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Existence Results for Multivalued Semilinear Functional Differential Equations

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1. Introduction

This note is concerned with the existence of mild solutions defined on a compact real interval for first and second order semilinear functional differential inclusions. In Section 3 we consider the following class of semilinear functional differential inclusions

$$y' - Ay \in F(t, y_t), \quad a.e.\ t \in [0, T] \tag{1}$$

$$y(t) = \phi(t),\ t \in [-r, 0] \tag{2}$$

where $F : J \times C([-r, 0], E) \longrightarrow \mathcal{P}(E)$ is a multivalued map, A is the infinitesimal generator of a strongly continuous semigroup $T(t), t \geq 0$, $\phi \in C([-r, 0], E)$, $\mathcal{P}(E)$ is the family of all subsets of E and E is a real separable Banach space with norm $|\cdot|$. Section 4 is devoted to the study of the following second order semilinear functional differential inclusions

$$y'' - Ay \in F(t, y_t), \quad a.e.\ t \in J := [0, T] \tag{3}$$

$$y(t) = \phi(t), \quad t \in [-r, 0],\ y'(0) = \eta, \tag{4}$$

where $F, \phi, \mathcal{P}(E), E$ are as in problem (1)-(2) and A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ and $\eta \in E$.

For any continuous function y defined on the interval $[-r,T]$ and any $t \in [0,T]$, we denote by y_t the element of $C([-r,0],E)$ defined by

$$y_t(\theta) = y(t+\theta), \quad \theta \in [-r,0].$$

Here $y_t(\cdot)$ represents the history of the state from time $t-r$, up to the present time t.

As a model for this class of equations, one may take the following partial integrodifferential Volterra equation with delay

$$\frac{\partial}{\partial t}w(\xi,t) = \frac{1}{\pi^2}\frac{\partial^2}{\partial \xi^2}w(\xi,t) + \int_0^t q(t,s,\xi,w(\xi,s))ds + h(t), t \geq 0, 0 \leq \xi \leq 1,$$

with boundary condition

$$w(0,t) = w(1,t) = 0, \ \ t \geq 0,$$

and initial condition

$$w(\xi,\theta) = \phi(\xi,\theta), \ \ 0 \leq \xi \leq 1, \ -r \leq \theta \leq 0,$$

where q, h and ϕ are appropriated functions. As the space E for this equation we choose $L^2([0,1])$. The operator A is given by $Az := \frac{1}{\pi^2}\frac{\partial^2 z}{\partial \xi^2}$ with domain

$$D(A) = \left\{ z \in E : \frac{\partial^2 z}{\partial \xi^2} \in E, \frac{\partial z}{\partial \xi}(0) = \frac{\partial z}{\partial \xi}(1) = 0 \right\}$$

and it is well known that this operator generates a semigroup.

In the last two decades several authors have paid attention to the problem of existence of mild solutions to initial and boundary value problems for semilinear evolution equations. We refer the interested reader to the monographs by Goldstein [8], Heikkila and Lakshmikantham [10] and Pazy [16] and to the paper by Heikkila and Lakshmikantham [11]. In [14], [15] existence theorems of mild solutions for semilinear evolution inclusions are given by Papageorgiou. Recently, by means of a fixed point argument and the semigroup theory existence theorems of mild solutions on infinite intervals for first and second order semilinear differential inclusions with a convex valued-right hand side are obtained by the first author in [1], [2]. Here we shall extend the above results to functional differential inclusions with a nonconvex valued right hand side. The method we are going to use is to reduce the existence of solutions to problems (1)-(2) and (3)-(4) to the search for fixed points of a suitable multivalued map on the Banach space $C([-r,T],E)$. In order to prove the existence of fixed points, we shall rely on a fixed point theorem for contraction multivalued maps due to Covitz and Nadler [5] (see also Deimling [6]). This method was applied recently by the authors in [3], in the case when $A = 0$. Notice that by using the noncompactness measure, existence results for semilinear differential and functional differential inclusions with nonconvex valued right hand side were considered in the very recent book of Kamenskii et al [13] and in the references given there.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this note.

