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# A note on isomorphisms of quantum systems

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*Abstract*: We consider the question as to whether a quantum system is uniquely determined by all values of all its observables. For this, we consider linearly nuclear GB$^*$-algebras over W$^*$-algebras as models of quantum systems.

*Key words*: quantum system, observables, GB$^*$-algebra, Jordan homomorphism.

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## 1. Introduction

The main objective of this paper is to determine whether all values of all observables in a quantum system are sufficent to determine the quantum system uniquely. To answer this question, we first have to find a suitable mathematical framework in which to reformulate the question.

In the well known formalism of Haag and Kastler, a quantum system takes on the following form: The observables of the system are self-adjoint elements of a $*$-algebra $A$ with identity element 1, and the states of the system are positive linear functionals $\phi$ of $A$ for which $\phi(1) = 1$. This is well in agreement with the Hilbert space formalism, where the observables are linear operators on a Hilbert space $H$, and all states are unit vectors in $H$. Since observables are generally unbounded linear operators on a Hilbert space (such as position and momentum operators, which are unbounded linear operators on the Hilbert space $L^2(\mathbb{R})$), one requires the $*$-algebra $A$ above to at least partly consist of unbounded linear operators on some Hilbert space. The question is then what $*$-algebra of unbounded linear operators one must take to house the observables of the quantum system under consideration. A candidate can be found among the elements in the class of GB$^*$-algebras, which are locally

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convex $*$-algebras serving as generalizations of C*-algebras, and were first studied by G.R. Allan in [2], and later by P.G. Dixon in [6] to include non locally convex $*$-algebras (see Section 2 for the definition of a GB*-algebra). Every GB*-algebra $A[\tau]$ contains a C*-algebra $A[B_0]$ which is dense in $A$ (see Section 2).

In [13], the author motivated why one can model a quantum system as a GB*-algebra $A[\tau]$ which is nuclear as a locally convex space (referred to as a *linearly nuclear* GB*-algebra for here on). In addition to this, it would be useful to have that $A[B_0]$ is a W*-algebra (i.e., a von Neumann algebra): Since $A[\tau]$ is also assumed to be locally convex, $A$ can be faithfully represented as a $*$-algebra $B$ of closed densely defined linear operators on a Hilbert space (see [6, Theorem 7.11] or [10, Theorem 6.3.5]). If we denote this $*$-isomorphism by $\pi : A \to B$, then $\pi(A[B_0]) = B_b$, where $B_b$ is the $*$-algebra of all bounded linear operators in $B$, and is a von Neumann algebra (this follows from [6, Theorem 7.11], or [10, Theorem 6.3.5]). Let $x \in A$ be self-adjoint. Then $\pi(x)$ is a self-adjoint element of $B$ and $\pi(x) = \int_{\sigma(\pi(x))} \lambda d\ P_\lambda$.

Now $(1 + y^*y)^{-1} \in B_b$ for all $y \in B$. By [7, Proposition 2.4], it follows that all $y \in B$ are affiliated with $B_b$. Therefore, $P_\lambda \in B_b$ for all $\lambda \in \sigma(\pi(x))$. The spectral projections, $P_\lambda$, $\lambda \in \sigma(\pi(x))$, are important for determining the probability of a particle in a certain set (see [8, Postulate 4, p. 13]).

So far, the observables of a quantum mechanical system are self-adjoint elements of a locally convex $*$-algebra $A[\tau]$ (more specifically, in our case, a linearly nuclear GB*-algebra with $A[B_0]$ a W*-algebra). In fact, one can sharpen this by noting that if $x, y \in A$ are self-adjoint (i.e., observables), then $x \circ y = \frac{1}{2}(xy + yx)$ is again self-adjoint, i.e., an observable. In 1932, J. von Neumann and collaborators proposed that a Jordan algebra be used to house the observables of a quantum system (see [5, Introduction]). A linear mapping $\phi : A \to A$ with $\phi(x \circ y) = \phi(x) \circ \phi(y)$ for all $x, y \in A_s$, where $A_s$ denotes the set of self-adjoint elements of $A$, is called a *Jordan homomorphism*. We note here that $A_s$ is a Jordan algebra with respect to the operation $\circ$ above. If, in addition, $\phi$ is a bijection, then $\phi$ is called a *Jordan isomorphism*, i.e., an isomorphism of Jordan algebras. A Jordan isomorphism is therefore an isomorphism of quantum systems (see [5, Introduction]). Observe that a linear map $\phi$ is a Jordan homomorphism if and only if $\phi(x^2) = \phi(x)^2$ for all $x \in A$.

