EXTRACTA MATHEMATICAE doi:10.17398/2605-5686.38.1.1
Vol. **38**, Num. 1 (2023), 1 – 15 Available online November 23, 2022

Topologies, posets and finite quandles

M. Elhamdadi [1], T. Gona [2], H. Lahrani [1]

[1] *Department of Mathematics and Statistics, University of South Florida Tampa, FL* 33620*, U.S.A.*

[2] *Department of Mathematics, University of California Berkeley, CA* 94720*, U.S.A.*

emohamed@math.usf.edu , gonatushar@berkeley.edu , lahrani@usf.edu

Received September 27, 2022 Presented by M. Mbekhta
Accepted October 31, 2022

Abstract: An Alexandroff space is a topological space in which every intersection of open sets is open. There is one to one correspondence between Alexandroff T_0-spaces and *partially ordered sets* (posets). We investigate Alexandroff T_0-topologies on finite quandles. We prove that there is a non-trivial topology on a finite quandle making right multiplications continuous functions if and only if the quandle has more than one orbit. Furthermore, we show that right continuous posets on quandles with n orbits are n-partite. We also find, for the even dihedral quandles, the number of all possible topologies making the right multiplications continuous. Some explicit computations for quandles of cardinality up to *five* are given.

Key words: quandles, topology, poset.

MSC (2020): 54E99, 57K12.

1. Introduction

Quandles are algebraic structures modeled on the *three* Reidemeister moves in classical knot theory. They have been used extensively to construct invariants of knots and links, see for example [6, 8, 10]. A Topological quandle is a quandle with a topology such that the quandle binary operation is compatible with the topology. Precisely, the binary operation is continuous and the right multiplications are *homeomorphisms*. Topological quandles were introduced in [11] where it was shown that the set of homomorphisms (called also the set of colorings) from the fundamental quandle of the knot to a topological quandle is an invariant of the knot. Equipped with the compact-open topology, the set of colorings is a topological space. In [5] a foundational account about topological quandles was given. More precisely, the notions of ideals, kernels, units, and inner automorphism group in the context of topological quandle were introduced. Furthermore, modules and quandle group bundles

ISSN: 0213-8743 (print), 2605-5686 (online)

over topological quandles were introduced with the purpose of studying central extensions of topological quandles. Continuous cohomology of topological quandles was introduced in [4] and compared to the algebraic theories. Extensions of topological quandles were studied with respect to continuous 2-cocycles, and used to show differences in second cohomology groups for some specific topological quandles. Nontriviality of continuous cohomology groups for some examples of topological quandles was shown. In [2] the problem of classification of topological Alexander quandle structures, up to isomorphism, on the real line and on the unit circle was investigated. In [7] the author investigated quandle objects internal to groups and topological spaces, extending the well-known classification of quandles internal to abelian groups [13]. In [14] quandle modules over quandles endowed with geometric structures were studied. The author also gave an infinitesimal description of certain modules in the case when the quandle is a regular s-manifold (smooth quandle with certain properties). Since any finite T_1-space is discrete, the category of finite T_0-spaces was considered in [12], where the point set topological properties of finite spaces were investigated. The homeomorphism classification of finite spaces was investigated and some representations of these spaces as certain classes of matrices was obtained.

This article arose from a desire to better understand the analogy of the work given in [12] in the context of *finite topological* quandles. It turned out that: there is no T_0-topology on any finite connected (meaning one orbit under the action of the Inner group) quandle X that makes X into a topological quandle (Theorem 4.4). Thus we were lead to consider topologies on finite quandles with more than *one* orbit. It is well known [1] that the category of Alexandroff T_0-spaces is equivalent to the category of *partially ordered sets* (posets). In our context, we prove that for a finite quandle X with more than one orbit, there exists a unique non trivial topology which makes right multiplications of X continuous maps (Proposition 4.6). Furthermore, we prove that if X be a finite quandle with two orbits X_1 and X_2 then any continuous poset on X is biparatite with vertex set X_1 and X_2 (Proposition 4.7). This article is organized as follows. In Section 2 we review the basics of topological quandles. Section 3 reviews some basics of posets, graphs and some hierarchy of separation axioms. In Section 4 the main results of the article are given. Section 5 gives some explicit computations based on some computer software (Maple and Python) of quandles up to order *five*.