$C([-r,0],E)$ is the Banach space of all continuous functions from $[-r,0]$ into E with the norm

$$\|\phi\| = \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}.$$

By $C([-r,T],E)$ we denote the Banach space of all continuous functions from $[-r,T]$ into E with the norm

$$\|y\|_{[-r,T]} := \sup\{|y(t)| : t \in [-r,T]\}.$$

$L^1([0,T],E)$ denotes the Banach space of measurable functions $y : [0,T] \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^T |y(t)|dt \quad \text{for all} \quad y \in L^1([0,T],E).$$

$B(E)$ denotes the Banach space of bounded linear operators from E into E with norm

$$\|N\|_{B(E)} = \sup\{\|N(y)\| : |y| = 1\}.$$

We say that a family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is a strongly continuous cosine family if:

(i) $C(0) = I$ (I is the identity operator in E),

(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s,t \in \mathbb{R}$,

(iii) the map $t \longmapsto C(t)y$ is strongly continuous for each $y \in E$.

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)y = \int_0^t C(s)yds, \quad y \in E, \ t \in \mathbb{R}.$$

The infinitesimal generator $A : E \longrightarrow E$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ay = \frac{d^2}{dt^2} C(t)y\Big|_{t=0}.$$

For more details on strongly continuous cosine and sine families, we refer the reader to the books of Goldstein [8], Fattorini [7], and to the papers of Travis and Webb [17], [18]. For properties of semigroup theory, we refer the interested reader to the books of Goldstein [8] and Pazy [16].

Let (X, d) be a metric space. We use the notations:

$P(X) = \{Y \in \mathcal{P}(X) : Y \neq \emptyset\}$, $P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}$.

Consider $H_d : P(X) \times P(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$, $d(a, B) = \inf_{b\in B} d(a, b)$.

Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space.

A multivalued map $N : J \longrightarrow P_{cl}(X)$ is said to be measurable if, for each $x \in X$, the function $Y : J \longrightarrow \mathbb{R}$, defined by

$$Y(t) = d(x, N(t)) = \inf\{d(x, z) : z \in N(t)\},$$

is measurable.

Definition 2.1. A multivalued operator $N : X \to P_{cl}(X)$ is called

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x, y), \quad \text{for each } x,\ y \in X,$$

b) contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

c) N has *a fixed point* if there is $x \in X$ such that $x \in N(x)$. The fixed point set of the multivalued operator N will be denoted by $\operatorname{Fix} N$.

For more details on multivalued maps and the proof of known results cited in this section we refer to the books of Deimling [6], Gorniewicz [9] and Hu and Papageorgiou [12].

Our considerations are based on the following fixed point theorem for contraction multivalued operators given by Covitz and Nadler in 1970 [5] (see also Deimling, [6] Theorem 11.1).

Lemma 2.1. *Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then* $\operatorname{Fix} N \neq \emptyset$.

3. First order semilinear FDIs

Now, we are able to state and prove our main theorem for the IVP (1)–(2). Before stating and proving this result, we give the definition of a mild solution of the IVP (1)–(2).

Definition 3.1. A function $y \in C([-r,T],E)$ is called a mild solution of (1)–(2) if there exists a function $v \in L^1(J,E)$ such that $v(t) \in F(t,y_t)$ a.e. on J, $y_0 = \phi$, and

$$y(t) = T(t)\phi(0) + \int_0^t T(t-s)v(s)ds.$$

Theorem 3.1. *Assume that:*

(H1) *A is the infinitesimal generator of a linear semigroup of bounded operators $T(t), t \geq 0$ with $\|T(t)\|_{B(E)} \leq M$;*

(H2) *$F : [0,T] \times C([-r,0],E) \longrightarrow P_{cl}(E)$ has the property that $F(\cdot,u) : [0,T] \to P_{cl}(E)$ is measurable for each $u \in C([-r,0],E)$;*

(H3) *there exists $l \in L^1(J,\mathbb{R})$ such that $H_d(F(t,u),F(t,\overline{u})) \leq l(t)\|u-\overline{u}\|$, for each $t \in J$ and $u, \overline{u} \in C([-r,0],E)$, and $d(0,F(t,0)) \leq l(t)$, for almost each $t \in J$.*

Then the IVP (1)-(2) *has at least one mild solution on* $[-r,T]$.

