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Operadores en espacios de Rochberg

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Contents

Resumen

Los espacios de Rochberg son simultáneamente sumas torcidas de espacios de Banach y espacios de interpolación generalizados. La presente tesis se centra en el estudio de operadores en espacios de Rochberg, y más concretamente, de los operadores definidos en los espacios de Rochberg $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$, obtenidos a partir de la pareja de interpolación (ℓ_∞, ℓ_1). La lista de tales espacios contiene al espacio de Hilbert $\ell_2 = \mathfrak{R}_1(\ell_\infty, \ell_1)_{1/2}$ y al espacio de Kalton-Peck $Z_2 = \mathfrak{R}_2(\ell_\infty, \ell_1)_{1/2}$, por lo que en el Capítulo 1 estudiamos operadores en el espacio de Kalton-Peck. En el Capítulo 3 presentamos la teoría de los espacios de Rochberg; la mayor parte de los resultados pueden verse como generalizaciones naturales de la teoría clásica de interpolación compleja. Combinando resultados de interpolación compleja y teoría de espacios de Banach con algunas técnicas propias de la homología y teoría de categorías, en el Capítulo 4 estudiamos las propiedades de $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$ y de los operadores definidos en éstos. Finalmente, en el Capítulo 5 consideramos la pareja $(\mathcal{T}_2, \mathcal{T}_2^*)$ formada por la 2-convexificación del espacio de Tsirelson y su dual, y mostramos que los correspondientes espacios de Rochberg $\mathfrak{R}_n(\mathcal{T}_2, \mathcal{T}_2^*)_{1/2}$ son débil Hilbert.

Abstract

Rochberg spaces are simultaneously twisted sums of Banach spaces and generalized interpolation spaces. This dissertation focuses on the study of operators on Rochberg spaces, and more specifically, on the operators defined on the Rochberg spaces $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$ obtained from the interpolation pair (ℓ_∞, ℓ_1). The list of such spaces contains the Hilbert space $\ell_2 = \mathfrak{R}_1(\ell_\infty, \ell_1)_{1/2}$ and the Kalton-Peck space $Z_2 = \mathfrak{R}_2(\ell_\infty, \ell_1)_{1/2}$, and so, in Chapter 1 we study operators on the Kalton-Peck space. On the other hand, Chapter 3 describes the theory of Rochberg spaces; most of the results can be regarded as natural generalizations of classical Complex Interpolation Theory. Combining techniques from Banach Space Theory and Complex Interpolation Theory with some homological and categorical ideas on Banach spaces, in Chapter 4 we study the properties of $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$ and the operators defined on them. Finally, in Chapter 5 we consider the couple $(\mathcal{T}_2, \mathcal{T}_2^*)$ formed by the 2-convexification of the Tsirelson space and its dual, showing that the associated Rochberg spaces $\mathfrak{R}_n(\mathcal{T}_2, \mathcal{T}_2^*)_{1/2}$ are weak Hilbert.

Introduction

In 1969, motivated by the study of $\mathcal{L}_p$ spaces and absolutely p-summing operators, Lindenstrauss and Rosenthal asked in [70, problem 4b, p. 347] whether there exists a Banach space X containing a subspace $E \subset X$ such that both E and X/E are isomorphic to Hilbert spaces but X is itself not isomorphic to a Hilbert space.

In homological language this is equivalent to the existence of a non-trivial short exact sequence

$$0 \longrightarrow \ell_2 \longrightarrow X \longrightarrow \ell_2 \longrightarrow 0$$

Any such Banach space X is called a non trivial *twisted Hilbert space*, and the first example was obtained by Enflo, Lindenstrauss and Pisier [47]. Based on their work, Kalton discovered in [58] that short exact sequences of quasi-Banach spaces can be described by using a certain type of non-linear maps called quasilinear maps.

A couple of years later, Kalton and Peck [62] devised a method to construct quasilinear maps on Banach spaces with unconditional bases and obtained several examples of non trivial twisted Hilbert spaces. Among them, the most important one is the now called *Kalton-Peck space* Z_2, generated by the quasilinear map $\mathsf{KP} : \ell_2 \to \ell_\infty$ defined by

$$\mathsf{KP}(x) = x \log \frac{|x|}{\|x\|_{\ell_2}}.$$

At the moment of its appearence, the Kalton-Peck space was found to be quite an exotic space and served as counterexample for some conjectures in Banach Space Theory. A problem that is still unsettled nowadays is whether Z_2 is isomorphic to its hyperplanes. This is known as the *Hyperplane Problem*, and appears in several papers [57, 59, 24].

At the initial stage of this work we studied operators on twisted Hilbert spaces. The main motivation was to acquire further insight on the Hyperplane Problem. At the begining of the 80's, examples of operators on Z_2 were scarce:

(i) If $\mathfrak{u} = (u_n)_{n\in\mathbb{N}} \subset \ell_2$ is a sequence of normalized blocks then the operator $T_2^{\mathfrak{u}} : Z_2 \to Z_2$ defined by

$$T_2^{\mathfrak{u}}(e_j, 0) = (u_j, 0) \quad \text{and} \quad T_2^{\mathfrak{u}}(0, e_j) = (\mathsf{KP}(u_j), u_j)$$

is an isometry. This operator receives the name of *block operator* and was used by Kalton [59] to show that every operator $T : Z_2 \to X$ is either strictly singular or an isomorphism on a complemented copy of Z_2.

(ii) If $T : \ell_2 \to \ell_2$ is bounded then $T_2 : Z_2 \to Z_2$ defined by $T_2(x, y) = (Tx, Ty)$ is bounded if and only if the commutator $[T, \mathsf{KP}] : \ell_2 \to \ell_2$ is bounded, i.e.,

$$\|T(\mathsf{KP}(x)) - \mathsf{KP}(Tx)\|_{\ell_2} \leq C\|x\|_{\ell_2}. \tag{1}$$

Examples of operators for which (1) does hold are the shift maps or the canonical projections associated to the unconditional basis of ℓ_2 [59, Section 4].

Some years later, in an apparently unrelated context, Rochberg and Weiss [83] developed the theory of *derived spaces*, a generalized framework for Complex Interpolation Theory that allowed to obtain commutator estimates as (1) in a systematic way. Their main result can be summarized as follows (see Chapter 2 for details):

- Given a couple (X_0, X_1) with associated differential map $\Omega_\theta : X_\theta \to X_0 + X_1$, if $T : X_0 + X_1 \to X_0 + X_1$ is interpolating then $[\Omega_\theta, T] : X_\theta \to X_\theta$ is bounded.

Some of the ideas of Rochberg and Weiss, such as the notion of differential map and derived space were subsequently generalized by Rochberg [82]. His main idea is to consider not just the evaluation of $f \in \mathcal{C}(X_0, X_1)$ and its first derivative f', but its truncated sequence of Taylor coefficients. More precisely, the n-th Rochberg space at θ is defined as

$$\mathfrak{R}_n(X_0, X_1)_\theta = \Big\{ \Big(\frac{1}{(n-1)!} f^{(n-1)}(\theta), \ldots, f'(\theta), f(\theta) \Big) \in (X_0 + X_1)^n \colon f \in \mathcal{C}(X_0, X_1) \Big\}$$

endowed with the quotient norm

$$\|(x_{n-1}, \ldots, x_0)\| = \inf\{\|f\|_{\mathcal{C}} \colon \frac{1}{i!} f^{(i)}(\theta) = x_i,\ 0 \le i \le n-1\}.$$

Therefore, first order Rochberg spaces are classical interpolation spaces and second order Rochberg spaces are the derived spaces. Moreover, each Rochberg space can be represented as a twisted sum of lower order Rochberg spaces. Thus, Rochberg spaces are simultaneously generalized interpolation spaces and twisted sums of increasing complexity.

The relationship between Rochberg spaces and the Kalton-Peck space was unraveled by Rochberg and Weiss [83, Section 3.D], who showed that Z_2 is the second Rochberg space at $\theta = 1/2$ of the couple (ℓ_∞, ℓ_1). This provided a cohesive theory that explained virtually any property known for Z_2 up to that moment; in particular, operators described in (ii) were obtained by the generalized Interpolating Principle for derived spaces, while block operators could be obtained by a generalized Stein Interpolation Principle.

We studied in [32] operators on Kalton-Peck space. However, we soon realized that most properties of operators on Z_2 could be extended to operators on the higher order Rochberg spaces $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$. The objective of this work is to show how this is the case and, more generally, to study operators on general Rochberg spaces, a topic we covered in the papers [33, 31, 30].

We now describe the structure of the work:

Chapter 1 is dedicated to study bounded operators on Z_2. The original results are based on the work

> [32] J.M.F. Castillo, M. González, R. Pino, *Operators on the Kalton-Peck space* Z_2, (to appear in) Israel J. Math. (2023).

Any linear map $T : Z_2 \to Z_2$ can be represented as a 2×2 matrix $\begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ where the entries are suitable linear maps on $\mathbb{K}^{\mathbb{N}}$. We prove in Theorem 1.2.1 that the linear map $\begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ is bounded on Z_2 if and only if the following maps are bounded:

$$\boxed{\begin{array}{c} \delta : \ell_2 \to \ell_2 \qquad \gamma : \ell_f \to \ell_2 \qquad \beta - \mathsf{KP} \circ \gamma : \ell_f \to \ell_2 \\ \delta + \gamma \circ \mathsf{KP}^{-1} : \ell_f^* \to \ell_2 \qquad \gamma + \delta \circ \mathsf{KP} : \ell_2 \to \ell_2 \\ \alpha + \beta \circ \mathsf{KP}^{-1} - \mathsf{KP}\left(\delta + \gamma \circ \mathsf{KP}^{-1}\right) : \ell_f^* \longrightarrow \ell_2 \end{array}}$$

Using this result we characterize bounded operators whose entries are diagonal operators; this extends a result due to Johnson, Lindenstrauss and Schechteman [57]. We also study the particular cases of triangular operators, namely when $\delta = 0$. For this class of operators we obtain Theorem 1.2.4:

$$\boxed{\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \text{ is bounded if and only if } \begin{cases} \alpha : \ell_2 \to \ell_2 \\ \gamma : \ell_2 \to \ell_2 \\ \alpha \mathsf{KP} - \mathsf{KP}\gamma + \beta : \ell_2 \to \ell_2 \end{cases} \text{ are bounded}}$$

We end the chapter discussing the so-called *Johnson-Lindenstrauss-Schechteman conjecture*: any operator on Z_2 is a strictly singular perturbation of an upper triangular operator. First formulated in [57], it is one of the most interesting approaches to the hyperplane problem.

In Chapter 2 we present the basic results of Complex Interpolation Theory for pairs of Banach spaces that we will need in the rest of this work. Most results of Chapter 2 are not new, but we have included some of the proofs to show the analogy between the case of interpolation spaces and the case of Rochberg spaces that we will study in the forthcoming Chapter 3.

We include at the end of Chapter 2 a brief motivation and discussion about the concept of *derived space*, which may be thought as a forerunner of Rochberg spaces. We close the chapter with the example of Kalton-Peck space, showing that it arises as the derived space of the couple (ℓ_∞, ℓ_1).

Chapter 3 presents the theory of Rochberg spaces initiated by Rochberg in [82] and later expanded in [12] by Cabello, Castillo and Kalton. The presentation closely follows the structure of the paper

> [31] J.M.F. Castillo, R.Pino, *The Rochberg garden*, Expositiones Math. 41 (2023), 333–397.

Our intention is to show the paralelism between Chapter 2 and Chapter 3 in such a way that each major result in Chapter 2 has an analogue in the case of Rochberg spaces:

- The Interpolation Principle for operators generalizes to Rochberg's Commutator Theorem: if $T : X_0 + X_1 \to X_0 + X_1$ is interpolating then

$$T_n = \begin{pmatrix} T & & & & \\ & \ddots & & & \\ & & T & & \\ & & & T & \\ & & & & T \end{pmatrix}$$

  is bounded on $\mathfrak{R}_n(X_0, X_1)_\theta$ for any $0 < \theta < 1$.

- Under suitable hypothesis, the identity $(X_0, X_1)^*_\theta = (X_0^*, X_1^*)_\theta$ generalizes to $\mathfrak{R}_n(X_0, X_1)^*_\theta = \mathfrak{R}_n(X_0^*, X_1^*)_\theta$.
- We have a Reiteration Theorem for Rochberg spaces: if $0 < \alpha < \theta < \beta < 1$ then $\mathfrak{R}_n(X_0, X_1)_{(1-\theta)\alpha+\theta\beta}$ and $\mathfrak{R}_n\big((X_0, X_1)_\alpha, (X_0, X_1)_\beta\big)_\theta$ are isomorphic.
- One has a generalization of Stein Interpolation Principle for Rochberg spaces: if $(T_z)_{z\in\overline{\mathbb{S}}}$ is an *interpolating family of operators* (cf. Section 2.5), then

$$\begin{pmatrix} T_\theta & \frac{1}{1!}\frac{dT_z}{dz}|_\theta & \frac{1}{2!}\frac{d^2T_z}{dz^2}|_\theta & \cdots & \frac{1}{(k-1)!}\frac{d^{k-1}T_z}{dz^{k-1}}|_\theta \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & T_\theta & \frac{1}{1!}\frac{dT_z}{dz}|_\theta & \frac{1}{2!}\frac{d^2T_z}{dz^2}|_\theta \\ 0 & 0 & 0 & T_\theta & \frac{1}{1!}\frac{dT_z}{dz}|_\theta \\ 0 & 0 & 0 & 0 & T_\theta \end{pmatrix}$$

  defines a bounded operator on $\mathfrak{R}_n(X_0, X_1)_\theta$.

The Commutator Theorem was proven by Rochberg in his seminal paper [82]. The Duality Theorem for Rochberg spaces was obtained by Cabello, Castillo and Corrêa [11]. To the best of our knowledge, the reiteration result for Rochberg spaces is new and we proved it in [31]. The extension of Stein's principle was first considered by Carro [16] and later by Castillo and Ferenczi [25]. We provide explicit proofs of all the previous results and discuss some of their consequences.

Chapter 4 is devoted to study the properties of the Rochberg spaces $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$. Taking into account that $\mathfrak{R}_1(\ell_\infty, \ell_1)_{1/2} = \ell_2$ and $\mathfrak{R}_2(\ell_\infty, \ell_1)_{1/2} = Z_2$, ths family of Rochberg spaces may be regarded as the natural generalizations of both ℓ_2 and the Kalton-Peck space. The contents of the chapter are based in the following papers:

[33] J.M.F. Castillo, M. González, R. Pino, *The structure of Rochberg spaces.* Submitted. Available at arXiv:2305.09845.

[30] J.M.F. Castillo, W. Cuellar, M. González, R. Pino, *On symplectic Banach spaces*, RACSAM 117, paper 56, 2023, http://dx.doi.org/10.1007/s13398-023-01389-8.

[31] J.M.F. Castillo, R.Pino, *The Rochberg garden*, Expositiones Math. 41 (2023), 333–397.

The Duality Theorem for Rochberg spaces implies that $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$ is isomorphic to its dual. This enables us to define a bilinear pairing $\omega_n : \mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2} \times \mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2} \to \mathbb{C}$ and an involution $+ : \mathcal{L}(\mathfrak{R}_n) \to \mathcal{L}(\mathfrak{R}_n)$ given by

$$\omega_n(T^+x, y) = \omega_n(x, Ty), \quad \text{for all } x, y \in \mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}.$$

We discuss several properties of this involution, which can be regarded as an analogue of the Hilbert space adjoint, and prove in Theorem 4.2.3 that block operators obtained in the previous chapter preserve ω_n. This seemingly harmless property will be key to prove the main original results of Chapter 4:

Thm. 4.1.1: Every seminormalized basic sequence in $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$ admits a subsequence equivalent to the canonical basis of some Orlicz space ℓ_{f_j} generated by the function $f_j(t) = t^2 \log^{2j} t$, for $0 \leq j \leq n-1$. Therefore, $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$ admits exactly n types of non-equivalent basic sequences.

