dagger a^\dagger)^* = a^{\dagger^*} b^{\dagger^*}$, it is clear that

$$ab = abb^* a^* a^{\dagger^*} b^{\dagger^*}, \qquad a^{\dagger^*} b^{\dagger^*} = abb^\dagger a^\dagger a^{\dagger^*} b^{\dagger^*},$$

which is equivalent to

$$asrb^{\dagger^*} = arsb^{\dagger^*}, \qquad aqpb^{\dagger^*} = apqb^{\dagger^*},$$

which in turn is equivalent to the following identities:

$$a(pq - qp)b^{\dagger^*} = 0, \qquad a(rs - sr)b^{\dagger^*} = 0.$$

Next suppose that the second statement of the theorem holds. Then, it is clear that

$$a^\dagger apqb^{\dagger^*} b^* = a^\dagger aqpb^{\dagger^*} b^*, \qquad a^\dagger arsb^{\dagger^*} b^* = a^\dagger asrb^{\dagger^*} b^*.$$

However, according again to the third statement of Proposition 2.4, and to the fact that $s = s^*$ and $p = p^*$, these equations can be rewritten as

$$spqp = a^\dagger(aa^\dagger a^{\dagger^*})b(b^\dagger b^{\dagger^*} b^*) = a^\dagger a^{\dagger^*} bb^\dagger = qp,$$
$$srsp = (a^\dagger aa^*)a^{\dagger^*}(bb^* b^{\dagger^*})b^* = a^* a^{\dagger^*} bb^* = sr.$$

Finally suppose that the third statement of the theorem holds. Then, since $p = p^*$ and $s = s^*$, it is clear that

$$a^\dagger abb^\dagger a^\dagger a^{\dagger^*} b^{\dagger^*} b^* = a^\dagger a^{\dagger^*} bb^\dagger,$$
$$a^\dagger abb^* a^* a^{\dagger^*} b^{\dagger^*} b^* = a^* a^{\dagger^*} bb^*,$$

which implies that

$$(aa^\dagger a)bb^\dagger a^\dagger a^{\dagger^*}(b^{\dagger^*}b^*b^{\dagger^*}) = (aa^\dagger a^{\dagger^*})(bb^\dagger b^{\dagger^*}),$$
$$(aa^\dagger a)bb^* a^* a^{\dagger^*}(b^{\dagger^*}b^*b^{\dagger^*}) = (aa^* a^{\dagger^*})(bb^* b^{\dagger^*}).$$

However, according to the third statement of Proposition 2.4 and to the fact that $a^\dagger$ and $b^\dagger$ are the Moore-Penrose inverse of a and b respectively, the previous equations are equivalent to

$$ab(b^\dagger a^\dagger)(b^\dagger a^\dagger)^* = (b^\dagger a^\dagger)^*,$$
$$ab(ab)^*(b^\dagger a^\dagger)^* = ab,$$

which, according again to the third statement of Proposition 2.4 implies that

$$(ab)^\dagger = b^\dagger a^\dagger.$$

∎

Note that in a C^*-algebra, under the same conditions of Theorem 3.1, when instead of generalized inverses Moore-Penrose inverses are considered, the characterization of Theorem 3.1 in [14] determines if $b^\dagger a^\dagger$ is a normalized generalized inverse of ab. However, in order to characterize the reverse order law for the Moore-Penrose inverse, another equation is necessary as well as new elements must be introduced.

The next three theorems provide characterizations which are equivalent to the one in Theorem 3.1. However, for sake of completeness they are included.

Theorem 3.2. *Under the same conditions and notations of Theorem* 3.1, *the following statements are equivalent:*

(i) $(ab)^\dagger = b^\dagger a^\dagger$,
(ii) $b^\dagger(qp - pq)a^* = 0$, *and* $b^\dagger(sr - rs)a^* = 0$,
(iii) $pqps = pq$, *and* $psrs = rs$.

Proof. The proof is similar to the one of Theorem 3.1. However, instead of the third statement of Proposition 2.4, the second statement of the mentioned proposition must be used. ∎

Theorem 3.3. *Under the same conditions and notations of Theorem* 3.1, *the following statements are equivalent:*

(i) $(ab)^\dagger = b^\dagger a^\dagger$,

(ii) $b^*(q^\dagger p - pq^\dagger)a^\dagger = 0$, *and* $b^*(sr^\dagger - r^\dagger s)a^\dagger = 0$,
(iii) $pq^\dagger ps = pq^\dagger$, *and* $psr^\dagger s = r^\dagger s$.

