EXTRACTA MATHEMATICAE Vol. **18**, Núm. 2, 201 – 208 (2003)

On Cross-Section Measures in Minkowski Spaces

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(Presented by J. Alonso)

AMS *Subject Class.* (2000): 52A20, 52A21 *Received January* 15, 2003

1. Introduction

One-dimensional cross-section measures are one-dimensional section and projection measures of convex bodies in $\mathbb{E}^d$, $d \geq 2$. More precisely, this notion is used to describe, in a unified way, the *width function* and the *maximal chord-length function* of a convex body $K \subset \mathbb{E}^d$ (see the monograph [1, §30 and §33], where also the notions of outer one-quermass and inner one-quermass are used). References regarding these cross-section measures are collected in [6] and [3, Chapter 3]. Both these concepts are closely related to further important tools and notions from convexity, such as Minkowski addition, difference bodies, central symmetrals, affine diameters, antipodality, circum- and inradius, cf. the references above as well as [4, §19.3], [8, Chapters 3 and 7] and the survey [7]. In particular, the minima and maxima of the width function and the maximal chord-length function coincide and are usually called *thickness* (*minimal width*) and *diameter*, respectively. Also, these notions are closely related to *bodies of constant width* and *reduced bodies*, see [2] and [5].

We will consider the analogues of both these functions for convex bodies in Minkowski space, and therefore provide some analytical and geometrical descriptions of their corresponding extensions (a basic reference to the geometry of Minkowski spaces is [9]). Collecting various approaches to these functions from the literature, we give a unified way to represent them in different terms, and prove some theorems about Minkowskian diameter, thickness, in- and circumradius. Since we also discuss the cases when the convex bodies under consideration have only finitely many extreme points (i.e., are polytopes), our results are partially discrete in nature.

2. Basic notations and background material

As usual, $\mathbb{E}^d$ denotes the d-dimensional Euclidean space with origin o, while $\langle\, .\, ,\, .\, \rangle$ and $|\,.\,|$ denote the scalar product and the norm in $\mathbb{E}^d$, respectively. The unit ball $\left\{x \in \mathbb{E}^d : |x| \le 1\right\}$ in $\mathbb{E}^d$ is denoted by B_E. We shall always use the letter K for an arbitrary *convex body* in $\mathbb{E}^d$, i.e., a compact, convex set with nonempty interior. The notations $\operatorname{ext} K$ and $\exp K$ denote the set of all *extreme* and *exposed points* of a convex body K, respectively. Furthermore we write $\operatorname{bd} K$ for the *boundary* of K. If X is a set from $\mathbb{E}^d$, then $\operatorname{cl} X$ is its *closure*. Given points x, y in $\mathbb{E}^d$, $[x, y]$ denotes the line segment with endpoints x and y. For this and further notions from convexity we refer to [1], [8] and [10].

A real finite-dimensional Banach space of dimension $d \ge 2$ is said to be a d-dimensional *Minkowski space*. It is well-known that each convex body $B \subset \mathbb{E}^d$ centered at the origin defines the Minkowski space $\mathcal{M}^d(B)$ with the norm $\|\,.\,\|_B$, in which the unit ball $\left\{x \in \mathbb{E}^d : \|x\|_B \le 1\right\}$ coincides with B. Trivially, the converse is also true, i.e., every Minkowski space is isometric to $\mathcal{M}^d(B)$ with appropriately chosen d and B. A basic reference for the geometry in Minkowski spaces is [9].

For a convex body K containing the origin in its interior, the body

$$K^* := \left\{ y \in \mathbb{E}^d : \langle y, x\rangle \le 1 \text{ for every } x \in K \right\}$$

is called the (*dual* or) *polar* body of K. Notice that the space $\mathcal{M}^d(B^*)$ is dual to $\mathcal{M}^d(B)$, as it is usually written also in functional analysis.

The *difference body* DK of K is introduced by $DK := \{x - y : x, y \in K\}$. The body DK is convex since it is the Minkowski sum of the convex bodies K and $-K := \{-x : x \in K\}$.