Thm. 4.3.1: Every operator $T : \mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2} \to X$ is either strictly singular or an isomorphism on a subspace $E \subset \mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$ isomorphic to $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$ and complemented on $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$. If moreover $X = \mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$ then $T(E)$ is also complemented in $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$.

Thm. 4.3.2: $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$ is not a complemented subspace of a Banach lattice (or a space with l.u.st) and does not contain complemented subspaces that are Banach lattices (or spaces with l.u.st).

Thm. 4.3.3: Strictly singular and strictly cosingular operators on $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$ coincide and they form the unique maximal operator ideal of $\mathcal{L}\big(\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}\big)$. As a consequence, we get a complete solution to the Perturbation Classes Problem for semi-Fredholm operators in $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$.

Thm. 4.3.4: The composition of n strictly singular operators on $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$ is compact, while the composition of $n-1$ operators is not necessarily compact (see the comments after Corollary 4.3.7).

Thm. 4.3.6: Every semi-Fredholm operator on $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$ has complemented kernel and range. In particular, every copy of $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$ in $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$ is complemented.

Thm. 4.3.7: We prove a generalized version of the "difference-compactness" theorem for operators on Z_2 obtained in [24] (see Proposition 1.2.5). As a consequence, we obtain that any Fredholm operator on $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$ which is upper triangular has as Fredholm index a multiple of n.

Thm. 4.4.1: We prove that $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$ are non-trivial symplectic Banach spaces. For even n this comes rather naturally since the natural bilinear pairing ω_n defines a symplectic structure, but for odd n one requires to modify such pairing by means of a complex structure in order to define a symplectic structure.

The last chapter is devoted to study the Rochberg spaces $\mathfrak{R}_n\big(\mathcal{T}_2, \mathcal{T}_2^*\big)_{1/2}$ defined by the couple formed by the 2-convexification of Tsirelson space and its dual. The case $n = 2$ was considered by Suárez in [91], who showed that the derived space $\mathfrak{R}_2\big(\mathcal{T}_2, \mathcal{T}_2^*\big)_{1/2}$ is a weak Hilbert space, providing the first example of non trivial twisted Hilbert weak Hilbert space.

In the initial part of Chapter 5 we extend Suárez's results proving that there is a function f_n depending only on the local type 2-constants $a_{m,2}(X_0)$, $a_{m,2}(X_1)$ and $a_{m,2}(X_\theta)$ such that

$$a_{m,2}\big(\mathfrak{R}_n(X_0, X_1)_\theta\big) \leq f_n\big(a_{m,2}(X_0), a_{m,2}(X_1), a_{m,2}(X_\theta)\big).$$

Using this estimate, the proof of Suárez for $n = 2$ can be translated almost verbatim to the general case, and thus we show in Theorem 5.2.1 that $\mathfrak{R}_n\big(\mathcal{T}_2, \mathcal{T}_2^*\big)_{1/2}$ is a weak Hilbert space for all $n \geq 1$, Unfortunately, we have been unable to decide whether these Rochberg spaces are non-isomorphic for different values of n.

The last part of this work are three Appendices. In the first one we introduce the basic theory of quasilinear maps and twisted sums. The second one is devoted to Operator Theory. The last one contains some results that belong to the Local Theory of Banach Spaces, which will be mainly used in Chapter 5.

About notation and preliminary results

The study of operators on Rochberg spaces is a conjunction of Banach Space Theory, Complex Interpolation Theory, Operator Theory and Homology of Banach spaces. We have been unable to follow a unique and unifying source for all the preliminary results needed for this work. Instead, we considered some basic references for each area:

- For homological methods, quasilinear maps and twisted sums of (Quasi-)Banach spaces we follow [10].
- For Banach Space Theory we use the books [3, 56, 71].
- For Complex Interpolation Theory we consider [7, 80, 61].
- For Operator Theory and Fredholm operators we employ [78, 50].

Most of our notation follows the previously cited sources, but there are two relevant exceptions:

- The notation of the Calderón space appearing in Interpolation Theory: for a given couple (X_0, X_1) we will denote it by $\mathcal{C}(X_0, X_1)$ instead of the usual $\mathcal{F}(X_0, X_1)$.
- It is common to use the "mathfrak" capital letters $\mathfrak{K}, \mathfrak{L}, \mathfrak{S}$ to denote operator ideals. We have decide to use "mathcal" capital letters $\mathcal{K}, \mathcal{L}, \mathcal{SS}$ to denote them; the assignment of these letters can be looked up in the Appendix B.

Given two real valued functions $f, g : X \to \mathbb{R}$, by $f \sim g$ we denote that there exist positive constants C, M such that $Cf(x) \leq g(x) \leq Mf(x)$ for every $x \in X$.

Given a real valued sequence $(x_n)_{n\in\mathbb{N}}$ and a positive function $f : \mathbb{N} \to \mathbb{R}$, we denote by $(x_n)_{n\in\mathbb{N}} = \mathcal{O}(f(n))$ that there exists an absolute constant $C > 0$ and $n_0 \in \mathbb{N}$ such that $|x_n| \leq Cf(n)$ for all $n \geq n_0$.

Chapter 1

Operators on Z_2

We recall the definition of the Kalton-Peck space Z_2 and its basic properties. We briefly discuss the inverse representation of Z_2 and the hyperplane problem. In the second part of the chapter we will study bounded on operators on Kalton-Peck space. See the Appendix A for the unexplained notation and results used in this chapter.

1.1 The Kalton-Peck space

In [62] Kalton and Peck produced a systematic method to construct quasilinear maps on certain Banach spaces with unconditional basis. Define a *sequence space* as a quasi-Banach space X for which the canonical vectors $(e_n)_{n\in\mathbb{N}}$ form a normalized unconditional basis. Examples of such spaces include ℓ_p spaces for all $0 < p < \infty$, the Schreier space $\mathcal{S}$ and Tsirelson space $\mathcal{T}$.
Consider the class L of lipschitz functions $\varphi : \mathbb{R} \to \mathbb{R}$ vanishing for all $t \leq 0$ and define the map

$$\Omega_\varphi(x) = \sum_{i=1}^{\infty} x_i \varphi\Big(-\log \frac{|x_i|}{\|x\|_X}\Big) e_n, \quad \text{for all } x = \sum_{i=1}^{\infty} x_i e_i \in X. \tag{1.1}$$

Then $\Omega_\varphi : X \curvearrowright X$ is quasilinear [62, pp. 11] (see also [10, Prop. 3.2.6]). Moreover, if X does not contain c_0, Kalton and Peck proved [62, Th. 4.2] the astonishing fact that Ω_φ is trivial if and only if $\varphi : \mathbb{R} \to \mathbb{R}$ is bounded (cf. [10, Prop. 3.2.7]).
The choice is $\varphi(t) = -t$ defines the so-called *Kalton-Peck map* $\mathsf{KP} : \ell_2 \curvearrowright \ell_2$ given by

$$\mathsf{KP}(x) = \sum_{i=1}^{\infty} x_i \log \frac{|x_i|}{\|x\|_{\ell_2}} e_n = x \log \frac{|x|}{\|x\|_2}, \quad \text{for all } x = \sum_{i=1}^{\infty} x_i e_i \in \ell_2. \tag{1.2}$$

The Kalton-Peck map defines the twisted Hilbert space $Z_2 = \ell_2 \oplus_{\mathsf{KP}} \ell_2$ called the *Kalton-Peck space*. The space ℓ_∞ can be regarded as the ambient space for the Kalton-Peck map, and thus Z_2 can be explicitely described as the space of pairs $(x, y) = \Big(\sum x_i e_i, \sum y_i e_i\Big) \in \ell_\infty \times \ell_2$ such that

$$\|(x,y)\| = \|x - \mathsf{KP}(y)\|_2 + \|y\|_2 = \left(\sum_{i=1}^{\infty} \Big(x_i - y_i \log \frac{|y_i|}{\|y\|} \Big)^2 \right)^{1/2} + \Big(\sum_{i=1}^{\infty} |y_i|^2 \Big)^{1/2} < \infty$$

This produces a short exact sequence

$$0 \longrightarrow \ell_2 \xrightarrow{\ i\ } Z_2 \xrightarrow{\ q\ } \ell_2 \longrightarrow 0 \tag{1.3}$$

where $i(x) = (x, 0)$ and $q(x, y) = y$. Note that Z_2 posseses all 3-space properties that ℓ_2 has, such as being superreflexive or to be ℓ_2-saturaded. Specific properties of Z_2 are:

(i) The sequence of 2-dimensional subspaces $E_n = \mathrm{span}\{(e_n, 0), (0, e_n)\}$ defines an unconditional finite dimensional decomposition (UFDD) for Z_2. In particular (cf. [18, Prop. 6.5]), the sequence $(u_n)_n$ defined by

$$u_{2n-1} = (e_n, 0) \quad \text{and} \quad u_{2n} = (0, e_n)$$

is a basis.

(ii) The dual quasilinear map KP^* can be identified with $-\mathsf{KP}$ [62, Th. 5.1]. In particular, Z_2^* can be identified with $\ell_2 \oplus_{-\mathsf{KP}} \ell_2$. The duality between the two spaces is given, for all finitely supported vectors $(x, y) \in Z_2$ and $(z, w) \in \ell_2 \oplus_{-\mathsf{KP}} \ell_2$, by

$$\langle (x, y), (z, w) \rangle_{Z_2} = \langle x, w \rangle_{\ell_2} + \langle y, z \rangle_{\ell_2},$$

Thus Z_2 is isomorphic to its dual via the map $(x, y) \in Z_2 \mapsto (x, -y) \in \ell_2 \oplus_{-\mathsf{KP}} \ell_2$.

(iii) Every normalized basic sequence on Z_2 has a subsequence equivalent to the canonical basis of either ℓ_2 or the Orlicz space ℓ_f defined by the Orlicz function $f(t) = (t|\log t|)^2$ (see [62, Th. 5.4]).

Using these three properties Kalton and Peck [62, Theorem 6.4] obtained the following cornerstone result:

Proposition 1.1.1. *The quotient map q of (1.3) is strictly singular and the embedding i is strictly cosingular.*

In the terminology of Section A.1.1 this means that KP is singular. This particular behaviour of the Kalton-Peck map it is used in [62, Section 6] to show that:

Proposition 1.1.2.

- *Every operator $T : Z_2 \to Z_2$ is either strictly singular or invertible on an isomorphic copy of Z_2.*
- *Z_2 contains no complemented subspace with unconditional basis. In particular, no copy of ℓ_2 is complemented.*

Shortly thereafter, Johnson, Lindenstrauss and Schechtman [57] showed that Z_2 lacks l.u.st (see the definition in [56]). A finer analysis of the Johnson-Lindenstrauss-Schechtman results provided by Kalton [59] yields the following striking extension of Proposition 1.1.2:

Theorem 1.1.1.

- *Given any Banach space X, every operator $T : Z_2 \to X$ is either strictly singular or invertible on an complemented copy of Z_2.*
- *Z_2 contains no complemented subspace with l.u.st. In particular, no copy of ℓ_2 is complemented and Z_2 is not complemented in a Banach lattice.*

Further discussion and extensions of Theorem 1.1.1 will be given in Section 4.3.

1.1.1 Inverse representation of Z_2

We have another short exact sequence

$$0 \longrightarrow \mathrm{Dom}(\mathsf{KP}) \xrightarrow{\ j\ } Z_2 \xrightarrow{\ p\ } \mathrm{Ran}(\mathsf{KP}) \longrightarrow 0$$

where $j(x) = (0, x)$ and $p(x, y) = x$. Kalton and Peck identified $\mathrm{Dom}(\mathsf{KP})$ as the Orlicz space ℓ_f generated by the Orlicz function $f(t) = t^2|\log t|^2$. Since KP^* can be identified with $-\mathsf{KP}$ and taking into account that ℓ_f is dense in ℓ_2, it follows from Proposition A.1.5 that $\mathrm{Ran}(\mathsf{KP}) = \mathrm{Dom}(\mathsf{KP})^* = \ell_f^*$. By [71, Example 4.c.1] ℓ_f^* can be identified with the Orlicz space ℓ_g where $g(t) = t^2\frac{1}{|\log t|^2}$ and thus we have a chain of strict continuous inclusions

$$\ell_f \subset \ell_2 \subset \ell_f^*.$$

The preceding discussion yields the short exact sequence

$$0 \longrightarrow \ell_f \xrightarrow{\ j\ } Z_2 \xrightarrow{\ p\ } \ell_f^* \longrightarrow 0 \tag{1.4}$$

generated by the quasilinear map $\mathsf{KP}^{-1} : \ell_f^* \curvearrowright \ell_f$ with ambient space ℓ_2, that we will call the *inverse Kalton-Peck map.*

1.1.2 The hyperplane problem

It was asked by Banach [4, pp. 153] if every Banach space X is isomorphic to its hyperplanes, i.e., whether X is isomorphic to $X \oplus \mathbb{R}$. When Z_2 appeared, some results suggested that it could be a natural counterexample. This question has appeared in several papers along the years (see [57, 59, 52, 24, 21]).

Despite nowadays we know explicit counterexamples to Banach's question [52, 67], this specific problem for the Kalton-Peck space has become a long standing conjecture:

Problem 1 (Hyperplane problem). *Is Z_2 isomorphic to its hyperplanes?*

The first thing to note is the connection between the hyperplane problem and the structure of operators on Z_2: since any isomorphism of Z_2 with its hyperplanes is necessarily a Fredholm operator of odd index (cf. Section B.1), Problem 1 can be reformulated as:

Problem 2. *Does there exist a Fredholm operator on Z_2 with odd index?*

1.2 Bounded operators on Z_2

Motivated by the Hyperplane problem and some closely related conjectures, we studied in [32] bounded operators in Z_2; in this section we present the main results obtained.

The two natural representations (1.3) and (1.4) of Z_2 yield four operators:

(i) The embeddings i, j.

(ii) The quotient maps q, p.

Any linear map $T : Z_2 \to Z_2$ can be described by a 2×2 matrix

$$T(x,y) = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (\alpha x + \beta y, \delta x + \gamma y), \tag{1.5}$$

given by four linear maps $\alpha, \beta, \delta, \gamma : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$. If T is bounded on Z_2 then we have that

$$\alpha = pTi : \ell_2 \to \ell_f^*, \quad \beta = pTj : \ell_f \to \ell_f^* \quad \delta = qTi : \ell_2 \to \ell_2, \quad \text{and } \gamma = qTj : \ell_f \to \ell_2$$

are bounded. Moreover, since $\ell_2 \oplus \ell_f$ is dense in Z_2, the four operators $\alpha, \beta, \delta, \gamma$ above uniquely define T. The boundness of $T : Z_2 \to Z_2$ can be reduced to conditions that the entries of the 2×2 matrix in (1.5) must satisfy:

Lemma 1.2.1. *Let $T = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ be a bounded operator on Z_2. Then the following conditions are satisfied:*

(d,g) *$\delta : \ell_2 \to \ell_2$ and $\gamma : \ell_f \to \ell_2$ are bounded.*

(b-Kg) *$(\beta - \mathsf{KP} \circ \gamma) : \ell_f \to \ell_2$ is bounded.*

(d+gK') *$\delta + \gamma \circ \mathsf{KP}^{-1} : \ell_f^* \to \ell_2$ is bounded.*

(g+dK) *$\gamma + \delta \circ \mathsf{KP} : \ell_2 \to \ell_2$ is bounded.*

Proof. (b-Kg) Let $x \in \ell_f$ and note that

$$\|Tjx\|_{Z_2} = \|T(0,x)\|_{Z_2} = \|(\beta x, \gamma x)\|_{Z_2} = \|(\beta - \mathsf{KP} \circ \gamma)x\|_2 + \|\gamma x\|_2 \leq \|Tj\| \, \|x\|_{\ell_f}.$$

Thus $\beta - \mathsf{KP} \circ \gamma : \ell_f \to \ell_2$ is bounded.

