Scattering by Arbitrary Cross-Section Cylinders Based on the T-Matrix Approach and Cylindrical to Plane Waves Transformation

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Abstract**—**Multiple scattering of parallel cylinders with arbitrary cross-section is computed using the T-matrix of each single scatterer and the general translational matrix for cylindrical waves. Usually, the recommended golden rule to compute the translational matrix is the Graf's addition theorem. However, this approach cannot be properly implemented for some geometries, like in a two-cylinder case when the center of one of them falls within the minimum circular cylinder that circumscribes the other one. In order to overcome this limitation, a transformation between cylindrical waves and plane waves, followed by propagation of the latter, is proposed. The new approach succeeds thanks to an adequate truncation of the evanescent plane wave spectrum. This strategy is demonstrated studying the scattering of three infinite elliptic metallic cylinders for different electrical sizes and observing the convergence of the results as a function of the truncated spectrum. Finally, to conclusively shown the interest and applicability of the approach, two more complex problems are treated: a group of infinite elliptic metallic cylinders where two different sizes are combined, and a practical real-life filter in SIW technology including several groups of rectangular dielectric cylinders.

Index Terms**— Graf's Addition Theorem, Multiple Scattering, Cylinders, Plane Wave Expansion, Cylindrical Wave Expansion.**

I. Introduction

SCATTERING of multiple parallel cylinders is a recurrent topic in electromagnetism. Since the pioneer paper by Twersky [1], many works have been published on this subject (see, for example, references in [2] and [3]).

The problem of scattering by multiple cylinders can be obviously studied by analyzing directly the whole problem [2], [4-5]. However, a much more efficient and well-known approach consists in studying first the scattering of the isolated cylinders and then connecting these solutions to obtain the response of the whole system. This connection implies transferring the scattering field between different sets of local coordinates [1], [6-12]. Scattering by each isolated cylinder is usually described in terms of amplitudes of standing cylindrical waves (containing all possible incident waves) and the amplitudes of the scattered waves. These complex amplitudes are related through a transition matrix (or T-matrix), a formalism that has been widely used in the context of spherical vector waves [13].

The T-matrix of an isolated cylinder can be calculated analytically in case of circular metallic or homogenous dielectric cylinders [14], whereas for arbitrary cross-section cylinders, the T-matrix calculation needs approximate or numerical methods, such as the null-field methods [2], finite-differences in frequency-domain (FDFD) [15], finite elements (FEM) [16], field matching methods [17], or modelling the scatterer by a certain number of circular cylinders [18]. But regardless of the selected approach, the subsequent coupling and combination of the isolated T-matrices, to obtain the scattering of a group of cylinders, has been always done in the literature using Graf's addition theorem, that relates the scattered field by one cylinder to the incoming field on the next one [1][6-12].

Unfortunately, Graf's addition theorem fails to provide accurate results for some geometries that will be described later. This work proposes a more general mathematical approach to transfer the field scattered between cylinders, based on the transformation between cylindrical and plane waves [19][20]. Such a transformation has previously been used successfully to solve the interaction of the scattered fields by circular cross-section cylinders with flat or rough boundaries [21-24].

It is worth mentioning at this point that there are other similar approaches, also based on the idea of studying the response of

Manuscript received April XX, 2021. This work was supported in part by Ministerio de Economía y Competitividad (MINECO)/Agencia Estatal de Investigación (AEI)/Fondo Europeo de Desarrollo Regional (FEDER), Unión Europea (UE), Spain, under projects TEC2017-83352-C2-2-P and TEC2017-83352-C2-1-P. Project co-financed by FEDER and Junta de Extremadura (GR21072). *(Corresponding Author Rafael Gómez-Alcalá.)*

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isolated elements and then connecting them. One of these is the so-called CMMoM, a combination on Characteristic Modes and the Method of Moments that can be applied to the calculation of multiple scattering by Arbitrary Cross-Section Cylinders [25]. A detailed comparison of the efficiencies of these methods is not easy, since it would require their implementation in the same programming language, the same solver and the same computer. For those cases where the Graf's addition theorem is applicable, the method in [25] seems to show similar performances to the T-matrix based method for multiple scattering, except that coupling matrices between isolated elements are all computed numerically in the CMMoM. However, as far as the authors know, there is no known example in the literature on the application of CMMoM to geometries where the Graf's addition theorem fails to provide accurate results.

The paper starts reviewing, for the sake of completeness, the classical treatment of multiple scattering by cylinders, based on the T-Matrix approach for individual cylinders and the Graf's addition theorem. The drawbacks of this approach are briefly outlined. Next, an alternative procedure to Graf's addition theorem, using the well-known transformations between cylindrical and plane waves and translation of the latter is given in detail. Finally, the capabilities of the proposed method are shown using two illustrative examples where the classic approach fails to produce accurate results: a group of very close elongated cylinders and a practical Substrate Integrated Waveguide (SIW) circuit, including cylinders delimited by metallic planes.

II. Theory

In this section, the time factor is taken to be exp{-jωt}, that is the one usually adopted in classical literature [19],[26-27].

A. Analysis of Multiple Cylinders Based on Translation of Cylindrical Waves

The T-Matrix ($\mathbf{T}_i$) of a single cylinder *i* relates the amplitudes $\mathbf{a}_i$ of standing (regular) cylindrical waves in which any incident wave can be expanded, to the amplitudes $\mathbf{b}_i$ of scattered waves by the cylinder, in the same local coordinate system

$$\mathbf{T}_i \mathbf{a}_i = \mathbf{b}_i, \tag{1}$$

here $\mathbf{a}_i$ and $\mathbf{b}_i$ are column vectors containing, respectively, the complex amplitudes of standing and scattered cylindrical waves in a local coordinate system for this cylinder. It should be noted that the T-Matrix is the source scattering matrix of a pure scatterer as defined by Yaghjian [26], which is related to the classical scattering matrix $\mathbf{S}_i$, given in terms of amplitudes of incoming and outgoing waves, as follows [27]

$$\mathbf{T}_i = \frac{1}{2}(\mathbf{S}_i - \mathbf{I}), \tag{2}$$

with $\mathbf{I}$ being the identity matrix. Although the number of cylindrical waves in the expansion is theoretically infinite, in practice it is truncated until a certain precision is achieved.

