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On Jordan ideals with left derivations in 3-prime near-rings

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Abstract: We will extend in this paper some results about commutativity of Jordan ideals proved in [2] and [6]. However, we will consider left derivations instead of derivations, which is enough to get good results in relation to the structure of near-rings. We will also show that the conditions imposed in the paper cannot be removed.

Key words: 3-prime near-rings, Jordan ideals, Left derivations.

MSC (2020): 16N60; 16W25; 16Y30.

1. Introduction

A right (resp. left) near-ring $\mathcal{A}$ is a triple $(\mathcal{A}, +, .)$ with two binary operations " $+$ " and "." such that:

(i) $(\mathcal{A}, +)$ is a group (not necessarily abelian),

(ii) $(\mathcal{A}, .)$ is a semigroup,

(iii) $(r+s).t = r.t + s.t$ (resp. $r.(s+t) = r.s + r.t$) for all $r; s; t \in \mathcal{A}$.

We denote by $Z(\mathcal{A})$ the multiplicative center of $\mathcal{A}$, and usually $\mathcal{A}$ will be 3-prime, that is, for $r, s \in \mathcal{A}$, $r\mathcal{A}s = \{0\}$ implies $r = 0$ or $s = 0$. A right (resp. left) near-ring $\mathcal{A}$ is a zero symmetric if $r.0 = 0$ (resp. $0.r = 0$) for all $r \in \mathcal{A}$, (recall that right distributive yields $0r = 0$ and left distributive yields $r.0 = 0$). For any pair of elements $r, s \in \mathcal{A}$, $[r, s] = rs - sr$ and $r \circ s = rs + sr$ stand for Lie product and Jordan product respectively. Recall that $\mathcal{A}$ is called 2-torsion free if $2r = 0$ implies $r = 0$ for all $r \in \mathcal{A}$. An additive subgroup J of $\mathcal{A}$ is said to be Jordan left (resp. right) ideal of $\mathcal{A}$ if $r \circ i \in J$ (resp. $i \circ r \in J$) for all $i \in J$, $r \in \mathcal{A}$ and J is said to be a Jordan ideal of $\mathcal{A}$ if $r \circ i \in J$ and $i \circ r \in J$ for all $i \in J$, $r \in \mathcal{N}$. An additive mapping

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$H : \mathcal{A} \to \mathcal{A}$ is a multiplier if $H(rs) = rH(s) = H(r)s$ for all $r, s \in \mathcal{A}$. An additive mapping $d : \mathcal{A} \to \mathcal{A}$ is a left derivation (resp. Jordan left derivation) if $d(rs) = rd(s) + sd(r)$ (resp. $d(r^2) = 2rd(r)$) holds for all $r, s \in \mathcal{A}$. The concepts of left derivations and Jordan left derivations were introduced by Breşar et al. in [7], and it was shown that if a prime ring $\mathcal{R}$ of characteristic different from 2 and 3 admits a nonzero Jordan left derivation, then $\mathcal{R}$ must be commutative. Obviously, every left derivation is a Jordan left derivation, but the converse need not be true in general (see [9, Example 1.1]). In [1], M. Ashraf et al. proved that the converse statement is true in the case when the underlying ring is prime and 2-torsion free. The study of left derivation was developed by S.M.A. Zaidi et al. in [9] and they showed that if J is a Jordan ideal and a subring of a 2-torsion-free prime ring R admits a nonzero Jordan left derivation and an automorphism T such that $d(r^2) = 2T(r)d(r)$ holds for all $r \in J$, then either $J \subseteq Z(\mathcal{R})$ or $d(J) = \{0\}$. Recently, there have been many works concerning the Jordan ideals of near-rings involving derivations; see, for example, [4], [5], [6], etc. For more details, in [6, Theorem 3.6 and Theorem 3.12], we only manage to show the commutativity of the Jordan ideal, but we don't manage to show the commutativity of our studied near-rings, hence our goal to extend these results to the left derivations.

2. Some preliminaries

To facilitate the proof of our main results, the following lemmas are essential.

Lemma 2.1. *Let $\mathcal{N}$ be a 3-prime near-ring.*

- (i) [3, Lemma 1.2 (iii)] *If $z \in Z(\mathcal{N}) \setminus \{0\}$ and $xz \in Z(\mathcal{N})$ or $zx \in Z(\mathcal{N})$, then $x \in Z(\mathcal{N})$.*
- (ii) [2, Lemma 3 (ii)] *If $Z(\mathcal{N})$ contains a nonzero element z of $\mathcal{N}$ which $z + z \in Z(\mathcal{N})$, then $(\mathcal{N}, +)$ is abelian.*
- (iii) [5, Lemma 3] *If $J \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

Lemma 2.2. ([8, Theorem 3.1]) *Let $\mathcal{N}$ be a 3-prime right near-ring. If $\mathcal{N}$ admits a nonzero left derivation d, then the following properties hold true:*

- (i) *If there exists a nonzero element a such that $d(a) = 0$, then $a \in Z(\mathcal{N})$,*
- (ii) *$(\mathcal{N}, +)$ is abelian, if and only if $\mathcal{N}$ is a commutative ring.*