Proof. The proof is similar to the one of Theorem 3.1. However, instead of the third statement of Proposition 2.4, the forth statement of the mentioned proposition must be used. In addition, in order to compute $q^\dagger$ and $r^\dagger$, Theorem 7 of [10] must be considered. ∎

THEOREM 3.4. *Under the same conditions and notations of Theorem* 3.1, *the following statements are equivalent:*

(i) $(ab)^\dagger = b^\dagger a^\dagger$,
(ii) $a^{\dagger^*}(pq^\dagger - q^\dagger p)b = 0$, *and* $a^{\dagger^*}(r^\dagger s - sr^\dagger)b = 0$,
(iii) $spq^\dagger p = q^\dagger p$, *and* $sr^\dagger sp = sr^\dagger$.

Proof. The proof is similar to the one of Theorem 3.1. However, instead of the third statement of Proposition 2.4, the fifth statement of the mentioned proposition must be used. In addition, in order to compute $q^\dagger$ and $r^\dagger$, Theorem 7 of [10] must be considered. ∎

Remark 3.5. Consider a C^*-algebra A, and two regular elements of A, a and b, such that ab is also regular. It is well-known that the reverse order law for the product ab is equivalent to the conditions

$$a^\dagger abb^* = bb^*a^\dagger a, \qquad bb^\dagger a^*a = a^*abb^\dagger,$$

see for example Theorem 2 of [7], which was proved for matrices but whose proof remains valid in a C^*-algebra, Proposition 4.4 of [18], Remark 3.2 of [4], Corollary 3.11 of [12], and also Theorem 5 of [13]. However, the above conditions are equivalent to the equalities

$$pq = qp, \qquad rs = sr.$$

In fact, the first condition is exactly

$$rs = sr.$$

As regard the second condition, since $bb^\dagger$ commutes with a^*a, according to Theorem 5 of [10], $bb^\dagger$ commutes with $(a^*a)^\dagger$, which, according to Theorem 7 of [10], proves that $pq = qp$.

On the other hand, if $bb^\dagger$ commutes with $a^\dagger a^{\dagger *}$, according again to Theorem 5 of [10], $bb^\dagger$ commutes with the Moore-Penrose inverse of $a^\dagger a^{\dagger *}$. In particular, according to Theorem 7 of [10], $bb^\dagger$ commutes with a^*a.

Furthermore, note that, according to Theorems 5 of [10], and to the fact that p, q, r and s are hermitian elements of A, the above conditions and equalities are equivalent to

$$q^\dagger p = pq^\dagger, \qquad r^\dagger s = sr^\dagger.$$

Consequently, the second condition of Theorems 3.1 and 3.2 (resp. Theorems 3.3 and 3.4) could have been replaced by the commutativity of p and q, and of r and s (resp. the commutativity of $q^\dagger$ and p and of $r^\dagger$ and s), however, this has not been done for two reasons. In first place, the conditions (ii) in the aforesaid Theorems are weaker, but above all, Theorems 3.1 - 3.4 have been presented in a way that they provide a characterization of the reverse order law for the Moore-Penrose inverse analogue to the one of Theorem 3.1 of [14] for generalized inverses.

4. Moore-Penrose hermitian elements

In first place, the main notion of this and the following section is introduced.

Definition 4.1. Consider a C^*-algebra A. A regular element $a \in A$ will be called Moore-Penrose hermitian, if $a^\dagger = a$.

Next follow the basic facts regarding the concept just introduced. In the next section Moore-Penrose hermitian elements will be fully characterize.

Proposition 4.2. *Consider a C^*-algebra A and an element $a \in A$. Then, the following statements hold:*

- (i) *Necessary and sufficient for a to be a Moore-Penrose hermitian element is $a = a^3$ and $(a^2)^* = a^2$.*
- (ii) *If a is a Moore-Penrose hermitian element, then a^n also is, $n \in \mathbb{N}$.*
- (iii) *The element a is Moore-Penrose hermitian if and only if a^* is.*
- (iv) *If a is a Moore-Penrose hermitian element, then $\sigma(a) \subseteq \{0, -1, 1\}$, where $\sigma(a)$ denotes the spectrum of a.*

Proof. Definition 4.1 and the equations defining the Moore-Penrose inverse prove the first point, which in turn proves the second.

The third point is clear, and the fourth is a consequence of the fact that $p = a^2$ is a hermitian idempotent.

5. Characterizations of Moore-Penrose hermitian elements

This section begins with the characterization of Moore-Penrose hermitian C^*-algebra elements. In first place, some notation is given.

