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Commutativity Criterions In Locally M-Convex Algebras

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1. Introduction

In this paper we define the notions of semicommutativity and semicommutativity modulo a linear subspace. We prove some results regarding the semicommutativity or semicommutativity modulo a linear subspace of a sequentially complete m-convex algebra. We show how can be applied such results in order to obtain commutativity criterions for locally m-convex algebras.

Let A be a complex m-convex algebra, whose topology is defined by a separating family $(p_\alpha)_{\alpha\in I}$ of submultiplicative seminorms.

The unitization of A over $\mathbb{C}$, denoted by A_1 is the m-convex algebra consisting of the set $\mathbb{C}\times A$ with addition, scalar multiplication and product defined (for all $x, y \in A$ and $\alpha, \beta \in \mathbb{C}$) by

$$(\alpha, x) + (\beta, y) = (\alpha+\beta, x+y)$$
$$\beta(\alpha, x) = (\beta\alpha, \beta x)$$
$$(\alpha, x)(\beta, y) = (\alpha\beta, xy + \alpha y + \beta x)$$

and with the seminorms $(q_\alpha)_{\alpha\in I}$, defined by

$$q_\alpha((\lambda, x)) = |\lambda| + p_\alpha(x)$$

for all $\alpha \in I$, $\lambda \in \mathbb{C}$ and $x \in A$; A_1 is an m-convex algebra with unit element $(1,0)$, $q_\alpha((1,0)) = 1$ for all $\alpha \in I$, and the mapping $a \to (0,a)$ is an isomorphism of A onto a subalgebra of A_1. It is a routine matter to verify that A_1

is sequentially complete when A is sequentially complete (i.e., every Cauchy sequence converges).

Recall that for an element x of an unital algebra A, the set

$$\sigma(x) = \{\lambda \in \mathbb{C} \,:\, \lambda 1 - x \notin G(A)\}$$

is called the spectrum of x, and

$$\rho(x) = \sup\{|\lambda| \,:\, \lambda \in \sigma(x)\}$$

is called the spectral radius of x, 1 being the unit element of A and $G(A)$ the set of all invertible elements of A.

If A_1 is a sequentially complete m-convex algebra, then (see [7])

$$\rho(y) = \sup_{\alpha \in I} \lim_{n\to\infty} q_\alpha(y^n)^{\frac{1}{n}}$$

for any $y \in A_1$. In particular, we have

$$\begin{aligned}\rho((0,x)) &= \sup_{\alpha \in I} \lim_{n\to\infty} q_\alpha((0,x)^n)^{\frac{1}{n}} \\ &= \sup_{\alpha \in I} \lim_{n\to\infty} q_\alpha((0,x^n))^{\frac{1}{n}} = \sup_{\alpha \in I} \lim_{n\to\infty} p_\alpha(x^n)^{\frac{1}{n}}\,.\end{aligned} \tag{1}$$

In the remainder of this paper we assume that A is a complex sequentially complete m-convex algebra with topology defined by a separating family $(p_\alpha)_{\alpha\in I}$ of submultiplicative seminorms.

Definition 1. A is said to be commutative iff $xy = yx$ for any $x, y \in A$, and A is said to be semicommutative iff $xyz = zxy$ for any $x, y, z \in A$. Given a linear subspace E of A, A is said to be commutative modulo E iff $xy - yx \in E$ for any $x, y \in A$, and A is said to be semicommutative modulo E iff $xyz - zxy \in E$ for any $x, y, z \in A$.

For example, in the set $\mathcal{M}_4(\mathbb{C})$ of all square matrices with four columns and complex elements, we consider the subset

$$A = \{X \in \mathcal{M}_4(\mathbb{C}) \,:\, X = \alpha M + \beta N + \gamma MN, \alpha, \beta, \gamma \in \mathbb{C}\}$$

where

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

If we define on A the family consisting of the submultiplicative seminorm

$$p(X) = |\alpha| + |\beta| + |\gamma|,$$

A become an m-convex algebra. This m-convex algebra is semicommutative but not commutative: on the one hand, $NM = 0$ (the null matrix) and

$$MN = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

on the other hand, using the fact that $MNM = NMN = 0$ it is a routine matter to verify that $XYZ = ZXY = 0$ for any $X, Y, Z \in A$, and so we obtain the semicommutativity of A.