2. Review of Quandles and Topological Quandles

A quandle is a set X with a binary operation $*$ satisfying the following three axioms:

(1) For all x in X, $x * x = x$,

(2) For all $y, z \in X$, there exists a unique x such that $x * y = z$,

(3) For all $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

These three conditions come from the axiomatization of the three Reidemeister moves on knot diagrams. The typical examples of quandles are: (i) Any Group G with conjugation $x*y = y^{-1}xy$, is a quandle called the *conjugation quandle* and (ii) Any group G with operation given by $x * y = yx^{-1}y$, is a quandle called the *core quandle*.

Let X be a quandle. For an element $y \in X$, left multiplication L_y and right multiplication R_y by an element y are the maps from X to X given respectively by $L_y(x) := y*x$ and $R_y(x) = x*y$. A function $f : (X, *) \to (X, *)$ is a quandle *homomorphism* if for all $x, y \in X, f(x*y) = f(x)*f(y)$. If furthermore f is a bijection then it is called an *automorphism* of the quandle X. We will denote by Aut(X) the automorphism group of X. The subgroup of Aut(X), generated by the automorphisms R_x, is called the *inner* automorphism group of X and denoted by $\mathrm{Inn}(X)$. If the group $\mathrm{Inn}(X)$ acts *transitively* on X, we then say that X is connected quandle meaning it has only one orbit. Since we do not consider topological connectedness in this article, then through the whole article, the word connected quandle will stand for algebraic connectedness. For more on quandles refer to [6, 8, 10, 3]. Topological quandles have been investigated in [2, 5, 11, 4]. Here we review some basics of topological quandles.

Definition 2.1. A *topological quandle* is a quandle X with a topology such that the map $X \times X \ni (x, y) \longmapsto x * y \in X$ is a continuous, the right multiplication $R_x : X \ni y \longmapsto y * x \in X$ is a homeomorphism, for all $x \in X$, and $x * x = x$.

It is clear that any finite quandle is automatically a topological quandle with respect to the discrete topology.

Example 2.2. [2] Let $(G, +)$ be a topological abelian group and let σ be a continuous automorphism of G. The continuous binary operation on G given by $x * y = \sigma(x) + (Id - \sigma)(y), \forall x, y \in G$, makes $(G, *)$ a topological quandle called *topological Alexander quandle.* In particular, if $G = \mathbb{R}$ and $\sigma(x) = tx$

for non-zero $t \in \mathbb{R}$, we have the topological Alexander structure on $\mathbb{R}$ given by $x * y = tx + (1-t)y$.

Example 2.3. The following examples were given in [11, 5]. The unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ with the binary operation $x*y = 2(x \cdot y)y - x$ is a topological quandle, where $\cdot$ denotes the inner product of $\mathbb{R}^{n+1}$. Now consider λ and μ be real numbers, and let $x, y \in S^n$. Then

$$\lambda x * \mu y = \lambda[2\mu^2(x \cdot y)y - x].$$

In particular, the operation

$$\pm x * \pm y = \pm(x * y)$$

provides a structure of topological quandle on the quotient space that is the projective space $\mathbb{RP}^n$.

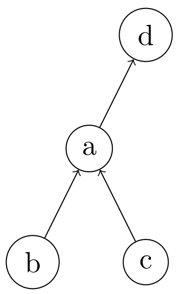
3. Review of topologies on finite sets, posets and graphs

Now we review some basics of directed graphs, posets and T_0 and T_1 topologies.

Definition 3.1. A *directed* graph G is a pair (V, E) where V is the set of vertices and E is a list of directed line segments called edges between pairs of vertices.

An edge from a vertex x to a vertex y will be denoted symbolically by $x < y$ and we will say that x and y are *adjacent*. The following is an example of a directed graph.

Example 3.2. Let $G = (V, E)$ where $V = \{a, b, c, d\}$ and $E = \{b < a, c < a, a < d\}$.

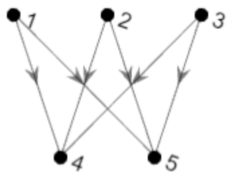


Definition 3.3. An *independent set* in a graph is a set of pairwise non-adjacent vertices.

Definition 3.4. A (directed) graph $G = (V, E)$ is called biparatite if V is the union of two disjoint independent sets V_1 and V_2.

Definition 3.5. A (directed) graph G is called *complete biparatite* if G is bipartite and for every $v_1 \in V_1$ and $v_2 \in V_2$ there is an edges in G that joins v_1 and v_2.