LEMMA 2.3. ([4, LEMMA 2.2]) *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero Jordan ideal J, then $j^2 \neq 0$ for all $j \in J \setminus \{0\}$.*

LEMMA 2.4. ([4, THEOREM 3.1]) *Let $\mathcal{N}$ be a 2-torsion free 3-prime right near-ring and J a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero left multiplier H, then the following assertions are equivalent:*

(i) $H(J) \subseteq Z(\mathcal{N})$;

(ii) $H(J^2) \subseteq Z(\mathcal{N})$;

(iii) *$\mathcal{N}$ is a commutative ring.*

LEMMA 2.5. ([5, THEOREM 1]) *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J a nonzero Jordan ideal of $\mathcal{N}$. Then $\mathcal{N}$ must be a commutative ring if J satisfies one of the following conditions:*

(i) *$i \circ j \in Z(\mathcal{N})$ for all $i, j \in J$.*

(ii) *$i \circ j \pm [i, j] \in Z(\mathcal{N})$ for all $i, j \in J$.*

LEMMA 2.6. *Let $\mathcal{N}$ be a left near-ring. If $\mathcal{N}$ admits a left derivation d, then we have the following identity:*

$$xyd(y^n) = yxd(y^n) \qquad \text{for all } \; n \in \mathbb{N}, \; x, y \in \mathcal{N}.$$

Proof. Using the definition of d. On one hand, we have

$$\begin{aligned} d(xy^{n+1}) &= xd(y^{n+1}) + y^{n+1}d(x) \\ &= xy^n d(y) + xyd(y^n) + y^{n+1}d(x) \quad \text{for all } \; n \in \mathbb{N}, \; x, y \in \mathcal{N}. \end{aligned}$$

On the other hand

$$\begin{aligned} d(xy^{n+1}) &= xy^n d(y) + yd(xy^n) \\ &= xy^n d(y) + yxd(y^n) + y^{n+1}d(x) \quad \text{for all } \; n \in \mathbb{N}, \; x, y \in \mathcal{N}. \end{aligned}$$

Comparing the two expressions, we obtain the required result. ∎

3. Results characterizing left derivations in 3-prime near-rings

In [2], the author proved that if $\mathcal{N}$ is a 3-prime 2-torsion-free near-ring which admits a nonzero derivation D for which $D(\mathcal{N}) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring. In this section, we investigate possible analogs of these results, where D is replaced by a left derivation d and by integrating Jordan ideals.

Theorem 3.1. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a left derivation d, then the following assertions are equivalent:*

(i) $d(J) \subseteq Z(\mathcal{N})$;

(ii) $d(J^2) \subseteq Z(\mathcal{N})$;

(iii) *$\mathcal{N}$ is a commutative ring or $d = 0$.*

Proof. Case 1: $\mathcal{N}$ is a 3-prime *right* near-ring. It is obvious that (iii) implies (i) and (ii). Therefore we only need to prove (i) ⇒ (iii) and (ii) ⇒ (iii).

(i) ⇒ (iii): Suppose that $Z(\mathcal{N}) = \{0\}$, then $d(J) = \{0\}$. From Lemma 2.2 (i), we get $J \subseteq Z(\mathcal{N})$ and by Lemma 2.1 (i), we conclude that $\mathcal{N}$ is a commutative ring. In this case, and by using the definition of d together with the 2-torsion freeness of $\mathcal{N}$, the above equation leads to

$$jd(n) = 0 \qquad \text{for all } j \in J,\ n \in \mathcal{N}. \tag{3.1}$$

Taking $j \circ m$ of j, where $m \in \mathcal{N}$ in (3.1) and using it, we get $J\mathcal{N}d(n) = \{0\}$ for all $n \in \mathcal{N}$. Since $\mathcal{N}$ is 3-prime and $J \neq \{0\}$, then $d = 0$.

Now suppose $Z(\mathcal{N}) \neq \{0\}$. By assumption, we have $d(j \circ j) \in Z(\mathcal{N})$ for all $j \in J$, which gives $(4j)d(j) \in Z(\mathcal{N})$ for all $j \in J$, that is $(d(4j))j \in Z(\mathcal{N})$ for all $j \in J$. Invoking Lemma 2.1 (i) and Lemma 2.2 (i) together with the 2-torsion freeness of $\mathcal{N}$, we obtain $J \subseteq Z(\mathcal{N})$, and Lemma 2.4 (i) forces that $\mathcal{N}$ is a commutative ring.

(ii) ⇒ (iii): Suppose that $Z(\mathcal{N}) = \{0\}$, then $d(J^2) = \{0\}$, which implies $J^2 \subseteq Z(\mathcal{N})$ by Lemma 2.2 (ii), hence $\mathcal{N}$ is a commutative ring by Lemma 2.4 (ii). Now using assumption, then we have $d(j^2) = 0$ for all $j \in J$. By the 2-torsion freeness of $\mathcal{N}$, it follows $jd(j) = 0$ for all $j \in J$. Since $\mathcal{N}$ is a commutative ring, we can write $jnd(j) = 0$ for all $j \in J$, $n \in \mathcal{N}$, which implies that $j\mathcal{N}d(j) = \{0\}$ for all $j \in J$. By the 3-primeness of $\mathcal{N}$, we conclude that $d(J) = \{0\}$. Using the same techniques as we have used in the proof of (i) ⇒ (iii) one can easily see that $d = 0$.