Definition 2. If A has unit element, the radical of A, denoted by $\operatorname{Rad} A$ is the set

$$\operatorname{Rad} A = \{x \in A : 1 - xy \in G(A) \text{ for any } y \in A\}.$$

If A hasn't unit element then we define the radical of A as

$$\operatorname{Rad} A = \{x \in A : (0, x) \in \operatorname{Rad} A_1\}.$$

2. Some commutativity criterions

Theorem 1. *If A is semicommutative then A is commutative modulo $\operatorname{Rad} A$.*

Proof. We have to prove that $(0, xy - yx) \in \operatorname{Rad} A_1$ for any $x, y \in A$. We prove first that

$$(xy)^n = x^n y^n \tag{2}$$

for any $n \in \mathbb{N}^*$ and $x, y \in A$. Indeed, for $n = 2$, from semicommutativity we have

$$(xy)^2 = (xy)(xy) = ((xy)x)y = x(yx)y = xy^2x = x^2y^2$$

for any $x, y \in A$, and by induction we obtain the equality (2).

Let now $x, y \in A$. For $\alpha \in I$ and $n \in \mathbb{N}^*$, we have

$$p_\alpha((xy)^n)^{\frac{1}{n}} = p_\alpha(x^n y^n)^{\frac{1}{n}} \leq p_\alpha(x^n)^{\frac{1}{n}} p_\alpha(y^n)^{\frac{1}{n}},$$

and by (1) we obtain

$$\rho((0,x)(0,y)) \leq \rho((0,x))\rho((0,y))\,. \tag{3}$$

Using again the semicommutativity of A, we find that $(xy-yx)^2 = 0$ for any $x, y \in A$. This implies that

$$\rho((0, xy-yx)) = \sup_{\alpha\in I} \lim_{n\to\infty} p_\alpha((xy-yx)^n)^{\frac{1}{n}} = 0 \tag{4}$$

for any $x, y \in A$. From (3) and (4) it follows that

$$\rho((0, xy-yx)(0,z)) = 0 \tag{5}$$

for any $x, y, z \in A$.

We prove now that for $u \in A$ with the properties

$$\rho(0,u) = 0 \qquad \text{and} \qquad \rho((0,u)(0,z)) = 0 \quad \text{for any} \quad z \in A,$$

the equality

$$\rho((0,u)(\lambda, v)) = 0 \tag{6}$$

hold for any $(\lambda, v) \in A_1$. Indeed, using the semicommutativity of A we obtain that $(0, \lambda u)$ and $(0, uv)$ are permutable elements of A_1. It is a well known fact that for permutable elements the spectral radius is submultiplicative so we have

$$\rho((0,u)(\lambda,v)) = \rho((0, \lambda u + uv)) \leq |\lambda|\rho((0,u)) + \rho((0,u)(0,v)) = 0$$

for any $(\lambda, v) \in A_1$.

From (4), (5) and (6) it follows that $\rho((0, xy-yx)t) = 0$ for any $x, y \in A$ and $t \in A_1$. We deduce that $1 \notin \sigma((0, xy-yx)t)$ for any $x, y \in A$ and $t \in A_1$. So $(1,0)-(0,xy-yx)t \in G(A_1)$ for any $x, y \in A$ and $t \in A_1$, and consequently

$$(0, xy-yx) \in \operatorname{Rad} A_1$$

for any $x, y \in A$, and this completes the proof. ∎

Definition 3. If $(q_\alpha)_{\alpha\in I}$ is a family of seminorms on A, the kernel of family $(q_\alpha)_{\alpha\in I}$, denoted by $\operatorname{Ker}((q_\alpha)_{\alpha\in I})$, is the set

$$\operatorname{Ker}((q_\alpha)_{\alpha\in I}) = \bigcap_{\alpha\in I} \operatorname{Ker}(q_\alpha)\,.$$