When considering a group of *N* parallel cylinders, the incident field on the cylinder *i* is given by the superposition of the field coming from outside the group of cylinders, and the contribution of the field scattered by the other cylinders, referred to the local coordinate system for cylinder *i*. These fields can be expressed in terms of complex amplitudes of cylindrical waves

$$\mathbf{a}_i = \mathbf{a}_{oi} + \sum_{\substack{k=1 \\ i\neq k}}^{N} \mathbf{G}_{ik} \mathbf{b}_k, \tag{3}$$

where $\mathbf{a}_{oi}$ accounts for the field coming from outside, and $\mathbf{G}_{ik}$ is the general translational matrix, that relates scattered waves by cylinder *k* in its coordinate system with standing waves in cylinder *i* in its coordinate system.

The elements of $\mathbf{G}_{ik}$ can be directly obtained by using the addition theorem for cylindrical harmonics (Graf's addition theorem) [28]:

$$H_n^{(1)}(\kappa d_k) e^{jn\varphi_k} = \sum_{m=-\infty}^{\infty} H_{n-m}^{(1)}(\kappa d_{ki}) e^{j(n-m)(\varphi_{ki})} \mathrm{J}_m(\kappa d_i) e^{jm(\varphi_i)}, d_i < d_{ki} \tag{4}$$

where (*m,n*) are the integer indexes of the cylindrical harmonics related, respectively, to the *i*-th and *k*-th cylinders; $H_n^{(1)}$ is the Hankel function corresponding to outward propagation, J_m is the Bessel function; κ is the radial wavenumber of the cylindrical waves, (d_k, φ_k) and (d_i, φ_i) are polar coordinates of the same point referred to local coordinate systems centered at two different positions O_k and O_i respectively; and (d_{ki}, φ_{ki}) are the polar coordinates of O_i with respect to O_k (see Fig. 1).

With these notations, Graf's addition theorem allows us to write a generic element of $\mathbf{G}_{ik}$ under the form

$$\mathrm{g}_{ik}(m,n) = e^{jm(\varphi_{ri}-\varphi_{ki})} H_{n-m}^{(1)}(\kappa d_{ki}) e^{jn(\varphi_{ki}-\varphi_{rk})}, \tag{5}$$

where φ_{ri} and φ_{rk} are the rotation angles defining local rotations of both cylinders.

Once the matrix $\mathbf{G}_{ik}$ is fully defined, (3) can be introduced into (1) thus arriving to the following equation for the global set of *N* parallel cylinders

$$\mathbf{Ta} + \mathbf{TGb} = \mathbf{b}, \tag{6}$$

where $\mathbf{a}$ and $\mathbf{b}$ are column vectors formed by $\mathbf{a}_{oi}$ and $\mathbf{b}_i$, respectively, T=diag($\mathbf{T}_i$) is a diagonal block-matrix and $\mathbf{G}$ is a block matrix whose elements are the individual $\mathbf{G}_{ik}$ matrices

$$\mathbf{G} = \begin{pmatrix} \mathbf{0} & \mathbf{G}_{12} & \cdots & & \mathbf{G}_{1N} \\ \mathbf{G}_{21} & \mathbf{0} & & \mathbf{G}_{ik} & \\ \vdots & & \ddots & & \vdots \\ & \mathbf{G}_{ki} & \cdots & \mathbf{0} & \mathbf{G}_{N-1N} \\ \mathbf{G}_{N1} & & & \mathbf{G}_{NN-1} & \mathbf{0} \end{pmatrix}. \tag{7}$$

Equation (6) can be formally solved by a direct method

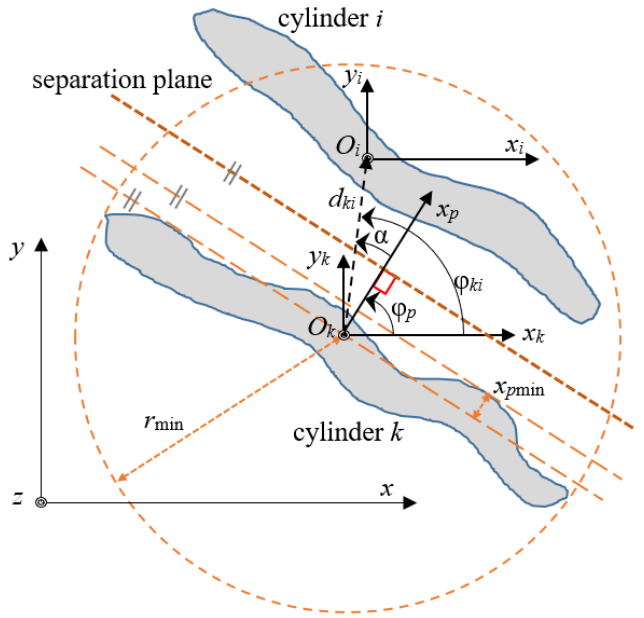

Fig. 1. Geometric description for the transformation between cylindrical and plane waves. Cross-section view.

$$\mathbf{b} = (\mathbf{I} - \mathbf{TG})^{-1}\mathbf{Ta}, \tag{8}$$

or, alternatively, by recursive [1], iterative [10] or accelerated [12] methods.