We already know that all possible values of an observable, when considered as a self-adjoint unbounded linear operator on a Hilbert space, are in the spectrum of the observable. An interesting question is therefore if a quantum system is uniquely determined by the values/measurements of its observables.

To answer this question in our setting of a linearly nuclear GB*-algebra $A[\tau]$ with $A[B_0]$ a W*-algebra (an abstract algebra of unbounded linear operators), one requires a notion of spectrum of an element which is an analogue of the notion of spectrum of a self-adjoint unbounded linear operator on a Hilbert space. The required notion is the Allan spectrum of an element of a locally convex algebra. The values/measurements of a self-adjoint element $x \in A$ (i.e., an observable) are therefore in $\sigma_A(x)$, the Allan spectrum of $x$, as defined in Definition 2.3 below in Section 2. If no confusion arises, we write $\sigma(x)$ instead of $\sigma_A(x)$.

The above question can be reformulated as follows: Let $A[\tau]$ be a linearly nuclear GB*-algebra with $A[B_0]$ a W*-algebra. Let $\phi : A \to A$ be a bijective self-adjoint linear map such that $\sigma(\phi(x)) = \sigma(x)$ for all $x \in A_s$, where $A_s$ is the set of all self-adjoint elements of $A$. Is $\phi$ a Jordan isomorphism?

Below, in Corollary 3.6, we answer this question affirmatively for the case where $A[\tau]$ has the additional property of being a Fréchet algebra, i.e., a complete and metrizable algebra. We do not require the GB*-algebra to be linearly nuclear in this result.

The above result is similar to results which are partial answers to a special case of an unanswered question of I. Kaplansky: If $A$ and $B$ are Banach algebras with identity, and $\phi : A \to B$ is a bijective linear map such that $\mathrm{Sp}_B(\phi(x)) = \mathrm{Sp}_A(x)$ for all $x \in A$, is it true that $\phi$ is a Jordan isomorphism? Here, $\mathrm{Sp}_A(x)$ refers to the spectrum of $x$, which is the set $\{\lambda \in \mathbb{C} : \lambda 1 - x \text{ is not invertible in } A\}$. The answer to this question remains unresolved for C*-algebras, but it has been shown, by B. Aupetit, to have an affirmative answer if $A$ and $B$ are von Neumann algebras (see [3, Theorem 1.3]). For the physical problem under consideration, we have to replace the spectrum of $x$ in Kaplansky's question with the Allan spectrum of $x$, as explained above. We refer the reader to [4] for an excellent introduction to Kaplansky's problem.

Section 2 of this paper contains all the background material required to understand the discussion in Section 3, where the main result is presented.

## 2. Preliminaries

In this section, we give all background material on generalized GB*-algebras (GB*-algebras, for short) which is required to understand the main results of this paper. GB*-algebras were introduced in the late sixties by G.R. Allan in [2], and taken further, in the early seventies, by P.G. Dixon in

[6, 7]. Recently, the author, along with M. Fragoulopoulou, A. Inoue and I. Zarakas, published a monograph on GB$^*$-algebras [10] containing much of the developed theory on this topic. Almost all concepts and results in this section are due Allan and Dixon, and can be found in [1, 2, 6]. We will, however, use [10] as a reference.

A *topological algebra* is an algebra which is a topological vector space and in which multiplication is separately continuous. If a topological algebra is equipped with a continuous involution, then it is called a *topological $*$-algebra*. A *locally convex $*$-algebra* is a topological $*$-algebra which is locally convex as a topological vector space. We say that a topological algebra is a *Fréchet algebra* if it is complete and metrizable.

Definition 2.1. ([10, Definition 3.3.1]) Let $A[\tau]$ be a unital topological $*$-algebra and let $\mathcal{B}^*$ denote a collection of subsets $B$ of $A$ with the following properties:

(i) $B$ is absolutely convex, closed and bounded;
(ii) $1 \in B$, $B^2 \subset B$ and $B^* = B$.

For every $B \in \mathcal{B}^*$, denote by $A[B]$ the linear span of $B$, which is a normed algebra under the gauge function $\|\cdot\|_B$ of $B$. If $A[B]$ is complete for every $B \in \mathcal{B}^*$, then $A[\tau]$ is called *pseudo-complete*.