Example 3.6. Let $V = V_1 \cup V_2$ where $V_1 = \{4, 5\}$ and $V_2 = \{1, 2, 3\}$. Then the directed graph $G = (V, E)$ is complete biparatite graph.



Now we recall the definition of partially ordered set.

Definition 3.7. A partially ordered set (poset) is a set X with an order denoted $\leq$ that is reflexive, antisymmetric and transitive.

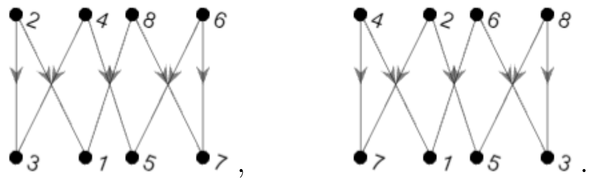
Example 3.8. For any set X, the power set of X ordered by the set inclusion relation $\subseteq$ forms a poset $(\mathcal{P}(X), \subseteq)$

Definition 3.9. Two partially ordered sets $P = (X, \leq)$ and $Q = (X, \leq')$ are said to be isomorphic if there exist a bijection $f : X \to X'$ such that $x \leq y$ if and only if $f(x) \leq' f(y)$.

Definition 3.10. A poset $(X, \leq)$ is connected if for all $x, y \in X$, there exists sequence of elements $x = x_1, x_2, \ldots, x_n = y$ such that every two consecutive elements x_i and x_{i+1} are comparable (meaning $x_i < x_{i+1}$ or $x_{i+1} < x_i$).

Notation: Given an order $\leq$ on a set X, we will denote $x < y$ whenever $x \neq y$ and $x \leq y$. Finite posets $(X, \leq)$ can be drawn as directed graphs where the vertex set is X and an arrow goes from x to y whenever $x \leq y$. For simplicity, we will not draw loops which correspond to $x \leq x$. We will then use the notation $(X, <)$ instead of $(X, \leq)$ whenever we want to ignore the reflexivity of the partial order.

Example 3.11. . Let $X = \mathbb{Z}_8$ be the set of integers modulo 8. The map $f : X \to X$ given by $f(x) = 3x - 2$ induces an isomorphism between the following two posets $(X, <)$ and $(X, <')$.



Definition 3.12. A chain in a poset $(X, <)$ is a subset C of X such that the restriction of $<$ to C is a total order (i.e. every two elements are comparable).

Now we recall some basics about topological spaces called T_0 and T_1 spaces.

Definition 3.13. A topological space X is said to have the property T_0 if for every pair of distinct points of X, at least one of them has a neighborhood not containing the other point.

Definition 3.14. A topological space X is said to have the property T_1 if for every pair of distinct points of X, each point has a neighborhood not containing the other point.

Obviously the property T_1 implies the property T_0. Notice also that this definition is equivalent to saying singletons are closed in X. Thus a T_1-topology on a *finite* set is a discrete topology.

Since any finite T_1-space is discrete, we will focus on the category of finite T_0-spaces. First we need some notations.

Let X be a finite topological space. For any $x \in X$, we denote

$$U_x := \text{the smallest open subset of } X \text{ containing } x.$$

It is well known [1] that the category of T_0-spaces is isomorphic to the category of posets. We have $x \leq y$ if and only if $U_y \subseteq U_x$ which is equivalent to $C_x \subset C_y$, where C_v is the complement U_v^c of U_v in X. Thus one obtain that $U_x = \{w \in X;\ x \leq w\}$ and $C_x = \{v \in X;\ v < x\}$. Under this correspondence of categories, the subcategory of finite posets is equivalent to the category of finite T_0-spaces.

Through the rest of this article we will use the notation of $x < y$ in the poset whenever $x \neq y$ and $x \leq y$.