Thus, the strategy just described above seems to be general and valid for all geometries. Unfortunately, this is not always true, due the restriction $d_i < d_{ki}$, that is intrinsic to Graf's addition theorem formulation in (4) [2]. While this restriction does not affect the solution of problems involving only cylinders of circular section (where it is always satisfied), this is not the case for a pair of arbitrary cross-section cylinders, where the minimum-size circle circumscribing one of the cylinder section includes the center of the other cylinder section. In such cases, the Hankel function becomes singular in (4) for the highest orders because its order is high in relation to its argument. This justifies the need for seeking alternate approaches.

B. General Translational Matrix Based on the Transformation between Cylindrical and Plane Waves

A possible alternative to Graf's addition theorem for transferring the scattered fields, expressed in terms of cylindrical waves, between two sets of local coordinates associated to two different cylinders, is to use a transformation between cylindrical and plane waves, followed by a translation of the latter [19]. For a pair of arbitrary cross-shaped cylinders i and k, as shown in Fig.1, the sequence of operations is as follows:

(a) Express the field scattered by the cylinder k in terms of cylindrical waves.

(b) Expand each previous cylindrical wave in terms of plane waves in the x_p-axis direction of the cylindrical wave expansion. This axis is chosen so that it will be orthogonal to a separation plane between cylinders i and k.

(c) Translate the plane waves from the cylinder k to the cylinder i, according to the propagation theory of plane waves, in the x_p-axis direction.

(d) Expand each plane wave in terms of standing cylindrical waves in the cylinder i, so that the local x_p-axis in cylinder i be parallel to the local x_p-axis of cylinder k.

The main drawback of this approach compared to the one based on Graf's addition theorem is that in step (b) it is necessary to compute a numerical integration of the plane wave spectrum. However, the plane wave expansion converges in the region where $x_p > x_{pmin}$, with x_{pmin} being the largest x_p-coordinate of the cylinder. This region is different from the convergence region of the expansion in cylindrical waves, given by $r > r_{min}$, with r_{min} being the radius of the minimum circular cylinder that circumscribes the cylinder. It is important to note that $|x_{pmin}| \leq r_{min}$. Consequently, translation of cylindrical waves by means of transformation to plane waves can be carried out when the cylinders are separated by a plane that does not intersect them, even if $d_i > d_{ki}$.

In this way, the sequence of operations give rise to the following alternate expression for the elements of $\mathbf{G}_{ik}$

$$g_{ik}(m,n) = e^{\mathrm{j}m(\varphi_{ri}-\varphi_p)}\mathrm{W}(\kappa d_{ki},\alpha)e^{\mathrm{j}n(\varphi_p-\varphi_{rk})}, \tag{9}$$

where φ_p is the angle defined by the local x-axis for cylinder k, parallel to the global x-axis, and the x_p-axis, that defines the direction of propagation of the plane waves, and α is the angle defined by the x_p-axis and the line connecting the centers, as shown in Fig. 1.

According to [19], $\mathrm{W}(\kappa d_{ki},\alpha)$ for parallel cylinders in z-direction and plane wave propagation in the direction defined by the x_p-axis, can be calculated as

$$\mathrm{W}(\kappa d_{ki},\alpha) = 2\int_{\Gamma^+} \mathrm{D}_n(\gamma)e^{\mathrm{j}\kappa\hat{\gamma}\boldsymbol{d}_{ki}}\mathrm{D}_m^{\dagger}(\gamma)d\gamma \tag{10}$$

with $\hat{\gamma} = (\cos\gamma, \sin\gamma)$ being the direction of propagation, which can be complex, $\boldsymbol{d}_{ki}$ is a vector with modulus equal to d_{ki} and direction from center of cylinder k to center of cylinder i. The integration path Γ^+ in the complex plane γ goes from $\mathrm{j}\infty - \frac{\pi}{2}$ to $-\mathrm{j}\infty + \frac{\pi}{2}$, passing through the origin of coordinates, and D_n is the transformation function between cylindrical and plane waves [19][29]

$$\mathrm{D}_n(\gamma) = \frac{1}{\sqrt{2\pi}}(\mathrm{j})^{-n}e^{-\mathrm{j}n\gamma} \tag{11}$$

In the "daggered" version (†), all explicit j are set to (-j). Therefore, (10) can be expressed as

$$\mathrm{W}(\kappa d_{ki},\alpha) = \frac{1}{\pi}\int_{\mathrm{j}\infty-\frac{\pi}{2}}^{-\mathrm{j}\infty+\frac{\pi}{2}} (\mathrm{j})^{-(n-m)}e^{\mathrm{j}(\kappa d_{ki}\cos(\gamma-\alpha)-(n-m)\gamma)}d\gamma. \tag{12}$$

By making $\gamma = \beta - \frac{\pi}{2}$, (12) can also be written as

$$\mathrm{W}(\kappa d_{ki}, \alpha) = \frac{1}{\pi} \int_{\mathrm{j}\infty}^{-\mathrm{j}\infty+\pi} e^{\mathrm{j}(\kappa d_{ki}\sin(\beta-\alpha)-(n-m)\beta)} d\beta, \quad (13)$$

so that

$$\mathrm{W}(\kappa d_{ki}, \alpha) = \mathrm{W}_0 + \mathrm{W}_1 + \mathrm{W}_2 \quad (14)$$

with

$$\mathrm{W}_0 = \frac{1}{\pi} \int_{0}^{\pi} e^{\mathrm{j}(\kappa d_{ki}\sin(\beta-\alpha)-(n-m)\beta)} d\beta, \quad (15)$$

$$\mathrm{W}_1 = \frac{1}{\pi} \int_{\mathrm{j}\infty}^{0} e^{\mathrm{j}(\kappa d_{ki}\sin(\beta-\alpha)-(n-m)\beta)} d\beta, \quad (16)$$

and

$$\mathrm{W}_2 = \frac{1}{\pi} \int_{\pi}^{-\mathrm{j}\infty+\pi} e^{\mathrm{j}(\kappa d_{ki}\sin(\beta-\alpha)-(n-m)\beta)} d\beta. \quad (17)$$