Definition 4. If p, q are seminorms on A, q is said to be p-continuous if there exists $k > 0$ such that

$$q(x) \leq kp(x) \qquad \text{for any } \; x \in A\,.$$

Theorem 2. *If $(q_\alpha)_{\alpha \in I}$ is a family of submultiplicative seminorms on A such that, for any $\alpha \in I$, q_α is a p_α-continuous seminorm and there exists $k_\alpha > 0$ such that*

$$q_\alpha(xy) \leq k_\alpha p_\alpha(yx)$$

for any $x, y \in A$, then A is semicommutative modulo $\mathrm{Ker}\,((q_\alpha)_{\alpha \in I})$.

Proof. Let $x, y \in A$, $z \in A_1$, and $f : \mathbb{C} \to A$ given by

$$f(\lambda) = \exp(\lambda z) xy \exp(-\lambda z)$$

for any $\lambda \in \mathbb{C}$, where

$$\exp(\lambda z) = (1, 0) + \sum_{n=1}^{\infty} \frac{1}{n!} \lambda^n z^n\,.$$

The function f is well defined, because A is an ideal of A_1. We consider the linear space $A/_{\mathrm{Ker}((q_\alpha)_{\alpha \in I})}$ and the family of seminorms $(p'_\alpha)_{\alpha \in I}$ given by

$$p'_\alpha(\widehat{x}) = q_\alpha(x)$$

for any $\widehat{x} \in A/_{\mathrm{Ker}((q_\alpha)_{\alpha \in I})}$ and $\alpha \in I$. We immediately obtain that the seminorms p'_α are well defined and that $(p'_\alpha)_{\alpha \in I}$ is a separating family. So the linear space $A/_{\mathrm{Ker}((q_\alpha)_{\alpha \in I})}$ endowed with the family $(p'_\alpha)_{\alpha \in I}$ is a locally convex space. We will denote this space by $\widetilde{A}$.

Let now $\widetilde{f} : \mathbb{C} \to \widetilde{A}$ defined by $\widetilde{f}(\lambda) = \widehat{f(\lambda)}$ for any $\lambda \in \mathbb{C}$. The function $\widetilde{f}$ is differentiable on $\mathbb{C}$. Indeed, for any $\alpha \in I$, from the fact that q_α is a p_α-continuous seminorm, we get the existence of a constant β_α such that

$$\begin{aligned} p'_\alpha\left(\frac{\widetilde{f}(\lambda) - \widetilde{f}(\lambda_0)}{\lambda - \lambda_0} - \widehat{f'(\lambda_0)}\right) &= p'_\alpha\left(\widehat{\frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} - f'(\lambda_0)}\right) \\ &= q_\alpha\left(\frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} - f'(\lambda_0)\right) \\ &\leq \beta_\alpha p_\alpha\left(\frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} - f'(\lambda_0)\right) \end{aligned}$$

because f is a differentiable function as a product of differentiable functions. It follows that $\widetilde{f}$ is diferentiable on $\mathbb{C}$ and

$$\left(\widetilde{f}\right)'(\lambda) = \widehat{f'(\lambda)} \qquad \text{for any} \quad \lambda \in \mathbb{C}\,.$$

Let $\alpha \in I$ and $\lambda \in \mathbb{C}$. We have

$$\begin{aligned} p'_\alpha\left(\widetilde{f}(\lambda)\right) &= p'_\alpha\left(\widehat{f(\lambda)}\right) = q_\alpha(f(\lambda)) \\ &= q_\alpha(\exp(\lambda z)xy\exp(-\lambda z)) \le k_\alpha p_\alpha(yz)\,. \end{aligned}$$