4. Topologies on non-connected quandles

As we mentioned earlier, since T_1-topologies on a finite set are discrete, we will focus in this article on T_0-topologies on *finite quandles*. A map on finite spaces is continuous if and only if it preserves the order. It turned out that on a finite quandle with a T_0-topology, left multiplications can not be continuous as can be seen in the following theorem

Theorem 4.1. *Let X be a finite quandle endowed with a T_0-topology. Assume that for all $z \in X$, the map L_z is continuous, then $x \leq y$ implies $L_z(x) = L_z(y)$.*

Proof. We prove this theorem by contradiction. Let X be a finite quandle endowed with a T_0-topology. Assume that $x \leq y$ and $L_z(x) \neq L_z(y)$. If $x = y$, then obviously $L_z(x) = L_z(y)$. Now assume $x < y$, then for all $a \in X$, the continuity of L_a implies that $a * x \leq a * y$. Assume that there exist $a_1 \in X$ such that, $z_1 := a_1 * x = L_{a_1}(x) < a_1 * y = L_{a_1}(y)$. The invertibility of right multiplications in a quandle implies that there exist unique a_2 such that $a_2 * x = a_1 * y$ hence $a_1 * x < a_2 * x$ which implies $a_1 \neq a_2$. Now we have $a_1 * x < a_2 * x \leq a_2 * y = z_2$. We claim that $a_2 * x < a_2 * y$. if $a_2 * y = a_2 * x$ and since $a_2 * x = a_1 * y$ we will have $a_2 * y = a_2 * x = a_1 * y$ hence $a_2 * y = a_1 * y$ but $a_1 \neq a_2$, thus contradiction. Now that we have proved $a_2 * x < a_2 * y$, then there exists a_3 such that $a_2 * y = a_3 * x$ we get, $a_2 * x < a_3 * x$ repeating the above argument we get, $a_3 * x < a_3 * y$. Notice that a_1, a_2 and a_3 are all pairwise disjoint elements of X. Similarly, we construct an *infinite* chain, $a_1 * x < a_2 * x < a_3 * x < \cdots$, which is impossible since X is a finite quandle. Thus we obtain a contradiction. ∎

We have the following Corollary

Corollary 4.2. *Let X be a finite quandle endowed with a T_0-topology. If C is a chain of X as a poset then any left continuous function L_x on X is a constant function on C.*

Definition 4.3. A quandle with a topology in which right multiplications (respectively left multiplications) are continuous is called *right topological quandle* (respectively *left topological quandle*).

In other words, right topological quandle means that for all $x, y, z \in X$,

$$x < y \quad \Rightarrow \quad x * z < y * z.$$

and, since left multiplications are not necessarily bijective maps, left topological quandle means that for all $x, y, z \in X$,

$$x < y \quad \Rightarrow \quad z * x \leq z * y.$$

THEOREM 4.4. *There is no T_0-topology on a finite connected quandle X that makes X into a right topological quandle.*

Proof. Let $x < y$. Since X is connected quandle, there exists $\phi \in \mathrm{Inn}(X)$ such that $y = \phi(x)$. Since X is finite, ϕ has a finite order m in the group $\mathrm{Inn}(X)$. Since ϕ is a continuous automorphism then $x < \phi(x)$ implies $x < \phi^m(x)$ giving a contradiction. ∎

COROLLARY 4.5. *There is no T_0-topology on any latin quandle that makes it into a right topological quandle.*

Thus Theorem 4.4 leads us to consider quandles X that are not connected, that is $X = X_1 \cup X_2 \cup \ldots X_k$ as orbit decomposition, search for T_0-topology on X and investigate the continuity of the binary operation.

PROPOSITION 4.6. *Let X be a finite quandle with orbit decomposition $X = X_1 \cup \{a\}$, then there exist unique non trivial T_0-topology which makes X right continuous.*

Proof. Let $X = X_1 \cup \{a\}$ be the orbit decomposition of the quandle X. For any $x, y \in X_1$, there exits $\phi \in \mathrm{Inn}(X)$ such that $\phi(x) = y$ and $\phi(a) = a$. Declare that $x < a$, then $\phi(x) < a$. Thus for any $z \in X_1$ we have $z < a$. Uniqueness is obvious. ∎

The T_0-topology in Proposition 4.6 is precisely given by $x < a$ for all $x \in X_1$.

PROPOSITION 4.7. *Let X be a finite quandle with two orbits X_1 and X_2. Then any right continuous poset on X is biparatite with vertex set X_1 and X_2.*

Proof. We prove this proposition by contradiction. For every $x_1, y_1 \in X_1$ such that $x_1 < y_1$. We know that there exist $\phi \in \mathrm{Inn}(X)$ such that $\phi(x_1) = y_1$. Hence, $x_1 < \phi(x_1)$ implies $x_1 < \phi^m(x_1) = x_1$, where m is the order of ϕ in $\mathrm{Inn}(X)$. Thus we have a contradiction. ∎