It should be noted that the entire spectrum of propagating plane waves is considered in (15) since β is real. Eq. (15), except for the case where α = 0 that will be discussed later, should be carefully evaluated since it has an oscillating kernel. An efficient method for its computation has been reported in [30]

By using $B = \pi - \beta$, W_2 is expressed as

$$\mathrm{W}_2 = \frac{(-1)^{(n-m)}}{\pi} \int_{\mathrm{j}\infty}^{0} e^{\mathrm{j}(\kappa d_{ki}\sin(B+\alpha)+(n-m)B)} dB. \quad (18)$$

Now, the sum of W_1 and W_2, denoted as W_s, is written as a function of the propagation constant of evanescent plane waves in the direction of propagation. As a result, by making $\kappa_{xp} = -\mathrm{j}\kappa\sin(\beta)$ in (16) and (18)

$$\mathrm{W}_s = -\mathrm{j}\frac{2}{\pi} \int_{0}^{\infty} e^{-\kappa_{xp} d_{ki}\cos(\alpha)} \sinh\left((n-m)\operatorname{asinh}\left(\frac{\kappa_{xp}}{\kappa}\right) - \mathrm{j}\sqrt{{\kappa_{xp}}^2 + \kappa^2}\sin(\alpha) \right) \frac{1}{\sqrt{{\kappa_{xp}}^2 + \kappa^2}} d\kappa_{xp}, \quad (n-m) \text{ odd} \quad (19)$$

$$\mathrm{W}_s = -\mathrm{j}\frac{2}{\pi} \int_{0}^{\infty} e^{-\kappa_{xp} d_{ki}\cos(\alpha)} \cosh\left((n-m)\operatorname{asinh}\left(\frac{\kappa_{xp}}{\kappa}\right) - \mathrm{j}\sqrt{{\kappa_{xp}}^2 + \kappa^2}\sin(\alpha) \right) \frac{1}{\sqrt{{\kappa_{xp}}^2 + \kappa^2}} d\kappa_{xp}, \quad (n-m) \text{ even} \quad (20)$$

If (19) and (20) were computed correctly and accurately, extending the integration limits to infinity, this would be just the result provided by using Graf's addition theorem. However, the integral expressions developed in this work provide a most welcome additional degree of freedom.

In the situations where a particular geometry produces a diverging cylindrical wave expansion in the near field zone, one way to avoid such divergence when directly applying the Graf's addition theorem would be to eliminate the contribution of the higher index cylindrical waves in the expansion (4). But this would strongly limit the maximum achievable precision, as it is clearly shown in the first example of the next section.

On the other hand, integral expressions (19-20) offer a straightforward way to truncate the spectrum of evanescent plane waves and remove only the waves that are attenuated faster when the plane wave propagation is chosen in the x_p direction, orthogonal to the separation plane. This is achieved in practice by normalizing the variable of integration κ_{xp} as $\bar{\kappa}_{xp} = \frac{\kappa_{xp}}{\kappa}$ and by truncating the upper limit of integration (and hence the spectrum of plane waves) from infinity to a maximum value of $\bar{\kappa}_{xp}$ equal to $\bar{\kappa}_{xtr}$. The criteria for the choice of $\bar{\kappa}_{xtr}$ will be discussed in the first example of the next section.

The same idea has been successfully applied for the case of a transformation between spherical waves and plane waves in the near field [31][32].

With this strategy,

$$\mathrm{W}_s = -\mathrm{j}\frac{2}{\pi} \int_{0}^{\bar{\kappa}_{xtr}} \frac{e^{-\bar{\kappa}_{xp}\kappa d_{ki}\cos(\alpha)}}{\sqrt{{\bar{\kappa}_{xp}}^2 + 1}} \sinh\left((n-m)\operatorname{asinh}(\bar{\kappa}_{xp}) - \mathrm{j}\kappa d_{ki}\sqrt{{\bar{\kappa}_{xp}}^2 + 1}\sin(\alpha) \right) d\bar{\kappa}_{xp}, \quad (n-m) \text{ odd} \quad (21)$$

$$\mathrm{W}_s = -\mathrm{j}\frac{2}{\pi} \int_{0}^{\bar{\kappa}_{xtr}} \frac{e^{-\bar{\kappa}_{xp}\kappa d_{ki}\cos(\alpha)}}{\sqrt{{\bar{\kappa}_{xp}}^2 + 1}} \cosh\left((n-m)\operatorname{asinh}(\bar{\kappa}_{xp}) - \mathrm{j}\kappa d_{ki}\sqrt{{\bar{\kappa}_{xp}}^2 + 1}\sin(\alpha) \right) d\bar{\kappa}_{xp}, \quad (n-m) \text{ even} \quad (22)$$

In (21), $\mathrm{W}_s(-(n-m)) = (\mathrm{W}_s\ (n-m))^*$, and in (22) $\mathrm{W}_s(-(n-m)) = (-\mathrm{W}_s\ (n-m))^*$, where '*' denotes the conjugated. Therefore, W_s with $(n-m)<0$ can be immediately obtained from W_s with $(n-m)>0$.

A particular case of interest arises when the direction of propagation for plane waves can be chosen so that it matches the direction between cylinder centers. In such a case, $\alpha = 0°$ and $\varphi_p = \varphi_{ki}$. Consequently, by making $\alpha = 0°$ and $\phi = -\gamma$ in (12), the expression given by Sommerfeld for $H_{n-m}^{(1)}(\kappa d_{ki})$

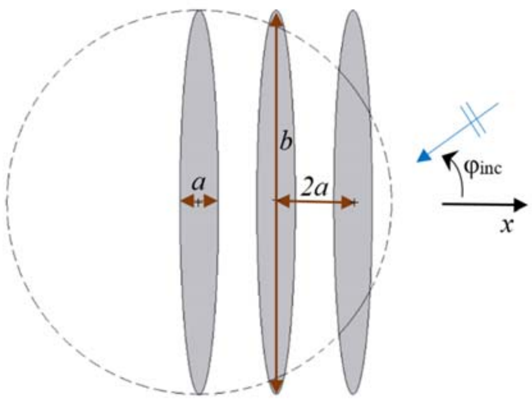

Fig. 2. Cross-section of three elliptic cylinders with $b = 10a, d = 2a$.