Now suppose $Z(\mathcal{N}) \neq \{0\}$. By our hypothesis, we have $d((j \circ j^2)j) \in Z(\mathcal{N})$ for all $j \in J$, and by a simplification, we find $d((j^2 \circ j)j) = (j^2)d(4j^2)$ for all $j \in J$:

$$\begin{aligned} d((j^2 \circ j)j) &= d((j^3 + j^3)j) = d(j^4 + j^4) = d(2j^2j^2) \\ &= 2j^2d(j^2) + j^2d(2j^2) = 2j^2d(j^2) + d(2j^2)j^2 \\ &= 2j^2d(j^2) + 2j^2d(j^2) = 4j^2d(j^2) = j^2d(4j^2). \end{aligned}$$

Hence, $j^2d(4j^2) \in Z(\mathcal{N})$ for all $j \in J$, which implies $j^2d((4j)(j)) \in Z(\mathcal{N})$ for all $j \in J$. Invoking Lemma 2.1 (i), then $j^2 \in Z(\mathcal{N})$ or $4d(j^2) = 0$ for all $j \in J$. In view of the 2-torsion freeness of $\mathcal{N}$ together with Lemma 2.2 (i), we can assure that

$$j^2 \in Z(\mathcal{N}) \qquad \text{for all } j \in J. \tag{3.2}$$

Applying the definition of d together with our hypothesis, and (3.2), we have for all $j \in J$ and $x \in \mathcal{N}$:

$$\begin{aligned} d(xj^4) &= d(xj^2j^2) = xj^2d(j^2) + j^2d(xj^2) \\ &= xj^2d(j^2) + d(xj^2)j^2 = xj^2d(j^2) + xj^2d(j^2) + j^4d(x) \\ &= j^2d(j^2)x + j^2d(j^2)x + j^4d(x) = (2j^2d(j^2))x + j^4d(x), \\ d(xj^4) &= xd(j^4) + j^4d(x) = x(2j^2d(j^2)) + j^4d(x). \end{aligned}$$

Comparing the two expressions, we obtain

$$x(2j^2d(j^2)) = (2j^2d(j^2))x \qquad \text{for all } j \in J,\ x \in \mathcal{N}.$$

Consequently, $2j^2d(j^2) \in Z(\mathcal{N})$ for all $j \in J$. According to Lemma 2.1 (i) and Lemma 2.2 (i), that follows $2j^2 \in Z(\mathcal{N})$ for all $j \in J$, which implies $(\mathcal{N}, +)$ is abelian by Lemma 2.1 (ii), and Lemma 2.2 (ii) assures that $\mathcal{N}$ is a commutative ring.

Case 2: $\mathcal{N}$ is a 3-prime *left* near-ring. It is obvious that (iii) implies (i) and (ii).

(i) ⇒ (iii): Suppose that $Z(\mathcal{N}) = \{0\}$. Using our hypothesis, then we have $d(j \circ n) = 0$ for all $j \in J$, $n \in \mathcal{N}$. Applying definition of d and using our assumption with the 2-torsion freeness of $\mathcal{N}$, we get

$$jd(n) = 0 \qquad \text{for all } n \in \mathcal{N}. \tag{3.3}$$

Replacing n by jnm in (3.3) and using it, then we get $j^2nd(m) = 0$ for all $j \in J$, $n, m \in \mathcal{N}$, which implies that $j^2\mathcal{N}d(m) = \{0\}$ for all $j \in J$, $m \in \mathcal{N}$. Using Lemma 2.3 together with the 3-primeness of $\mathcal{N}$, it follows that $d = 0$.

Now assuming that $Z(\mathcal{N}) \neq \{0\}$. By Lemma 2.6, we can write $jnd(j) = njd(j)$ for all $j \in J$, $n \in \mathcal{N}$, which reduces to $d(j)\mathcal{N}[j,m] = \{0\}$ for all $j \in J$, $m \in \mathcal{N}$ and by the 3-primeness of $\mathcal{N}$, we conclude that

$$j \in Z(\mathcal{N}) \text{ or } d(j) = 0 \qquad \text{for all } j \in J. \tag{3.4}$$

Suppose that there is $j_0 \in J$ such that $d(j_0) = 0$. Using our hypothesis, then we have $d(j_0(j_0 \circ n)) \in Z(\mathcal{N})$ for all $n \in \mathcal{N}$. Applying the definition of d and using our assumption, we get $j_0 d((j_0 \circ n)) \in Z(\mathcal{N})$ for all $n \in \mathcal{N}$. By Lemma 2.1 (i), we conclude

$$j_0 \in Z(\mathcal{N}) \text{ or } d((j_0 \circ n)) = 0 \qquad \text{for all } n \in \mathcal{N}. \tag{3.5}$$

If $d((j_0 \circ n)) = 0$ for all $n \in \mathcal{N}$, using the 2-torsion freeness of $\mathcal{N}$, we get

$$j_0 d(n) = 0 \qquad \text{for all } n \in \mathcal{N}. \tag{3.6}$$

Replacing n by $j_0 nm$ in (3.6) and using it, then we get $j_0^2 nd(m) = 0$ for all $n, m \in \mathcal{N}$. Since $d \neq 0$, the 3-primeness of $\mathcal{N}$ gives $j_0^2 = 0$, which is a contradiction with Lemma 2.3. Then (3.4) becomes $J \subseteq Z(\mathcal{N})$, which forces that $\mathcal{N}$ is commutative ring by Lemma 2.1 (iii).