(d+gK') A bounded homogeneous lifting $L_q : \ell_f^* \to Z_2$ for $q : Z_2 \to \ell_f^*$ is given by $L_q(\omega) = (\omega, \mathsf{KP}^{-1}\omega)$. Then for every $\omega \in \ell_f^*$ one has

$$\|pTL_q(\omega)\|_2 = \|(\delta + \gamma \circ \mathsf{KP}^{-1})\omega\|_2 \leq \|pT\| \, \|L_q\| \, \|\omega\|_{\ell_f^*}.$$

Hence $\delta + \gamma \circ \mathsf{KP}^{-1} : \ell_f^* \to \ell_2$ is bounded.

(g+dK) Same argument as in (d+gK') using instead the lifting $L_p(x) = (\mathsf{KP}x, x)$ for $p : Z_2 \to \ell_2$. $\square$

A surprising fact is that if we add an additional condition, these properties characterize the boundness of T.

Theorem 1.2.1. *The operator $T = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ is bounded on Z_2 if and only if the four necessary conditions in Lemma 1.2.1 as well as $(\star)$ hold, where*

$$(\star) \qquad \alpha + \beta \circ \mathsf{KP}^{-1} - \mathsf{KP}\left(\delta + \gamma \circ \mathsf{KP}^{-1}\right) : \ell_f^* \longrightarrow \ell_2 \quad \text{is a bounded map.}$$

Proof. Condition $(\star)$ is necessary: if T is a bounded operator then

$$\|\alpha\omega + \beta x - \mathsf{KP}(\delta\omega + \gamma x)\|_2 \leq \|T\| \|(\omega, x)\|_{Z_2}$$

and the choice $(\omega, \mathsf{KP}^{-1}\omega) \in Z_2$ yields

$$\|\alpha\omega + \beta\mathsf{KP}^{-1}\omega - \mathsf{KP}(\delta\omega + \gamma\mathsf{KP}^{-1}\omega)\|_2 \leq \|T\| \|(\omega, \mathsf{KP}^{-1}\omega)\|_{Z_2} \leq \|T\| \|\omega\|_{\ell_f^*}.$$

To prove the converse we will show that the following three conditions hold:

(1) $\delta\omega + \gamma x \in \ell_2$;

(2) $\alpha\omega + \beta x - \mathsf{KP}(\delta\omega + \gamma x) \in \ell_2$;

(3) $\|\alpha\omega + \beta x - \mathsf{KP}(\delta\omega + \gamma x)\|_2 + \|\delta\omega + \gamma x\|_2 \leq C\|(\omega, x)\|_{Z_2}$.

We shall use repeatedly the fact that the quasinorms induced by the representations (1.3) and (1.4) of Z_2 as twisted sum are equivalent (see Subsection 1.1.1), i.e., there exist a constant $M > 0$ such that

$$\frac{1}{M}\|(\omega, x)\|_{Z_2} \leq \|x - \mathsf{KP}^{-1}\omega\|_{\ell_f} + \|\omega\|_{\ell_f^*} \leq M\|(\omega, x)\|_{Z_2}. \tag{$\heartsuit$}$$

(1) Given $(\omega, x) \in Z_2$, note that, by definition, $\omega - \mathsf{KP}x \in \ell_2$ and $x - \mathsf{KP}^{-1}\omega \in \ell_f$; on the other hand, by assumption the maps $\gamma + \delta \circ \mathsf{KP} : \ell_2 \to \ell_2$ and $\delta + \gamma \circ \mathsf{KP}^{-1} : \ell_f^* \to \ell_2$ are bounded. Thus

$$\delta\omega + \gamma x = \frac{1}{2}\Big(\delta(\omega - \mathsf{KP}x) + \gamma(x - \mathsf{KP}^{-1}\omega) + (\gamma + \delta \circ \mathsf{KP})x + (\delta + \gamma \circ \mathsf{KP}^{-1})\omega\Big) \in \ell_2$$

Using ($\heartsuit$) yields

$$\begin{aligned}\|\delta\omega + \gamma x\|_2 &\leq \frac{1}{2}\Big(\|\delta\|\,\|\omega - \mathsf{KP}x\|_2 + \|\gamma\|\,\|x - \mathsf{KP}^{-1}\omega\|_{\ell_f} + \|\gamma + \delta \circ \mathsf{KP}\|\|x\|_2 \\ &\quad + \|\delta + \gamma \circ \mathsf{KP}^{-1}\|\,\|\omega\|_{\ell_f^*}\Big) \\ &\leq \frac{1}{2}\Big(\|\delta\| + \|\gamma\|\,M + \|\gamma + \delta \circ \mathsf{KP}\| + \|\delta + \gamma \circ \mathsf{KP}^{-1}\|\,M\Big)\|(\omega, x)\|_{Z_2}.\end{aligned}$$

To prove (2) we decompose $\alpha\omega + \beta x - \mathsf{KP}(\delta\omega + \gamma x)$ in three pieces:

$$\begin{aligned}&\alpha\omega + \beta \circ \mathsf{KP}^{-1}\omega - \mathsf{KP}(\delta + \gamma \circ \mathsf{KP}^{-1})\omega \\ &+\beta(x - \mathsf{KP}^{-1}\omega) - \mathsf{KP}(\gamma x - \gamma \circ \mathsf{KP}^{-1}\omega) \\ &+\mathsf{KP}(\gamma x - \gamma \circ \mathsf{KP}^{-1}\omega) + \mathsf{KP}(\delta + \gamma \circ \mathsf{KP}^{-1})\omega - \mathsf{KP}(\delta\omega + \gamma x).\end{aligned}$$

The first piece is bounded by ($\star$). For the third piece recall that $\mathsf{KP} : \ell_2 \curvearrowright \ell_2$ is quasilinear, hence $\mathsf{KP}(x + y) - \mathsf{KP}(x) - \mathsf{KP}(y) \in \ell_2$ and satisfies the estimate

$$\|\mathsf{KP}(x + y) - \mathsf{KP}(x) - \mathsf{KP}(y)\|_2 \leq \|\mathsf{KP}\|(\|x\|_2 + \|y\|_2)$$

for all $x, y \in \ell_2$. Since $\gamma : \ell_f \to \ell_2$ and $\delta + \gamma \circ \mathsf{KP}^{-1} : \ell_f^* \to \ell_2$ are bounded, all elements in which the Kalton-Peck is defined in the third piece are elements in ℓ_2; setting $x = \gamma(x - \mathsf{KP}^{-1}\omega)$ and $y = (\delta + \gamma \circ \mathsf{KP}^{-1})\omega$, we can apply the quasilinear estimate to obtain

$$\begin{aligned}&\|\mathsf{KP}\,(\delta\omega + \gamma x) - \mathsf{KP}\,\big(\delta\omega + \gamma x - \delta\omega - \gamma\mathsf{KP}^{-1}\omega\big) - \mathsf{KP}\,\big(\delta + \gamma\mathsf{KP}^{-1}\big)\,\omega\|_2 \\ &\leq \|\mathsf{KP}\|\big(\|\delta\omega + \gamma\mathsf{KP}^{-1}\omega\|_2 + \|\gamma x - \gamma\mathsf{KP}^{-1}\omega\|_2\big) \\ &\leq \|\mathsf{KP}\|\big(\|\delta + \gamma \circ \mathsf{KP}^{-1}\|\,\|\omega\|_{\ell_f^*} + \|\gamma\|\,\|x - \mathsf{KP}^{-1}\omega\|_{\ell_f}\big) \\ &\leq \|\mathsf{KP}\|(\|\gamma\| + \|\delta + \gamma \circ \mathsf{KP}^{-1}\|)(\omega, x)\|_{Z_2}.\end{aligned}$$

For the second piece note that $\beta(x - \mathsf{KP}^{-1}\omega) - \mathsf{KP}(\gamma x - \gamma \circ \mathsf{KP}^{-1}\omega)$ is equal to

$$\beta(x - \mathsf{KP}^{-1}\omega) - \mathsf{KP} \circ \gamma(x - \mathsf{KP}^{-1}\omega) = (\beta - \mathsf{KP} \circ \gamma)(x - \mathsf{KP}^{-1}\omega).$$

Since by assumption $(\beta - \mathsf{KP} \circ \gamma) : \ell_f \to \ell_2$ is bounded, we conclude that

$$\begin{aligned}\|\beta(x - \mathsf{KP}^{-1}\omega) - \mathsf{KP}(\gamma x - \gamma \circ \mathsf{KP}^{-1}\omega)\|_2 &\leq \|(\beta - \mathsf{KP} \circ \gamma)\| \, \|(x - \mathsf{KP}^{-1}\omega)\|_{\ell_f} \\ &\leq M\|\beta - \mathsf{KP} \circ \gamma\| \, \|(\omega, x)\|_{Z_2}\end{aligned}$$

Finally, (3) follows by the bounds obtained in both (1) and (2). □

The previous characterization has consequences for the structure of operators in Z_2. The following one will be very useful:

Proposition 1.2.1.

$$I_{\ell_f^*} - \mathsf{KP} \circ \mathsf{KP}^{-1} : \ell_f^* \to \ell_2 \quad \textit{and} \quad I_{\ell_2} - \mathsf{KP}^{-1} \circ \mathsf{KP} : \ell_2 \to \ell_f \quad \textit{are bounded maps.}$$

Proof. The first part is just condition $(\star)$ applied to the identity operator $T = I_{Z_2}$ on Z_2. To deduce the second assertion compose $I_{\ell_f^*} - \mathsf{KP} \circ \mathsf{KP}^{-1} : \ell_f^* \to \ell_2$ at the right with $\mathsf{KP} : \ell_2 \to \ell_f^*$. This gives that

$$\mathsf{KP} - \mathsf{KP} \circ \mathsf{KP}^{-1} \circ \mathsf{KP} : \ell_2 \to \ell_2 \quad \text{is bounded.} \tag{1.6}$$

Now observe that

$$\begin{aligned}\mathsf{KP}(I_{\ell_2} - \mathsf{KP}^{-1} \circ \mathsf{KP}) = &\big[\mathsf{KP}(I_{\ell_2} - \mathsf{KP}^{-1} \circ \mathsf{KP}) - \mathsf{KP} + \mathsf{KP}(\mathsf{KP}^{-1} \circ \mathsf{KP})\big] \\ &+ \big[\mathsf{KP} - \mathsf{KP} \circ \mathsf{KP}^{-1} \circ \mathsf{KP}\big]\end{aligned}$$

Using (1.6) and that KP is quasilinear on ℓ_2 we conclude that

$$\mathsf{KP}(I_{\ell_2} - \mathsf{KP}^{-1} \circ \mathsf{KP}) : \ell_2 \to \ell_2 \quad \text{is bounded.}$$

Since $\mathrm{Dom}(\mathsf{KP}) = \ell_f$, this can only occur if $I_{\ell_2} - \mathsf{KP}^{-1} \circ \mathsf{KP} : \ell_2 \to \ell_f$. □

Condition $(\star)$ in Theorem 1.2.1 is quite subtle since it involves both KP and KP^{-1}. Taking into account that no explicit expression for KP^{-1} is avaliable at this moment, we present a cleaner and equivalent condition which only involves KP. To do this we will use Proposition 1.2.1 to slighlty modify the conditions of Theorem 1.2.1.

Proposition 1.2.2. *Conditions* $(\star)$ *and* (b-Kg) *in Theorem 1.2.1 can be replaced by*

(a-dK) $\alpha - \delta \circ \mathsf{KP} : \ell_2 \to \ell_2$ *is bounded.*

($\square$) $\alpha \circ \mathsf{KP} + \beta - \mathsf{KP}\big(\delta \circ \mathsf{KP} + \gamma\big) : \ell_2 \to \ell_2$ *is bounded.*

Proof. Compose $(\star)$ at the right by $\mathsf{KP} : \ell_2 \to \ell_f^*$ to obtain that

$$\alpha \circ \mathsf{KP} + \beta \circ \mathsf{KP}^{-1} \circ \mathsf{KP} - \mathsf{KP}\big(\delta \circ \mathsf{KP} + \gamma \circ \mathsf{KP}^{-1} \circ \mathsf{KP}\big) : \ell_2 \to \ell_2 \quad \text{is bounded.} \tag{1.7}$$

Now note that

$$(\square) = \alpha \circ \mathsf{KP} + \beta - \mathsf{KP}\big(\delta \circ \mathsf{KP} + \gamma\big) = (1.7) + \beta(I_{\ell_2} - \mathsf{KP}^{-1} \circ \mathsf{KP}) \\ - \mathsf{KP}\big(\delta \circ \mathsf{KP} + \gamma\big) + \mathsf{KP}\big(\delta \circ \mathsf{KP} + \gamma \circ \mathsf{KP}^{-1} \circ \mathsf{KP}\big).$$

Since we are assuming the other conditions in Theorem 1.2.1, we have that $\delta \circ \mathsf{KP} : \ell_2 \to \ell_2$ and $\delta + \gamma \circ \mathsf{KP}^{-1} : \ell_f^* \to \ell_2$ are bounded. Hence $\delta \circ \mathsf{KP} + \gamma \circ \mathsf{KP}^{-1} \circ \mathsf{KP}$ is bounded on ℓ_2. Thus we can use that KP is quasilinear on ℓ_2 to rewrite the last term of previous equation:

$$\begin{aligned}(1.7) + \beta(I_{\ell_2} - \mathsf{KP}^{-1} \circ \mathsf{KP}) - \mathsf{KP}\big(\delta \circ \mathsf{KP} + \gamma\big) + \mathsf{KP}\big(\delta \circ \mathsf{KP} + \gamma \circ \mathsf{KP}^{-1} \circ \mathsf{KP}\big) \\ = (1.7) + \beta(I_{\ell_2} - \mathsf{KP}^{-1} \circ \mathsf{KP}) - \mathsf{KP}(\gamma(I_{\ell_2} - \mathsf{KP}^{-1} \circ \mathsf{KP})) \\ = (1.7) + (\beta - \mathsf{KP} \circ \gamma)(I_{\ell_2} - \mathsf{KP}^{-1} \circ \mathsf{KP}).\end{aligned}$$

Now, by Proposition 1.2.1 we have that $I_{\ell_2} - \mathsf{KP}^{-1} \circ \mathsf{KP} : \ell_2 \to \ell_f$ boundedly. Using condition (b-Kg) and (1.7) we obtain ($\square$).

The converse use the same ideas: composition of ($\square$) at the right by $\mathsf{KP}^{-1} : \ell_f^* \to \ell_2$ and the fact that KP is quasilinear on ℓ_2 gives

$$(\star) = \Big(\alpha \circ \mathsf{KP} \circ \mathsf{KP}^{-1} + \beta \circ \mathsf{KP}^{-1} - \mathsf{KP}\big(\delta \circ \mathsf{KP} \circ \mathsf{KP}^{-1} + \gamma \circ \mathsf{KP}^{-1}\big)\Big) \\ + (\alpha - \mathsf{KP} \circ \delta)(I_{\ell_f^*} - \mathsf{KP} \circ \mathsf{KP}^{-1}).$$

By Proposition 1.2.1 and condition (a-dK) we are done. □

Using this last result we can characterize bounded operators on Z_2 excluding the inverse representation:

Theorem 1.2.2. *The operator* $T = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ *is bounded on* Z_2 *if and only if:*

(d,g) $\delta : \ell_2 \to \ell_2$ *and* $\gamma : \ell_f \to \ell_2$ *are bounded.*

(a-dK) $(\alpha - \delta \circ \mathsf{KP}) : \ell_2 \to \ell_2$ *is bounded.*

(g+dK) $\gamma + \delta \circ \mathsf{KP} : \ell_2 \to \ell_2$ *is bounded.*

($\square$) $\alpha \circ \mathsf{KP} + \beta - \mathsf{KP}\big(\delta \circ \mathsf{KP} + \gamma\big) : \ell_2 \to \ell_2$ *is bounded.*

Proof. This is direct using Proposition 1.2.2 and Theorem 1.2.1 above. Condition (d+gK') is the only one which is aparently lacking, but we can show that, in this case, it is equivalent to (g+dK) due to Proposition 1.2.1: if $\delta + \gamma \circ \mathsf{KP}^{-1} : \ell_f^* \to \ell_2$ is bounded then

$$\delta \circ \mathsf{KP} + \gamma = (\delta \circ \mathsf{KP} + \gamma \circ \mathsf{KP}^{-1} \circ \mathsf{KP}) + \gamma(I_{\ell_2} - \mathsf{KP}^{-1} \circ \mathsf{KP}).$$

Since $\gamma : \ell_f \to \ell_2$, we are done by Proposition 1.2.1.

