EXTRACTA MATHEMATICAE Vol. **18**, Núm. 1, 65 – 80 (2003)

On the Class of Continuous Images of Valdivia Compacta †

Ondřej F.K. Kalenda

Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic
e-mail: kalenda@karlin.mff.cuni.cz

(Presented by J.M.F. Castillo)

AMS *Subject Class.* (2000): 46B26, 54D30, 54C05 *Received February* 7, 2002

1. Introduction

A compact space K is called *Corson* [8] if it is homeomorphic to a subset of

$$\Sigma(\Gamma) = \{x \in \mathbf{R}^\Gamma : \operatorname{supp} x \text{ is countable}\}$$

for a set Γ. A compact space K is called *Valdivia* [2] if it is homeomorphic to some $K' \subset \mathbf{R}^\Gamma$ with $K' \cap \Sigma(\Gamma)$ dense in K'. A subset $A \subset K$ is called Σ*-subset* if there is a homeomorphic injection $h : K \to \mathbf{R}^\Gamma$ with $A = h^{-1}(\Sigma(\Gamma))$. In this setting K is Valdivia if and only if it has a dense Σ-subset (cf. [6, Section 1.1]).

Corson and Valdivia compacta are useful for studying the structure of non-separable Banach spaces, they are closely related with projectional resolutions of the identity and Markushevich bases, see e.g. [10], [11], [12], [2], [6]. There are studied also the associated Banach spaces.

A subspace $S \subset X^*$ is called a Σ*-subspace* if there is a one-to-one linear weak* continuous mapping $T : X^* \to \mathbf{R}^\Gamma$ with $S = T^{-1}(\Sigma(\Gamma))$. A Banach space X is called *weakly Lindelöf determined* (or shortly WLD) if X^* is a Σ-subspace of itself. The space X is called *Plichko* (1-*Plichko*) if X^* admits a norming (1-norming, respectively) Σ-subspace.

The class of Valdivia compact spaces is not closed to continuous images [13], [6, Theorem 3.21] and the class of 1-Plichko spaces is not closed to subspaces [6, Sections 4.5 and 5.2]. However, continuous images of Valdivia compacta enjoy many properties of Valdivia ones, see for example [6, Theorem

†This work was partially supported by Research grants GAUK 277/2001, GAČR 201/00/1466 and MSM 113200007.

3.27]. In the present paper we study the class of continuous images of Valdivia compacta and the associated class of subspaces of 1-Plichko spaces. We show some analogues and differences between these classes and the classical ones. We also introduce subclasses of weak Corson compact spaces and weakly WLD Banach spaces which play a similar role as Corson compacta and WLD spaces.

In the second section we study two classes of countably compact spaces which are a generalization of Corson compacta. Third section is devoted to the classes of compact spaces. In the fourth section we investigate associated Banach spaces. The last section contains some open problems.

All topological spaces are assumed to be Hausdorff and completely regular.

A subspace A of a topological space X is called *countably closed* if $\overline{C} \subset A$ for each $C \subset A$ countable. A space X is said to have *countable tightness* (to be *Fréchet-Urysohn*) if, whenever $A \subset X$ and $x \in \overline{A}$ then there is a countable subset $C \subset A$ with $x \in \overline{C}$ (there is a sequence $x_n \in A$ with $x_n \to x$, respectively).

2. Corson and weakly Corson countably compact spaces

We say that a countably compact space X is *Corson* if there is a continuous injection of X into some $\Sigma(\Gamma)$. Such an injection is necessarily a homeomorphism onto its image [6, Lemma 1.8]. Within the class of compact spaces our definition is just the classical definition of Corson compact spaces [8]. Corson compact spaces are stable to continuous images [8], more generally Corson countably compact spaces are stable to quotient mappings [4]. However, it is easy to see (cf. Theorem 2.5 below) that Corson countably compact spaces are not stable to continuous images. Countably compact spaces which are continuous images of Corson countably compact spaces will be called *weakly Corson*. We begin by some easy or well-known stability properties of Corson and weakly Corson spaces.

Lemma 2.1. *The class of Corson countably compact spaces is closed to taking countably closed subspaces, quotient images, countable products, finite topological sums, "one-point countably compact modifications" of arbitrarily large topological sums.*

By a "one-point countably-compact modification" of a topological sum of countably compact spaces we mean any countably compact space containing that topological sum as its (topological) subspace with one-point complement.

An example is the one-point compactification of the topological sum of a family of compact spaces. Among one-point countably compact modifications there is a maximal element. This is the topological sum extended by a point ∞ such that neighborhood basis of ∞ is formed by complements of closed countably compact subsets of the topological sum.