For a convex body K and a variable u ranging over $\mathbb{E}^d \setminus \{o\}$ we introduce the following well-known functions:

$$\begin{aligned}
&h_K(u) := \max\{\langle x, u\rangle : x \in K\} && (\textit{support function}),\\
&w_K(u) := h_K(u) + h_K(-u) && (\textit{width function}),\\
&r_K(u) := \max\{\alpha : \alpha u \in K\}\ \ (u \in \mathbb{E}^d \setminus \{o\}) && (\textit{radius function}),\\
&l_K(u) := \max\{\alpha : \alpha > 0,\ \alpha u \in DK\} && (\textit{maximal chord-length function}).
\end{aligned}$$

The functions $h_K(u)$ and $w_K(u)$ are homogeneous of order one, while $r_K(u)$ and $l_K(u)$ are homogeneous of order -1. Furthermore, it turns out that these functions are continuous (see, for instance, [3, pp.16,18]). Hence, whenever we consider the supremum or the infimum over a compact set of algebraic expressions constructed by the functions defined above, we can always replace them by maximum and minimum, respectively.

The relations $w_K(u) = h_{DK}(u)$ and $l_K(u) = r_{DK}(u)$ follow directly from the definitions.

We notice that for a convex body B symmetric with respect to the origin we obtain $h_B(u) = \|u\|_{B^*}$ just applying the definition of the support function and the fact that $\|\,.\,\|_{B^*}$ is the norm of the space dual to $\mathcal{M}^d(B)$.

In the sequel we need several trivial characterizations of inclusion. Namely, if B_1 and B_2 are convex bodies in $\mathbb{E}^d$ centered at the origin, then the condition $B_1 \subset B_2$ is equivalent to $B_2^* \subset B_1^*$. Furthermore, the latter two conditions are equivalent to any of the two inequalities $r_{B_1}(u) \le r_{B_2}(u)$ and $h_{B_1}(u) \le h_{B_2}(u)$ supposed to be fulfilled for an arbitrary direction u.

Further on, we introduce the notion of *cross-section measures* in Minkowski spaces, starting with the respective generalization of the width function. Following Chakerian and Groemer [2, p.52] we define the *Minkowskian width* $w_{K,B}(u)$ of a convex body K in Minkowski space $\mathcal{M}^d(B)$ by $w_{K,B}(u) := w_K(u)/\,h_B(u)$. The corresponding generalization of the maximal chord-length function goes analogously. Namely, we consider a Minkowski space $\mathcal{M}^d(B)$, a measured body K and a direction u. Then the length of the longest (in Minkowskian sense) chord of K of direction u is said to be the value of the Minkowskian maximal chord-length function at direction u (notation: $l_{K,B}(u)$). Taking into account the geometrical meanings of $l_K(u)$ and $r_B(u)$ (for u lying on the Euclidean unit sphere) we can easily derive the representation $l_{K,B}(u) = l_K(u)/\,r_B(u)$.

A chord $[x, y]$ of a convex body K is said to be an *affine diameter* of K if there exist different parallel supporting hyperplanes of K, say H_1 and H_2, such that x belongs to H_1 and y to H_2. It is well-known that the following three conditions are equivalent: *(i)* the chord $[x, y]$ of K is an affine diameter; *(ii)* the point $(x - y)$ lies in $\operatorname{bd} DK$; *(iii)* $\|x - y\|_B = l_{K,B}(u)$.

3. Representations of diameter and thickness in Minkowski spaces

A convex body K_1 in $\mathbb{E}^d$ is said to be *inscribed* in a convex body K_2 if K_1 is contained in K_2 and any larger homothetical copy $\alpha K_1 + p$ of K_1, with $\alpha > 1$ and $p \in \mathbb{E}^d$, is not contained in K_2. Exactly in this case we shall also say that K_2 is *circumscribed* to K_1. If K_1 is symmetric with respect to some point, i.e., a ball $\alpha B,\ \alpha > 0$, of some Minkowski space $\mathcal{M}^d(B)$, we say that $K_1 = \alpha B$ is an *inball* of K_2 in this space, and the radius $\underline{r}_B(K_2) := \alpha$ of K_1 is called the *inradius* of K_2. The notions *circumball* and *circumradius* $\overline{r}_B(K_2)$ are introduced analogously.