An element $x \in A$ is called *bounded*, if for some nonzero complex number $\lambda$, the set $\{(\lambda x)^n : n = 1, 2, 3, \dots\}$ is bounded in $A$. We denote by $A_0$ the set of all bounded elements in $A$.

A unital topological $*$-algebra $A[\tau]$ is called *symmetric* if, for every $x \in A$, the element $(1 + x^*x)^{-1}$ exists and belongs to $A_0$.

Definition 2.2. ([10, Definition 3.3.2]) A symmetric pseudo-complete locally convex $*$-algebra $A[\tau]$, such that the collection $\mathcal{B}^*$ has a greatest member, denoted by $B_0$, is called a *GB$^*$-algebra* over $B_0$.

Every C$^*$-algebra is a GB$^*$-algebra. An example of a GB$^*$-algebra, which generally need not be a C$^*$-algebra, is a pro-C$^*$-algebra. By a *pro-C$^*$-algebra*, we mean a complete topological $*$-algebra $A[\tau]$ for which the topology $\tau$ is defined by a directed family of C$^*$-seminorms.

Another example of a GB$^*$-algebra which is not a pro-C$^*$-algebra is the locally convex $*$-algebra $L^\omega([0,1]) = \cap_{p\geq 1} L^p([0,1])$ defined by the family of seminorms $\{\|\cdot\|_p : p \geq 1\}$, where $\|\cdot\|_p$ is the $L^p$-norm on $L^p([0,1])$ for all $p \geq 1$.

If $A$ is commutative, then $A_0 = A[B_0]$ [10, Lemma 3.3.7(ii)]. In general, $A_0$ is not a $*$-subalgebra of $A$, and $A[B_0]$ contains all normal elements of $A_0$, i.e., all $x \in A$ such that $xx^* = x^*x$ [10, Lemma 3.3.7(i)].

DEFINITION 2.3. ([10, DEFINITION 2.3.1]) Let $A[\tau]$ be topological algebra with identity element 1 and $x \in A$. The set $\sigma_A(x)$ is the subset of $\mathbb{C}^*$, the one-point compactification of $\mathbb{C}$, defined as follows:

(i) if $\lambda \neq \infty$, then $\lambda \in \sigma_A(x)$ if $\lambda 1 - x$ has no bounded inverse in $A$;

(ii) $\infty \in \sigma_A(x)$ if and only if $x \notin A_0$.

We define $\rho_A(x)$ to be $\mathbb{C}^* \setminus \sigma_A(x)$.

If there is no risk of confusion, then we write $\sigma(x)$ to denote $\sigma_A(x)$.

PROPOSITION 2.4. ([10, THEOREM 3.3.9, THEOREM 4.2.11]) *If $A[\tau]$ is a GB$^*$-algebra, then the Banach $*$-algebra $A[B_0]$ is a C$^*$-algebra, which is sequentially dense in $A$. Moreover, $(1 + x^*x)^{-1} \in A[B_0]$ for every $x \in A$ and $B_0$ is the unit ball of $A[B_0]$.*

The next proposition has to do with extensions of characters of the commutative C$^*$-algebra $A[B_0]$ to the GB$^*$-algebra $A$, which could be infinite valued.

PROPOSITION 2.5. ([10, PROPOSITION 2.5.4]) *Let $A[\tau]$ be a commutative pseudocomplete locally convex $*$-algebra with identity. Then, for any character $\phi$ on $A_0$, there exists a $\mathbb{C}^*$-valued function $\phi'$ on $A$ having the following properties:*

(i) *$\phi'$ is an extension of $\phi$;*

(ii) *$\phi'(\lambda x) = \lambda \phi'(x)$ for all $\lambda \in \mathbb{C}$ (with the convention that $0.\infty = 0$);*

(iii) *$\phi'(x + y) = \phi'(x) + \phi'(y)$ for all $x, y \in A$ for which $\phi'(x)$ and $\phi'(y)$ are not both $\infty$;*

(iv) *$\phi'(xy) = \phi'(x)\phi'(y)$ for all $x, y \in A$ for which $\phi'(x)$ and $\phi'(y)$ are not both $0, \infty$ in some order;*

(v) *$\phi'(x^*) = \overline{\phi'(x)}$ for all $x \in A$ (with the convention that $\overline{\infty} = \infty$).*

## 3. The main result

The following example is an example of a linearly nuclear GB$^*$-algebra over a W$^*$-algebra, which is not a C$^*$-algebra.