Proof. Transform the problem into a fixed point problem. Consider the multivalued operator, $N : C([-r,T],E) \to \mathcal{P}(C([-r,T],E))$ defined by:

$$N(y) := \left\{ h \in C([-r,T],E) : h(t) = \left\{ \begin{array}{ll} \phi(t), & \text{if } t \in [-r,0] \\ T(t)\phi(0) & \\ +\int_0^t T(t-s)g(s)ds, & \text{if } t \in [0,T] \end{array} \right. \right\}$$

where

$$g \in S_{F,y} = \Big\{ g \in L^1([0,T],E) : g(t) \in F(t,y_t) \ \text{ for a.e. } \ t \in [0,T] \Big\}.$$

Remark 3.1. (i) It is clear that the fixed points of N are solutions to (1)-(2).
(ii) For each $y \in C([-r,T],E)$ the set $S_{F,y}$ is nonempty since by (H2) F has a measurable selection (see [4], Theorem III.6).

We shall show that N satisfies the assumptions of Lemma 2.1. The proof will be given in two steps.

Step 1: $N(y) \in P_{cl}(C[-r,T],E)$ for each $y \in C([-r,T],E)$.

Indeed, let $(y_n)_{n\geq 0} \in N(y)$ such that $y_n \longrightarrow \tilde{y}$ in $C([-r,T],E)$. Then $\tilde{y} \in C([-r,T],E)$ and

$$y_n(t) \in T(t)\phi(0) + \int_0^t T(t-s)F(s,y_s)ds \ \text{ for each } \ t \in [0,T].$$

Using the closedness property of the values of F and the second part of (H3) we can prove that $\int_0^t T(t-s)F(s,y_s)ds$ is closed for each $t \in [0,T]$. Then

$$y_n(t) \longrightarrow \tilde{y}(t) \in T(t)\phi(0) + \int_0^t T(t-s)F(s,y_s)ds, \quad \text{for} \ \ t \in [0,T].$$

So $\tilde{y} \in N(y)$.

Step 2: $H_d(N(y_1),N(y_2)) \leq \gamma\|y_1-y_2\|_{[-r,T]}$ for each $y_1,y_2 \in C([-r,T],E)$ (where $\gamma < 1$).

Let $y_1,y_2 \in C([-r,T],E)$ and $h_1 \in N(y_1)$. Then there exists $g_1(t) \in F(t,y_{1t})$ such that

$$h_1(t) = T(t)\phi(0) + \int_0^t T(t-s)g_1(s)ds, \quad t \in [0,T].$$

From (H3) it follows that

$$H_d(F(t,y_{1t}),F(t,y_{2t})) \leq l(t)\|y_{1t}-y_{2t}\|.$$

Hence there is $w \in F(t,y_{2t})$ such that

$$\|g_1(t)-w\| \leq l(t)\|y_{1t}-y_{2t}\|, \quad t \in [0,T].$$

Consider $U : [0,T] \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : \|g_1(t)-w\| \leq l(t)\|y_{1t}-y_{2t}\|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t,y_{2t})$ is measurable (see Proposition III.4 in [4]) there exists $g_2(t)$ a measurable selection for V. So, $g_2(t) \in F(t,y_{2t})$ and

$$\|g_1(t)-g_2(t)\| \leq l(t)\|y_{1t}-y_{2t}\|, \quad \text{for each} \ \ t \in J.$$

Let us define for each $t \in J$

$$h_2(t) = T(t)\phi(0) + \int_0^t T(t-s)g_2(s)ds.$$

Then we have

$$\begin{aligned}
|h_1(t) - h_2(t)| &\le M \int_0^t \|g_1(s) - g_2(s)\|\, ds \le M \int_0^t l(s)\|y_{1s} - y_{2s}\| ds \\
&= M \int_0^t l(s)e^{-\tau L(s)} e^{\tau L(s)} \|y_{1s} - y_{2s}\|\, ds \\
&\le M\|y_1 - y_2\|_B \int_0^t l(s)e^{\tau L(s)} ds \\
&= M\|y_1 - y_2\|_B \frac{1}{\tau} \int_0^t (e^{\tau L(s)})' ds \\
&\le M \frac{\|y_1 - y_2\|_B}{\tau} e^{\tau L(t)},
\end{aligned}$$