The following lemma gives several equivalent definitions for inscribed bodies in case when both K_1 and K_2 are centered at the same point.

LEMMA 1. *Let B_1 and B_2 be convex bodies symmetric with respect to the origin and let B_1 be contained in B_2. Then the following conditions are equivalent.*

(i) *The body B_1 is inscribed in B_2.*

(ii) *For every $\alpha > 1$, αB_1 is not contained in B_2.*

(iii) *For some direction $u_0 \in \mathbb{E}^d \setminus \{o\}$ we have $r_{B_1}(u_0) = r_{B_2}(u_0)$.*

(iv) *For some direction $u_0 \in \mathbb{E}^d \setminus \{o\}$ we have $h_{B_1}(u_0) = h_{B_2}(u_0)$.*

(v) *The body B_2^* is inscribed in B_1^*.*

(vi) *There exists an $x \in B_1$ with $\|x\|_{B_2} = 1$.*

(vii) *There exists an $x \in \operatorname{cl} \exp B_1$ with $x \in \operatorname{bd} B_2$.*

Proof. The implication from (i) to (ii) is obvious. Let us prove the converse implication by contradiction. We suppose that there exists a body $\beta B_1 + p$ with $\beta > 1$ and some $p \in \mathbb{E}^d \setminus \{o\}$ which is contained in B_2. Then, by symmetry of B_1 and B_2, also $\beta B_1 - p$ is contained in B_2. Therefore $\beta B_1 \subset \operatorname{conv}\{(\beta B_1 + p) \cup (\beta B_1 - p)\} \subset B_2$, a contradiction. The equivalence of (ii) and (iii)-(vii) is, in each case, more or less trivial. ∎

We notice that Condition (ii) of Lemma 1 implies the uniqueness of both the inball and the circumball which have the same center of symmetry as the body.

As in Euclidean spaces, the *diameter* $\operatorname{diam}_B(K)$ of a convex body K in Minkowski space $\mathcal{M}^d(B)$ is defined to be the largest distance occurring between points of K, i.e.: $\operatorname{diam}_B(K) := \max\{\|x-y\|_B : x, y \in K\}$. We denote the maximum and the minimum of the function $w_{K,B}(u)$ by $\overline{w}_B(K)$ and $\underline{w}_B(K)$, respectively, and call these values the *maximal* and the *minimal width* of K in $\mathcal{M}^d(B)$, respectively. For the maximum and the minimum of the function $l_{K,B}(u)$ we introduce the notations $\bar{l}_B(K)$ and $\underline{l}_B(K)$, respectively.

Theorems 2 and 3, presented below, provide various geometrical and analytical representations of Minkowskian diameter and thickness, respectively.

THEOREM 2. *For a convex K body in a Minkowski space $\mathcal{M}^d(B)$ the Minkowskian diameter of K is equal to*

(i) *the maximal width of K in $\mathcal{M}^d(B)$,*

(ii) *the maximum of the maximal chord-length function of K in $\mathcal{M}^d(B)$,*

(iii) *the Minkowskian circumradius of DK,*

(iv) *the Minkowskian length of the longest affine diameter of K,*

(v) *the analytical expression* $\max\{\langle x, u\rangle : x \in DK,\ u \in B^*\}$.

Proof. In view of Lemma 1 we obtain that DK is inscribed in the bodies $\overline{w}_B(K) \cdot B$, $\overline{l}_B(K) \cdot B$, and $\overline{r}_B(DK) \cdot B$. Consequently $\overline{w}_B(K) = \overline{l}_B(K) = \overline{r}_B(DK)$. Further on, $\operatorname{diam}_B(K) = \max\{\|x\|_B : x \in DK\} = \overline{r}_B(DK)$. It is left to obtain (iv) and (v). Part (iv) is obvious. To prove (v) we derive $\operatorname{diam}_B(K) = \max\{\|x\|_B : x \in DK\} = \max\{h_{B^*}(x) : x \in DK\}$ and apply simply the definition of the support function. ∎

THEOREM 3. *For a convex body K in a Minkowski space $\mathcal{M}^d(B)$ the following values are equal:*

(i) *the minimal width of K in $\mathcal{M}^d(B)$,*

(ii) *the minimum of the Minkowskian maximal chord-length function,*

(iii) *the Minkowskian inradius of DK,*

(iv) *the Minkowskian length of the shortest affine diameter of K,*

(v) *the analytical expression* $\left(\max\{\langle x, u\rangle : u \in (DK)^*,\ x \in B\}\right)^{-1}$.