So $\widetilde{f}$ is differentiable and bounded on $\mathbb{C}$ and using Liouville Theorem we get that $\widetilde{f}$ is a constant function. This implies that $\left(\widetilde{f}\right)'(\lambda) = \widehat{0}$ for any $\lambda \in \mathbb{C}$. We have

$$\left(\widetilde{f}\right)'(\lambda) = \widehat{f'(\lambda)} = z\exp(\lambda z)xy\exp(-\lambda \widehat{z) - \exp}(\lambda z)xy\exp(-\lambda z)z$$

for any $\lambda \in \mathbb{C}$. For $\lambda = 0$ we obtain $\widehat{0} = zx\widehat{y - x}yz$. So $zxy - xyz \in \operatorname{Ker}((q_\alpha)_{\alpha\in I})$ for any $x, y, z \in A$ and this completes the proof. ∎

Corollary 1. *If there exists a separating family $(q_\alpha)_{\alpha\in I}$ of submultiplicative seminorms on A with the properties that for any $\alpha \in I$, q_α is a p_α-continuous seminorm and there exists $k_\alpha > 0$ such that*

$$q_\alpha(xy) \le k_\alpha p_\alpha(yx) \qquad \text{for any} \quad x, y \in A\,,$$

then A is semicommutative.

Proof. From the fact that $(q_\alpha)_{\alpha\in I}$ is a separating family, we have

$$\operatorname{Ker}((q_\alpha)_{\alpha\in I}) = \bigcap_{\alpha\in I} \operatorname{Ker}(q_\alpha) = \{0\}$$

and now we use Theorem 2. ∎

Corollary 2. *If A has unit element and for any $\alpha \in I$ there exists $k_\alpha > 0$ such that*

$$p_\alpha(xy) \le k_\alpha p_\alpha(yx) \qquad \text{for any} \quad x, y \in A\,,$$

then A is commutative.

Remark. If the conditions of Corollary 1 or Corollary 2 are satisfied then A is commutative modulo $\operatorname{Rad} A$. In addition, if A has unit element, then A is commutative.

Now we consider that A has unit element. We denote by $S = S(A)$ the set of all states on A, i.e., the set of all continuous functionals s on A with the properties that $s(1) = 1$ and there exists $\alpha \in I$ such that

$$|s(x)| \leq p_\alpha(x) \qquad \text{for any} \quad x \in A\,.$$

Recall that, for an element $x \in A$, the set

$$V(x) = \{s(x) \,:\, s \in S\}$$

is called the numerical range of x, and

$$v(x) = \sup\{|s(x)| \,:\, s \in S\}$$

is called the numerical radius of x. We recall the generalization of Bohnenblust and Karlin theorem for m-convex algebras (see [3], [4]). Let A be an unital m-convex algebra and $x \in A$. Then

$$\frac{1}{e} \sup_{\alpha \in I} p_\alpha(x) \leq v(x) \leq \sup_{\alpha \in I} p_\alpha(x)\,.$$

COROLLARY 3. *If A has unit element and for any $\alpha \in I$ there exists $k_\alpha > 0$ such that*

$$v(xy) \leq k_\alpha p_\alpha(yx) \qquad \text{for any} \quad x, y \in A\,,$$

then A is commutative.

Proof. From the generalization of Bohnenblust and Karlin theorem we have

$$\frac{1}{e} \sup_{\alpha \in I} p_\alpha(xy) \leq v(xy) \qquad \text{for any} \quad x, y \in A\,.$$

Now using Corollary 2 it follows that A is a commutative algebra. ∎

THEOREM 3. *If A has unit element and for any $\alpha \in I$ there exists $k_\alpha > 0$ such that*

$$p_\alpha(x)^2 \leq k_\alpha p_\alpha(x^2) \qquad \text{for any} \quad x \in A\,,$$

then A is commutative.