PROPOSITION 4.8. *Let X be a finite quandle with two orbits X_1 and X_2. Then the complete bipartite graph with vertex set X_1 and X_2 forms a right continuous poset.*

Proof. Let X be a finite quandle with two orbits X_1 and X_2. If $x \in X_1$ and $y \in X_2$ then for every $\phi \in \mathrm{Inn}(X)$ we have $\phi(x) \in X_1$ and $\phi(y) \in X_2$. Proposition 4.7 gives that the graph is bipartite and thus $x < y$. We then obtain $\phi(x) < \phi(y)$ giving the result. ∎

Remark 4.9. By Proposition 4.8 and Theorem 4.1, there is a non-trivial T_0-topology making X right continuous if and only if the quandle has more than one orbit.

Notice that Proposition 4.8 can be generalized to n-paratite complete graph.

The following table gives the list of right continuous posets on some even dihedral quandles. In the table, the notation (a, b) on the right column means $a < b$.

Table 1: Right continuous posets on dihedral quandles

Quandle	Posets
R_4	((0,1),(2,1),(0,3),(2,3)) .
R_6	((0, 1), (0, 5), (2, 1), (2, 3), (4, 3), (4, 5)) ; ((0,3), (2, 5), (4, 1)) .
R_8	(2, 7), (4, 7), (6, 1), (6, 3), (0, 5), (2, 5), (4, 1), (0, 3)) ; ((0, 1), (6, 7), (4, 5), (0, 7), (2, 1), (2,3), (4, 3), (6, 5)) .
R_{10}	((0, 1), (6, 7), (4, 5), (2, 1), (8, 9), (2, 3), (4, 3), (8, 7), (0, 9), (6, 5)) ; ((4, 7), (6, 9), (2, 9), (8, 1), (8, 5), (0, 7), (6, 3), (2, 5), (4, 1), (0, 3)) ; ((2, 7), (8, 3), (0, 5), (4, 9), (6, 1)) .

Notice that in Table 1, the dihedral quandle R_4 has only one right continuous poset $((0,1),(2,1),(0,3),(2,3))$ which is complete biparatite. While the dihedral quandle R_6 has two continuous posets $((0,1),(0,5),(2,1),(2,3),(4,3),(4,5))$ and $((0,3),(2,5),(4,1))$ illustrated below.

Moreover, in Table 1, for R_8 the bijection f given by $f(k) = 3k-2$ makes the two posets isomorphic. The same bijection gives isomorphism between the first two posets of R_{10}. The following Theorem characterizes non complete biparatite posets on dihedral quandles.

Theorem 4.10. *Let R_{2n} be a dihedral quandle of even order. Then R_{2n} has $s+1$ right continuous posets, where s is number of odd natural numbers less than n and relatively non coprime with n*

Proof. Let $X = R_{2n}$ be the dihedral quandle with orbits $X_1 = \{0, 2, \ldots, 2n-2\}$ and $X_2 = \{1, 3, \ldots, 2n-1\}$. For every $x \in X_2$, we construct a partial order $<_x$ on R_{2n}, such that for all $y \in X$, we have $2y <_x 2y - x$ and $2y <_x 2y + x$. Then $<_x$ is clearly right continuous partial order since $2y < 2y - x$ and $2y < 2y + x$ for all y imply that $2z - 2y < 2z - (2y - x)$. In other words we obtain $2y * z < (2y - x) * z$. From the definition of the order $<_x$ it is clear the two partial orders $<_x$ and $<_{2n-x}$ are the same. Hence we obtain the following distinct partial orders $<_1, <_3, \ldots$. Now we check which ones are isomorphic. If m is odd and $gcd(n, m) = 1$ then $f(k) = mk - 2$ is a bijective function making $<_1$ and $<_m$ isomorphic. Now let m be odd and $gcd(m, n) = k > 1$. The two posets $<_1$ and $<_m$ are non isomorphic since $<_1$ is connected poset, as in Definition 3.10, and $<_m$ is not connected poset. We show that these are the only right continuous posets. Given a right continuous poset on R_{2n} then $a < b$ can be written as $a < a - (a - b)$ which implies that $a <_x b$ where $x = a - b$. Now if $a < b$ then by Proposition 4.7, we have $a \in X_1$, $b \in X_2$. Now let $a = 2\alpha$ and $b = 2\beta + 1$ then $a - b = 2(\alpha - \beta) - 1 \in X_2$. This ends the proof. ∎

Corollary 4.11. *For the dihedral quandle R_{2^n} with 2^n elements, there is a unique right continuous poset.*

5. Some computer calculations

In this section we give non-trivial right and left continuous posets on the finite quandles of order up to 5 based on Maple [9] and Python computations.