Proof of Lemma 2.1. The stability to countably closed subspaces and to finite topological sums is trivial, that to quotient images is a result of [4]. That to countable products follows from [6, Lemma 3.28]. Let us show the stability to one-point countably compact modifications of topological sums. Let X_α be a Corson countably compact space for $\alpha \in A$ and $X = \bigoplus_{\alpha\in A} X_\alpha \cup \{\infty\}$ be countably compact. Fix $f_\alpha : X_\alpha \to \Sigma(\Gamma_\alpha)$ continuous one-to-one mappings. Put

$$\Gamma = \{(\alpha,\gamma) : \alpha \in A, \gamma \in \Gamma_\alpha\} \cup \{(\alpha, A) : \alpha \in A\}$$

and define $f : X \to \mathbf{R}^\Gamma$ by the formula

$$f(x)(\alpha,\gamma) = \begin{cases} f_\alpha(x)(\gamma), & x \in X_\alpha,\ \gamma \in \Gamma_\alpha, \\ 1, & x \in X_\alpha,\ \gamma = A, \\ 0, & \text{otherwise.} \end{cases}$$

This is clearly a continuous one-to-one mapping satisfying $f(X) \subset \Sigma(\Gamma)$. Hence X is Corson. ∎

Lemma 2.2. *The class of weakly Corson countably compact spaces is stable to all operations mentioned in Lemma 2.1 and, moreover, to continuous images and finite unions.*

Proof. The stability to closed subspaces, countable products and finite topological sums follows easily from Lemma 2.1 and definitions. The stability to continuous images is trivial, that to finite unions follows from the observation that any finite union is a continuous image of a finite topological sum. It remains to show the stability to one-point countably compact modifications of topological sums. Let X_α be a weakly Corson countably compact space for any $\alpha \in A$ and $X = \bigoplus_{\alpha\in A} X_\alpha \cup \{\infty\}$ be countably compact. By the definitions there is, for each $\alpha \in A$, a Corson countably compact space Y_α and a continuous onto mapping $f : Y_\alpha \to X_\alpha$. Let Y be the maximal one-point countably compact modification of the topological sum of all Y_α's. Then Y is Corson by Lemma 2.1. Define $f : Y \to X$ by the formula

$$f(y) = \begin{cases} f_\alpha(y), & y \in Y_\alpha, \\ \infty, & y = \infty. \end{cases}$$

Then f is an onto mapping. Moreover, it is continuous. Indeed, let $U \subset X$ be open and $V = f^{-1}(U)$. Obviously $V \cap Y_\alpha$ is open for each α, so $V \setminus \{\infty\}$ is open. If $\infty \notin U$ then $\infty \notin V$ and we are done. Suppose now $\infty \in U$. Then it remains to prove that V is a neighborhood of ∞ in Y. To this end it is enough to show that $Y \setminus V$ is countably compact. As each Y_α is countably compact, it suffices to observe that $Y_\alpha \subset V$ for all but finitely many α. And this is true because $X_\alpha \subset U$ for all but finitely many α as $X \setminus U$ is countably compact. ∎

Now we are going to introduce a topological operation, which will yield another stability property. This will be the $[0,\eta)$-*sum*, where η is an ordinal.

Suppose that X_α is a topological space for each isolated ordinal $\alpha < \eta$. The $[0,\eta)$-sum of spaces X_α is the set

$$X = \{(\alpha, x) : x \in X_\alpha,\ \alpha < \eta \text{ isolated}\} \cup \{(\alpha,\alpha) : \alpha < \eta \text{ limit}\}$$

equipped with the following topology. The sets $\{\alpha\} \times X_\alpha$ for α isolated are canonically homeomorphic to X_α and clopen in X. A neighborhood basis for (α,α) with α limit is formed by sets

$$X(\gamma,\alpha] = \{(\beta,x) \in X : \gamma < \beta \leq \alpha\}, \quad \gamma < \alpha.$$

It is easy to check that X is a Hausdorff (regular, completely regular) topological space whenever each X_α is Hausdorff (regular, completely regular, respectively).

LEMMA 2.3. • *If η is either an isolated ordinal or an ordinal with uncountable cofinality, then any $[0,\eta)$-sum of countably compact spaces is countably compact.*

• *Any $[0,\omega_1)$-sum of Corson (weakly Corson) countably compact spaces is again Corson (weakly Corson).*

Proof. To show the first part, let (α_n, x_n) be any sequence in X. Then there is a subsequence α_{n_k} which is either constant or strictly increasing. If $\alpha_{n_k} = \alpha$ for each k and α is isolated, then $x_{n_k} \in X_\alpha$ and hence it has a cluster point $x \in X_\alpha$. Then (α, x) is a cluster point of the sequence x_n. If α is limit, then $x_{n_k} = \alpha$, hence (α,α) is a cluster point of x_n. It remains to consider the case when α_{n_k} is increasing. Let α denote the supremum of this sequence. By the assumptions on η we have $\alpha < \eta$. It follows from the definition of the topology on $[0,\eta)$-sum that (α_{n_k}, x_{n_k}) converges to (α,α), so the sequence x_n has a cluster point.