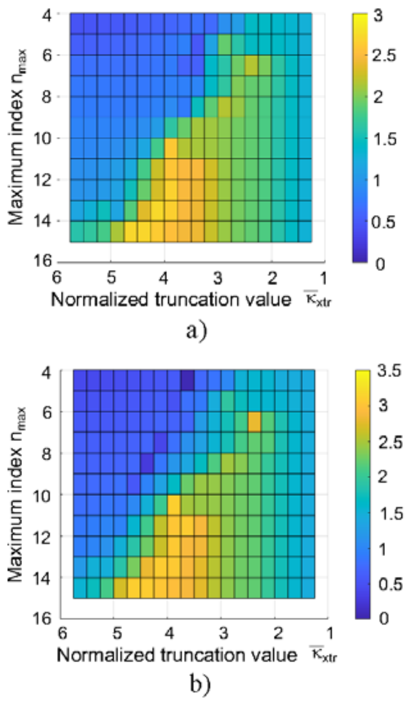

Fig. 3. Accuracy in terms of approximate number of decimal digits as a function of the maximum index n_{max} in the cylindrical wave expansion and the chosen value for $\bar{\kappa}_{xtr}$. *b*=0.5λ. a) *d*=*2a*. b) *d*=*3a*.

[33] [29, eq. (6.8.39)] is obtained

$$\mathrm{W}(\kappa d_{ki}, 0) = \frac{1}{\pi} \int_{\mathrm{j}\infty-\frac{\pi}{2}}^{-\mathrm{j}\infty+\frac{\pi}{2}} (\mathrm{j})^{-(n-m)} e^{\mathrm{j}\kappa d_{ki} cos(\phi)+\mathrm{j}(n-m)\phi} d\phi = H_{n-m}^{(1)}(\kappa d_{ki}) \qquad (23)$$

In order to avoid the singularity of the Hankel function, the spectrum of evanescent plane waves should be truncated. Therefore, $\mathrm{W}(\kappa d_{ki}, 0)$ will be again calculated as the sum of W_0 and W_s with $\alpha = 0°$

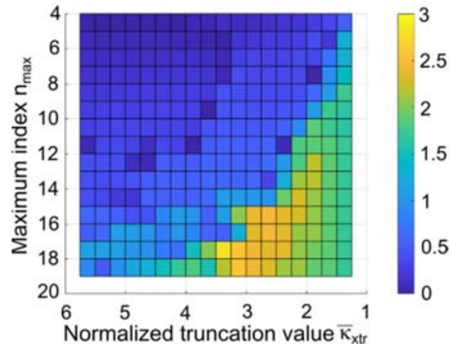

Fig. 4. Accuracy in terms of approximate number of decimal digits as a function of the maximum index n_{max} in the cylindrical wave expansion and the chosen value for $\bar{\kappa}_{xtr}$. *b*=λ. *d*=*2a*.

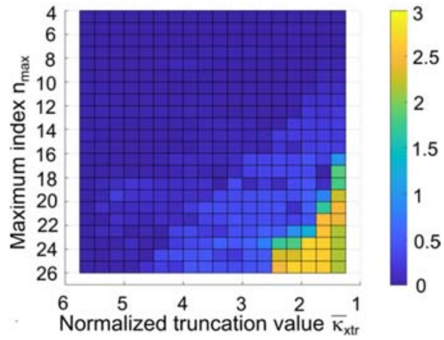

Fig. 5. Accuracy in terms of approximate number of decimal digits as a function of the maximum index n_{max} in the cylindrical wave expansion and the chosen value for $\bar{\kappa}_{xtr}$. *b*=2λ. *d*=*2a*.

$$\mathrm{W}_0(\alpha = 0) = \frac{1}{\pi} \int_0^{\pi} e^{\mathrm{j}(\kappa d_{ki} \sin(\beta) - (n-m)\beta)} d\beta \qquad (24)$$

$$\mathrm{W}_\mathrm{s}(\alpha = 0) = -\mathrm{j}\frac{2}{\pi} \int_0^{\bar{\kappa}_{xtr}} \frac{e^{-\bar{\kappa}_{xp}\kappa d_{ki}}}{\sqrt{\bar{\kappa}_{xp}{}^2+1}} \sinh\left((n-m)\mathrm{asinh}(\bar{\kappa}_{xp})\right) d\bar{\kappa}_{xp}, \quad (n-m) \text{ odd} \qquad (25)$$

$$\mathrm{W}_\mathrm{s}(\alpha = 0) = -\mathrm{j}\frac{2}{\pi} \int_0^{\bar{\kappa}_{xtr}} \frac{e^{-\bar{\kappa}_{xp}\kappa d_{ki}}}{\sqrt{\bar{\kappa}_{xp}{}^2+1}} \cosh\left((n-m)\mathrm{asinh}(\bar{\kappa}_{xp})\right) d\bar{\kappa}_{xp}, \quad (n-m) \text{ even} \qquad (26)$$

Equation (24) can be decomposed as follows

$$\mathrm{W}_0(\alpha = 0) = \mathrm{J}_{n-m}(\kappa d_{ki}) - \mathrm{j}\mathrm{E}_{n-m}(\kappa d_{ki}) \qquad (27)$$

with J_{n-m} being the Bessel function of the first kind and integer order, and E_{n-m} the Weber function of integer order, since

$$\mathrm{J}_{n-m}(\kappa d_{ki}) = \frac{1}{\pi} \int_0^{\pi} \cos((n-m)\beta - \kappa d_{ki} \sin(\beta))\, d\beta \qquad (28)$$

and

$$\mathrm{E}_{n-m}(\kappa d_{ki}) = \frac{1}{\pi}\int_0^{\pi} \sin((n-m)\beta - \kappa d_{ki}\sin(\beta))\,d\beta\,, \quad (29)$$

that can be quickly computed from recurrence formulas [34]. The procedure described above will be used only between pairs of cylinders where Graf's addition theorem is no longer applicable. For the rest of pairs, Graf's addition theorem is more efficient.