Proof. The equality of the values described in (i)-(ii) can be proved just in the same way as we did it for the values in (i)-(ii) of Theorem 2. In order to prove (iv) we use Lemma 1 which implies that the shortest affine diameter has length $\gamma := \min\{\|x\|_B : x \in \operatorname{bd} DK\}$. But then γB is inscribed in DK, which yields that γ is equal to the inradius of DK. At last we prove the equality of the inradius of DK and the value in (v). Since $\underline{r}_B(DK) \cdot B$ is inscribed in DK, we get that B is inscribed in $\frac{1}{\underline{r}_B(DK)} \cdot DK$. This implies that $\max\{\|x\|_{DK} : x \in B\} = \frac{1}{\underline{r}_B(DK)}$. Notice that in the latter equality we use the norm of the Minkowski space constructed by the body itself, namely $\|\,.\,\|_{DK}$. Let us replace this norm by the corresponding support function, yielding $\frac{1}{\underline{r}_B(DK)} = \max\left\{h_{(DK)^*}(x) : x \in B\right\}$. Taking the power -1 of the latter equality and applying the definition of the support function, we derive that $\underline{r}_B(DK)$ is equal to the value in (v), and the proof is finished. ∎

In view of Theorem 3 we introduce the *thickness* $\Delta_B(K)$ of a convex body K in $\mathcal{M}^d(B)$ to be any of the equal values described by Parts (i)-(v) of this theorem.

Parts (i)-(iv) of Theorems 2 and 3, applied to the Euclidean space, present well-known relations from the theory of Euclidean cross-section measures (see, for instance, [10, Section 7.6]). However, in the literature on Minkowski spaces these relations and their proofs have never been given in a unified way. It is one of the aims of our paper to fill this gap. It should be noticed that the representations of diameter and thickness given in Parts (v) of the theorems above are new.

Whenever we consider a scalar product of a convex combination $(1-\lambda)x_1+\lambda x_2$, where $x_1, x_2 \in \mathbb{E}^d$ and $\lambda \in [0,1]$, and a point y from $\mathbb{E}^d$, we can estimate this from above as follows: $\langle (1-\lambda)x_1 + \lambda x_2, y\rangle \le \max\{\langle x_1, y\rangle, \langle x_2, y\rangle\}$. Thus, in the expressions given in Parts (v) of the last two theorems we can ignore non-extreme points of the sets over which we take the maxima (not changing these maxima). Leaving only the points belonging to the closure of exposed points we obtain the representations

$$\operatorname{diam}_B(K) = \max\{\langle x, u\rangle : x \in \operatorname{cl}\exp DK,\ u \in \operatorname{cl}\exp B^*\}, \tag{3.1}$$

$$\Delta_B(K) = \Big(\max\{\langle x, u\rangle : u \in \operatorname{cl}\exp (DK)^*,\ x \in \operatorname{cl}\exp B\}\Big)^{-1}. \tag{3.2}$$

Suppose that both K and B are *polytopes*. Then the latter relations become discrete and can be investigated in the spirit of *computational geometry*. In both the formulae we have to compute the quantity $\max\{\langle x, u\rangle : x \in X, y \in Y\}$ with $X = \operatorname{cl}\exp DK$, $Y = \operatorname{cl}\exp B^*$ for (3.1) and $X = \operatorname{cl}\exp(DK^*)$, $Y = \operatorname{cl}\exp B$ for (3.2). If X and Y can be given explicitly, we have to solve a simple task, i.e., among $\frac{1}{2}\cdot\operatorname{card} X\cdot\operatorname{card} Y$ values we have to find the maximal one (card stands for the cardinality of a set). If one of these sets, say X, is given explicitly and the convex hull of the other one is defined by a system of linear inequalities, we can also compute the corresponding cross-section measure, but in this case the problem is much harder, since it is reduced to $\frac{1}{2}\cdot\operatorname{card} X$ linear programming tasks. It is important that all these tasks have the same domain of definition and therefore can be solved altogether.