Now suppose that X_α is a Corson countably compact space for $\alpha < \omega_1$ isolated. Let X be the $[0, \omega_1)$-sum. By the previous paragraph X is countably compact. As X_α is Corson, there is a continuous injection $h_\alpha : X_\alpha \to \Sigma(\Gamma_\alpha)$. Put

$$\Gamma = \{(\alpha, \gamma) : \gamma \in \Gamma_\alpha,\ \alpha < \omega_1 \text{ isolated}\} \cup \{(\omega_1, \alpha) : \alpha < \omega_1\}$$

and define $h : X \to \mathbf{R}^\Gamma$ by the formula

$$h(\alpha, x)(\beta, \gamma) = \begin{cases} h_\alpha(x)(\gamma) & \alpha = \beta < \omega_1 \text{ isolated}, \\ 0 & \alpha \neq \beta < \omega_1,\ \beta \text{ isolated}, \\ 1 & \beta = \omega_1,\ \gamma < \alpha, \\ 0 & \beta = \omega_1,\ \gamma \geq \alpha. \end{cases}$$

Then h is continuous, one-to-one and maps X into $\Sigma(\Gamma)$. So X is Corson.

Finally suppose that X_α is a weakly Corson countably compact space for $\alpha < \omega_1$ isolated. For each α there is a Corson countably compact space Y_α and a continuous surjection φ_α of Y_α onto X_α. Let X and Y be the $[0, \omega_1)$-sums of all X_α's (all Y_α's, respectively). The space Y is Corson by the previous paragraph. Define a function $\varphi : Y \to X$ by the formula

$$\varphi(\alpha, y) = \begin{cases} (\alpha, \varphi_\alpha(y)) & \alpha \text{ isolated}, \\ (\alpha, \alpha) & \alpha \text{ limit}. \end{cases}$$

It is clear that φ is a mapping of Y onto X. Moreover, it follows from the definition of the $[0, \omega_1)$-sum that it is continuous. Hence X is weakly Corson.

∎

The following lemma characterizes Corson countably compact spaces among weakly Corson ones.

Lemma 2.4. *Let X be a weakly Corson countably compact space. The following assertions are equivalent.*

(1) *X is Corson.*
(2) *X is Fréchet-Urysohn.*
(3) *X has countable tightness.*

Proof. The implication $1 \Rightarrow 2$ follows from [9, Theorem 2.1], see also [6, Lemma 1.6].

The implication $2 \Rightarrow 3$ is trivial.

$3 \Rightarrow 1$ Let Y be a Corson countably compact space and $f : Y \to X$ a continuous onto mapping. Let $F \subset Y$ be closed. We claim that $f(F)$ is closed in X. Let $x \in \overline{f(F)}$. Then there is $C \subset f(F)$ countable with $x \in \overline{C}$. Choose a countable set $D \subset F$ such that $f(D) = C$. Then $\overline{D} \subset F$ and, moreover, $\overline{D}$ is compact, so $f(\overline{D})$ contains x. As $f(\overline{D}) \subset f(F)$ we get $x \in f(F)$. Therefore $f(F)$ is closed. So f is a closed mapping, in particular it is a quotient mapping, so X is Corson by Lemma 2.1. ∎

THEOREM 2.5.
- *Let X be a countably compact set of ordinals. Then X is weakly Corson if and only if X has cardinality at most $\aleph_1$. Further, X is Corson if and only if* $\operatorname{card} X \leq \aleph_1$ *and any ordinal of uncountable cofinality contained in X is an isolated point of X.*
- *The space $[0, \eta]$ is Corson (weakly Corson) if and only if $\eta < \omega_1$ ($\eta < \omega_2$, respectively).*

Proof. Suppose that X is a countably compact set of ordinals with $\operatorname{card} X \leq \aleph_1$ and $\theta = \sup X$. Let Y denote the closure of X in $[0, \theta]$. Then clearly $\operatorname{card} Y \leq \aleph_1$ and Y is a well-ordered compact space, hence Y is homeomorphic to $[0, \eta]$ for some $\eta < \omega_2$. As X is countably closed in Y, it is enough to show (due to Lemma 2.2) that $[0, \eta]$ is weakly Corson for any $\eta < \omega_2$.

We will show it by transfinite induction. $[0, 0]$ is clearly even Corson. Suppose that $\eta < \omega_2$ is such that $[0, \gamma]$ is weakly Corson for every $\gamma < \eta$. There are three possibilities.

(a) $\eta = \xi + 1$ – then $[0, \eta]$ is the topological sum of $[0, \xi]$ and $\{\eta\}$, and so it is weakly Corson by Lemma 2.2.

(b) η is a limit ordinal of countable cofinality. So there are $\eta_n < \eta$ such that $\eta_n \nearrow \eta$. Then $[0, \eta]$ is the one-point compactification of the topological sum of weakly Corson compact spaces $[0, \eta_1]$, $(\eta_1, \eta_2]$, $(\eta_2, \eta_3]$, …, hence it is weakly Corson by Lemma 2.2.