(ii) ⇒ (iii): Suppose that $Z(\mathcal{N}) = \{0\}$, then $d(j^2) = 0$ for all $j \in J$, by the 2-torsion freeness of $\mathcal{N}$, we get

$$jd(j) = 0 \qquad \text{for all } j \in J. \tag{3.7}$$

Using Lemma 2.6, we can write $jnd(j) = njd(j)$ for all $j \in J$, $n \in \mathcal{N}$, from (3.7), we get $jnd(j) = 0$ for all $j \in J$, $n \in \mathcal{N}$, which implies $j\mathcal{N}d(j) = \{0\}$ for all $j \in J$, $n \in \mathcal{N}$ and by the 3-primeness of $\mathcal{N}$, we deduce that $d(J) = \{0\}$. Using the same techniques as used in the proof of (i)⇒(iii), we conclude that $d = 0$.

Assuming that $Z(\mathcal{N}) \neq \{0\}$. By Lemma 2.5, we can write

$$jnd(j^2) = njd(j^2) \qquad \text{for all } x, y \in \mathcal{N}, \tag{3.8}$$

which implies that

$$d(j^2)\mathcal{N}[j,m] = \{0\} \qquad \text{for all } j \in J,\ m \in \mathcal{N}.$$

By the 3-primeness of $\mathcal{N}$, we conclude that

$$j \in Z(\mathcal{N}) \text{ or } d(j^2) = 0 \qquad \text{for all } j \in J. \tag{3.9}$$

If there exists $j_0 \in J$ such that $d(j_0^2) = 0$, using the definition of d and the 2-torsion freeness of $\mathcal{N}$, then we have

$$j_0 d(j_0) = 0. \tag{3.10}$$

By Lemma 2.6, we can write $j_0 \mathcal{N} d(j_0) = \{0\}$. In view of the 3-primeness of $\mathcal{N}$, that follows $d(j_0) = 0$. Using our hypothesis, we have $d(j_0(2i^2)) \in Z(\mathcal{N})$ for all $i \in J$. Applying the definition of d and using our assumption, we get $j_0 d(2i^2) \in Z(\mathcal{N})$ for all $i \in J$. By the 2-torsion freeness of $\mathcal{N}$ and Lemma 2.1 (i) we conclude

$$j_0 \in Z(\mathcal{N}) \text{ or } id(i) = 0 \qquad \text{for all } i \in J. \tag{3.11}$$

If $id(i) = 0$ for all $i \in J$. Using the same techniques as used in the proof of (ii) ⇒ (iii), we conclude that $d = 0$. Then (3.9) becomes

$$J \subseteq Z(\mathcal{N}) \text{ or } d = 0.$$

∎

Corollary 3.2. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a left derivation d, then the following assertions are equivalent:*

(i) $d(\mathcal{N}) \subseteq Z(\mathcal{N})$;

(ii) $d(\mathcal{N}^2) \subseteq Z(\mathcal{N})$;

(iii) *$\mathcal{N}$ is a commutative ring or $d = 0$.*

The following example proves that the 3-primeness of $\mathcal{N}$ in Theorem 3.1 cannot be omitted.

Example 3.3. Let $\mathcal{R}$ be a 2-torsion right or left near-ring which is not abelian. Define $\mathcal{N}$, J and d by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ r & 0 & 0 \\ s & t & 0 \end{pmatrix} : r, s, t, 0 \in \mathcal{R} \right\}, J = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p & 0 & 0 \end{pmatrix} : p, 0 \in \mathcal{R} \right\},$$

$$d \begin{pmatrix} 0 & 0 & 0 \\ r & 0 & 0 \\ s & t & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & t & 0 \end{pmatrix}.$$

Then $\mathcal{N}$ is a right or left near-ring which is not 3-prime, J is a nonzero Jordan ideal of $\mathcal{N}$ and d is a nonzero left derivation of $\mathcal{N}$ which is not a derivation. It is easy to see that

(i) $d(J) \subseteq Z(\mathcal{N})$.

(ii) $d(J^2) \subseteq Z(\mathcal{N})$.

However, neither $d = 0$ nor $\mathcal{N}$ is a commutative ring.

4. Some polynomial identities in right near-rings involving left derivations

This section is motivated by [6, Theorem 3.6 and Theorem 3.12]. Our aim in the current paper is to extend these results of Jordan ideals on 3-prime near-rings admitting a nonzero left derivation.