In the following tables we have excluded the trivial and connected quandles.

Table 2: Continuous posets on quandles of order 3

Quandle for $n = 3$	Right continuous posets	Left continuous poset
$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix}$	((0,2),(1,2)) .	((0,1)) .

As seen in Table 2 for $n = 3$, there exist a unique right continuous poset and a unique left continuous poset.

Table 3: Continuous posets on quandles of order 4

Quandles for $n = 4$	Right continuous poset	Left continuous poset
$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 3 \end{bmatrix}$	((0, 3)) ; ((0, 1), (0, 2), (0, 3)) ; ((0, 1), (0, 3), (1, 2)) ; ((0, 1), (0, 2), (1, 3), (2, 3)) ; ((2, 3), (1, 3)) ; ((2, 3), (1, 3), (0, 3)) .	((0,1),(1,2)) and ((1,2)) .
$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 3 \end{bmatrix}$	((0,3),(1,3),(2,3)) .	((0,1),(1,2)) and ((1,2)) .
$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}$	((0,2),(1,2),(0,3),(1,3),(2,3)) ; ((0,2),(1,2),(0,3),(1,3)) ; ((0,2),(1,2)) ; ((2,3)) .	((0,1),(2,3)) and ((2,3)) .
$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 2 \\ 2 & 3 & 2 & 1 \\ 3 & 2 & 1 & 3 \end{bmatrix}$	((0,1),(0,2),(0,3)) .	None .
$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 3 & 3 & 2 & 2 \\ 2 & 2 & 3 & 3 \end{bmatrix}$	((0,2),(0,3),(1,2),(1,3)) .	((0,1),(2,3)) .

Table 4: Continuous posets on quandles of order 5, Part I

Quandles for $n=5$	Right continuous poset	Left continuous poset
$\begin{bmatrix} 0&0&0&0&0\\ 1&1&1&1&1\\ 2&2&2&2&3\\ 3&3&3&3&2\\ 4&4&4&4&4 \end{bmatrix}$	((0,1),(1,2),(1,3),(0,4)) ; ((0,2),(0,3),(1,2),(1,3),(4,2),(4,3)) ; ((0,2),(0,3),(1,2),(1,3),(2,4),(3,4)) ; ((0,1),(1,4),(4,2),(4,3)) .	((0,1),(1,2),(2,3)) ; ((0,1),(1,2)) ; ((1,2)) .
$\begin{bmatrix} 0&0&0&0&0\\ 1&1&1&1&2\\ 2&2&2&2&3\\ 3&3&3&3&1\\ 4&4&4&4&4 \end{bmatrix}$	((0,1), (0,2), (0,3), (2,4), (3,4), (1,4)) ; ((0,4)) ; ((0,1), (0,2), (0,3)) ; ((0,4),(4,1),(4,2),(4,3)) .	((0,1), (1,2), (2,3)) ; ((0,1), (1,2)) ; ((2,3)) .
$\begin{bmatrix} 0&0&0&0&1\\ 1&1&1&1&0\\ 2&2&2&2&3\\ 3&3&3&3&2\\ 4&4&4&4&4 \end{bmatrix}$	((1,2),(0,3),(2,4),(3,4)) ; ((1,2),(0,2),(1,3),(0,3),(2,4),(3,4)) ; ((1,4),(0,4)) ; ((1,2),(0,2),(1,3),(0,3)) .	((1,2),(0,1),(2,3)) ; ((0,1),(0,2)) ; ((0,2),(1,2)) .
$\begin{bmatrix} 0&0&0&0&1\\ 1&1&1&1&2\\ 2&2&2&2&3\\ 3&3&3&3&0\\ 4&4&4&4&4 \end{bmatrix}$	((0,4),(1,4),(2,4),(3,4)) .	((1,2),(0,1),(2,3)) ; ((0,1),(0,2)) ; ((0,2),(1,2)) .
$\begin{bmatrix} 0&0&0&0&0\\ 1&1&1&1&1\\ 2&2&2&4&3\\ 3&3&4&3&2\\ 4&4&3&2&4 \end{bmatrix}$	((0,2),(0,3),(0,4)) ; ((0,2),(0,3),(0,4),(1,2),(1,3),(1,4)) ; ((0,1)) ; ((0,2),(0,3),(0,4),(0,1),(1,2),(1,3),(1,4)).	((0,1),(0,2),(0,3)) ; ((0,1),(0,2)) ; ((0,1)) .
$\begin{bmatrix} 0&0&0&0&0\\ 1&1&1&2&2\\ 2&2&2&1&1\\ 3&3&3&3&3\\ 4&4&4&4&4 \end{bmatrix}$	((1,3),(2,3)) ; ((1,3),(2,3),(1,4),(2,4)) ; ((0,4)) ; ((3,2),(3,1)) ; ((1,3),(2,3),(4,1),(4,2)) .	((0,1),(1,2),(3,4)) ; ((0,1),(1,2)) ; ((0,1),(3,4)) ; ((0,1)) .
$\begin{bmatrix} 0&0&0&0&0\\ 1&1&1&2&2\\ 2&2&2&1&1\\ 3&4&4&3&3\\ 4&3&3&4&4 \end{bmatrix}$	((0,1),(0,2),(0,3),(0,4)) ; ((0,1),(0,2)) ; ((0,1),(0,2),(2,3),(2,4),(1,3),(1,4)) ; ((1,3),(1,4),(2,3),(2,4)).	((0,1),(1,2),(3,4)) ; ((0,1),(1,2)) ; ((0,1),(3,4)) ; ((0,1)).