Conversely, assume that $\delta \circ \mathsf{KP} + \gamma : \ell_2 \to \ell_2$ is bounded. Then

$$\delta + \gamma \circ \mathsf{KP}^{-1} = (\delta \circ \mathsf{KP} \circ \mathsf{KP}^{-1} + \gamma \circ \mathsf{KP}^{-1}) + \delta(I_{\ell_f^*} - \mathsf{KP} \circ \mathsf{KP}^{-1}).$$

Taking into account that $\delta : \ell_2 \to \ell_2$, Proposition 1.2.1 finishes the proof. □

1.2.1 Linearization conditions

To give specific examples of operators on Z_2, condition ($\square$) is quite problematic because the term $\mathsf{KP}(\delta \circ \mathsf{KP} + \gamma)$ is dificult to work with. To overcome this we shall consider a further condition. We say that the linear map $T = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ satisfies the *linearization condition* if

$$\delta : \ell_f^* \to \ell_2 \quad \text{is bounded} \qquad \text{and} \qquad \gamma : \ell_2 \to \ell_2 \quad \text{is bounded.} \tag{$\mathcal{L}$}$$

Note that this is stronger than condition (d,g); indeed, ($\mathcal{L}$) is just the fact that both maps of condition (g+dK) are separately continuous. Moreover, if the operator T is bounded, the two conditions in ($\mathcal{L}$) are equivalent (use precisely that $\delta \circ \mathsf{KP} + \gamma$ is bounded on ℓ_2). To give an example, any bounded upper triangular operator on Z_2 (those for which $\delta = 0$) trivially satisfies ($\mathcal{L}$). The name given comes from the fact that this property will enable us to treat KP as a linear map on condition ($\square$), as we show now:

Proposition 1.2.3. *Under ($\mathcal{L}$), condition ($\square$) is equivalent to*

$$\alpha \circ \mathsf{KP} + \beta - \mathsf{KP} \circ \delta \circ \mathsf{KP} - \mathsf{KP} \circ \gamma : \ell_2 \to \ell_2 \quad \textit{is bounded.} \tag{$\diamondsuit$}$$

Proof. By hypothesis, both $\delta \circ \mathsf{KP} : \ell_2 \to \ell_2$ and $\gamma : \ell_2 \to \ell_2$ are bounded. Since KP is quasilinear on ℓ_2, it follows that

$$\mathsf{KP}(\delta \circ \mathsf{KP} + \gamma) - \mathsf{KP}(\delta \circ \mathsf{KP}) - \mathsf{KP} \circ \gamma : \ell_2 \to \ell_2 \quad \text{is bounded.} \tag{1.8}$$

To end the proof just note that ($\diamondsuit$) = ($\square$) + (1.8). $\square$

The linearization condition allow us to simplify the hypothesis in Theorem 1.2.2:

Corollary 1.2.1. *Let* $T = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ *be a linear map satisfying ($\mathcal{L}$). Then T is bounded if and only if:*

(a-Kd) $\alpha - \mathsf{KP} \circ \delta : \ell_2 \to \ell_2$ *is bounded.*

($\diamondsuit$) $\alpha \circ \mathsf{KP} + \beta - \mathsf{KP} \circ \delta \circ \mathsf{KP} - \mathsf{KP} \circ \gamma : \ell_2 \to \ell_2$ *is bounded.*

Proof. This is direct using ($\mathcal{L}$), Theorem 1.2.2 and Proposition 1.2.3. $\square$

1.2.2 Decompositions of operators

If $T = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ is bounded then condition ($\diamondsuit$) can be used to decompose T in simpler operators. Think for instance in the case where $\beta : \ell_2 \to \ell_2$ is bounded; then condition ($\diamondsuit$) implies that

$$\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$$

is bounded. Hence we can decompose T as

$$T = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \delta & \gamma \end{pmatrix} + \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$$

where both operators are bounded on Z_2. The general idea for decomposing operators in this way is that all the process is governed by self-improving properties of the entries α, β, δ and γ. More precisely, there are three levels in which those operators act

$$\ell_f \subset \ell_2 \subset \ell_f^*$$

By self-improving property in this context we mean that an operator, which would be bounded a priori in ℓ_2 or ℓ_f^*, actually takes values into a smaller space. We note that if $\alpha : \ell_f^* \to \ell_2$ or $\gamma : \ell_2 \to \ell_f$ are bounded, then both

$$\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix}$$

are bounded on Z_2, respectively. Thus, self-improving properties lead to further decomposition of the original operator T.

The remaining term in ($\diamondsuit$) is $\mathsf{KP} \circ \delta \circ \mathsf{KP}$, which is the most difficult one. Since $\mathsf{KP} : \ell_2 \to \ell_f^*$ and $\mathsf{KP} : \ell_f \to \ell_2$ are bounded, to bound $\mathsf{KP} \circ \delta \circ \mathsf{KP}$ on ℓ_2 one needs that $\delta : \ell_f^* \to \ell_f$, which would yield the decomposition

$$\begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix}$$

This condition on δ is somewhat unnatural and suggest that δ can be regarded as some sort of residual term.

We can illustrate these ideas regarding self-improving properties with the example of diagonal operators on sequence spaces, i.e., maps D_a of the form $\sum x_n e_n \mapsto \sum a_n x_n e_n$ where $a = (a_n)_{n\in\mathbb{N}} \in \ell_\infty$. If we consider a bounded operator $\begin{pmatrix} D_a & D_b \\ D_d & D_c \end{pmatrix}$ then the self-improvement properties of D_a, D_b, D_c and D_d adopt the following form:

(i) Since $b \in \ell_\infty$, the multiplication $D_b : \ell_2 \to \ell_2$ is directly bounded and no condition on b is imposed.

(ii) On the other hand, the self-improving properties of $D_a : \ell_f^* \to \ell_2$ and $D_c : \ell_2 \to \ell_f$ means that a and c must "cancel out" a logaritmic perturbation, hence both sequences $(a_n)_n$ and $(c_n)_{n\in\mathbb{N}}$ must converge to 0 roughly as $\mathcal{O}\left(\frac{1}{\log n}\right)$. A more detailed argument is outlined below in Corollary 1.2.2.

(iii) Finally, in the case of $D_d : \ell_f^* \to \ell_f$, one must get rid of a quadratic logarithmic perturbation, and thus $(d_n)_n = \mathcal{O}\left(\frac{1}{\log^2 n}\right)$.

1.2.3 Characterization of operator with diagonal entries

Let us give a quick application of Corollary 1.2.1 to operators on Z_2 whose entries are diagonal operators: assume that $a, b, c, d \in \ell_\infty$ are bounded sequences and consider on Z_2 the linear map

$$D = \begin{pmatrix} D_a & D_b \\ D_d & D_c \end{pmatrix}$$

Corollary 1.2.2. *D is bounded on Z_2 if and only if*

$$D_d : \ell_f^* \to \ell_2 \quad \textit{and} \quad D_a - D_c - \mathsf{KP} \circ D_d : \ell_f^* \to \ell_2 \quad \textit{are bounded.}$$

Proof. First note that, in this case, D_c is already bounded on ℓ_2. Thus, if D is bounded then the necessary condition (d+gK') reduces to $D_d : \ell_f^* \to \ell_2$. In particular, D satisfies the linearization condition and, applying Corollary 1.2.1, we deduce that $(\diamondsuit)$ holds. Furthermore, observe that in this case $(\diamondsuit)$ is equivalent to

$$(D_a - D_c)\mathsf{KP} - \mathsf{KP} \circ D_d \circ \mathsf{KP} : \ell_2 \to \ell_2 \quad \text{is bounded.} \tag{1.9}$$

Indeed, since $b \in \ell_\infty$, the multiplication D_b is bounded on ℓ_2, so we can ignore it. On the other hand, it is known (see [10, Proposition 3.12.5]) that

$$\|\mathsf{KP}(D_\theta x) - D_\theta \mathsf{KP}(x)\|_2 \leq C\, \|\theta\|_\infty\, \|x\|_2, \quad \text{for all } \theta \in \ell_\infty,\, x \in \ell_2. \tag{1.10}$$

A proof of (1.10) will be given at the end of Chapter 2 using Interpolation Theory. Now the equivalence between (1.9) and $(\diamondsuit)$ is direct by (1.10). Moreover, Corollary 1.2.1 gives by condition (a-Kd) that $D_a - \mathsf{KP} \circ D_d : \ell_2 \to \ell_2$ is bounded. Using this last fact we can show that (1.9) is equivalent to the second enunciated condition. To this end, we compose (1.9) at the right by $\mathsf{KP}^{-1} : \ell_f^* \to \ell_2$ to obtain that

$$(D_a - D_c)\mathsf{KP} \circ \mathsf{KP}^{-1} - \mathsf{KP} \circ D_d \circ \mathsf{KP} \circ \mathsf{KP}^{-1} : \ell_f^* \to \ell_2 \quad \text{is bounded.} \tag{1.11}$$

Now we decompose

$$(D_a - D_c) - \mathsf{KP} \circ D_d = (1.11) + (D_a - D_c - \mathsf{KP} \circ D_d)(I_{\ell_f^*} - \mathsf{KP} \circ \mathsf{KP}^{-1}).$$

By Proposition 1.2.1 we have that $I_{\ell_f^*} - \mathsf{KP} \circ \mathsf{KP}^{-1} : \ell_f^* \to \ell_2$ is bounded; taking into account that both $D_a - \mathsf{KP} \circ D_d$ and D_c are bounded on ℓ_2, we conclude that $(D_a - D_c - \mathsf{KP} \circ D_d)(I_{\ell_f^*} - \mathsf{KP} \circ \mathsf{KP}^{-1}) : \ell_f^* \to \ell_2$ is bounded. Thus, using (1.11) it follows that

$$(D_a - D_c) - \mathsf{KP} \circ D_d : \ell_f^* \to \ell_2 \quad \text{is bounded.}$$

Conversely, to see that the two stated conditions are sufficient just note that if $D_a - D_c - \mathsf{KP} \circ D_d : \ell_f^* \to \ell_2$ is bounded, then composition at the right by $\mathsf{KP} : \ell_2 \to \ell_f^*$ gives (1.9). Moreover, if $D_d : \ell_f^* \to \ell_2$, then D satisfies $(\mathcal{L})$ and therefore (1.9) is equivalent to $(\diamondsuit)$ by the same arguments we used in the first part of the proof. Furthermore, since D_a is bounded on ℓ_2, it follows that $D_a - \mathsf{KP} \circ D_d : \ell_2 \to \ell_2$ is bounded. Hence D is bounded by Corollary 1.2.1. $\square$

We can reinterpret the two conditions of last corollary in terms of the asymptotic behaviour of the sequences a, c, d in the form

$$(d_n)_{n\in\mathbb{N}} = \mathcal{O}\Big(\frac{1}{\log n}\Big) \quad \text{and} \quad (a_n - c_n - (\log n) d_n)_{n\in\mathbb{N}} = \mathcal{O}\Big(\frac{1}{\log n}\Big). \tag{1.12}$$

Indeed, since ℓ_f^* can be identified with the Orlicz space defined by $f = t^2|\log t|^{-2}$, $(d_n)_n = \mathcal{O}\Big(\frac{1}{\log n}\Big)$ is equivalent to the boundness of the diagonal operator $D_d : \ell_f^* \to \ell_2$. The second condition in (1.12) is more subtle: if $(d_n)_n = \mathcal{O}\Big(\frac{1}{\log^2 n}\Big)$ then $D_d : \ell_f^* \to \ell_f$, hence $(\log n) d_n = \mathcal{O}\Big(\frac{1}{\log n}\Big)$ and we conclude that

$$(a_n - c_n)_n = \mathcal{O}\Big(\frac{1}{\log n}\Big).$$

In particular, $D_a - D_c : \ell_2 \to \ell_2$ is compact. The converse is also true: if $(a_n - c_n)_n = \mathcal{O}\Big(\frac{1}{\log n}\Big)$ then $\big((\log n)\, d_n\big)_n = \mathcal{O}\Big(\frac{1}{\log n}\Big)$, and so $(d_n)_n = \mathcal{O}\Big(\frac{1}{\log^2 n}\Big)$.

We can deduce from Corollary 1.2.2 the following result of Johnson, Lindenstrauss and Schechtman [57] that completely describes operators on Z_2 whose entries are scalars (see also [6, Lemma 16.15]):

Corollary 1.2.3. *If $\alpha, \beta, \delta, \gamma$ are scalars, then the linear map $\begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ is bounded on Z_2 if and only if $\delta = 0$ and $\alpha = \gamma$.*

Proof. Multiplication by a scalar c is the same as the operator $D_{\hat{c}}$ with $\hat{c} = (c, c, c, \ldots) \in \ell_\infty$. Hence Corollary 1.2.2 applies and we obtain that $D_{\hat{\delta}} : \ell_f^* \to \ell_2$, hence $\delta = 0$ since ℓ_2 and ℓ_f^* do not have equivalent norms. The remaining condition reduces to $D_{\hat{\alpha}} - D_{\hat{\gamma}} : \ell_f^* \to \ell_2$, which, by the same reasons, implies that $\alpha = \gamma$. $\square$

The Johnson-Lindenstrauss-Schechtman Theorem hints at something very deep about the structure of operators in twisted sum spaces. We will see that Corollary 1.2.3 admit two generalizations on the Kalton-Peck space:

(i) In the case of triangular operators, if we replace the scalar entries by general operators, we have that $\alpha - \gamma : \ell_2 \to \ell_2$ is compact, which extends the condition $\alpha = \gamma$. See Proposition 1.2.5 below.

(ii) The *Johnson-Lindenstrauss-Schechtman conjecture*: "Every bounded operator on Z_2 is a strictly singular perturbation of an upper triangular operator". This cojecture generalize the condition $\delta = 0$ in previous results, as we will explain in forthcoming Section 1.2.6.