Theorem 4.1. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero left derivation d and a multiplier H satisfying $d(x \circ j) = H(x \circ j)$ for all $j \in J$, $x \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

Proof. Assume that $d(x \circ j) = H(x \circ j)$ for all $j \in J$, $x \in \mathcal{N}$. If $H = 0$, the last equation becomes $d(x \circ j) = 0$ for all $j \in J$, $x \in \mathcal{N}$. And recalling Lemma 2.2 (ii), then $(x \circ j) \in Z(\mathcal{N})$ for all $j \in J$, $x \in \mathcal{N}$, so $\mathcal{N}$ is a commutative ring by Lemma 2.5 (i).

Now assume that $H \neq 0$ and $d(x \circ j) = H(x \circ j)$ for all $j \in J$, $x \in \mathcal{N}$. Replacing x by xj and using the fact that $(xj \circ j) = (x \circ j)j$, we get

$$d((x \circ j)j) = H((x \circ j)j) \qquad \text{for all } i, j \in J,\ x \in \mathcal{N}.$$

By the definition of d and H, we obtain

$$(x \circ j)d(j) + jd(x \circ j) = H(x \circ j)j \qquad \text{for all } i, j \in J,\ x \in \mathcal{N}.$$

Replacing j by $(y \circ i)$, where $i \in J$, $y \in \mathcal{N}$, in the preceding expression, we can see that

$$(x \circ (y \circ i))d((y \circ i)) + (y \circ i)d(x \circ (y \circ i)) = H(x \circ (y \circ i))(y \circ i)$$

for all, $i, j \in J$, $x, y \in \mathcal{N}$.

By a simplification, we thereby obtain

$$(y \circ i)H(x \circ (y \circ i)) = 0 \qquad \text{for all } i, j \in J,\ x, y \in \mathcal{N}. \tag{4.1}$$

Applying H on (4.1), it follows that

$$(y \circ i)H(H(x \circ (y \circ i))) = 0 \qquad \text{for all } i, j \in J,\ x, y \in \mathcal{N}. \tag{4.2}$$

Applying d on (4.1) and recalling (4.2), we get

$$H(x \circ (y \circ i))H(y \circ i) = 0 \qquad \text{for all } x, y \in \mathcal{N}, \tag{4.3}$$

which gives

$$xH(y \circ i)H(y \circ i) = -H(y \circ i)xH(y \circ i) \qquad \text{for all } x, y \in \mathcal{N}.$$

Substituting xz instead of x in preceding equation and applying it, we obviously obtain

$$\begin{aligned} xzH(y \circ i)H(y \circ i) &= (-H(y \circ i))xzH(y \circ i) \\ &= x(-H(y \circ i))zH(y \circ i) \qquad \text{for all } x, y, z \in \mathcal{N}. \end{aligned}$$

This forces that

$$[x, (-H(y \circ i))]zH(y \circ i) = 0 \qquad \text{for all } x, y, z \in \mathcal{N}.$$

Then $[x, (-H(y \circ i))]\mathcal{N}H(y \circ i) = \{0\}$ for all $x, y \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$, we get

$$(-H(y \circ i)) \in Z(\mathcal{N}) \qquad \text{for all } i \in J,\ y \in \mathcal{N}. \tag{4.4}$$

Substituting yi instead y in (4.4), $(-H(y \circ i))i \in Z(\mathcal{N})$ for all $i \in J$, $y \in \mathcal{N}$. It follows that Lemma 2.1 (i)

$$H(y \circ i) = 0 \ \text{ or } \ i \in Z(\mathcal{N}) \qquad \text{for all } i \in J,\ y \in \mathcal{N}. \tag{4.5}$$

Suppose that there exists an element $i_0 \in J$ such that

$$H(y \circ i_0) = 0 \qquad \text{for all } y \in \mathcal{N}, \tag{4.6}$$

which implies $(-i_0)H(y) = H(y)i_0$ for all $y \in \mathcal{N}$. Replacing y by xyz in the last equation, we get

$$(-i_0)H(xyz) = H(xyz)i_0 \qquad \text{for all } x, y, z \in \mathcal{N},$$

which means that

$$(-i_0)xyH(z) = x(-i_0)yH(z) \qquad \text{for all } x, y, z \in \mathcal{N},$$

so $[x, -i_0]\mathcal{N}H(z) = \{0\}$ for all $x, z \in \mathcal{N}$. Since $H \neq 0$ and $\mathcal{N}$ is 3-prime, we get $-i_0 \in Z(\mathcal{N})$. Now substituting $-i_0$ instead i in (4.4), we obtain

$-H(y \circ (-i_0)) \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$, which implies $(-H(2y))(-i_0) \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$, using Lemma 2.1 (i), we get $-2H(y) \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$ or $i_0 = 0$. Thus (4.5) becomes

$$-2H(y) \in Z(\mathcal{N}) \text{ for all } y \in \mathcal{N} \quad \text{or} \quad J \subseteq Z(\mathcal{N}). \tag{4.7}$$

CASE 1: If $-2H(y) \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$. Replacing y by zt in the last equation, we obtain $(-2H(z))t \in Z(\mathcal{N})$ for all $z, t \in \mathcal{N}$. Since $\mathcal{N}$ is 2-torsion free and $H \neq 0$, we obtain $\mathcal{N} \subseteq Z(\mathcal{N})$ by Lemma 2.1 (ii). Which assures that $\mathcal{N}$ is a commutative ring by Lemma 2.1 (iii).