(c) η is a limit ordinal of cofinality ω_1. Then there is an increasing long sequence of ordinals $(\eta_\gamma : \gamma < \omega_1)$ such that η_α for limit α is the supremum of the preceding η_γ's and the supremum of the whole family is η. Then $[0, \eta_0]$ and $(\eta_\gamma, \eta_{\gamma+1}]$ for $\gamma < \omega_1$ are weakly Corson by the induction hypothesis and their $[0, \omega_1)$-sum, which is the interval $[0, \eta)$, is weakly Corson by Lemma 2.3. Finally, $[0, \eta]$ is the union of $[0, \eta)$ and $\{\eta\}$, and so it is weakly Corson by Lemma 2.2.

Now suppose that X is a weakly Corson countably compact set of ordinals with $\operatorname{card} X \geq \aleph_2$. Let Y be the set of those $x \in X$ which are the limit of a sequence of isolated points of X. Then clearly Y is countably closed in X, so it is weakly Corson by Lemma 2.2. Further, Y is Fréchet-Urysohn, so it is Corson by Lemma 2.4. As clearly $\operatorname{card} Y \geq \aleph_2$, we can without loss of generality suppose that X is Corson.

Let $\theta = \sup X$ and Z denote the closure of X in $[0, \theta]$. Then Z is a well-ordered compact space and so it is homeomorphic to $[0, \eta]$ for some $\eta \geq \omega_2$. This homeomorphism maps X onto the set of all ordinals from $[0, \eta]$ which are either isolated or of countable cofinality. Hence, by Lemma 2.1, the set

$$X_0 = \{\alpha < \omega_2 : \alpha \text{ isolated or of countable cofinality}\}$$

is Corson. Let $h : X_0 \to \Sigma(\Gamma)$ be a one-to-one continuous mapping. If $\gamma \in \Gamma$, then $h_\gamma : \alpha \mapsto h(\alpha)(\gamma)$ is continuous and so there is $\beta(\gamma) < \omega_2$ such that h_γ is constant on $(\beta(\gamma), \omega_2) \cap X_0$. Put

$$I = \{\gamma \in \Gamma : h_\gamma = 0 \text{ on } (\beta(\gamma), \omega_2) \cap X_0\}.$$

If $\Gamma \setminus I$ is uncountable, choose $J \subset \Gamma \setminus I$ with $\operatorname{card} J = \aleph_1$ and put $\beta_0 = \sup\{\beta(\gamma) : \gamma \in J\}$. Then $\beta_0 < \omega_2$ and hence there is $\alpha \in (\beta_0, \omega_2) \cap X_0$. Then $h_\gamma(\alpha) \neq 0$ for each $\gamma \in J$, thus $h(\alpha) \notin \Sigma(\Gamma)$, a contradiction.

Therefore $\Gamma \setminus I$ is countable. Due to the previous paragraph there is a unique continuous extension $\tilde{h} : X_0 \cup \{\omega_2\} \to \Sigma(\Gamma)$ of the mapping h. As $\tilde{h}|X_0$ is one-to-one, there is at most one $\alpha \in X_0$ such that $\tilde{h}(\alpha) = \tilde{h}(\omega_2)$. Hence there is some $\alpha < \omega_2$ such that $\tilde{h}$ is one to one on $X_1 = (X_0 \cap (\alpha, \omega_2)) \cup \{\omega_2\}$. Therefore X_1 is Corson, which is a contradiction with Lemma 2.4 as X_1 is not Fréchet-Urysohn.

Thus we proved the characterization of weakly Corson countably compact sets of ordinals. That of Corson ones follows now easily from Lemma 2.4. The case of $[0, \eta]$ is just a special case of the countably compact case. ∎

3. Weakly Corson and weakly Valdivia compact spaces

Let us call a compact space K *weakly Valdivia* if it has a dense countably compact subset which is weakly Corson. This is not a new class, as says the following proposition.

Proposition 3.1. *A compact space K is weakly Valdivia if and only if it is a continuous image of a Valdivia compact space.*

Proof. Suppose there is a Valdivia compact space L and a continuous surjection $\varphi : L \to K$. Let A be a dense Σ-subset of L. Then A is a Corson countably compact space and $\varphi(A)$ is weakly Corson and dense in K. Thus K is weakly Valdivia.

Conversely let B be a dense weakly Corson countably compact subset of K. Then there is $A \subset \Sigma(\Gamma)$ countably compact and $\varphi : A \to B$, a continuous surjection. As $\overline{A}^{\mathbf{R}^\Gamma} = \beta A$ [6, Lemma 1.8 and Proposition 1.9], the space βA is Valdivia. The continuous extension $\beta\varphi : \beta A \to K$ witnesses that K is a continuous image of a Valdivia compact space. ∎

The next proposition sums stability properties of weak Valdivia compact spaces.

Proposition 3.2. *The class of weakly Valdivia compact spaces is closed to arbitrary products, continuous images, finite unions, one-point compactifications of arbitrary topological sums. Moreover, if K is weakly Valdivia and L is a subset of K which can be written as the closure of the union of an arbitrary family of G_δ subsets of K, then L is weakly Valdivia as well.*

Proof. The stability to arbitrary products and one-point compactifications of topological sums follows, using Proposition 3.1, from the respective stability properties of Valdivia compact spaces, see [6, Theorem 3.29 and Theorem 3.35]. The stability to continuous images is trivial by Proposition 3.1, that to finite unions follows from the fact that a finite union is a continuous image of a finite topological sum.