With the exception of Corollary 1.2.2, where we used that KP commutes with multiplication operators, the previous results do not actually depend on Z_2 or KP. They are general facts about quasilinear maps on Banach spaces. Theorem 1.2.2 has the following analogue:

Theorem 1.2.3. *Assume that $F : X \curvearrowright X$ is an unbounded quasilinear map. Then the operator $\begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ is bounded on $X \oplus_F X$ if and only if:*

(d,g) *$\delta : X \to X$ and $\gamma : \mathrm{Dom}(F) \to X$ are bounded.*

(a-Kd) *$(\alpha - F \circ \delta) : X \to X$ is bounded.*

(g+dK) *$\gamma + \delta \circ F : X \to X$ is bounded.*

($\square'$) *$\alpha \circ F + \beta - F\big(\delta \circ F + \gamma\big) : X \to X$.*

The proof is the same as Theorem 1.2.2. Corollary 1.2.3 still holds:

Corollary 1.2.4 (General Johnson-Lindenstrauss-Schechtman condition)**.** *Let $F : X \curvearrowright X$ be an unbounded quasilinear map. If $\alpha, \beta, \delta, \gamma$ are scalars, then the linear map $\begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ is bounded on $X \oplus_F X$ if and only if $\delta = 0$ and $\alpha = \gamma$.*

Proof. Since we have a continuous inclusion $\mathrm{Dom}(F) \subset X$, Theorem 1.2.3 implies that both conditions are sufficient. To prove that are also necessary: if $\delta \neq 0$ then condition $\gamma + \delta \circ F : X \to X$ implies that $F : X \to X$ is bounded, which is imposible. On the other hand, since $\delta = 0$ and F is homogeneous, condition $(\square')$ reduces to

$$(\alpha - \gamma)F : X \to X \quad \text{is bounded,}$$

and therefore yields $\alpha = \gamma$. $\square$

1.2.4 Upper triangular operators

Let us set $\delta = 0$ and consider the resulting upper triangular operator $\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$. Then α carries the canonical copy $\ell_2 \subset Z_2$ into itself, and thus, γ is the induced operator in the quotient space $Z_2/\ell_2 = \ell_2$, hence both $\alpha, \gamma : \ell_2 \to \ell_2$ are bounded. On the other hand, condition $(\diamondsuit)$ is reduced to

$$\alpha \circ \mathsf{KP} - \mathsf{KP} \circ \gamma + \beta : \ell_2 \to \ell_2 \quad \text{is bounded.}$$

By Proposition 1.2.3 we deduce that such conditions are sufficient:

Theorem 1.2.4. *The operator $\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$ is bounded on Z_2 if and only if $\alpha, \gamma : \ell_2 \to \ell_2$ and*

$$\alpha \circ \mathsf{KP} - \mathsf{KP} \circ \gamma + \beta : \ell_2 \to \ell_2 \quad \textit{is bounded.} \tag{$\spadesuit$}$$

Note that the generalized commutator $[\alpha, \mathsf{KP}, \gamma] = \alpha \circ \mathsf{KP} - \mathsf{KP} \circ \gamma$ is a quasilinear map on ℓ_2, and the preceding condition just means that $[\alpha, \mathsf{KP}, \gamma]$ is *trivial*, i.e., it can be approximated by the linear map β. Thus, upper triangular operators on Z_2 correspond to trivial twisted Hilbert spaces generated by certain pullbacks and pushouts of KP. Even if we assume $\alpha = \gamma$, it is not true that $[\alpha, \mathsf{KP}] = [\alpha, \mathsf{KP}, \alpha]$ is trivial on ℓ_2 for a given operator $\alpha : \ell_2 \to \ell_2$ (see [24, Th. 3.1] or [25, Prop. 8.5] for some examples). Despite this, we have the following fact proved in [24, Prop. 5.8]:

Lemma 1.2.2. *For every operator $\alpha : \ell_2 \to \ell_2$ and every block subspace $W \subset \ell_2$, the commutator $[\alpha, \mathsf{KP}]$ is trivial on some infinite dimensional subspace of W.*

Althought rather simple due to previous calculations, we state now the characterization of diagonal operators to the case of upper triangular operators. This will be useful to set up the next results regardind the compact difference between diagonal entries. Thus, we assume that $a, b, c \in \ell_\infty$ are bounded sequences and consider on Z_2 the linear map

$$D = \begin{pmatrix} D_a & D_b \\ 0 & D_c \end{pmatrix}$$

Then Corollary 1.2.2 directly gives that:

Proposition 1.2.4. *D is bounded if and only if $(a_n - c_n)_n = \mathcal{O}\Big(\frac{1}{\log n}\Big)$. In particular, $D_a - D_c : \ell_2 \to \ell_2$ is compact.*

A rather deep extension of this result to general twisted Hilbert spaces is given in [13] by F. Cabello and R. García using different methods.

Proposition 1.2.4 admits the following generalization proved in [24, Cor. 5.9]:

Proposition 1.2.5. *Let $\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$ be a bounded operator on Z_2. Then $\alpha - \gamma : \ell_2 \to \ell_2$ is compact.*

Proof. By Theorem 1.2.4 we have that $\alpha\mathsf{KP} - \mathsf{KP}\gamma : \ell_2 \curvearrowright \ell_2$ is trivial. Then note that

$$\begin{aligned} \Big[\alpha\mathsf{KP} - \mathsf{KP}\gamma\Big] &= \Big[\alpha\mathsf{KP} - \mathsf{KP}\alpha\Big] + \mathsf{KP}\alpha - \mathsf{KP}\gamma \\ &= [\alpha, \mathsf{KP}] + (-\mathsf{KP}(\alpha - \gamma) + \alpha\mathsf{KP} - \mathsf{KP}\gamma) \\ &+ \mathsf{KP}(\alpha - \gamma). \end{aligned}$$

If $\alpha - \gamma$ is not compact, then it is invertible on on some infinite dimensional subspace $W \subset \ell_2$. Passing to a further subspace if necessary, it follows by Lemma 1.2.2 that $[\alpha, \mathsf{KP}]$ is trivial on some infinite dimensional subspace $Y \subset W$. Using that KP is quasilinear on ℓ_2, we deduce that $\mathsf{KP}(\alpha - \gamma) : \ell_2 \curvearrowright \ell_2$ is trivial on Y. Hence, KP is trivial on $(\alpha - \gamma)(Y) \subset \ell_2$, which is impossible since KP is singular. $\square$

The previous result provides a simple way to check if an upper triangular operator is not bounded: let S_+ and S_- be the right and left shift operators on ℓ_2, respectively. Consider the upper triangular map $S = \begin{pmatrix} S_+ & I_{\ell_2} \\ 0 & S_- \end{pmatrix}$ on Z_2. Then S is not bounded by Proposition 1.2.5 since, if it were, then $S_+ - S_- : \ell_2 \to \ell_2$ should be compact, which is absurd.

1.2.5 Diagonal operators

Condition (♠) for upper triangular operators states that the quasilinear map $\alpha \circ \mathsf{KP} - \mathsf{KP} \circ \gamma : \ell_2 \to \ell_2$ has to be trivial. If, moreover $\beta = 0$, it has to be *bounded*:

Corollary 1.2.5. *The map $T = \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix}$ is bounded on Z_2 if and only if $\alpha, \gamma : \ell_2 \to \ell_2$ and*

$$\alpha \circ \mathsf{KP} - \mathsf{KP} \circ \gamma : \ell_2 \to \ell_2 \quad \textit{is bounded.}$$

The following commutative diagram resumes the situation:

$$\begin{tikzcd}
\ell_2 \arrow[r] \arrow[d, "\alpha"'] & Z_2 \arrow[r] \arrow[d, "T"] & \ell_2 \arrow[ll, bend right, "\mathsf{KP}"'] \arrow[d, "\gamma"] \\
\ell_2 \arrow[r] & Z_2 \arrow[r] & \ell_2 \arrow[ll, bend left, "\mathsf{KP}"]
\end{tikzcd}$$

For instance, the maps

$$\begin{pmatrix} S_+ & 0 \\ 0 & S_+ \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} S_- & 0 \\ 0 & S_- \end{pmatrix}$$

define bounded operators on Z_2. Indeed, KP commutes with S_+:

$$S_+\mathsf{KP}(x) = \sum_{n=1}^{\infty} x_{n-1} \log \frac{|x_{n-1}|}{\|x\|} e_n = \mathsf{KP}\Big((0, x_1, x_2, x_3, \ldots)\Big) = \mathsf{KP}(S_+x).$$

Hence, $\begin{pmatrix} S_+ & 0 \\ 0 & S_+ \end{pmatrix}$ is bounded by Corollary 1.2.5. On the other hand, using the identification of Z_2 with $\ell_2 \oplus_{-\mathsf{KP}} \ell_2$ it can be shown (see Subsection 4.2.1 for a precise statement) that

$$\begin{pmatrix} S_- & 0 \\ 0 & S_- \end{pmatrix}^* \text{ is bounded if and only if } \begin{pmatrix} S_-^* & 0 \\ 0 & S_-^* \end{pmatrix} = \begin{pmatrix} S_+ & 0 \\ 0 & S_+ \end{pmatrix} \text{ is bounded.}$$

Corollary 1.2.6. *The operator* $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ *is bounded on* Z_2 *if and only if* $\alpha : \ell_2 \to \ell_2$ *and*

$$[\alpha, \mathsf{KP}] = \alpha \circ \mathsf{KP} - \mathsf{KP} \circ \alpha : \ell_2 \to \ell_2 \quad \textit{is bounded.}$$

Note that if $T = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ is bounded on Z_2 then the operator α itself must be bounded on both the range space ℓ_f^* and on the domain space ℓ_f. Indeed, recall that the domain space is just the subspace of Z_2 formed by vectors of the form $(0, x) \in Z_2$, in which T clearly acts boundedly. Thus, $\alpha(\ell_f) \subset \ell_f$ and since $\ell_f^* = Z_2/\ell_f$, it follows that α is also bounded on ℓ_f^*. This gives direct ways to show that some diagonal operators on Z_2 can not be bounded. Take any norm one $x \in \ell_2$ such that $x \notin \ell_f$ and consider the isometry U on ℓ_2 sending e_1 to x. Then U is not bounded on the aforementioned Orlicz spaces, hence $\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$ is not bounded on Z_2. Checking if the commutator map is bounded for arbitrary operators can be tricky. Consider the Cesàro and Hilbert operator, denoted respectively by C and H; those are bounded operators on ℓ_2 whose matrix representation is given by

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.13}$$

Thus $C(x) = \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)_{n\in\mathbb{N}}$ and $H(x) = \left(\sum_{k=1}^{\infty} \frac{x_k}{k+n-1}\right)_{n\in\mathbb{N}}$. The corresponding commutator estimates are

$$\left(\sum_{n=1}^{\infty} \Big| \frac{1}{n} \sum_{k=1}^{n} x_k \Big[\log \frac{\left|\frac{1}{n}\sum_{k=1}^{n} x_k\right|}{\|C(x)\|} - \log \frac{|x_k|}{\|x\|} \Big] \Big|^2 \right)^{1/2} \leq C\|x\|_{\ell_2} \tag{1.14}$$

and

$$\left(\sum_{n=1}^{\infty} \Big| \sum_{k=1}^{\infty} \frac{x_k}{n+k-1} \Big[\log \frac{\left|\sum_{k=1}^{\infty} \frac{x_k}{n+k-1}\right|}{\|H(x)\|} - \log \frac{|x_k|}{\|x\|} \Big] \Big|^2 \right)^{1/2} \leq C'\|x\|_{\ell_2} \tag{1.15}$$

The fundamental result to be explained in Chapter 2 is that the commutator estimates (1.14) and (1.15) not only are true, but they are direct consequences of the behaviour of C and H on the interpolation scale of ℓ_p spaces.

Until now, we put special focus in formulating conditions to bound an operator on Z_2 in terms of KP. One can express the conditions using KP^{-1}. All such inverse conditions are obtained much in the same way so we may skip the proofs. The only remarkable thing is that in order to deduce the analogous condition to ($\diamondsuit$) we need to introduce the following companion to ($\mathcal{L}$):

$$\alpha : \ell_f^* \to \ell_f^* \quad \text{is bounded} \qquad \text{and} \qquad \beta : \ell_2 \to \ell_f^* \quad \text{is bounded.} \tag{$\mathcal{L}^*$}$$

The key here is that ($\mathcal{L}^*$) linearizes KP^{-1} in the same sense that ($\mathcal{L}$) linearizes KP. For the sake of completeness, we summarize all conditions in the following table:

Boundness conditions for operators on Z_2		
Operator	Canonical representation	Inverse representation
General	• $\alpha\mathsf{KP} + \beta - \mathsf{KP}(\delta\mathsf{KP} + \gamma) : \ell_2 \to \ell_2$	• $\mathsf{KP}^{-1}(\alpha + \beta\mathsf{KP}^{-1}) - \delta - \gamma\mathsf{KP}^{-1} : \ell_f^* \to \ell_f$
($\mathcal{L}$)	• $\alpha\mathsf{KP} - \mathsf{KP}\gamma + \beta - \mathsf{KP}\delta\mathsf{KP} : \ell_2 \to \ell_2$	———
($\mathcal{L}^*$)	———	• $\gamma\mathsf{KP}^{-1} - \mathsf{KP}^{-1}\alpha + \delta - \mathsf{KP}^{-1}\beta\mathsf{KP}^{-1} : \ell_f^* \to \ell_f$
$\delta = 0$	• $\alpha\mathsf{KP} - \mathsf{KP}\gamma + \beta : \ell_2 \to \ell_2$	• $\gamma\mathsf{KP}^{-1} - \mathsf{KP}^{-1}\alpha - \mathsf{KP}^{-1}\beta\mathsf{KP}^{-1} : \ell_f^* \to \ell_f$
$\beta = 0$	• $\alpha\mathsf{KP} - \mathsf{KP}\gamma - \mathsf{KP}\delta\mathsf{KP} : \ell_2 \to \ell_2$	• $\gamma\mathsf{KP}^{-1} - \mathsf{KP}^{-1}\alpha + \delta : \ell_f^* \to \ell_f$
$\delta, \beta = 0$	• $\alpha\mathsf{KP} - \mathsf{KP}\gamma : \ell_2 \to \ell_2$	• $\gamma\mathsf{KP}^{-1} - \mathsf{KP}^{-1}\alpha : \ell_f^* \to \ell_f$
$\alpha = \gamma$	• $[\alpha, \mathsf{KP}] : \ell_2 \to \ell_2$	• $[\alpha, \mathsf{KP}^{-1}] : \ell_f^* \to \ell_f$

1.2.6 The Johnson-Lindenstrauss-Schechtman conjecture

In Proposition 1.2.5 we saw that condition $\alpha = \gamma$ of Corollary 1.2.3 could be generalized to the fact that $\alpha - \gamma : \ell_2 \to \ell_2$ is compact for any upper triangular operator $\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$. Now we focus on how to generalize the fact that $\delta = 0$, which, as we will see, is closely connected to the hyperplane problem. The first thing to note is that Z_2 is isomorphic to its subspaces of finite *even codimension*. This can be achieved using the bounded operators

$$\begin{pmatrix} S_+ & 0 \\ 0 & S_+ \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} S_- & 0 \\ 0 & S_- \end{pmatrix}$$

and its powers. One of the most interesting approaches to the hyperplane problem was given by Johnson, Lindenstrauss and Schechtman in [57]. They reasoned the following way: since the quotient map $q : Z_2 \to \ell_2$ is strictly singular, given any operator $T : Z_2 \to Z_2$ of the form

$$\begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$$

the composition $\delta = qTi : \ell_2 \to \ell_2$ is strictly singular on ℓ_2, hence compact. This means that every operator $T : Z_2 \to Z_2$ carries, up to a residual factor indicated by δ, the canonical copy $\ell_2 \subset Z_2$ into itself. Therefore, they posed the following conjecture:

Conjecture 1 (JLS Conjecture)**.** *Every operator $T : Z_2 \to Z_2$ is a strictly singular perturbation of an upper triangular operator.*

Every Fredholm operator of the form $T = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} + S$ with S strictly singular has even index. Indeed, recall that the index of a Fredholm operator is invariant under strictly singular perturbations (cf. Proposition B.1.2). Thus, if T is Fredholm then $U = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$ is Fredholm with the same index. In this case, U is Fredholm if and only if both α and γ are Fredholm on ℓ_2 (see [103]), in which case $ind(U) = ind(\alpha) + ind(\gamma)$. But Proposition 1.2.5 gives that $\alpha - \gamma \in \mathcal{K}(\ell_2)$; using again the invariance of the index under compact perturbations we obtain that $ind(\alpha) = ind(\gamma + (\alpha - \gamma)) = ind(\gamma)$. Therefore, $ind(T) = ind(U) = ind(\alpha) + ind(\gamma)$ is even. Hence, an isomorphism between Z_2 and its hyperplanes is something impossible if the representation provided by JLS-Conjecture is true.