CASE 2: If $J \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring by virtue of Lemma 2.1 (iii). ∎

The next result is an immediate consequence of Theorem 3.1, just to take $H = id_{\mathcal{N}}$ in Theorem 4.1.

COROLLARY 4.2. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero left derivation d such that $d(x \circ j) = x \circ j$ for all $j \in J$, $x \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

THEOREM 4.3. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero right Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a left derivation d and a nonzero multiplier H satisfying any one of the following identities:*

(i) $d(H(J)) = \{0\}$;

(ii) $d(H(J^2)) = \{0\}$;

(iii) $d(H(n \circ j)) = d(H([n, j]))$ *for all* $j \in J$, $n \in \mathcal{N}$;

(iv) $d(H(nj)) = H(j)d(n)$ *for all* $j \in J$, $n \in \mathcal{N}$,

then $d = 0$.

Proof. (i) Assume that $d(H(J)) = \{0\}$. Therefore, by Lemma 2.2 (i) and Lemma 2.4 (i), $\mathcal{N}$ is a commutative ring. Using our hypothesis and by the 2-torsion freeness of $\mathcal{N}$, we can see $d(H(j)n) = 0$ for all $j \in J$, $n \in \mathcal{N}$. Applying the definition of d, we obtain

$$H(j)d(n) = 0 \quad \text{for all } j \in J, \ n \in \mathcal{N}. \tag{4.8}$$

Replacing j by $j \circ m$, where $m \in \mathcal{N}$ in (4.8) and using it, we can easily arrive at $H(J)\mathcal{N}d(n) = \{0\}$ for all $n \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$, we conclude that $d(\mathcal{N}) = \{0\}$ or $H(J) = \{0\}$. If $H(J) = \{0\}$, then $H((j \circ m) \circ n)) = 0$ for all $j \in J$, $n, m \in \mathcal{N}$. In view of the 2-torsion freeness of $\mathcal{N}$, we get $J\mathcal{N}H(n) = \{0\}$ and by the 3-primeness of $\mathcal{N}$, we obtain $J = \{0\}$ or $H(n) = \{0\}$, that would contradict with our hypothesis, then $d = 0$.

(ii) Suppose that $d\left(H(J^2)\right) = \{0\}$, according to Lemma 2.2 (i) and Lemma 2.4 (i), $\mathcal{N}$ is a commutative ring. Now using our hypothesis, $d(H(i(j \circ n))) = 0$ for all $i, j \in J$, $n \in \mathcal{N}$, by the 2-torsion freeness of $\mathcal{N}$, we can see $d(H(ijn)) = 0$ for all $i, j \in J$, $n \in \mathcal{N}$. Applying the definition of d, we obtain

$$iH(j)d(n) = 0 \qquad \text{for all } i, j \in J, \ n \in \mathcal{N}. \tag{4.9}$$

Substituting $j \circ m$ for j, where $m \in \mathcal{N}$ and $i \circ t$ for j, where $t \in \mathcal{N}$ in (4.9) and using it, we can easily arrive at $J\mathcal{N}H(J)\mathcal{N}d(n) = \{0\}$ for all $n \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$, we conclude that $d(\mathcal{N}) = \{0\}$ or $H(J) = \{0\}$ or $J = \{0\}$. If $H(J) = \{0\}$, using the same techniques as we have used in the proof of (i), one can easily find $d = 0$.

(iii) Suppose that $d(H(n \circ j)) = d(H([n, j]))$ for all $j \in J$, $n \in \mathcal{N}$. Taking nj instead of n, we obtain

$$d(H((n \circ j)j)) = d(H([n, j]j)) \qquad \text{for all } j \in J, \ n \in \mathcal{N}.$$

Using the definition of d, we get

$$H(n \circ j)d(j) + jd(H(n \circ j)) = H([n, j])d(j) + jd(H([n, j]))$$

for all $j \in J$, $n \in \mathcal{N}$.

By a simplification, we can rewrite this equation as

$$2jH(n)d(j) = 0 \qquad \text{for all } j \in J, \ n \in \mathcal{N}.$$

Substituting zyt for n, where $x, y, z \in \mathcal{N}$ in last equation, we can see

$$2jyH(z)td(j) = 0 \qquad \text{for all } j \in J, \ y, z, t \in \mathcal{N}.$$

By the 2-torsion freeness of $\mathcal{N}$, the above equation becomes $j\mathcal{N}H(z)\mathcal{N}d(j) = \{0\}$ for all $j \in J, z \in \mathcal{N}$. Since $\mathcal{N}$ is 3-prime and $H \neq 0$, it follows that $d(J) = \{0\}$, which forces that $d = 0$ by (i).