where $L(t) = \int_0^t l(s)ds$, $\tau > M$ and $\|\cdot\|_B$ is the Bielecki-type norm on $C([-r,T],E)$ defined by

$$\|y\|_B = \max_{t\in[-r,T]} \{|y(t)|e^{-\tau L(t)}\}.$$

Then

$$\|h_1 - h_2\|_B \le \frac{M}{\tau}\|y_1 - y_2\|_B.$$

By the analogous relation, obtained by interchanging the roles of y_1 and y_2, it follows that

$$H_d(N(y_1), N(y_2)) \le \frac{M}{\tau}\|y_1 - y_2\|_B.$$

So, N is a contraction and thus, by Lemma 2.1, it has a fixed point y, which is a mild solution to (1)-(2). ∎

4. Second order semilinear FDIs

Definition 4.1. A function $y \in C([-r,T],E)$ is called a mild solution of (3)–(4) if there exists a function $v \in L^1(J,E)$ such that $v(t) \in F(t, y_t)$ a.e. on J, $y_0 = \phi$, and

$$y(t) = C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s)v(s)ds.$$

THEOREM 4.1. *Assume that* (H2), (H3) *and*

(H4) *A is an infinitesimal generator of a given strongly continuous and bounded cosine family $\{C(t) : t \in J\}$ with $\|C(t)\|_{B(E)} \le \overline{M}$;*

are satisfied. Then the IVP (3)-(4) *has at least one mild solution on* $[-r, T]$.

Proof. Transform the problem into a fixed point problem. Consider the multivalued operator, $N_1 : C([-r,T],E) \to \mathcal{P}(C([-r,T],E))$ defined by:

$$N_1(y) := \left\{ h \in C([-r,T],E) : h(t) = \left\{ \begin{array}{ll} \phi(t), & \text{if } t \in [-r,0] \\ C(t)\phi(0) + S(t)\eta + & \\ \int_0^t S(t-s)g(s)ds & \text{if } t \in [0,T] \end{array} \right. \right\}$$

where

$$g \in S_{F,y} = \Big\{ g \in L^1([0,T],E) : g(t) \in F(t,y_t) \ \text{ for a.e. } \ t \in [0,T] \Big\}.$$

We shall show that N_1 satisfies the assumptions of Lemma 2.1. The proof will be given in two steps.

Step 1 *: $N_1(y) \in P_{cl}(C[-r,T],E)$ for each $y \in C([-r,T],E)$.*

Indeed, let $(y_n)_{n\ge 0} \in N_1(y)$ such that $y_n \longrightarrow \tilde{y}$ in $C([-r,T],E)$. Then $\tilde{y} \in C([-r,T],E)$ and

$$y_n(t) \in C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s)F(s,y_s)ds \ \text{ for each } \ t \in [0,T].$$

Using the closedness property of the values of F and the second part of (H3) we can prove that $\int_0^t S(t-s)F(s,y_s)ds$ is closed for each $t \in [0,T]$. Then

$$y_n(t) \longrightarrow \tilde{y}(t) \in C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s)F(s,y_s)ds, \quad \text{for } \ t \in [0,T].$$

So $\tilde{y} \in N(y)$.

Step 2: *$H_d(N_1(y_1), N_1(y_2)) \le \gamma\|y_1 - y_2\|_{[-r,T]}$ for each $y_1, y_2 \in C([-r,T],E)$ (where $\gamma < 1$).*

Let $y_1, y_2 \in C([-r,T],E)$ and $h_1 \in N_1(y_1)$. Then there exists $g_1(t) \in F(t, y_{1t})$ such that

$$h_1(t) = C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s)g_1(s)ds, \quad t \in [0,T].$$