There are obvious situations for which this conjecture does hold. Think for instance when $\delta : \ell_f^* \to \ell_f$ is bounded. We saw in Section 1.2.2 that, in this case, such condition leads to the decomposition

$$\begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix} \tag{1.16}$$

in which both operators are bounded. Moreover, the last operator is always strictly singular on Z_2: the canonical quotient map $p : Z_2 \to \ell_f^*$ is strictly singular by Theorem 1.1.1 since, if it not were, Z_2 could be embedded in ℓ_f^*, which is impossible due to type/cotype constraints. Then Proposition A.1.4 gives that $S = \begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix}$ is strictly singular if and only if its restriction to $\ell_f = \ker(p)$ is strictly singular. Since $S|_{\ell_f} = 0$ it follows that (1.16) is strictly singular. If we drop the previous condition on δ, even if we assume the quite stringent condition

$$\delta : \ell_f^* \to \ell_2 \quad \text{and} \quad \delta : \ell_2 \to \ell_f \quad \text{are bounded,} \tag{Δ}$$

it is not clear how to obtain a decomposition as indicated by the (JLS)-conjecture. In this case the factorization (1.16) is not allowed since the last term would not be bounded, and thus one has to perturb also the remaining entries of the matrix to induce something of the form

$$\begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} = \begin{pmatrix} \alpha' & \beta' \\ 0 & \gamma' \end{pmatrix} + \begin{pmatrix} \alpha - \alpha' & \beta - \beta' \\ \delta & \gamma - \gamma' \end{pmatrix}$$

Condition (Δ) implies the linearization condition $(\mathcal{L})$, and since both operators must be bounded, Corollary 1.2.5 and Theorem 1.2.4 forces that

$$\alpha' \circ \mathsf{KP} - \mathsf{KP} \circ \gamma' + \beta' : \ell_2 \to \ell_2$$

and

$$(\alpha - \alpha') \circ \mathsf{KP} - \mathsf{KP} \circ (\gamma - \gamma') + (\beta - \beta') - \mathsf{KP} \circ \delta \circ \mathsf{KP} : \ell_2 \to \ell_2$$

are bounded maps. Therefore, in this case the problem can be reduced to the simultaneous approximation of two naturally defined quasilinear maps:

Problem 3. *Given three bounded operators $\alpha, \gamma : \ell_2 \to \ell_2$ and $\delta : \ell_2 \to \ell_f$ such that the quasilinear map*

$$\alpha \circ \mathsf{KP} - \mathsf{KP} \circ \gamma - \mathsf{KP} \circ \delta \circ \mathsf{KP}$$

is trivial on ℓ_2, there exist two bounded operators $\alpha', \gamma' : \ell_2 \to \ell_2$ satisfying the following conditions:

(1) $\alpha' \circ \mathsf{KP} - \mathsf{KP} \circ \gamma' : \ell_2 \to \ell_2$ *is trivial.*

(2) $(\alpha - \alpha') \circ \mathsf{KP} - \mathsf{KP} \circ (\gamma - \gamma') - \mathsf{KP} \circ \delta \circ \mathsf{KP} : \ell_2 \to \ell_2$ *is trivial.*

It is clear by the preceding discussion that upper and lower triangular operators play a key role in the study of the hyperplane problem. This leads to the question of whether there exist a natural and systematic way to construct triangular operators on Z_2. It turns out that there is a general method to obtain upper triangular operators, and it depends on the fundamental fact that the Kalton-Peck space appears naturally in the (apparently unrelated) context of Complex Interpolation Theory of Banach spaces. The whys and the hows are the main content of next two chapters.

Chapter 2

Complex interpolation for pairs

In this chapter we present the basics of complex interpolation for pairs of Banach spaces. The main source for us will be the classical [7], but we shall use at times the other works [80, 61]. The origins of complex interpolation of Banach spaces can be traced back to the work of Riesz and Thorin and their *convexity theorem* for L_p spaces:

Theorem 2.0.1 (Riesz-Thorin Theorem)**.** *Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. Let T be an operator such that both*

$$T : L_{p_0} \to L_{q_0} \quad \textit{and} \quad T : L_{p_1} \to L_{q_1}$$

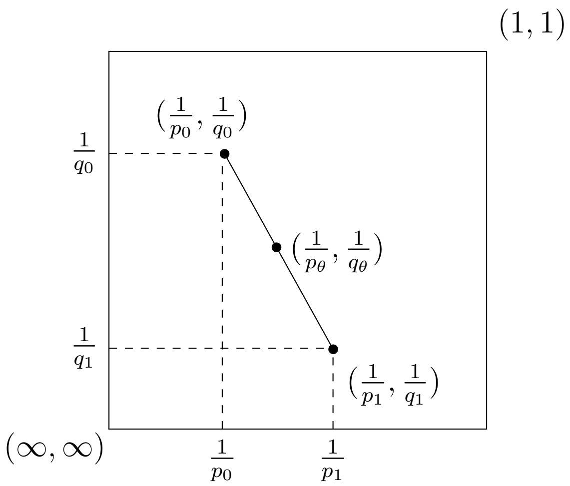
are bounded. Then $T : L_{p_\theta} \to L_{q_\theta}$ is bounded with norm

$$\left\| T : L_{p_\theta} \to L_{q_\theta} \right\| \leq \left\| T : L_{p_0} \to L_{q_0} \right\|^{1-\theta} \left\| T : L_{p_1} \to L_{q_1} \right\|^{\theta}, \tag{2.1}$$

where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \textit{and} \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Equivalently, the function $M(p, q) = \|T : L_p \to L_q\|$ is log-convex on the following square:



2.1 The complex method for pairs on the strip

The starting point of abstract complex interpolation theory seeks to mimicry the situation for L_p spaces to general Banach spaces: given two Banach spaces X_0 and X_1, produce a family $(X_\theta)_{\theta\in\Theta}$ of Banach spaces such that

$$T : X_\theta \to X_\theta \text{ is bounded whenever } T : X_0 \to X_0 \quad \text{and} \quad T : X_1 \to X_1 \text{ are.} \tag{2.2}$$

In this case X_θ is referred to as an *interpolation space* of X_0 and X_1.

Let $\mathcal{U}$ be a Hausdorff topological vector space. A pair (X_0, X_1) of Banach spaces is called $(\mathcal{U}, i_0, i_1)$-compatible if there exist continuouos injections $i_0 : X_0 \to \mathcal{U}$ and $i_1 : X_1 \to \mathcal{U}$. We will identify X_0 and X_1 with $i_0(X_0)$ and $i_1(X_1)$, so that i_j becomes the inclusion map $X_j \subset \mathcal{U}$ for each $j = 0, 1$. When the space $\mathcal{U}$ and the given injections are clear from context, we will refer to (X_0, X_1) as a *Banach couple* or simply as a couple.

Given a couple (X_0, X_1) one can form the sum space $X_0 + X_1$ endowed with the norm

$$\|x\|_{X_0+X_1} = \inf_{x=x_0+x_1} \|x_0\|_{X_0} + \|x_1\|_{X_1}$$

and the intersection space $X_0 \cap X_1$ with the maximum norm $\|x\|_{X_0\cap X_1} = \max\{\|x\|_{X_0}, \|x\|_{X_1}\}$. Both $X_0 \cap X_1$ and $X_0 + X_1$ are Banach spaces [7, Lemma 2.3.1] for which there exist obvious continuous inclusions

$$X_0 \cap X_1 \subset X_j \subset X_0 + X_1$$

for each $j = 0, 1$. Any Banach space X such that $X_0 \cap X_1 \subset X \subset X_0 + X_1$ with continuous inclusions will be called an *intermediate space* for the Banach couple (X_0, X_1).

We describe now the complex method developed by Calderón in [14]. We denote by $\mathbb{S}$ the strip $\{z \in \mathbb{C} \colon 0 < Re(z) < 1\}$ and by $\overline{\mathbb{S}}$ its closure. Let (X_0, X_1) be a fixed couple and define the *Calderón space* $\mathcal{C}(X_0, X_1)$ as the vector space of all bounded and continuous functions $f : \overline{\mathbb{S}} \to X_0 + X_1$ which are analytic on $\mathbb{S}$ and such that

$$\text{the map } \; t \in \mathbb{R} \mapsto f(j + it) \in X_j \quad \text{is bounded and continuous for each } j = 0, 1. \tag{2.3}$$

Lemma 2.1.1 (Vector-valued Hadamard three-line theorem)**.**
Let B be any Banach space and $f : \overline{\mathbb{S}} \to B$ a continuous and bounded function which is analytic on $\mathbb{S}$. Then for any $0 < \theta < 1$ we have that

$$\|f(\theta)\|_B \le \Big(\sup_t \|f(it)\|_B\Big)^{1-\theta} \Big(\sup_t \|f(1 + it)\|_B\Big)^{\theta}$$

Note that all funtions $f \in \mathcal{C}(X_0, X_1)$ satisfy the hypotheses of Lemma 2.1.1. Thus, the expression

$$\|f\|_{\mathcal{C}} = \max_{j=0,1}\{\sup_{t\in\mathbb{R}} \|f(j + it)\|_{X_j}\}$$

defines a norm on $\mathcal{C}(X_0, X_1)$ which makes it a Banach space [7, Lemma 4.1.1].

Moreover, (2.3) sets the Banach spaces X_0 and X_1 as boundary conditions for the functions $f \in \mathcal{C}(X_0, X_1)$. Thus the evaluations $f(\theta)$ define a space X_θ whose properties depend on X_0 and X_1.

Precisely, given any $0 < \theta < 1$ define the complex interpolation space $X_\theta = (X_0, X_1)_\theta$ as the vector space

$$\Big\{x \in X_0 + X_1 \colon x = f(\theta),\ f \in \mathcal{C}(X_0, X_1)\Big\}$$

endowed with the quotient norm

$$\|x\|_\theta = \inf\{\|f\|_{\mathcal{C}} \colon f(\theta) = x,\ f \in \mathcal{C}(X_0, X_1)\}.$$

By Lemma 2.1.1 the evaluation map $\delta_\theta : \mathcal{C}(X_0, X_1) \to X_0 + X_1$ given by $\delta_\theta(f) = f(\theta)$ is bounded, and we can identify X_θ with the quotient space $\mathcal{C}(X_0, X_1)/\ker \delta_\theta$. Continuity of the evaluation map also yields the continuous inclusion $X_\theta \subset X_0 + X_1$ and using constant functions one also has $X_0 \cap X_1 \subset X_\theta$, which gives the chain

$$X_0 \cap X_1 \subset X_\theta \subset X_0 + X_1.$$

It only remains to show whether the interpolation principle (2.2) does hold for X_θ.

Let us call an operator $T : X_0 + X_0 \to X_0 + X_1$ *interpolating* if the operator $T_{\mathcal{C}} : \mathcal{C}(X_0, X_1) \to \mathcal{C}(X_0, X_1)$ given by

$$T_{\mathcal{C}}(f)(z) = T(f(z))$$

is bounded.
If $T : X_0 \to X_0$ and $T : X_1 \to X_1$ are bounded operators, then the linear map $T_\Sigma : X_0 + X_1 \to X_0 + X_1$ defined for $x = x_0 + x_1 \in X_0 + X_1$ by

$$T_\Sigma(x) = T(x_0) + T(x_1)$$

is bounded. Indeed,

$$\|T_\Sigma(x)\|_{X_0+X_1} = \|Tx_0 + Tx_1\|_{X_0+X_1} \le \|T : X_0 \to X_0\|\|x_0\|_{X_0} + \|T : X_1 \to X_1\|\|x_1\|_{X_1}.$$

Taking the infimum on both sides over all representations $x = x_0 + x_1$ we deduce that

$$\|T_\Sigma : X_0 + X_1 \to X_0 + X_1\| \le \max\{\|T : X_0 \to X_0\|, \|T : X_1 \to X_1\|\}.$$

The hypothesis in (2.2) is subsumed in:

Proposition 2.1.1. *Let $T : X_0 \to X_0$ and $T : X_1 \to X_1$ be bounded operators. Then $T_\Sigma : X_0 + X_1 \to X_0 + X_1$ is interpolating.*

Proof. We must show that the induced map $T_{\mathcal{C}} : \mathcal{C}(X_0, X_1) \to \mathcal{C}(X_0, X_1)$ is well defined and bounded:

(i) Given $f \in \mathcal{C}(X_0, X_1)$, the map $z \in \mathbb{S} \mapsto T_{\mathcal{C}}(f)(z) = T_\Sigma(f(z))$ is continuous (being the composition of two continuous maps) and bounded since $\|T_\Sigma(f(z))\|_{X_0+X_1} \le \|T_\Sigma : X_0 + X_1 \to X_0 + X_1\|\,\|f(z)\|_{X_0+X_1}$.

$$\begin{tikzcd} \mathbb{S} \arrow[r, "f"] \arrow[dr, "T_{\mathcal{C}}(f)"'] & X_0 + X_1 \arrow[d, "T_\Sigma"] \\ & X_0 + X_1 \end{tikzcd}$$

(ii) $T_{\mathcal{C}}(f)$ is analytic on $\mathbb{S}$: given any $\theta \in \mathbb{S}$ one has that

$$\delta'_\theta\Big(T_{\mathcal{C}}(f)\Big) = \lim_{z\to\theta} \frac{T_\Sigma(f(z)) - T_\Sigma(f(\theta))}{z-\theta}.$$

Since T_Σ is linear and continuous we conclude that $\delta'_\theta(T_{\mathcal{C}}(f)(z)) = T_\Sigma(f'(z))$. In fact, the same arguments provide the analogous result for higher order derivatives:

$$\delta_\theta^{(k)}\Big(T_{\mathcal{C}}(f)\Big) = T_\Sigma\big(\delta_\theta^{(k)}(f)\big) \quad \text{for any } k \in \mathbb{N}. \tag{2.4}$$

(iii) Thus $T_{\mathcal{C}}\Big(\mathcal{C}(X_0,X_1)\Big) \subset \mathcal{C}(X_0,X_1)$ and, moreover,

$$\begin{aligned}\|T_{\mathcal{C}}(f)\|_{\mathcal{C}} &= \max\{\sup_t \|T_\Sigma(f(it))\|_{X_0}, \sup_t \|T_\Sigma(f(1+it))\|_{X_1}\} \\ &\le \max\{\|T: X_0 \to X_0\|, \|T: X_1\to X_1\|\}\|f\|_{\mathcal{C}}\end{aligned}$$

for any $f \in \mathcal{C}(X_0,X_1)$.

□

Interpolating operators $T : X_0 + X_1 \to X_0 + X_1$ satisfy the conclussion of (2.2):

Theorem 2.1.1 (Interpolation principle for operators)**.**
Let $T : X_0 + X_1 \to X_0 + X_1$ be an interpolating operator. Then $T : X_\theta \to X_\theta$ is bounded.

Proof. All information is contained in the following commutative diagram

$$\begin{tikzcd}
\mathcal{C}(X_0,X_1) \arrow[r, "\delta_\theta"] \arrow[d, "T_{\mathcal{C}}"'] & X_0 + X_1 \arrow[d, "T"] \\
\mathcal{C}(X_0,X_1) \arrow[r, "\delta_\theta"] & X_0 + X_1
\end{tikzcd}$$

Let $x \in X_\theta$ be fixed and select $f \in \mathcal{C}(X_0,X_1)$ such that $f(\theta) = x$ and $\|f\|_{\mathcal{C}} \le (1+\varepsilon)\|x\|_\theta$. Then

$$\begin{aligned}\|Tx\|_\theta &= \|T\big(f(\theta)\big)\|_\theta = \|T_{\mathcal{C}}(f)(\theta)\|_\theta \\ &\le \|T_{\mathcal{C}}(f)\|_{\mathcal{C}} \le \|T_{\mathcal{C}}\|\,\|f\|_{\mathcal{C}} \le (1+\varepsilon)\|T_{\mathcal{C}}\|\,\|x\|_\theta.\end{aligned}$$ □

If $T : X_0 \to X_0$ and $T : X_1 \to X_1$ are bounded, then we can obtain an explicit bound for $\|T_{\mathcal{C}}\|$: given any $f \in \mathcal{C}(X_0,X_1)$ such that $f(\theta) = x$ and $\|f\|_{\mathcal{C}} \le (1+\varepsilon)\|x\|_\theta$, define the analytic function

$$h(z) = \frac{f(z)}{\|T: X_0 \to X_0\|^{1-z}\,\|T : X_1 \to X_1\|^z}.$$

Then $h \in \mathcal{C}(X_0,X_1)$ with $\|h\|_{\mathcal{C}} \le 1$. A direct calculation shows that $\|T_{\mathcal{C}}(h)\|_{\mathcal{C}} \le \|f\|_{\mathcal{C}}$, hence

$$\frac{\|Tx\|_\theta}{\|T: X_0\to X_0\|^{1-\theta}\,\|T : X_1 \to X_1\|^\theta} = \|T_{\mathcal{C}}(h)(\theta)\|_\theta \le \|f\|_{\mathcal{C}} \le (1+\varepsilon)\|x\|_\theta.$$

It follows that

$$\|T : X_\theta \to X_\theta\| \le \|T : X_0 \to X_0\|^{1-\theta}\,\|T : X_1 \to X_1\|^\theta, \tag{2.5}$$

which is the log-convexity bound analogous to (2.1).