Now suppose that K is weakly Valdivia and L is a subset of K which can be written as the closure of the union of arbitrary family of G_δ subsets of K. Let $A \subset K$ be a dense weakly Corson countably compact subset. It follows from [6, Lemma 1.11] that $L \cap A$ is dense in L. As $L \cap A$ is weakly Corson by Lemma 2.2, L is weakly Valdivia. ∎

The class of weakly Valdivia compact spaces is not stable to taking closed subsets. In fact any compact space is homeomorphic to a subset of $[0,1]^\Gamma$, which is even Valdivia and not all compact spaces are weakly Valdivia.

Proposition 3.3. *There is a closed subset of $\{0,1\}^{\omega_1}$ which is not weakly Valdivia. Any weakly Corson compact is hereditarily weakly Valdivia.*

Proof. Let $K \subset [0,1]$ be the Cantor set and $A \subset K$ be any subset of cardinality $\aleph_1$ consisting of both-sided accumulation points. Let K_A be the compact from [5]. Then it is not weakly Valdivia by [6, Example 1.18(ii)] and it can be found in $\{0,1\}^{\omega_1}$ as it is zero-dimensional and has weight $\aleph_1$.

The second part is obvious. ∎

The following proposition is an analogue of Lemma 2.3 for the compact case.

PROPOSITION 3.4. *Any* $[0, \omega_1 + 1)$*-sum of (weakly) Valdivia compacta is again a (weakly) Valdivia compactum.*

Proof. First let us note that $[0, \eta)$-sum of compact spaces is again compact whenever η is isolated. Now we are going to prove the statement for Valdivia compacta.

Let K_α be a Valdivia compactum for $\alpha < \omega_1 + 1$ isolated. Fix a dense Σ-subset A_α of K_α. Denote by K the $[0, \omega_1 + 1)$-sum of K_α's. Put

$$A = \bigcup_{\alpha < \omega_1 + 1 \text{ isolated}} \{\alpha\} \times A_\alpha \cup \{(\alpha, \alpha) : \alpha < \omega_1 \text{ limit}\}.$$

Then A is dense in K and it is the $[0, \omega_1)$-sum of A_α's. As each A_α is a Corson countably compact space, A has the same property by Lemma 2.3. By [6, Proposition 1.4 and Lemma 1.9] it now suffices to prove that $K = \beta A$. Let f be a real continuous function of A. We will describe the continuous extension of f on K.

As $K_\alpha = \beta A_\alpha$, the restriction $f|\{\alpha\} \times A_\alpha$ can be continuously extended on $\{\alpha\} \times K_\alpha$. Moreover, $\{(\alpha, \alpha) : \alpha < \omega_1 \text{ limit}\}$ is homeomorphic to $[0, \omega_1)$ and hence f is constant on $\{(\alpha, \alpha) : \alpha \in (\beta, \omega_1) \text{ limit}\}$ for some $\beta < \omega_1$. So we extend f to the point (ω_1, ω_1) by this common value.

In this way we have extended f to all points of K. It follows easily from the definition of the topology on K that this extension is continuous.

The case of weakly Valdivia compacta is even more easy. Let A_α be a dense weakly Corson countably compact subset of K_α. Define A as above. Then A is weakly Corson by Lemma 2.3 and therefore K is weakly Valdivia. ∎

Now we give a characterization of weakly Valdivia compacta among ordinal segments.

THEOREM 3.5. *Let* η *be an ordinal. Then the following are equivalent.*

(i) $\eta < \omega_2$.
(ii) $[0, \eta]$ *is Valdivia.*
(iii) $[0, \eta]$ *is weakly Valdivia.*
(iv) $[0, \eta]$ *is weakly Corson.*

Proof. (i)⇒(iv) follows from Theorem 2.5.

(i)⇒(ii) follows by transfinite induction from Proposition 3.4. This is also proved in [6, Theorem 3.7].

The implications (ii)⇒(iii) and (iv)⇒(iii) are trivial.

(iii)⇒(i) Suppose $\eta \geq \omega_2$. If $[0,\eta]$ is weakly Valdivia, there is $A \subset [0,\eta]$ dense weakly Corson countably compact space. But clearly $\operatorname{card} A \geq \omega_2$, which is a contradiction with Theorem 2.5. ∎

The following theorem sums up duality properties of weakly Valdivia compacta. It is an analogue of [6, Theorems 5.2 and 5.3].

THEOREM 3.6. *Let K be a compact space. Consider the following assertions.*

(1) *K is weakly Valdivia.*
(2) *$\left(B_{C(K)^*}, w^*\right)$ is weakly Valdivia.*
(3) *$P(K)$ is weakly Valdivia.*

Then $1 \Rightarrow 2 \Leftrightarrow 3$. If K has a dense set of G_δ points, then all three assertions are equivalent.