Table 5: Continuous posets on quandles of order 5, Part II

Quandles for $n = 5$	Right continuous poset	Left continuous poset
$\begin{bmatrix} 0&0&0&1&1\\ 1&1&1&0&0\\ 2&2&2&2&2\\ 3&3&4&3&3\\ 4&4&3&4&4 \end{bmatrix}$	((0,2), (1,2) (2,3),(2,4)) ; ((0,2), (1,2)) ; ((2,3),(2,4)) ; ((0,3),(0,4),(1,3),(1,4)) .	((0,1),(1,2),(3, 4)) ; ((3,4)) ; ((0,1),(1,2)) .
$\begin{bmatrix} 0&0&0&1&1\\ 1&1&1&2&2\\ 2&2&2&0&0\\ 3&3&3&3&3\\ 4&4&4&4&4 \end{bmatrix}$	((0, 3),(1,3), (2,3), (3,4)) ; ((0,3),(1,3),(2, 3)) ; ((3,4)) .	((0,1),(1,2),(3,4)) ; ((3,4)) ; ((0,1),(1,2)) .
$\begin{bmatrix} 0&0&0&1&2\\ 1&1&1&2&0\\ 2&2&2&0&1\\ 3&3&3&3&3\\ 4&4&4&4&4 \end{bmatrix}$	((0,3),(1,3),(2,3),(0,4),(1,4),(2,4)) ; ((0,3),(1,3),(2,3)) ; ((3,4)).	((0,1),(1,2)) ; ((0,1)) .
$\begin{bmatrix} 0&0&0&0&0\\ 1&1&1&1&1\\ 2&2&2&2&2\\ 4&4&4&3&3\\ 3&3&3&4&4 \end{bmatrix}$	((0,1),(0,2)) ; ((0,1),(1,3),(1,4)) ; ((0,1),(1,2),(2,3),(2,4)) .	((0,1),(0,2)) ; ((0,1),(1,2),(3,4)) ; ((0,1),(3,4)) ; ((3,4)) .
$\begin{bmatrix} 0&0&0&0&0\\ 1&1&1&2&2\\ 2&2&2&1&1\\ 4&4&4&3&3\\ 3&3&3&4&4 \end{bmatrix}$	((0,1),(0,2)) ; ((0,1),(1,3),(1,4)) ; ((1,3),(1,4),(2,3),(2,4)) .	((0,1),(0,2)) ; ((0,1),(1,2),(3,4)) ; ((0,1),(3,4)) ; ((3,4)) .
$\begin{bmatrix} 0&0&0&1&1\\ 1&1&1&2&2\\ 2&2&2&0&0\\ 4&4&4&3&3\\ 3&3&3&4&4 \end{bmatrix}$	((0,4),(1,4),(2,4),(0,3),(1,3),(2,3)) .	((0,1),(0,2)) ; ((0,1),(1,2),(3,4)) ; ((0,1),(3,4)) ; ((3,4)) .
$\begin{bmatrix} 0&0&0&0&0\\ 1&1&4&2&3\\ 2&3&2&4&1\\ 3&4&1&3&2\\ 4&2&3&1&4 \end{bmatrix}$	((0,1),(0,2),(0,3),(0,4)) .	None .