2.1.1 Calderón spaces

An apparently innocuous but quite important tool for interpolation theory is the existence of alternative Calderón spaces that define the same interpolation spaces. Precisely, let us denote by $\mathcal{C}_0(X_0, X_1)$ the subspace of $\mathcal{C}(X_0, X_1)$ formed by all functions f such that

$$\text{for each } j = 0, 1, \quad \|f(j + it)\|_{X_j} \to 0, \text{ whenever } |t| \to \infty.$$

Observe that, given $f \in \mathcal{C}(X_0, X_1)$ and $\delta > 0$, the function $f_\delta(z) = e^{\delta z^2} f(z)$ belongs to $\mathcal{C}_0(X_0, X_1)$ with norm $\|f_\delta\|_{\mathcal{C}} \leq e^\delta \|f\|_{\mathcal{C}}$. Since $f(\theta) = e^{-\delta\theta^2} f_\delta(\theta)$, it is not hard to show that $\|\cdot\|_\theta$ can be computed using functions in $\mathcal{C}_0(X_0, X_1)$ (see [80, Remark 8.6]). Thus, we can replace $\mathcal{C}_0$ by $\mathcal{C}$ in the definition of the interpolation space X_θ.

The advantage of using $\mathcal{C}_0$ instead of the original Calderón space comes from the fact that $\mathcal{C}_0$ contains a dense subspace formed by very manageable functions: let $\mathcal{C}_{00}(X_0, X_1)$ be the space formed by all functions $f : \overline{\mathbb{S}} \to X_0 \cap X_1$ of the form

$$f(z) = \sum_{k=1}^{N} f_k(z) x_k, \quad \text{where } x_k \in X_0 \cap X_1$$

and such that $f_k \in \mathcal{C}(\mathbb{C}, \mathbb{C})$ vanishes at the infinity, i.e.,

$$\lim_{R\to\infty} \sup\{|f(z)| \colon z \in \overline{\mathbb{S}},\ |Im(z)| \geq R\} = 0.$$

Lemma 2.1.2. *$\mathcal{C}_{00}(X_0, X_1)$ is dense in $\mathcal{C}_0(X_0, X_1)$.*

See [80, Lemma 8.11] or [7, Lemma 4.2.3] for a proof. One can also substitute the elements $x_k \in X_0 \cap X_1$ by vectors belonging to any dense subset of $X_0 \cap X_1$. Explicit examples of functions on $\mathcal{C}_{00}$ are given by the family

$$f(z) = e^{\delta z^2} \sum_{k=1}^{N} e^{\lambda_k z} x_k, \quad \text{where } \delta > 0,\ \lambda_k \in \mathbb{R} \text{ and } x_k \in X_0 \cap X_1.$$

Two direct and useful consequences are:

Lemma 2.1.3. *For each $0 < \theta < 1$ the intersection space $X_0 \cap X_1$ is dense in X_θ. Moreover, for any $x \in X_0 \cap X_1$ we have that*

$$\|x\|_\theta = \inf\{\|f\|_{\mathcal{C}} \colon f \in \mathcal{C}_{00}(X_0, X_1),\ f(\theta) = x\}. \tag{2.6}$$

The first part is classical and already known by Calderón [14, 9.3], while the "moreover part" is due to Stafney [86, Lemma 2.5] (see also [80, Lemma 8.11]).

Lemma 2.1.4. *Let (X_0, X_1) be a Banach couple and denote for each $j = 0, 1$ the closure of $X_0 \cap X_1$ in X_j by A_j. Then for each $0 < \theta < 1$*

$$\mathcal{C}_0(X_0, X_1) = \mathcal{C}_0(A_0, A_1) \quad \textit{and} \quad (X_0, X_1)_\theta = (A_0, A_1)_\theta.$$

See [80, Remark 8.13] or [7, 4.2.2] for further comments. A simple example for which Lemma 2.1.4 applies is the sequence ℓ_p spaces: $(\ell_1, \ell_\infty)_\theta = (\ell_1, c_0)_\theta$.

Any function $f \in \mathcal{C}(X_0, X_1)$, being an analytic function on $\mathbb{S}$ which continuously extends to a bounded continuous function on $\overline{\mathbb{S}}$, admits a Poisson integral representation on $\mathbb{S}$, i.e., for every $z \in \mathbb{S}$ there exist a (unique) probability measure μ_z on $\partial\mathbb{S}$ such that

$$f(z) = \int_{\partial\mathbb{S}} f(\xi) d\mu_z(\xi)$$

for all $f \in \mathcal{C}(X_0, X_1)$ (see [80, pp. 301-302] or [6, Appendix I]). Moreover, if $0 < \theta < 1$, there exist two measures μ_θ^0 and μ_θ^1 supported on ∂_0 and ∂_1, respectively, so that $\mu_\theta = \mu_\theta^0 + \mu_\theta^1$. Thus

$$f(\theta) = \int_{\partial\mathbb{S}} f(\xi) d\mu_\theta(\xi) = \int_{\partial_0} f(it) d\mu_\theta^0(it) + \int_{\partial_1} f(1+it) d\mu_\theta^1(1+it). \tag{2.7}$$

For each $j = 0, 1$, the measure μ_θ^j is absolutely continuous with respect to the induced Lebesgue measure dt on ∂_j and the corresponding Radon-Nikodym derivative is given by the Poisson kernels on the strip (see [100]):

$$d\mu_\theta^j(j+it) = \frac{\sin(\pi\theta)}{2\big(\cosh(\pi t) + (-1)^{j+1}\cos(\pi\theta)\big)} dt = P_\theta^j(t) dt.$$

Moreover, since the representation (2.7) also holds if we assumed that f just were harmonic on $\mathbb{S}$, using the function $Re(z)$ and the fact that μ_θ is a probability measure it follows that

$$\int_{\partial_0} du_\theta^0(it) = 1 - \theta \quad \text{and} \quad \int_{\partial_1} du_\theta^1(1+it) = \theta. \tag{2.8}$$

Furthermore, for any $0 < \theta < 1$ it was shown by Calderón [14, 9.4] (see also [7, Lemma 4.3.2] or [80, Lemma 8.20]) that

$$\|f(\theta)\|_\theta \leq \int_{\partial_0} \|f(it)\|_{X_0} d\mu_\theta^0(it) + \int_{\partial_1} \|f(1+it)\|_{X_1} d\mu_\theta^1(1+it). \tag{2.9}$$

If we denote by $\|f\|_{\mathcal{C}_\theta}$ the right side of (2.9), then $\|\cdot\|_{\mathcal{C}_\theta}$ is a norm on Calderón space which satisfies by (2.8) that

$$\|f(\theta)\|_\theta \leq \|f\|_{\mathcal{C}_\theta} \leq \|f\|_{\mathcal{C}} \tag{2.10}$$

We will need (2.10) in Chapter 5, since these integral norms are more suitable to work with averages of analytic functions.

2.2 Some examples of interpolation scales

In this section we give several examples of complex interpolation scales, namely, we describe the spaces X_θ obtained for some couples (X_0, X_1). Some of these examples will be studied in forthcoming sections.

2.2.1 L_p spaces over a σ-finite measure space

Let (Ω, μ) be a σ-finite measure space and, for each $1 \leq p \leq \infty$, denote by $L_p(\mu)$ the corresponding Banach space of p-integrable functions. The cases of $(\mathbb{R}^n, \lambda_n)$ with the

Lebesgue measure and $(\mathbb{N}, \nu)$ with counting measure are the main examples which we will consider. Adapted to this context, the claim of Riesz-Thorin Theorem (see Theorem 2.0.1) translates into the interpolation formula

$$(L_{p_0}(\mu), L_{p_1}(\mu)) = L_{p_\theta}(\mu), \quad \text{where } 0 < \theta < 1 \quad \text{and} \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \tag{2.11}$$

The proof of this fact can be found in [7, Section 5.1] or [80, Th. 8.15], but it essentially reduces to obtaining an *homogeneous bounded selection* for the evaluation map $\delta_\theta : \mathcal{C}(L_{p_0}(\mu), L_{p_1}(\mu)) \to L_{p_\theta}(\mu)$, i.e., an homogenous map $B_{p_\theta} : L_{p_\theta}(\mu) \to \mathcal{C}(L_{p_0}(\mu), L_{p_1}(\mu))$ such that

$$B_{p_\theta}(x)(\theta) = x \quad \text{with} \quad \|B_{p_\theta}(x)\|_{\mathcal{C}} \leq C\|x\|_\theta, \quad \text{for each } x \in L_{p_\theta}(\mu).$$

An explicit expression for B_{p_θ} on a dense subspace of $L_{p_\theta}(\mu)$ can be obtained in the following way: define the function $p(z)^{-1} = (1-z)p_0^{-1} + zp_1^{-1}$ and note that given any norm one simple function $x \in L_{p_\theta}(\mu)$, the polar decomposition of x gives rise to the analytic function (observe that $sign(x) = x/|x|$)

$$f_x(z) = sign(x)|x|^{\frac{p_\theta}{p(z)}} = x|x|^{\frac{(1-z)p_\theta}{p_0} + \frac{zp_\theta}{p_1} - 1} \in \mathcal{C}(L_{p_0}(\mu), L_{p_1}(\mu)).$$

For an arbitrary simple function $x \in L_{p_\theta}$ we define $B_{p_\theta}(x)$ extending the previous map homogeneously as

$$B_{p_\theta}(x)(z) = \|x\|_{p_\theta} f_{\frac{x}{\|x\|}}(z) = x\Big(\frac{|x|}{\|x\|_{p_\theta}}\Big)^{\frac{(1-z)p_\theta}{p_0} + \frac{zp_\theta}{p_1} - 1} \in \mathcal{C}(L_{p_0}(\mu), L_{p_1}(\mu)). \tag{2.12}$$

The map B_{p_θ} is a selection for δ_θ by (2.11) and is isometric in the sense that $\|x\|_{p_\theta} = \|B_{p_\theta}(x)\|_{\mathcal{C}}$. As we shall see in forthcoming Section 2.6, the properties of B_{p_θ} will give further information about the scale of $L_p(\mu)$ spaces that is not obtaineable using just classical interpolation theory.

2.2.2 Weighted L_p spaces

Let (Ω, μ) be a measure space. A weight $w : \Omega \to \mathbb{R}$ is a positive μ-measurable function that is locally integrable. Given a weight w and $1 \leq p < \infty$, we denote by $L_p(w, \mu) = L_p(w)$ the corresponding *weighted* Lebesgue space, that is, the Banach space formed by $f \in L_0(\mu)$ such that $w^{1/p} f \in L_p(\mu)$ under the norm

$$\|f\|_{p^w} = \Big(\int_\Omega |f|^p w d\mu\Big)^{1/p} = \|w^{1/p} f\|_p.$$

Let w_0 and w_1 be two weights on (Ω, μ). Interpolation of two different weighted Lebesgue spaces $L_p(w_0)$ and $L_q(w_1)$ were studied by E. Stein and G. Weiss in [89] (see also [7, Section 5.5]). Here we are interested in the case $p = q$, for which one has the identification

$$(L_p(w_0, \mu), L_p(w_1, \mu))_\theta = L_p(w_0^{1-\theta} w_1^\theta, \mu), \quad \text{for all } 0 < \theta < 1. \tag{2.13}$$

The map

$$B_\theta(x)(z) = \Big(\frac{w_0}{w_1}\Big)^{\frac{(z-\theta)}{p}} x = \Big(\frac{w_0^{1/p}}{w_1^{1/p}}\Big)^{z-\theta} x, \quad \text{for each } 0 < \theta < 1. \tag{2.14}$$

defines a bounded homogeneous selection for the evaluation map $\delta_\theta : \mathcal{C}(L_p(w_0, \mu), L_p(w_1, \mu)) \to L_p(w_0^{1-\theta} w_1^\theta, \mu)$ [43, 22]. Note that, unlike the previous case of L_p spaces, the selection (2.14) is *linear* in x. If we denote by $w_\theta = w_0^{1-\theta} w_1^\theta$, it is readily verified that

$$\|B_\theta(x)(0 + it)\|_{p^{w_0}} = \|B_\theta(x)(1 + it)\|_{p^{w_1}} = \|(w_0^{1-\theta} w_1^\theta)^{1/p} x\|_p = \|x\|_{p^{w_\theta}}.$$

2.2.3 Hardy and Sobolev spaces

Let $\mathbb{T} \subset \mathbb{C}$ be the unit circle and denote by m be the normalized Lebesgue measure on $\mathbb{T}$. If $\mathcal{H}(\mathbb{D})$ stands for the space of complex valued analytic functions on the unit disk, then the Hardy space $H_p(\mathbb{D})$ is defined as the Banach space

$$H_p(\mathbb{D}) = \{f \in \mathcal{H}(\mathbb{D}) \colon \sup_{0<r<1} M_p(f,r) < \infty\},$$

where

$$M_p(f,r) = \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dm(t)\Big)^{1/p}, \quad \text{for each } 0<r<1,$$

if $1 \leq p < \infty$. In the case $p = \infty$ we consider the usual modification

$$M_\infty(f,r) = \sup_{0<t<2\pi} |f(re^{it})|.$$

There exists a close relationship between Hardy spaces $H_p(\mathbb{D})$ and Lebesgue spaces $L_p(\mathbb{T}, m)$. The celebrated Fatou's Theorem states that, given any function $F(z) = \sum_{n=0} a_n z^n \in H_p(\mathbb{D})$, then (see [101, Th. 8]):

(i) the radial limit $\lim_{r\to 1} F(re^{it}) = f(e^{it})$ exists almost everywhere for $t \in (0, 2\pi)$;

(ii) f admits a Fourier series representation of the form $f(t) = \sum_{n=0}^{\infty} a_n e^{int}$;

(iii) $f \in L_p(\mathbb{T})$ with norm $\|f\|_p = \sup_{0<r<1} M_p(F,r)$.

Conversely, let $p \geq 1$ and $f \in L_p(\mathbb{T})$ such that $f(t) = \sum_{n=0}^{\infty} a_n e^{int}$. If $P(r,\theta) = \frac{1-r^2}{1-2r\cos(\theta)+r^2}$ is the Poisson kernel on $\mathbb{D}$, then the Poisson formula

$$F(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P(r, \theta - t) f(e^{it}) dm(t)$$

allows us to recover an analytic function $F(z) = \sum_{n=0}^{\infty} a_n z^n \in H_p(\mathbb{D})$ whose boundary values are precisely given by f.

Thus, the Hardy space $H_p(\mathbb{D})$ can be identified with the closed subspace $H_p(\mathbb{T}) \subset L_p(\mathbb{T}, m)$ generated by the polynomials [46, Th. 3.3] (see also [101, 75]). Hence we can consider the orthogonal projection $R : L_2(\mathbb{T}) \to H_2(\mathbb{T})$ given by

$$f(t) = \sum_{n\in\mathbb{Z}} a_n e^{int} \in L_2(\mathbb{T}) \mapsto R(f)(t) = \sum_{n=0}^{\infty} a_n e^{int}.$$

Such operator can be extended to a bounded projection R for all $1 < p < \infty$ (cf. [101, Th. 10]), which receives the name of *Riesz projection.* Using this, one can readily identify the interpolation scale of Hardy spaces in the reflexive range $1 < p, q < \infty$ (see [102]):

$$(H_p(\mathbb{T}), H_q(\mathbb{T}))_\theta = H_r(\mathbb{T}), \quad \text{where } \frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}.$$

To obtain the corresponding selector for the scale of Hardy spaces we will need the following result proved by Boas [8] (see also [75, Th. 0.3]):

Proposition 2.2.1 (Boas). *The map $\mathcal{B} : L_p(\mathbb{T}) \to H_p(\mathbb{T})$ given by*

$$\mathcal{B}\Big(\sum_{n\in\mathbb{Z}} a_n e^{int}\Big) = a_0 + \sum_{n\geq 1} a_n e^{i(2n)t} + \sum_{n\geq 1} a_{-n} e^{i(2n-1)t}.$$

is an isomorphism for any $1 < p < \infty$.