III. Results

In this section, some examples of scattering of parallel cylinders are presented, where Graf's addition theorem does not work properly. A criterion for a proper truncation is shown.

A. Study of three infinite elliptic metallic cylinders

The first example consists of three perfect electric conductor (PEC) elliptic cylinders in a regular arrangement along the *x*-axis, with the minor axis *a* being just one tenth of the major axis

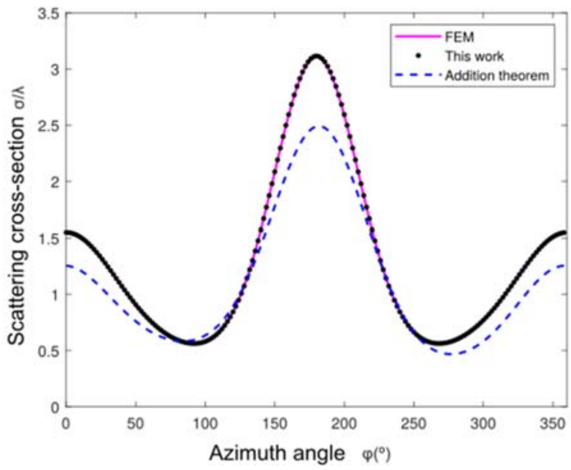

Fig. 6. Scattering cross-section pattern for φ_{inc}=0º with *b*=0.5λ.

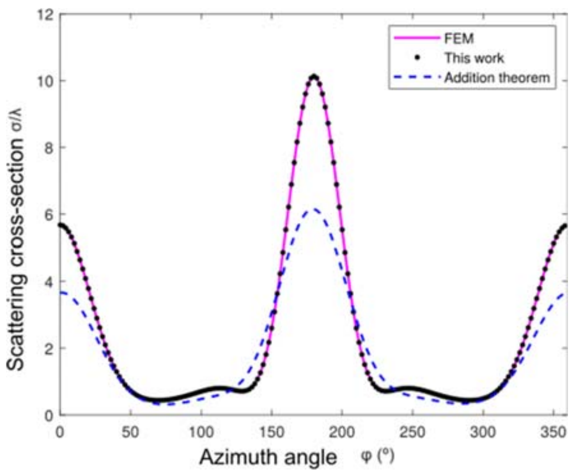

Fig. 7. Scattering cross-section pattern for φ_{inc}=0º with *b*=λ.

b. Fig. 2 shows a cross-section view of the problem and the incoming plane wave arriving at an incidence angle φ_{inc}. A Transverse Magnetic (TM) incident wave is considered. Also depicted in Fig.2 is the minimum circular cylinder that circumscribes one of the outer elliptic cylinders, to show that Graf's addition theorem cannot be properly applied here.

In order to compare the convergence of our solution as a function of both, the maximum index n_{max} in the cylindrical wave expansion and the chosen value for the truncation limit $\bar{\kappa}_{xtr}$ in (25) and (26), the same configuration is studied for three different electric sizes, *b*=0.5λ, *b*= λ and *b*= 2λ.

The T-matrix of an isolated elliptic cylinder is computed by means of the Finite Element Method (FEM) [16]. This method is also applied to compute the T-matrix of the array of cylinders for comparison purposes. It should be noted that this latter T-matrix cannot be directly compared with the result provided by (8), since in this expression the cylindrical waves are referred to the local coordinate system of each elliptic cylinder. Hence, (8) is properly post-processed by analytical translation to refer all the cylindrical waves to the same coordinate system. Then, both T-matrices are compared in terms of relative error (RE) [35]

$$\mathrm{RE} = \frac{\|\mathbf{T}_{\mathrm{a}} - \mathbf{T}_{\mathrm{FEM}}\|}{\|\mathbf{T}_{\mathrm{FEM}}\|}\,, \quad (30)$$

where $\mathbf{T}_{\mathrm{a}}$ is the T-matrix computed with the proposed method, with all the cylindrical waves referred to the same global coordinate system, and $\mathbf{T}_{\mathrm{FEM}}$ is the T-matrix of the array,

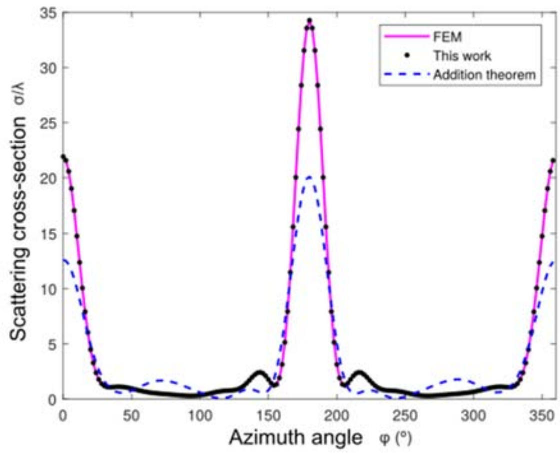

Fig. 8. Scattering cross-section pattern for φ_{inc}=0º with *b*=2λ.

computed with the FEM, that is taken as a reference solution. An accurate approximation of $\mathbf{T}_{\mathrm{a}}$ will provide scattered field results for any observation point located outside the minimum circular cylinder that contains the group of cylinders studied. Figs. 3, 4 and 5 show the accuracy in terms of approximate number of decimal digits, measured as -log(RE) [35], for *b*=0.5λ, *b*= λ and *b*= 2λ, respectively.