Proof. $1 \Rightarrow 2$ The space K is a continuous image of a Valdivia compactum L. The dual unit ball $\left(B_{C(L)^*}, w^*\right)$ is Valdivia by [6, Theorem 5.2] and $\left(B_{C(K)^*}, w^*\right)$ is a continuous image of $\left(B_{C(L)^*}, w^*\right)$, thus it is weakly Valdivia.

$2 \Rightarrow 3$ It is easy to check that $P(K)$ is weak* closed weak* G_δ subset of $\left(B_{C(K)^*}, w^*\right)$. Hence the assertion follows by Proposition 3.2.

$3 \Rightarrow 2$. Suppose $P(K)$ is weakly Valdivia. Then $P(K) \times P(K) \times [0,1]$ is weakly Valdivia as well, due to Proposition 3.2. And $(\mu,\nu,t) \mapsto t\mu - (1-t)\nu$ is a continuous mapping onto $\left(B_{C(K)^*}, w^*\right)$.

$3 \Rightarrow 1$ if K has a dense set of G_δ points. Let us consider K canonically embedded to $P(K)$. If k is a G_δ point of K then it is also a G_δ point of $P(K)$. Then use Proposition 3.2. ∎

It is unknown whether $3 \Rightarrow 1$ holds without the assumption that K has a dense set of G_δ points (cf. the analogous problem for Valdivia compacta, [6, Question 5.10]).

Before stating the next theorem let us recall that a compact space K has *property* (M) if any Radon probability measure in K has separable support.

THEOREM 3.7. *If K is weakly Corson and has property* (M), *then $\left(B_{C(K)^*}, w^*\right)$ is weakly Corson as well.*

Proof. Let A be a Corson countably compact space and $\varphi' : A \to K$ a continuous surjection. Put $L = \beta A$ and let $\varphi : L \to K$ be the continuous extension of φ'. Then, by [6, Proposition 1.9], A is a Σ-subset of L and

$$\tilde{A} = \{\mu \in B_{C(L)^*} : \operatorname{supp} \mu \text{ is a separable subset of } A\}$$

is a dense Σ-subset of $\left(B_{C(L)^*}, w^*\right)$ by [6, Proposition 5.1]. Let

$$T : \left(B_{C(L)^*}, w^*\right) \longrightarrow \left(B_{C(K)^*}, w^*\right)$$

be the canonical continuous surjection ($T(\mu) = \varphi(\mu)$). We claim that $T(\tilde{A}) = \left(B_{C(K)^*}, w^*\right)$.

As $\tilde{A}$ is convex symmetric and T affine, it is enough to check that $P(K) \subset T(\tilde{A})$. Let $\mu \in P(K)$. Put $H = \operatorname{supp} \mu$. By the property (M) the set H is separable, let $\{h_n : n \in \mathbf{N}\}$ be a dense subset. For each $n \in \mathbf{N}$ choose some $a_n \in A$ such that $\varphi(a_n) = h_n$ and put $M = \overline{\{a_n : n \in \mathbf{N}\}}^L$. Then M is a metrizable compact subset of A and $\varphi(M) = H$. Hence there is $\nu \in P(M)$ such that $T(\nu) = \mu$. As $\operatorname{supp} \nu \subset M$ is separable, $\nu \in \tilde{A}$ and so $\mu \in T(\tilde{A})$. This completes the proof. ∎

4. Associated Banach spaces

Let us call Banach space X *weakly Plichko* if it is isomorphic (or, equivalently, isometric) to a subspace of a Plichko space. A space X will be called *weakly 1-Plichko* if it is isometric to a subspace of a 1-Plichko space. It is not known whether any weakly Plichko space is already Plichko, while there are weakly 1-Plichko spaces which are not 1-Plichko, see e.g. [6, Section 4.5]. It is clear that a space is weakly Plichko if and only if it can be renormed to be weakly 1-Plichko.

PROPOSITION 4.1. *Let X be a Banach space. The following assertions are equivalent.*

(a) *X is weakly 1-Plichko.*
(b) *(B_{X^*}, w^*) contains a dense convex symmetric Corson countably compact subset.*
(c) *(B_{X^*}, w^*) is weakly Valdivia.*

Proof. (a)⇒(b) Let X be isometric to a subspace of a 1-Plichko space Y. Denote by i the injection of X into Y and by i^* the adjoint surjection of Y^*

onto X^*. Let A be a dense convex symmetric Σ-subset of (B_{Y^*}, w^*). Then $i^*(A)$ is the required subset of (B_{X^*}, w^*).

(b)⇒(c) is trivial.