Example 3.1. Consider a family $\{H_\alpha : \alpha \in \Lambda\}$ of finite dimensional Hilbert spaces. Then, for every $\alpha \in \Lambda$, we have that $B(H_\alpha)$ is a finite dimensional C$^*$-algebra, and hence a linearly nuclear space, with respect to the operator norm $\|\cdot\|_\alpha$. Let $A = \Pi_\alpha B(H_\alpha)$. Then $A$ is a pro-C$^*$-algebra in the product topology $\tau$, when all $B(H_\alpha)$ are equipped with their operators norms $\|\cdot\|_\alpha$ [9, Chapter 2]. Furthermore, $A[\tau]$ is linearly nuclear since it is a product of linearly nuclear spaces. Observe that $x\xi = (x_\alpha(\xi_\alpha))_\alpha$ for all $\xi = (\xi_\alpha)_\alpha \in H$, where $H$ is the direct sum of the Hilbert spaces $H_\alpha$. Note that $H$ is itself a Hilbert space. Now

$$\begin{aligned} A[B_0] &= \left\{x = (x_\alpha)_\alpha \in A : \sup_\alpha \|x_\alpha\|_\alpha < \infty\right\} \\ &= \oplus_\alpha B(H_\alpha), \end{aligned}$$

and this is a von Neumann algebra with respect to the norm $\sup_\alpha \|x_\alpha\|_\alpha$.

Lemma 3.2. *If $x$ is a self-adjoint element of a GB$^*$-algebra $A[\tau]$, then $x$ is a projection if and only if $\sigma(x) \subseteq \{0,1\}$.*

*Proof.* Let $x \in A$ be a projection and let $B$ be a maximal commutative $*$-subalgebra of $A$ containing $x$. Then $\sigma_B(x) = \sigma_A(x)$ (see [10, Proposition 2.3.2]) and $B$ is a GB$^*$-algebra over the C$^*$-algebra $B_b = A[B_0] \cap B$ (see [6]). Let $M_0$ denote the character space of the commutative C$^*$-algebra $B_b$. Then, by Proposition 2.5 and [10, Corollary 3.4.10], it follows that

$$\begin{aligned} \sigma_B(x) &= \left\{\widehat{x}(\phi) = \phi'(x) : \phi \in M_0\right\} \\ &= \left\{\phi(x) : \phi \in M_0\right\} \\ &\subseteq \{0,1\}. \end{aligned}$$

The second equality above follows from the fact that $x \in A[B_0]$, due to the fact that $x$ is a projection, and therefore $x \in B_b$. Therefore $\sigma_A(x) \subseteq \{0,1\}$.

Now assume that $\sigma_A(x) \subseteq \{0,1\}$. Let $B$ be a maximal commutative $*$-subalgebra of $A$ containing $x$. Then $\sigma_B(x) = \sigma_A(x)$. Like above, we have that

$$\left\{\widehat{x}(\phi) = \phi'(x) : \phi \in M_0\right\} = \sigma_B(x) = \sigma_A(x) \subseteq \{0,1\}$$

for all characters $\phi$ on $A[B_0]$. Therefore $\widehat{x}$ is an idempotent function. Since $x \mapsto \widehat{x}$ is an algebra $*$-isomorphism [10, Theorem 3.4.9], we get that $x$ is an idempotent element of $A$. Therefore $x$ is a projection because $x$ is self-adjoint. ∎

If $A$ and $B$ are $*$-algebras and $\phi : A \to B$ a linear map such that $\phi(x^2) = \phi(x)^2$ for all self-adjoint elements $x$ in $A$, then $\phi$ is a Jordan homomorphism [3, page 922]. We require this in the proof of Proposition 3.3 below.

Proposition 3.3. *Let $A[\tau]$ be a GB*-algebra with $A[B_0]$ a W*-algebra, and let $B$ be a topological $*$-algebra. Suppose further that the multiplications on $A$ and $B$ are jointly continuous. If $\phi : A \to B$ is a continuous linear mapping which maps projections to projections, then $\phi$ is a Jordan homomorphism.*

*Proof.* Let $s$ be a self-adjoint element in $A[B_0]$. By the spectral theorem, and the fact that $A[B_0]$ is a W*-algebra, there is a sequence $(s_n)$ of finite linear combinations of orthogonal projections in $A[B_0]$ such that $s_n \to s$ in norm [11, Theorem 5.2.2], and hence also with respect to the topology $\tau$ on $A$, since the restriction of the topology $\tau$ to $A[B_0]$ is weaker than the norm topology of $A[B_0]$. Therefore $\phi(s_n^2) = \phi(s_n)^2$ for every $n$. Hence, since $\phi$ is continuous, and since the multiplications on $A$ and $B$ are jointly continuous, it follows that

$$\phi(s^2) = \phi\Big(\lim_{n\to\infty} s_n^2\Big) = \phi\Big(\lim_{n\to\infty} s_n\Big)^2 = \phi(s)^2.$$

This holds for any self-adjoint element $s \in A[B_0]$. By the paragraph following Lemma 3.2, $\phi|_{A[B_0]}$ is a Jordan homomorphism.