(iv) Suppose that $d(H(nj)) = H(j)d(n)$ for all $j \in J$, $n \in \mathcal{N}$. From this equation we obtain

$$d(nH(j)) = H(j)d(n) \qquad \text{for all } j \in J, \ n \in \mathcal{N}.$$

Using the definition of d, we have

$$nd(H(j)) + H(j)d(n) = H(j)d(n) \qquad \text{for all } j \in J,\ n \in \mathcal{N}.$$

Then $nd(H(j)) = 0$ for all $j \in J$, $n \in \mathcal{N}$, which implies that $d(H(J)) = \{0\}$ by invoking the 3-primeness of $\mathcal{N}$, and consequently $d = 0$ by (i). ∎

The next result is an immediate consequence of Theorem 3.1, just to take $H = id_{\mathcal{N}}$ in Theorem 4.6.

Corollary 4.4. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero right Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a left derivation d and a nonzero multiplier H satisfying any one of the following identities:*

(i) $d(J) = \{0\}$;

(ii) $d(J^2) = \{0\}$;

(iii) $d(n \circ j) = d([n, j])$ *for all* $j \in J$, $n \in \mathcal{N}$,

(iv) $d(nj) = jd(n)$ *for all* $j \in J$, $n \in \mathcal{N}$;

then $d = 0$.

The following example proves that the 3-primeness of $\mathcal{N}$ in Theorem 4.1 and Theorem 4.3 cannot be omitted.

Example 4.5. Let $\mathcal{S}$ be a 2-torsion right near ring which is not abelian. Define $\mathcal{N}$, J, d and H by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 & p \\ 0 & q & 0 \\ 0 & 0 & 0 \end{pmatrix} : p, q, 0 \in \mathcal{S} \right\}, \quad J = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 0 \end{pmatrix} : s, 0 \in \mathcal{S} \right\},$$

$$d \begin{pmatrix} 0 & 0 & p \\ 0 & q & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad H \begin{pmatrix} 0 & 0 & p \\ 0 & q & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\mathcal{N}$ is a right near-ring which is not 3-prime, J is a nonzero Jordan ideal of $\mathcal{N}$, d is a nonzero left derivation of $\mathcal{N}$, and H is a nonzero multiplier of $\mathcal{N}$, such that

(i) $d(x \circ j) = H(x \circ j)$ for all $j \in J$, $x \in \mathcal{N}$;

(ii) $d(H(J)) = \{0\}$;

(iii) $d\left(H(J^2)\right) = \{0\}$;

(iv) $d(H(n \circ j)) = d(H([n, j]))$ for all $j \in J$, $n \in \mathcal{N}$;

(v) $d(H(nj)) = H(j)d(n)$ for all $j \in J$, $n \in \mathcal{N}$.

However, neither $d = 0$ nor $\mathcal{N}$ is a commutative ring.

THEOREM 4.6. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$ and let H a nonzero multiplier on $\mathcal{N}$. Then there is no nonzero left derivation d such that $d(x \circ j) = H([x, j])$ for all $j \in J$, $x \in \mathcal{N}$.*

Proof. Assume that

$$d(x \circ j) = H([x, j]) \qquad \text{for all } j \in J,\ x \in \mathcal{N}. \tag{4.10}$$

Replacing x by j, in (4.10), we get

$$2d(j^2) = d(j^2 + j^2) = d(j \circ j) = 0 \qquad \text{for all } j \in J.$$

By the 2-torsion freeness of $\mathcal{N}$, we get

$$0 = d(j^2) = 2jd(j) \qquad \text{for all } j \in J. \tag{4.11}$$

In view of the 2-torsion freeness of $\mathcal{N}$, this easily yields

$$jd(j) = 0 \qquad \text{for all } j \in J. \tag{4.12}$$

Replacing x by xj in (4.10), we get

$$d(xj \circ j) = H([xj, j]) \qquad \text{for all } j \in J,\ x \in \mathcal{N}.$$

Using the fact that $(xj \circ j) = (x \circ j)j$ and $[xj, j] = [x, j]j$, we obtain

$$d((x \circ j)j) = H([x, j]j) \qquad \text{for all } j \in J,\ x \in \mathcal{N}.$$

By the definition of d, the last equation is expressible as

$$(x \circ j)d(j) = [H([x, j]), j] \qquad \text{for all } j \in J,\ x \in \mathcal{N}.$$

Substituting xj instead x, it follows from (4.12) that

$$[H([xj, j]), j] = 0 \qquad \text{for all } j \in J,\ x \in \mathcal{N}. \tag{4.13}$$

Replacing x by $d(j)x$ in (4.13) and using (4.12), we can easily arrive at

$$[d(j)H(x)j^2, j] = 0 \qquad \text{for all } j \in J,\ x \in \mathcal{N}.$$

Which reduces to

$$d(j)H(x)j^3 = 0 \qquad \text{for all } j \in J,\ x \in \mathcal{N}.$$

Substituting rst instead x where $r, s, t \in \mathcal{N}$ in the last equation, we get $d(j)rH(s)tj^3 = 0$ for all $j \in J$, $r, s, t \in \mathcal{N}$, which implies $d(j)\mathcal{N}H(s)\mathcal{N}j^3 = \{0\}$ for all $j \in J$, $s \in \mathcal{N}$. Since $H \neq 0$ and using the 3-primeness hypothesis, it follows that

$$d(j) = 0 \ \text{ or } \ j^3 = 0 \qquad \text{for all } j \in J. \tag{4.14}$$

Suppose that there exists an element $j_0 \in J \setminus \{0\}$ such that $j_0^3 = 0$. Replacing j by j_0 and x by xj_0^2 in (4.10) and using (4.12), then

$$d(xj_0^2 \circ j_0) = H([xj_0^2, j_0] \qquad \text{for all } x \in \mathcal{N}.$$