Table 6: Continuous posets on quandles of order 5, Part III

Quandles for $n = 5$	Right continuous poset	Left continuous poset
$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 3 & 2 \\ 4 & 4 & 4 & 4 & 4 \end{bmatrix}$	((1,2),(0,2),(1,3),(0,3),(2,4),(3,4)) ; ((1,4),(0,4)) ; ((1,2),(0,2),(1,3),(0,3)) .	((0,1),(2,3)) ; ((0,1)) .
$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 4 & 3 \\ 3 & 3 & 4 & 3 & 2 \\ 4 & 4 & 3 & 2 & 4 \end{bmatrix}$	((0,2),(0,3),(0,4),(1,2),(1,3),(1,4)) .	((0,1),(2,3)) ; ((0,1)) .
$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 \\ 4 & 4 & 4 & 3 & 3 \\ 3 & 3 & 3 & 4 & 4 \end{bmatrix}$	((0,2),(0,3),(0,4),(1,2),(1,3),(1,4)) ; ((0,2),(1,2)) ; ((2,3),(2,4)) ; ((0,3),(0,4),(1,3),(1,4)) .	((0,1),(2,3)) ; ((0,1)) .
$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 3 & 4 & 2 & 4 & 3 \\ 4 & 2 & 4 & 3 & 2 \\ 2 & 3 & 3 & 2 & 4 \end{bmatrix}$	((0,2),(0,3),(0,4),(1,2),(1,3),(1,4)) .	None .

Acknowledgements

The authors would like to thank the referee for the comments which helped to improve the paper. Mohamed Elhamdadi was partially supported by Simons Foundation collaboration grant 712462.

References

[1] P.S. Alexandroff, Diskrete räume, *Mat. Sb.* **2** (1937), 501 – 518.

[2] Z. Cheng, M. Elhamdadi, B. Shekhtman, On the classification of topological quandles, *Topology Appl.* **248** (2018), 64 – 74. MR3856599

[3] M. Elhamdadi, Distributivity in quandles and quasigroups, in "Algebra, Geometry and Mathematical Physics", Springer Proc. Math. Stat., 85, Springer, Heidelberg, 2014 325 – 340. MR3275946

[4] M. Elhamdadi, M. Saito, E. Zappala, Continuous cohomology of topological quandles *J. Knot Theory Ramifications* **28** (6) (2019), 1950036, 22 pp. MR3956354

[5] M. Elhamdadi, El-Kaïoum M. Moutuou, Foundations of topological racks and quandles, *J. Knot Theory Ramifications* **25** (3) (2016), 1640002, 17 pp. MR3475069

[6] M. Elhamdadi, S. Nelson, "Quandles–an introduction to the algebra of knots", Student Mathematical Library, 74, American Mathematical Society, Providence, RI, 2015. MR3379534

[7] T. Grøsfjeld, Thesaurus racks: categorizing rack objects, *J. Knot Theory Ramifications* **30** (4) (2021), 2150019, 18 pp. MR4272643

[8] D. Joyce, A classifying invariant of knots, the knot quandle, *J. Pure Appl. Algebra* **23** (1) (1982), 37 – 65. MR638121 (83m:57007)

[9] Maple 15, Magma package-copyright by Maplesoft, a division of Waterloo Maple, Inc., 1981 – 2011.

[10] S.V. Matveev, Distributive groupoids in knot theory (russian), *Mat. Sb. (N.S.)* **119 (161)** (1) (1982), 78 – 88, 160. MR672410 (84e:57008)

[11] R.L. Rubinsztein, Topological quandles and invariants of links, *J. Knot Theory Ramifications* **16** (6) (2007), 789 – 808. MR2341318 (2008e:57012)

[12] R.E. Stong, Finite topological spaces, *Trans. Amer. Math. Soc.* **123** (1966), 325 – 340. MR195042

[13] M. Szymik, Permutations, power operations, and the center of the category of racks, *Comm. Algebra* **46** (1) (2018), 230 – 240. MR3764859

[14] N. Takahashi, Modules over geometric quandles and representations of Lie-Yamaguti algebras, *J. Lie Theory* **31** (4) (2021), 897 – 932. MR4327618