Recall that if A is a C^*-algebra and $a \in A$, then $L_a\colon A \to A$ is the map defined by left multiplication by a, that is

$$L_a(x) = ax, \qquad (x \in A).$$

In addition, the range and the null space of L_a will be denoted by $R(L_a) = aA$ and $N(L_a) = a^{-1}(0)$ respectively.

Theorem 5.1. *Consider a C^*-algebra A. Then, the following statements are equivalent:*

(i) *$a \in A$ is a Moore-Penrose hermitian element,*

(ii) *$aA = a^*A$, $a^{-1}(0) = a^{*-1}(0)$, $A = aA \oplus a^{-1}(0)$, and if $L = L_a\mid_{aA}\colon aA \to aA$ and $\tilde{L} = L_{a^*}\mid_{aA}\colon aA \to aA$, then $L^2 = \tilde{L}^2 = \tilde{I}$, where $\tilde{I}$ denotes the identity map of aA.*

Proof. Suppose that a is a Moore-Penrose hermitian element of A, and consider the map $L_{a^2}\colon A \to A$. Since a^2 is an idempotent, L_{a^2} is a projection defined in A. Consequently, $A = R(L_{a^2}) \oplus N(L_{a^2})$. However, since a is a Moore-Penrose hermitian element of A, an easy calculation proves that $R(L_{a^2}) = aA$ and $N(L_{a^2}) = a^{-1}(0)$. Moreover, since L_{a^2} is a projection, it is clear that $L^2 = \tilde{I}$.

In addition, according to the third statement of Proposition 4.2, a^* is a Moore-Penrose hermitian element. Moreover, according to the fifth statement of Proposition 2.4, $a = a^*a^*a$ and $a^* = aaa^*$, which implies that $aA = a^*A$. Furthermore, since according to the third statement of Proposition 2.4, $a = aa^*a^*$ and $a^* = a^*aa$, it follows that $a^{*-1}(0) = a^{-1}(0)$. However, since a^2 is hermitian, $\tilde{L}^2 = L^2 = \tilde{I}$.

Conversely, if the second statement holds, a straightforward calculation proves that $L_a = L_a^3$ and $L_{a^*}^2 = L_a^2$, which clearly implies that $a = a^3$ and $(a^2)^* = a^2$, that is a is a Moore-Penrose hermitian element. ∎

Next follows the characterization of Moore-Penrose hermitian elements in the frame of Hilbert spaces. However, firstly several notions and results need to be reviewed.

As in the case of a C^*-algebra element, a Hilbert space operator will be said *Moore-Penrose hermitian*, if it has a Moore-Penrose inverse $T^\dagger$ and

$$T^\dagger = T.$$

Recall that if $T\colon H \to H$ is a bounded linear operator defined on the Hilbert space H, then the Moore-Penrose inverse of T is the unique linear and continuous map $T^\dagger$ for which the following equations hold:

$$T = TT^\dagger T, \quad T^\dagger = T^\dagger T T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T.$$

Note that the operator T admits a generalized inverse in $A = L(H)$ if and only if $R(T)$ is closed, see Theorem 3.8.2 of [9]. However, when the operator T admits a generalized inverse, it can be chosen to be the Moore-Penrose inverse of T in $A = L(H)$, see Theorem 5 of [10]. Moreover, in this case it is unique, and it coincides with the Moore-Penrose inverse of T viewed as an operator defined on $L(H)$, see [10], [11], [14] and [15]. In addition, a bounded linear map which has a generalized inverse will be called a *regular operator*.

THEOREM 5.2. *Consider a Hilbert space H, and T a regular operator defined on H. Then the following statements are equivalent:*

(i) *T is a Moore-Penrose hermitian operator,*
(ii) *there exist two orthogonal Hilbert subspaces H_1 and H_2 such that $H = H_1 \oplus H_2$, $T \mid H_1 = 0$, $T(H_2) \subseteq H_2$, and if T_2 denotes the restriction of T to H_2, then $T_2^2 = I_2$, where I_2 denotes the identity map of H_2.*

Proof. Suppose that $T^\dagger = T$ and consider the self-adjoint projection $P = T^\dagger T = TT^\dagger = T^2$. In particular, the Hilbert space can be presented as the orthogonal direct sum $H = N(T^2) \oplus R(T^2)$. However, as in the case of a C^*-algebra, since T is a Moore-Penrose hermitian operator, a straightforward calculation proves that $N(T) = N(T^2)$ and $R(T) = R(T^2)$. Define $H_1 = N(T)$, $H_2 = N(T)^\perp = R(T)$, where $N(T)^\perp$ denotes the orthogonal subspace of $N(T)$. Then, it is clear that $T(H_2) \subseteq H_2$, and $T_2^2 = I_2$.