(c)⇒(a) Suppose (B_{X^*}, w^*) is weakly Valdivia. Then it is a continuous image of a Valdivia compact L. The space $C(L)$ is 1-Plichko by [6, Theorem 5.2], $C(B_{X^*}, w^*)$ is isometric to a subspace of $C(L)$ and X is isometric to a subspace of $C(B_{X^*}, w^*)$. ∎

Let us remark that the analogous statement for 1-Plichko spaces and Valdivia compacta is not true. While the analogue of (a)⇔(b) is true by [6, Theorem 2.7], there is, by [7], a Banach space with Valdivia dual unit ball which is not 1-Plichko. A significant subclass of weakly 1-Plichko spaces is formed by those Banach spaces whose dual unit ball is weakly Corson. Let us call such spaces *weakly* WLD.

The following proposition is an immediate consequence of Lemma 2.2.

PROPOSITION 4.2. *The class of weakly* WLD *Banach spaces is closed to isomorphisms, subspaces and quotients.*

As a corollary we get the following example.

EXAMPLE 4.3. If $\alpha < \omega_2$, then for any equivalent norm on $C[0, \alpha]$ the dual unit ball is a continuous image of a Valdivia compactum.

This example shows that there are non-WLD Banach spaces whose dual unit ball with respect to any equivalent norm is weakly Valdivia. This should be compared with [6, Theorem 4.22] which says that if the dual unit ball with respect to any equivalent norm is even Valdivia, then the space is already WLD. Now it is natural to ask which spaces are weakly 1-Plichko in any equivalent norm. Is any such space weakly WLD? We do not know the complete answer but some methods used for Valdivia compacta and 1-Plichko spaces can be applied in this situation, too. We continue by the following analogue of [6, Lemma 4.27].

PROPOSITION 4.4. *Let X be a Banach space such that X^* contains a convex weak* compact subset which is not weakly Valdivia. Then there is an equivalent norm on $X \times \mathbf{R}$ in which this space is not weakly 1-Plichko.*

Proof. Let $K \subset X^*$ be a convex weak* compact which is not weakly Valdivia. Put

$$B = \operatorname{conv}\left(\left(\frac{1}{2}B_{X^*} \times \left[-\frac{1}{2}, \frac{1}{2}\right]\right) \cup \big(K \times \{1\}\big) \cup \big((-K) \times \{-1\}\big)\right).$$

Then B is a dual unit ball of an equivalent norm on $X \times \mathbf{R}$. Further, $K \times \{1\}$ is a weak* closed weak* G_δ subset of B. If B were weakly Valdivia, K would have the same property by Proposition 3.2. So B is not weakly Valdivia. ∎

As a corollary we get the following general result on $C(K)$ spaces.

COROLLARY 4.5. *Let K be a compact space such that there is a closed subset $L \subset K$ with a dense set of (relatively) G_δ points which is not weakly Valdivia. Then there is an equivalent norm on $C(K)$ in which this space is not weakly* 1*-Plichko.*

Proof. By Theorem 3.6 the space $P(L)$ of probability measures on L is not weakly Valdivia. Now, $P(L)$ can be identified with a convex weak* compact subset of $P(K)$. Hence the result follows easily from Proposition 4.4. ∎

Another result is the following analogue of [3, Lemma 4(iii)].

PROPOSITION 4.6. *Let X be weakly* 1*-Plichko and the norm on X be Gâteaux smooth. Then X is weakly WLD.*

Proof. We follow the proof of [3, Lemma 4(iii)]. Let A be a dense weakly Corson countably compact subset of (B_{X^*}, w^*). If the norm on X is Gâteaux differentiable and $\xi \in S_{X^*}$ attains its norm at some point of S_X, then ξ is a weak* G_δ point of B_{X^*}, hence $\xi \in A$ by [6, Lemma 1.11]. By Bishop-Phelps theorem norm-attaining functionals are norm dense in S_{X^*}. As A is clearly closed to limits of sequences, $S_{X^*} \subset A$. Finally, by a corollary to Josefson-Nissenzweig theorem $B_{X^*} \subset A$. This completes the proof. ∎

In the following theorem we collect some interesting examples in which the previous methods can be applied.

THEOREM 4.7. *On the following Banach spaces X there is an equivalent norm in which X is not weakly* 1*-Plichko.*

(1) $X = C(K)$ *where K contains $[0, \omega_2]$. This is the case for example if K is a Valdivia compactum which is not "$\aleph_2$-Corson" (cf.* [6, Section 1.3]*).*
(2) $X = C(K)$ *where $K = [0,1]^\Gamma$ or $K = \{0,1\}^\Gamma$ for uncountable Γ.*
(3) $X = \ell_1(\Gamma)$ *for uncountable Γ.*

Proof. The assertion (1) follows immediately from Corollary 4.5 and Theorem 2.5.

The assertion (2) follows from Corollary 4.5 and the proof of Proposition 3.3.