Proof. Let $\alpha \in I$ and $x \in A$. An induction argument lead us to

$$p_\alpha(x) \le k_\alpha^{1-\frac{1}{2^n}} (p_\alpha(x^{2^n}))^{\frac{1}{2^n}} .$$

We denote

$$\rho_\alpha(x) = \lim_{n\to\infty} (p_\alpha(x^{2^n}))^{\frac{1}{2^n}} .$$

Letting $n \to \infty$, we obtain $p_\alpha(x) \le k_\alpha \rho_\alpha(x)$. As it is known, $\rho_\alpha(xy) = \rho_\alpha(yx)$ for any $x, y \in A$ (see [7]). So

$$p_\alpha(xy) \le k_\alpha \rho_\alpha(xy) = k_\alpha \rho_\alpha(yx) \le k_\alpha p_\alpha(yx)$$

for any $x, y \in A$, and using Corollary 2 we obtain that A is commutative. ∎

COROLLARY 4. (See [2]) *Let A be a complex Banach algebra with unit such that, for some $k > 0$,*

$$\|xy\| \le k\|yx\| \qquad \text{for any } \ x, y \in A .$$

Then A is commutative.

THEOREM 4. *If A has unit element and for any $\alpha \in I$ there exists $k_\alpha > 0$ such that*

$$p_\alpha(x) \le k_\alpha \rho(x) \qquad \text{for any } \ x \in A ,$$

then A is commutative.

Proof. Let $x, y \in A$ and $f : \mathbb{C} \to A$ given by

$$f(\lambda) = \exp(\lambda x) y \exp(-\lambda x)$$

for any $\lambda \in \mathbb{C}$. For any $\alpha \in I$ and $\lambda \in \mathbb{C}$, we have

$$p_\alpha(f(\lambda)) = p_\alpha(\exp(\lambda x) y \exp(-\lambda x)) \le k_\alpha \rho(\exp(\lambda x) y \exp(-\lambda x)) = k_\alpha \rho(y)$$

because the spectral radius has the property $\rho(xy) = \rho(yx)$ for any $x, y \in A$ (see [7]). So f is a bounded and differentiable function on $\mathbb{C}$ and using Liouville theorem we obtain that f is a constant function. This implies that $f'(\lambda) = 0$ for any $\lambda \in \mathbb{C}$ and consequently

$$x \exp(\lambda x) y \exp(-\lambda x) - \exp(\lambda x) y x \exp(-\lambda x) = 0 \qquad \text{for any } \ \lambda \in \mathbb{C} .$$

For $\lambda = 0$ we have $xy - yx = 0$, and this completes the proof. ∎

COROLLARY 5. (See [1] and [2]) *Let A be a complex Banach algebra with unit such that, for some $k > 0$,*

$$\|x\| \le k\rho(x) \qquad \text{for any } \ x \in A .$$

Then A is commutative.

References

[1] Aupetit, B., "Propriétés Spectrales des Algèbres de Banach", Lecture Notes Math. 735, Springer, Berlin, 1979.

[2] Bonsall, F.F., Duncan, J., "Complete Normed Algebras", New York, Springer-Verlag, 1973.

[3] Bonsall, F,F., Duncan, J., "Numerical Ranges of Operators and of Elements of Normed Algebras", LMS Lecture Note Series 2, Cambridge University Press, London, 1971.

[4] Giles, J.R., Koehler, D.O., On numerical ranges of elements of locally m-convex algebras, *Pacific J. Math.* **49** (1973), 79 – 91.

[5] Mallios, A., "Topological Algebras, Selected Topics", North-Holland Math. Studies 124, Amsterdam, 1986.

[6] Mocanu, Gh., On numerical radius of an element of a normed algebra, *Glasgow Math. J.* **15** (1974), 90 – 92.

[7] Źelazko, W., "Selected Topics in Topological Algebras", Lecture Note Series No. 31, Aarhus Universitet, Aarhus, 1971.