Using our assumption, we find that

$$d(j_0xj_0^2) = H(-j_0xj_0^2) \qquad \text{for all } x \in \mathcal{N}.$$

By the definition of d, we get

$$j_0d(xj_0^2) + xj_0^2d(j_0) = -j_0H(x)j_0^2 \qquad \text{for all } x \in \mathcal{N}.$$

In light of equation (4.12), it follows easily that

$$j_0d(xj_0^2) = -j_0H(x)j_0^2 \qquad \text{for all } x \in \mathcal{N}.$$

So, by (4.14) and (4.12), we get

$$-j_0H(x)j_0^2 = 0 \qquad \text{for all } x \in \mathcal{N}.$$

Substituting rst instead x gives $-j_0rH(s)tj_0^2 = 0$ for all $r, s, t \in \mathcal{N}$, which implies $(-j_0)\mathcal{N}H(s)\mathcal{N}j_0^2 = \{0\}$ for all $s \in \mathcal{N}$. Since $H \neq 0$, by the 3-primeness of $\mathcal{N}$ and Lemma 2.3, the preceding expression leads to $j_0 = 0$.

Hence, (4.14) becomes $d(J) = \{0\}$, which leads to $d = 0$ by Theorem 3.1 (i); a contradiction. ∎

Corollary 4.7. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. Then there is no nonzero left derivation d such that $d(x \circ j) = [x, j]$ for all $j \in J$, $x \in \mathcal{N}$.*

THEOREM 4.8. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. Then $\mathcal{N}$ admits no nonzero left derivation d such that $d([x,j]) = d(x)j$ for all $j \in J$, $x \in \mathcal{N}$.*

Proof. Assume that

$$d([x,j]) = d(x)j \qquad \text{for all } x \in \mathcal{N},\ j \in J. \tag{4.15}$$

Replacing x by j in (4.15), we get

$$d(j)j = 0 \qquad \text{for all } j \in J. \tag{4.16}$$

Substituting xj instead of x in (4.15), we obtain

$$d([xj,j]) = d(xj)j \qquad \text{for all } j \in J,\ x \in \mathcal{N}.$$

Notice that $[xj,j] = [x,j]j$, the last relation can be rewritten as

$$d([x,j]j) = (xd(j) + jd(x))j \qquad \text{for all } j \in J,\ x \in \mathcal{N}.$$

The definition of d gives us

$$[x,j])d(j) + jd([x,j]) = jd(x)j \qquad \text{for all } j \in J,\ x \in \mathcal{N}.$$

Using our assumption, we obviously obtain

$$xjd(j) = jxd(j) \qquad \text{for all } j \in J,\ x \in \mathcal{N}. \tag{4.17}$$

Replacing x by yt in (4.17) and invoking it, we can see that

$$yjtd(j) = jytd(j) \qquad \text{for all } j \in J,\ y,t \in \mathcal{N}.$$

The last equation gives us $[y,j]\mathcal{N}d(j) = \{0\}$ for all $j \in J$, $x \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$, we get

$$j \in Z(\mathcal{N}) \ \text{ or } \ d(j) = 0 \qquad \text{for all } j \in J. \tag{4.18}$$

If there exists $j_0 \in J$ such that $d(j_0) = 0$. Using Lemma 2.4, we obtain $j_0 \in Z(\mathcal{N})$. In this case, (4.18) becomes $J \subseteq Z(\mathcal{N})$ which forces that $\mathcal{N}$ is a commutative ring by Lemma 2.1 (i). Hence (4.6) implies that $d(x)j = 0$ for all $j \in J$, $x \in \mathcal{N}$. Replacing j by $j \circ t$ in the last equation, it is obvious that $2d(x)tj = 0$ for all $j \in J$, $t,x \in \mathcal{N}$. It follows from the 2-torsion freeness of $\mathcal{N}$ that $d(x)\mathcal{N}j = \{0\}$ for all $j \in J$, $x \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$, we conclude that $d = 0$ or $J = \{0\}$; a contradiction. ∎

5. Conclusion

In this paper, we study the 3-prime near-rings with left derivations. We prove that a 3-prime near-ring that admits a left derivation satisfying certain differential identities on Jordan ideals becomes a commutative ring. In comparison to some recent studies that used derivations, these results are considered more developed. In future research, one can discuss the following issues:

(i) Theorem 3.1, Theorem 4.1, Theorem 4.3 and Theorem 4.6 can be proven by replacing left derivation d by a generalized left derivation.

(ii) The study of 3-prime near-rings that admit generalized left derivations satisfying certain differential identities on Lie ideals is another interesting work for the future.

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