Let $x \in A$. Then there is a sequence $(x_n)$ in $A[B_0]$ such that $x_n \to x$. Since $\phi$ is continuous, $A[B_0]$ is dense in $A$, and the multiplications on $A$ and $B$ are jointly continuous, it follows that $\phi(x^2) = \phi(x)^2$. This holds for every $x \in A$, and therefore $\phi$ is a Jordan homomorphism. ∎

We say that an element $x$ in a GB*-algebra $A[\tau]$ is *positive* if there exists $y \in A$ such that $x = y^*y$. The following proposition is required to prove Theorem 3.5 below.

Proposition 3.4. ([12, Proposition 7]) *Let $A[\tau_1]$ and $B[\tau_2]$ be Fréchet GB*-algebras. If $\phi : A \to B$ is a linear mapping which maps positive elements of $A$ to positive elements of $B$, then $\phi$ is continuous.*

Theorem 3.5. *Let $A[\tau]$ be a Fréchet GB*-algebra with $A[B_0]$ a W*-algebra, and let $\phi : A \to A$ be a self-adjoint linear map such that $\sigma(\phi(x)) \subseteq \sigma(x)$ for all $x \in A_s$, where $A_s$ is the set of all self-adjoint elements of $A$. Then $\phi$ is a Jordan isomorphism.*

*Proof.* By hypothesis and [10, Proposition 6.2.1], it follows that if $x \in A$ is a positive element, then $\sigma(\phi(x)) \subset \sigma(x) \subseteq [0, \infty]$, and therefore $\phi(x)$ is a positive element in $A$. Therefore $\phi$ maps positive elements of $A$ to positive elements of $A$. By Proposition 3.4 and the fact that $A$ is a Fréchet GB*-algebra, it follows that $\phi$ is continuous.

We now show that if $p \in A$ is a projection, then $\phi(p)$ is also a projection in $A$. If $p \in A$ is a projection, then $p$ and $\phi(p)$ are self-adjoint elements in $A$. Therefore, by Lemma 3.2, $\sigma(p) \subseteq \{0, 1\}$. Since $\sigma(\phi(p)) \subseteq \sigma(p)$, we get that $\sigma(\phi(p)) \subseteq \{0, 1\}$. By Lemma 3.2 again, $\phi(p)$ is a projection.

Since $A[B_0]$ is a W*-algebra and the multiplication on $A$ is jointly continuous (because $A$ is a Fréchet algebra), it follows from Proposition 3.3 that $\phi$ is a Jordan homomorphism. ∎

The following corollary is the desired result of this section, and affirms that all quantum mechanical isomorphisms, in the context of Fréchet GB*-algebras, are Jordan isomorphisms.

Corollary 3.6. *Let $A[\tau]$ be a Fréchet GB*-algebra with $A[B_0]$ a W*-algebra, and let $\phi : A \to A$ be a bijective self-adjoint linear map such that $\sigma(\phi(x)) = \sigma(x)$ for all $x \in A_s$, where $A_s$ is the set of all self-adjoint elements of $A$. Then $\phi$ is a Jordan isomorphism.*

In [3], B. Aupetit proved that any bijective linear map $\phi : A \to B$ between von Neumann algebras $A$ and $B$, satisfying $\mathrm{Sp}_B(\phi(x)) = \mathrm{Sp}_A(x)$ for all $x \in A$, is a Jordan homomorphism. Observe that $\phi$ need not be self-adjoint. The proof of Aupetit's result in [3] is complicated and relies on a deep spectral characterization of idempotents in a semi-simple Banach algebra (see [3, Theorem 1.1]). If we additionally assume that $\phi$ is self-adjoint, then one has a much simpler proof of his result, namely, the proof of Corollary 3.6 for the case where $A[\tau]$ is a von Neumann algebra.

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