Let us prove the assertion (3). Let K be the compact used in the proof of Proposition 3.3. Then K is not weakly Valdivia, has weight $\aleph_1$ and has a dense set of G_δ points. So the dual unit ball $(B_{C(K)^*}, w^*)$ is not weakly Valdivia by Theorem 3.6. The Banach space $C(K)$ has density $\aleph_1$, and so it is isometric to a quotient of $\ell_1(\Gamma)$ whenever Γ is uncountable. Hence $(\ell_1(\Gamma))^*$ contains a copy of $(B_{C(K)^*}, w^*)$ as a weak* closed convex symmetric subset. So, by Proposition 4.4, there is an equivalent norm on $\ell_1(\Gamma) \times \mathbf{R}$ in which this space is not weakly 1-Plichko. It remains to conclude using the obvious fact that $\ell_1(\Gamma) \times \mathbf{R}$ is isomorphic to $\ell_1(\Gamma)$. ∎

Another set of problems is that concerning isomorphisms of $C(K)$ spaces, see [6, Section 5.3]. Suppose that $C(K)$ and $C(L)$ are isomorphic. It is known that L is Corson with property (M) whenever K is, see [1], and that K can be Valdivia without L being Valdivia, see [6, Theorem 5.18] for a general result. It is still unknown whether L is Corson whenever K is, cf. [1, Problem on p. 218] or [6, Question 5.22]. Here we give a generalization of the mentioned result of [1].

PROPOSITION 4.8. *Let K be a weakly Corson compactum with property* (M). *If L is a compact space such that $C(L)$ is isomorphic to $C(K)$, then L is weakly Corson.*

Proof. If K is weakly Corson with property (M), then $C(K)$ is weakly WLD by Theorem 3.7. Hence $C(L)$ is weakly WLD by Proposition 4.2. Therefore L is weakly Corson by Lemma 2.2. ∎

We do not know whether, in the previous proposition, L has necessarily property (M). Neither do we know whether the assumption that K has property (M) can be dropped. As an immediate consequence of the above proposition (together with Theorem 2.5) we get the following

EXAMPLE 4.9. If $C(L)$ is isomorphic to $C[0, \alpha]$ for some $\alpha < \omega_2$, then L is weakly Valdivia.

This example is a partial positive answer to [6, Question 5.22], see Question 4 below.

5. Open problems

In this final section we collect some open questions in this area.

Question 1. Let K be a weakly Corson compact space. Does K have a dense set of G_δ points?

Notice that any Corson compact space has a dense set of G_δ points (see e.g. [6, Theorem 3.3]).

Question 2. Let K be a compact space such that each closed subset of K is weakly Valdivia. Is K weakly Corson?

The analogous question on Valdivia and Corson compacta has negative answer by [6, Theorem 3.7].

Question 3. Let X be a Banach space which is weakly 1-Plichko in any equivalent norm. Is X weakly WLD?

The previous section contains several partial positive answers.

Question 4. Suppose $C(K)$ is isomorphic to $C(L)$ and K is weakly Valdivia. Is L weakly Valdivia as well?

A partial positive answer is given by Example 4.9.

Question 5. Suppose $C(K)$ is weakly Plichko (or even weakly 1-Plichko). Is K weakly Valdivia?

Theorem 3.6 answers this question positively in case that $C(K)$ is weakly 1-Plichko and K has a dense set of G_δ points.

References

[1] Argyros, S., Mercourakis, S., Negrepontis, S., Functional analytic properties of Corson compact spaces, *Studia Math.,* **89** (1988), 197 – 229.
[2] Deville, R., Godefroy, G., Some applications of the projective resolutions of the identity, *Proc. London. Math. Soc.,* **67** (1) (1993), 183 – 199.
[3] Fabian, M., Godefroy, G., Zizler, V., A note on Asplund generated Banach spaces, *Bull. Polish Acad. Sci. Math.,* **47** (3) (1999), 221 – 230.
[4] Gul'ko, S.P., Properties of sets that lie in Σ-products, *Dokl. Akad. Nauk SSSR,* **237** (3) (1977), 505 – 508 (in Russian).

[5] Kalenda, O., Stegall compact spaces which are not fragmentable, *Topol. Appl.,* **96** (2) (1999), 121 – 132.
[6] Kalenda, O., Valdivia compact spaces in topology and Banach space theory, *Extracta Math.,* **15** (1) (2000), 1 – 85.
[7] Kalenda, O., A new Banach space with Valdivia dual unit ball, *Israel J. Math.,* to appear.
[8] Michael, E., Rudin, M.E., A note on Eberlein compacts, *Pacific J. Math.,* **72** (2) (1977), 487 – 495.
[9] Noble, N., The continuity of functions on Cartesian products, *Trans. Amer. Math. Soc.,* **149** (1970), 187 – 198.
[10] Valdivia, M., Resolution of the identity in certain Banach spaces, *Collect. Math.,* **39** (1988), 127 – 140.
[11] Valdivia, M., Projective resolutions of identity in $C(K)$ spaces, *Arch. Math.,* **54** (1990), 493 – 498.
[12] Valdivia, M., Simultaneous resolutions of the identity operator in normed spaces, *Collect. Math.,* **42** (2) (1991), 265 – 284.
[13] Valdivia, M., On certain compact topological spaces, *Revista Matemática de la Universidad Complutense de Madrid,* **10** (1) (1997), 81 – 84.