From (H3) it follows that

$$H_d(F(t, y_{1t}), F(t, y_{2t})) \leq l(t)\|y_{1t} - y_{2t}\|.$$

Hence there is $w \in F(t, y_{2t})$ such that

$$\|g_1(t) - w\| \leq l(t)\|y_{1t} - y_{2t}\|, \quad t \in [0, T].$$

Consider $U : [0, T] \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : \|g_1(t) - w\| \leq l(t)\|y_{1t} - y_{2t}\|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, y_{2t})$ is measurable (see Proposition III.4 in [4]) there exists $g_2(t)$ a measurable selection for V. So, $g_2(t) \in F(t, y_{2t})$ and

$$\|g_1(t) - g_2(t)\| \leq l(t)\|y_{1t} - y_{2t}\|, \text{ for each } t \in J.$$

Let us define for each $t \in J$

$$h_2(t) = C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s)g_2(s)ds.$$

Then we have

$$\begin{aligned}
|h_1(t) - h_2(t)| &\leq T\overline{M} \int_0^t \|g_1(s) - g_2(s)\|\, ds \\
&\leq T\overline{M} \int_0^t l(s)\|y_{1s} - y_{2s}\| ds \\
&= T\overline{M} \int_0^t l(s)e^{-\tau L(s)}e^{\tau L(s)}\|y_{1s} - y_{2s}\|\, ds \\
&\leq T\overline{M}\|y_1 - y_2\|_B \int_0^t l(s)e^{\tau L(s)} ds \\
&= T\overline{M}\|y_1 - y_2\|_B \frac{1}{\tau} \int_0^t (e^{\tau L(s)})' ds \\
&\leq T\overline{M} \frac{\|y_1 - y_2\|_B}{\tau} e^{\tau L(t)}.
\end{aligned}$$

Then

$$\|h_1 - h_2\|_B \leq \frac{T\overline{M}}{\tau}\|y_1 - y_2\|_B.$$

By the analogous relation, obtained by interchanging the roles of y_1 and y_2, it follows that

$$H_d(N_1(y_1), N_1(y_2)) \leq \frac{T\overline{M}}{\tau}\|y_1 - y_2\|_B.$$

Let $\tau > T\overline{M}$. Then N_1 is a contraction and thus, by Lemma 2.1, it has a fixed point y, which is a mild solution to (3)-(4). ∎

Remark 4.1. The reasoning used above can be applied to obtain existence results for the following first and second order semilinear integrodifferential inclusions of Volterra type

$$y' - Ay \in \int_0^t k(t,s)F(s, y_s)ds, \quad a.e.\ t \in [0,T] \tag{5}$$

$$y(t) = \phi(t) \quad t \in [-r, 0], \tag{6}$$

and

$$y'' - Ay \in \int_0^t k(t,s)F(s, y_s)ds, \quad a.e.\ t \in [0,T] \tag{7}$$

$$y(t) = \phi(t) \quad t \in [-r, 0], \quad y'(0) = \eta, \tag{8}$$

where A, F, ϕ, η are as in problems (1)-(2) and (3)-(4) and $k : D \to \mathbb{R},\ D = \{(t,s) \in J \times J : t \geq s\}$.

We state only the results. We need the following assumptions:

(H5) for each $t \in [0,T]$, $k(t,s)$ is measurable on $[0,t]$ and

$$K(t) = \text{ess sup}\{|k(t,s)|, \quad 0 \leq s \leq t\},$$

is bounded on $[0,T]$;

(H6) the map $t \longmapsto k_t$ is continuous from $[0,T]$ to $L^\infty(J, \mathbb{R})$; here $k_t(s) = k(t,s)$;

Theorem 4.2. *Assume that* (H1)–(H3), (H5)–(H6) *are satisfied. Then the IVP* (5)-(6) *has at least one mild solution on* $[-r, T]$.

Theorem 4.3. *Assume that* (H2), (H3)–(H6) *are satisfied. Then the IVP* (7)-(8) *has at least one mild solution on* $[-r, T]$.

Remark 4.2. The contraction constant is given by $\gamma = \frac{TM}{\tau}\sup_{t\in[0,T]} K(t) < 1$ for Theorem 4.2 and by $\gamma = \frac{T^2\overline{M}}{\tau}\sup_{t\in[0,T]} K(t) < 1$ for Theorem 4.3.

Remark 4.3. The above results obtained on the compact interval $[-r, T]$ can be extended to the infinite interval $[-r, \infty)$ by using the same fixed point theorem. To this end, we consider the Banach space of continuous functions $y : [-r, \infty) \to E$ such that $t \to \exp(-\tau L(t))y(t)$ is bounded on $[-r, \infty)$. The norm on this space is given by

$$\|y\| = \sup_{t\in[-r,\infty)} \exp(-\tau L(t))|y(t)|.$$

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