These Figs. 3-5 show the existence of a combination of values of the maximum index n_{max} and of the truncation value $\bar{\kappa}_{xtr}$ able to provide good accuracy. In general, the range of good values of the truncation limit $\bar{\kappa}_{xtr}$ increases with the maximum index number n_{max}. On the other hand, as it is well known (see for example the discussion in [10]), the lower value of the maximum index n_{max} needed to achieve a good accuracy increases with the size of the scatterer. Such value must be at least the one that includes all propagating cylindrical modes given by the integer part of κr_{min} [26]. However, a greater value of n_{max} is usually needed because evanescent modes can also be generated, and they can reach other cylinders with

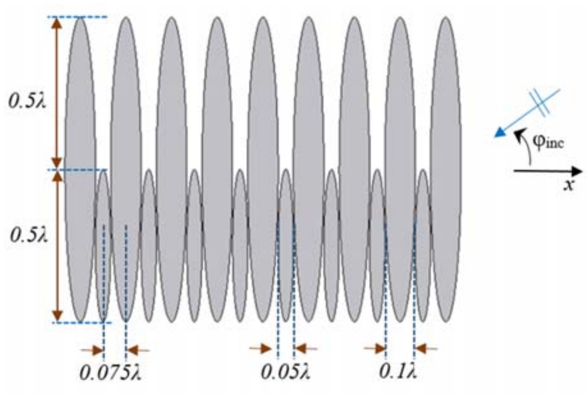

Fig. 9. Cross-section of an array of cylinders.

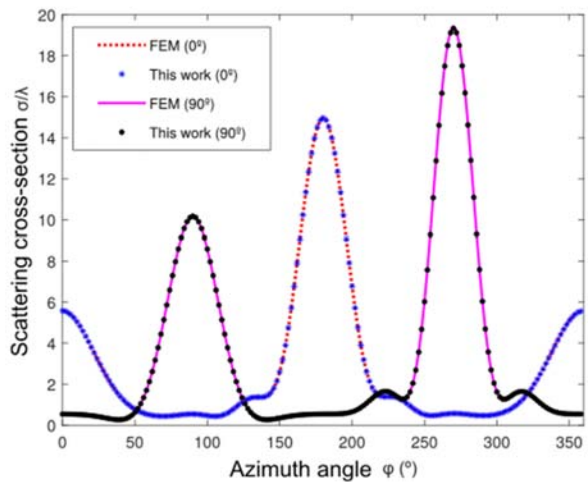

Fig. 10. Scattering cross-section pattern for φ_{inc}=0º and φ_{inc}=90º for the array of cylinders in Fig. 9, with TM polarization.

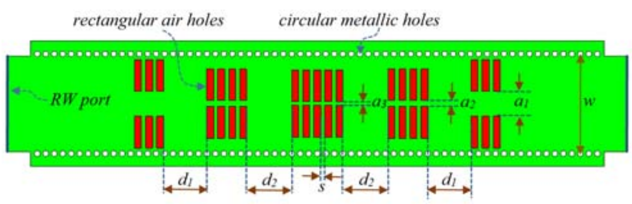

Fig. 11. SIW filter. Top view. d_1=7.634, d_2=7.964, a_1=4.32, a_2=1, a_3=0.63, s=0.734. (Dimensions in mm)

enough amplitude to be scattered. This is even more necessary in the case of overlap of the minimum circular cylinders that circumscribe arbitrary cross-section cylinders due to the close interactions between them. There is also a trend for the range of acceptable values of $\bar{\kappa}_{xtr}$ to slightly shrink for larger

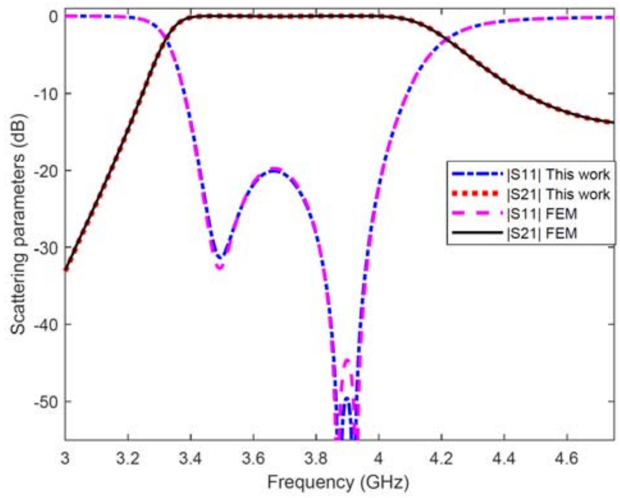

Fig. 12. Scattering parameters for the SIW filter of Fig. 11, compared with an in-house FEM.

scatterers. An exhaustive explanation of the phenomenon of relative convergence shown through figures 3-5, has been reported in [36] for the three-dimensional case. Such an explanation is directly applicable to the two-dimensional case treated in this paper.

Regarding the influence of the distance between the scatterers in the convergence region, it has been found that it does not change while the minimum circular cylinder circumscribing one of the cylinder sections includes the center of the other cylinder section. However, the accuracy will increase with distance, as can be seen by comparing Figs. 3a and 3b. In the first case, a spacing between scatterers centers of $d = 2a$ is used, whereas in the second, the spacing is increased to $d = 3a$. The main difference between them is the maximum precision achieved.