Conversely, it the second statement of the theorem holds, it is clear that $T^3 = T$ and T^2 is the orthogonal projection onto H_2, in particular T^2 is an hermitian projection. ∎

Next normal Moore-Penrose hermitian elements will be considered. However, first of all some preparation is necessary.

Given a C^*-algebra A, the *conorm* of an element $a \in A$ is defined by

$$c(a) = \inf\{\|ax\|\colon \operatorname{dist}(x, a^{-1}(0)) = 1,\ x \in A\},$$

see [11] and [14].

It is worth noticing that if a is a regular element, then

$$c(a) = \frac{1}{\|a^\dagger\|},$$

see Proposition 1.3 of [14] and Theorem 2 of [11].

Next consider a bounded linear operator $T\colon H \to H$, where H is a Hilbert space. Then, T is said a *partial isometry*, if T admits a Moore-Penrose inverse and $T^\dagger = T^*$, see [15] and Chapter 15 of [8]. In order to keep an analogy with the Hilbert space case, an element a of a C^*-algebra A will be called a *partial isometry*, if a is regular and $a^\dagger = a^*$.

It is clear that if $a \in A$ is a partial isometry, then a^*a is a hermitian idempotent. Conversely, consider $a \in A$ such that a^*a is a hermitian idempotent. Then, since each C^*-algebra has a faithful representation in a Hilbert space, see for example Theorem 7.10 of [6], according to Problem 127, Chapter 15, of [8], a straightforward calculation shows that a is a partial isometry. Furthermore, since a is a partial isometry if and only if a^* is, then necessary and sufficient for a to be a partial isometry is that aa^* is a hermitian idempotent. See [1] where an equivalent definition of the notion under consideration was considered.

In the following proposition a generalization of Corollary 3.2 of [15] will be proved. This result will be central for the characterization of normal Moore-Penrose hermitian elements.

Proposition 5.3. *Consider a C^*-algebra A, and a non-zero regular element $a \in A$. Then, necessary and sufficient for a to be a partial isometry is $c(a) = \|a\| = 1$.*

Proof. Let $a \in A$ be a non-zero regular element, and consider, according to Theorem 7.10 of [6], a Hilbert space H and $\pi\colon A \to L(H)$ a faithful representation of A. It is worth noticing that in this case $\pi(a) \in L(H)$ is regular and $\pi(a)^\dagger = \pi(a^\dagger)$.

Suppose that a is a partial isometry. Then $\pi(a) \in L(H)$ also is a partial isometry. Then, according to Corollary 3.2 of [15], $\|\pi(a)\| = 1$. In particular, $\|a\| = 1$. Moreover, according to Proposition 1.3 of [14],

$$c(a) = \frac{1}{\|a^*\|} = \frac{1}{\|a\|} = 1.$$

Conversely, suppose that a regular element $a \in A$ is such that $c(a) = \|a\| = 1$. Then, $\|\pi(a)\| = 1$, and according again to Proposition 1.3 of [14], $\|a^\dagger\| = 1$.

On the other hand, since $\pi(a^\dagger) = \pi(a)^\dagger$, according to Corollaries 2.3 and 3.2 of [15], $\pi(a)$ is a partial isometry. However, since $\pi\colon A \to L(H)$ is a faithful representation, a is a partial isometry. ∎

Theorem 5.4. *Consider a C^*-algebra A. Then, an element $a \in A$ is a normal Moore-Penrose hermitian element if and only if a is a hermitian partial isometry.*

Proof. Suppose that $a \in A$ is a normal Moore-Penrose hermitian element. Then, according to Theorem 2.9 of [6], to the fourth statement of Proposition 4.2, and to Corollary 1.6 of [14], $c(a) = 1$. Moreover, since a is a Moore-Penrose hermitian element, according to Proposition 1.3 of [14] or to Theorem 2 of [11], $\|a\| = 1$. Consequently, according to Proposition 5.3, a is a partial isometry. However, $a = a^\dagger = a^*$, that is a is hermitian.

The converse is clear. ∎

Note that if $T\colon H \to H$ is a linear and continuous Hilbert space map, then T is a normal Moore-Penrose hermitian operator if and only if the map T_2 in Theorem 5.2 is a hermitian unitary operator.

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