The following theorem shows in which directions we can expect the optimal values for lengths of affine diameters or the width of a convex body.

THEOREM 4. *For a convex body K in a Minkowski space $\mathcal{M}^d(B)$ the following statements hold true.*

I. *The width of K is maximal at one of the directions from $\operatorname{cl}\exp B^*$, i.e, we have*
$$\operatorname{diam}_B(K) = \max\{w_{K,B}(u) : u \in \operatorname{cl}\exp B^*\}.$$

II. *There exists an affine diameter of K having maximal length and being parallel to some $u \in \operatorname{cl}\exp DK$, i.e.,*
$$\operatorname{diam}_B(K) = \max\{l_{K,B}(u) : u \in \operatorname{cl}\exp DK\}.$$

III. *The width of a convex body is necessarily minimal at one of the directions from $\operatorname{cl}\exp(DK)^*$, i.e.,*
$$\Delta_B(K) = \min\{w_{K,B}(u) : u \in \operatorname{cl}\exp(DK)^*\}.$$

IV. *There exists an affine diameter of K having the minimal length and being parallel to some $u \in \operatorname{cl}\exp B$, i.e., we have*
$$\Delta_B(K) = \min\{l_{K,B}(u) : u \in \operatorname{cl}\exp B\}.$$

Proof. I. By Theorem 2, DK is inscribed in $\operatorname{diam}_B(K) \cdot B$. Consequently B^* is inscribed in $\operatorname{diam}_B(K) \cdot (DK)^*$. Therefore, for some $u \in \operatorname{cl}\exp B^*$ we have $u \in \operatorname{diam}_B(K) \cdot \operatorname{bd}(DK)^*$. Then $r_{B^*}(u) = \operatorname{diam}_B(K) \cdot r_{(DK)^*}(u)$. Simple equivalent transformations of the latter equality yield $\operatorname{diam}_B(K) = w_K(u)/h_B(u) = w_{K,B}(u)$.

II. Using again that DK is inscribed in $\operatorname{diam}_B(K) \cdot B$, we find a point u from $\operatorname{cl}\exp DK$ with $u \in \operatorname{diam}_B(K) \cdot \operatorname{bd} B$. Consequently $\operatorname{diam}_B(K) \cdot r_B(u) = r_{DK}(u)$, which is equivalent to $\operatorname{diam}_B(K) = l_{K,B}(u)$. Since u corresponds to some affine diameter of K having the same direction as the vector u, the proof is complete.

The remaining Parts III and IV can be proved analogously. ∎

Parts II and III of Theorem 4 are direct extensions of the corresponding Euclidean statements, while Parts I and IV are purely Minkowskian. More precisely, Part I really "works" when B is not smooth, and Part IV when B is not strictly convex. For instance, if B is a polygon with $2n$ vertices, we have to compute the width function for n different directions in order to find the Minkowskian diameter (see Figs. 1 and 2, where $n = 2$), and we have to evaluate the lengths of n affine diameters in order to find the Minkowskian thickness (see Figs. 2 and 3).

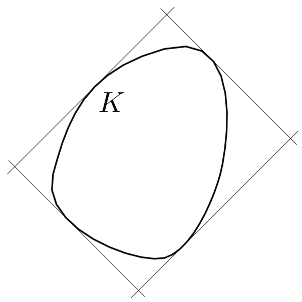

Figure 1

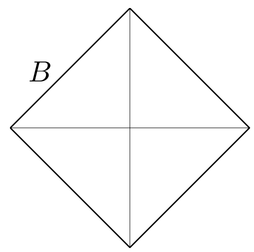

Figure 2

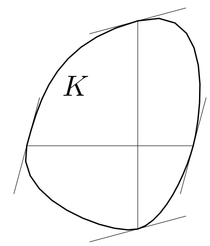

Figure 3

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