Figs. 6, 7 and 8 provide the scattering cross-section patterns for φ_{inc}=0º, for b=0.5λ, b= λ and b= 2λ, respectively. The results obtained by the proposed method with 2.5 decimal digits of

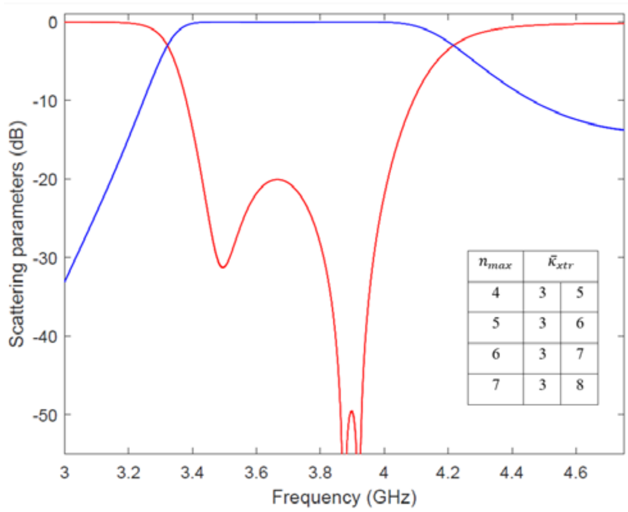

Fig. 13. Convergence of the proposed method of the scattering parameters. Results are superimposed for the values of n_{max} and $\bar{\kappa}_{xtr}$ given inside.

accuracy are compared with the FEM simulation for the array and the best result obtained by using Graf's addition theorem. It can be seen that the results obtained are indistinguishable from those of FEM, whereas those obtained by applying Graf's addition theorem deviate greatly from them. This is because these last results have been obtained with a small value of n_{max}, so that the influence of higher order modes is ignored.

On the other hand, this small value of n_{max} cannot be increased in the direct application of the Graf's theorem, because then the numerical instabilities, due to the Hankel function approaching its singular value, would spoil the quality of the results.

B. Analysis of an array of infinite elliptic metallic cylinders mixing two different sizes

Fig. 9 shows an array of 17 infinite elliptic metallic cylinders of two different sizes that correspond to the two first elliptic cylinders studied in the previous paragraph.

In this case, it is necessary to use (15), (19) and (20) to compute the general translational coefficients between cylinders of different sizes, since the direction of propagation for plane waves cannot be chosen so that it matches the direction between cylinder centers ($\alpha \neq 0°$). Another important difference with respect to the previous cases is that cylinders are very close to each other, so that they are almost tangent.

Fig. 10 shows the scattering cross-section pattern for φ_{inc}=0º and φ_{inc}=90º, with TM polarization. Results were obtained by using the value of n_{max} and $\bar{\kappa}_{xtr}$, that provided the best results found in Fig. 3 and Fig. 4. For the computation of the general translational coefficients between cylinders of different sizes, $\bar{\kappa}_{xtr}$ is chosen to be the most restrictive, i.e. the lower value that corresponds to Fig. 4. With this choice, the accuracy achieved in relation to the FEM results is two decimal digits, providing an excellent agreement between both methods.

C. Filter in SIW technology with periodic perforations

In this example, a filter in SIW technology is analyzed with the method proposed in [16], which is based on the Generalized Scattering Matrix of isolated elements, and the use of general translational matrices. This filter uses evanescent waveguide sections of reduced permittivity. It has been designed in [37, Fig. 5] and here it is implemented with rectangular air holes. Fig. 11 shows the filter analyzed in this work, where the metal vias (circular metallic holes) have a diameter of 1 mm and a longitudinal spacing of 1.5 mm, and the rectangular air holes are 1.216 mm by 5.6 mm. The value of *w*, given between the centers of the metal vias, is 17.502 mm, so that the width of the equivalent rectangular waveguide (RW) used to excite the filter is 16.8 mm [38]. A substrate with a thickness of 0.64 mm and relative dielectric permittivity of 10 is used.

In order to analyze the filter following the method proposed in [16], the T-matrix of a single rectangular air hole surrounded by the substrate is previously computed with FEM. As it can be seen in Fig. 11, the rectangular air holes are so close that Graf's addition theorem cannot properly applied. Therefore, the translational coefficients between them are calculated through the transformation to plane waves proposed in this work. Results are shown in Fig. 12, compared with those provided with an in-house FEM [39] and a very good agreement is verified. Fig. 13 shows the convergence of the problem by superimposing the results obtained for the set of values of n_{max} and $\bar{\kappa}_{xtr}$ given inside the figure. As can be observed, there are practically no differences between them despite the range of values represented. This is because the electrical size of the rectangular air hole in the substrate is small (less than 0.25λ) throughout the frequency band that is in agreement with the convergence studies results carried out in the first example.

IV. Conclusion

This work deals with the problem of solving multiple scattering by parallel cylinders using the T-matrix of each single scatterer and the general translational matrix for cylindrical waves. This method is widely used, since it is much more efficient and versatile than directly solving the whole problem with some numerically intensive methods such as FEM, Method of Moments or FDTD. However, its use is strongly limited to those cases and geometries where Graf's addition theorem can be applied. To alleviate this drawback, this paper replaces the direct application of Graf's addition theorem by a method based on the transformation between cylindrical and planar waves and vice versa.

The new approach, while equivalent from a strict mathematical point of view to the original Graf's addition theorem, allows a full control of the higher evanescent modes existing in the near field, that are responsible for the practical failure of that theorem in some geometries.

The validity of the new concepts has been proved through the analysis of some canonical examples, like a set of elongated elliptical cylinders, where Graf's addition theorem cannot be used. Then, the practical interest of the proposed method is convincingly demonstrated through the solution of a real-life example in the form of a SIW filter.

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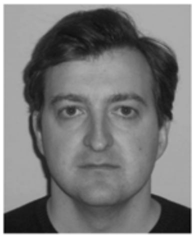

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