The map $\mathcal{B}$, called the *Boas isomorphism*, defines by Proposition 2.1.1 the bounded operator $\mathcal{B}_{\mathcal{C}} : \mathcal{C}(L_p(\mathbb{T}), L_{p^*}(\mathbb{T})) \to \mathcal{C}(H_p(\mathbb{T}), H_{p^*}(\mathbb{T}))$. Then the map $B^H_{1/2} : H_2(\mathbb{T}) \to \mathcal{C}(H_p(\mathbb{T}), H_{p^*}(\mathbb{T}))$ defined by

$$B^H_{1/2}(x)(z) = \mathcal{B}_{\mathcal{C}}\Big(B_{1/2}\big(\mathcal{B}^{-1}(x)\big)\Big)(z) \tag{2.15}$$

is a bounded homogeneous selection for $\delta_{1/2} : \mathcal{C}(H_p(\mathbb{T}), H_{p^*}(\mathbb{T})) \to H_2(\mathbb{T})$; that is, a selection for the Hardy spaces is given by the selection of L_p spaces conjugated by the Boas isomorphism. This is summarized in the diagram

$$\begin{tikzcd}
\mathcal{C}(L_p(\mathbb{T}), L_p^*(\mathbb{T})) \arrow[r, "\delta_{1/2}"'] \arrow[d, "\mathcal{B}_{\mathcal{C}}"'] & L_2(\mathbb{T}) \arrow[l, bend right, "B_{1/2}"'] \arrow[d, "\mathcal{B}"] \\
\mathcal{C}(H_p(\mathbb{T}), H_p^*(\mathbb{T})) \arrow[r, "\delta_{1/2}"] & H_2(\mathbb{T}) \arrow[l, bend left, "B^H_{1/2}"]
\end{tikzcd}$$

This previous facts will be important in later sections, as they will imply that the Rochberg spaces associated to the scale of Hardy spaces are isomorphic to the corresponding Rochberg spaces defined by the scale $(L_1(\mathbb{T}), L_\infty(\mathbb{T}))_\theta$, for $0 < \theta < 1$.

We focus now in the case of Sobolev spaces. General references for this topic are [72, 1], although we follow here the description given in [76].

For $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, the Sobolev space $W^k_p(\mathbb{R}^n)$ is the Banach space of all functions $f : \mathbb{R}^n \to \mathbb{C}$ whose distributional derivatives up to order k are in $L_p(\mathbb{R}^n)$. Precisely, given a finite sequence $\alpha = (\alpha_i)_{i=1}^n \in \mathbb{N}^n$ denote by $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1{}^{\alpha_1}\,\partial x_2^{\alpha_2}\cdots\partial x_n{}^{\alpha_n}}$ the usual partial derivative associated to the multi-index α, where $|\alpha| = \sum \alpha_i$ is the order of the derivative. A function $g : \mathbb{R}^n \to \mathbb{C}$ is said to be the α-th distributional partial derivative of a function $f : \mathbb{R}^n \to \mathbb{C}$, denoted by $g = D^\alpha f$, if satisfies the equation

$$\int_{\mathbb{R}^n} g\varphi\, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f D^\alpha \varphi\, dx,$$

when tested against all infinitely differentiable functions $\varphi : \mathbb{R}^n \to \mathbb{C}$ with compact support. The Sobolev space $W^k_p(\mathbb{R}^n)$ is, for $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, the Banach space

$$W^k_p(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C}\text{: } D^\alpha f \text{ exist and } D^\alpha f \in L_p(\mathbb{R}^n) \text{ for all } |\alpha| \leq k\}$$

endowed with the norm

$$\|f\|_{k,p} = \begin{cases} \left(\sum_{|\alpha|\le k} \int_{\mathbb{R}^n} |D^\alpha f(x)|^p \, dx \right)^{1/p}, & \text{for } 1 \le p < \infty \\ \max_{|\alpha|\le k} \operatorname{ess\,sup}_{x\in\mathbb{R}^n} |D^\alpha f(x)|, & \text{if } p = \infty \end{cases}$$

If we consider the space $\bigoplus_{|\alpha|\le k} L_p(\mathbb{R}^n)$ endowed with the norm

$$\|(f_\alpha)\|_p = \begin{cases} \left(\sum_{|\alpha|\le k} \|f_\alpha\|_p^p \right)^{1/p}, & \text{for } 1 \le p < \infty \\ \max_{|\alpha|\le k} \|f_\alpha\|_\infty, & \text{if } p = \infty \end{cases}$$

then, by the very definition we have that the map

$$f \in W_p^k(\mathbb{R}^n) \mapsto \mathcal{J}(f) = (D^\alpha f)_{|\alpha|\le k} \in \bigoplus_{|\alpha|\le k} L_p(\mathbb{R}^n)$$

defines an isometric embedding for all $1 \le p \le \infty$. In the particular case where $p = 2$, we have that $\bigoplus_{|\alpha|\le k} L_2(\mathbb{R}^n)$ is a Hilbert, hence $W_2^k(\mathbb{R}^n)$ is a Hilbert and there exist the orthogonal projection $\mathcal{S} : \bigoplus_{|\alpha|\le k} L_2(\mathbb{R}^n) \to W_2^k(\mathbb{R}^n)$. Just like the Riesz projection, $\mathcal{S}$ extends to a bounded projection $\mathcal{S} : \bigoplus_{|\alpha|\le k} L_p(\mathbb{R}^n) \to W_p^k(\mathbb{R}^n)$ for all $1 < p < \infty$ [76, Th. 4], called the *Sobolev projection*.

One has for all $k \in \mathbb{N}$ that

$$(W_p^k(\mathbb{R}^n), W_q^k(\mathbb{R}^n))_\theta = W_r^k(\mathbb{R}^n), \quad \text{where } \frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q} \quad \text{and } 1 < p, q < \infty.$$

The paper [73] contains a proof which also includes the endpoint space $W_1^k(\mathbb{R}^n)$.

Moreover, using the theory of Fourier multipliers it can be shown that $W_p^k(\mathbb{R}^n)$ is isomorphic to $L_p(\mathbb{R}^n)$ for each $1 < p < \infty$ through an explicit isomorphism (see [76, Th. 6] or [77]):

Proposition 2.2.2. *For each $k \in \mathbb{N}$ and $n \in \mathbb{N}$ there exist an isomorphism $\mathcal{T} : L_p(\mathbb{R}^n) \to W_p^k(\mathbb{R}^n)$ for all $1 < p < \infty$.*

The operator $\mathcal{T}$ does not depend on p (see [76, pp. 1372], the comments before Proposition 8). Therefore, reasoning like we did for the case of Hardy spaces, the map $B_{1/2}^W : W_2^k(\mathbb{R}^n) \to \mathcal{C}(W_p^k(\mathbb{R}^n), W_{p^*}^k(\mathbb{R}^n))$ given by

$$B_{1/2}^W(x)(z) = \mathcal{T}_{\mathcal{C}}\Big(B_{1/2}(\mathcal{T}^{-1}(x))\Big)(z)$$

is a bounded homogeneous selection for the evaluation map $\delta_{1/2} : \mathcal{C}(W_p^k(\mathbb{R}^n), W_{p^*}^k(\mathbb{R}^n)) \to W_2^k(\mathbb{R}^n)$.

2.3 Shifts

Let (X_0, X_1) be a Banach couple. Given an analytic function f on $\mathbb{S}$ and $0 < \theta < 1$, we denote by $\Delta_k^\theta(f) = \frac{1}{k!}\delta_\theta^{(k)}(f)$ the k-th Taylor coefficient of f at θ. Then the linear map

$\Delta_k^\theta : \mathcal{C}(X_0, X_1) \to X_0 + X_1$ is continuous. Indeed, applying the Cauchy Integral Theorem one gets the estimate

$$\|f^{(k)}(\theta)\|_{X_0+X_1} \le \frac{k!}{\min\{\theta, 1-\theta\}^k}\|f\|_{\mathcal{C}}, \tag{2.16}$$

and thus $\|\Delta_k^\theta\| \le \frac{1}{\min\{\theta,1-\theta\}^k}$ (cf. [22, Lemma 3.8]). When θ is fixed, we will denote Δ_k^θ simply by Δ_k.

Due to historical reasons (mainly the use of Three lines Theorem in the proof of Riesz-Thorin Theorem), the usual setting for complex interpolation is the unit strip $\overline{\mathbb{S}}$ on the complex plane. However, in the next chapter we will need to work with higher order derivaties of analytic functions on $\mathbb{S}$, and this domain is not as suitable to perform some of the involved techniques.

Precisely, in the unit disk $\mathbb{D}$, one has the well-known factorization result: if f is analytic on $\mathbb{D}$ and has a zero of order k at 0, then $f = z^k g$, where g is analytic on $\mathbb{D}$.

Therefore:

(i) Taylor coefficients can be shifted by multiplying by z^k:

$$\Delta_{m+k}(z^k g) = \Delta_m(g) \quad \text{for all } m, k \in \mathbb{N}.$$

We shall refer to $f \mapsto z^k f$ as the *shift map*.

(ii) If $\mathcal{A}(\mathbb{D}, X)$ denotes the vector valued disk algebra and $I_j = \{e^{i\theta} \colon j\pi < \theta < (j+1)\pi\} \subset \mathbb{T}$ for each $j = 0, 1$, then we may define for any Banach couple (X_0, X_1) the Banach space

$$\mathcal{C}^{\mathbb{D}}(X_0, X_1) = \{f \in \mathcal{A}(X_0 + X_1) \colon f|_{I_j} : I_j \to X_j \text{ is bounded and continuous, } j = 0, 1\},$$

under the norm $\|f\| = \max_{j=0,1} \sup_{\theta \in I_j} \|f(e^{i\theta})\|_j$. Note that this is an analogue of Calderón space where the domain used is the unit disk instead of the strip. Then the map

$$f \in \mathcal{C}^{\mathbb{D}}(X_0, X_1) \mapsto zf \in \mathcal{C}^{\mathbb{D}}(X_0, X_1)$$

is an isometric isomorphism onto the subspace of functions vanishing at 0.

We want to translate property (i) to the open strip $\mathbb{S}$ and the usual Calderón space $\mathcal{C}(X_0, X_1)$. In general, the shift $f \mapsto zf$ can not be applied directly since, being an unbounded domain, multiplication by z is not allowed on the unit strip.

Fix $0 < \theta < 1$ and pick a conformal mapping $\varphi : \mathbb{S} \to \mathbb{D}$ vanishing at θ. An example is given by

$$\varphi(z) = \frac{e^{i\pi z} - e^{i\pi\theta}}{e^{i\pi z} - e^{-i\pi\theta}}, \quad z \in \mathbb{S}.$$

Such map is unique up to a rotation by a result of Poincaré [5, 13.14 Lemma]. Moreover, $\varphi(z)$ is continuous on the closed strip and satisfies that $\varphi(\partial\mathbb{S}) \subset \partial\mathbb{D}$. See [80, Chapter 8] for some further properties of φ.

The map

$$f \in \mathcal{C}(X_0, X_1) \mapsto \varphi f \in \ker \delta_\theta \subset \mathcal{C}(X_0, X_1) \tag{2.17}$$

is an isometric linear map. Moreover, any $g \in \ker \delta_\theta$ can be expressed as $g = \varphi f$ for some unique $f \in \mathcal{C}(X_0, X_1)$ (see the proof of [17, Th. 4.1]). Thus the isometric map (2.17) is onto.

A simple calculation shows that this map does not shift the coefficients of the analytic functions $f \in \mathcal{C}(X_0, X_1)$. Indeed, by the Leibniz rule we have that

$$\Delta_n(\varphi f) = \sum_{i=0}^{n} \Delta_{n-k}(\varphi)\Delta_k(f). \tag{2.18}$$

We observe that, unless $\Delta_1(\varphi) = 1$ and $\Delta_k(\varphi) = 0$ for all $k \neq 1$, the map (2.17) will not be shifting Taylor coefficients.

Let us amend this. Let P be the polynomial $P = \sum_i a_i \varphi^i$. Since (2.17) is isometric, it follows that $Pf \in \mathcal{C}(X_0, X_1)$ for any $f \in \mathcal{C}(X_0, X_1)$ with $\|Pf\|_{\mathcal{C}} \leq \sum_i |a_i| \|f\|_{\mathcal{C}}$. The following result appears in [12, Lemma 1]:

Lemma 2.3.1. *Given $m \in \mathbb{N}$ and $0 \leq k \leq m$, there exist a polynomial $P_k(\varphi)$ of degree at most m such that for every $0 \leq i \leq m$,*

$$\Delta_i(P_k(\varphi)) = \delta_{ik}.$$

Using this we can construct the desired shift operators on $\mathcal{C}(X_0, X_1)$:

Proposition 2.3.1.

(1) For any $n, k \in \mathbb{N}$ there exist a bounded linear operator $S_-^k : \mathcal{C}(X_0, X_1) \to \bigcap_{j=0}^{k-1} \ker \Delta_k$ making the following diagram commutative

$$\begin{tikzcd}
\mathcal{C}(X_0, X_1) \arrow[r, "S_-^k"] \arrow[dr, "(\Delta_{n-1}, \dots, \Delta_0)"'] & \bigcap_{j=0}^{k-1} \ker \Delta_k \arrow[d, "(\Delta_{n+k-1}, \dots, \Delta_k)"] \\
& \Sigma^n
\end{tikzcd}$$

In particular, for any $f \in \mathcal{C}(X_0, X_1)$ one has

$$(\Delta_{n+k-1}, \dots, \Delta_k)(S_-^k(f)) = (\Delta_{n-1}, \dots, \Delta_0)(f). \tag{2.19}$$

(2) For any $n, k \in \mathbb{N}$ there exist a bounded linear operator $S_+^k : \bigcap_{j=0}^{k-1} \ker \Delta_k \to \mathcal{C}(X_0, X_1)$ making the following diagram commutative

$$\begin{tikzcd}
\mathcal{C}(X_0, X_1) \arrow[dr, "(\Delta_{n-1}, \dots, \Delta_0)"'] & \bigcap_{j=0}^{k-1} \ker \Delta_k \arrow[l, "S_+^k"'] \arrow[d, "(\Delta_{n+k-1}, \dots, \Delta_k)"] \\
& \Sigma^n
\end{tikzcd}$$

In particular, for any $f \in \bigcap_{j=0}^{k-1} \ker \Delta_j$ one has

$$(\Delta_{n+k-1}, \dots, \Delta_k)(f) = (\Delta_{n-1}, \dots, \Delta_0)(S_+(f)). \tag{2.20}$$

Proof.

(1) Let $f \in \mathcal{C}(X_0, X_1)$ be any function and fix $n, k \in \mathbb{N}$. Apply Lemma 2.3.1 to obtain a polynomial $P_k(\varphi)$ such that

$$\Delta_i(P_k(\varphi)) = \delta_{i,k}, \quad \text{for all } 0 \leq i \leq n+k-1.$$

Then the map $S_-^k(f) = P_k(\varphi)f$ defines a bounded operator from $\mathcal{C}(X_0, X_1)$ into $\bigcap_{j=0}^{k-1} \ker \Delta_j$ such that

$$\Delta_j(P_k(\varphi)f) = \sum_{l=0}^{m} \Delta_{j-l}(P_k(\varphi))\Delta_l(f) = \Delta_{j-k}(f)$$

for any $k \leq j \leq n+k-1$, which gives (2.19).

(2) On the other hand, assume that $f \in \bigcap_{j=0}^{k-1} \ker \Delta_j$ and fix $n, k \in \mathbb{N}$. Then, as we noted after (2.17), f is of the form $f = \varphi^k g$ where $g \in \mathcal{C}(X_0, X_1)$ (see also [22, Proposition 3.10]). Use Lemma 2.3.1 to define the polynomial

$$P(\varphi) = \sum_{j=0}^{n-1} \Delta_{k+j}(\varphi^k)P_j(\varphi).$$

Note that for each $0 \leq j \leq n-1$ we have that

$$\Delta_{j+k}(f) = \Delta_{j+k}(\varphi^k g) = \sum_{i=0}^{j+k} \Delta_{j+k-i}(\varphi^k)\Delta_i(g) = \sum_{i=0}^{j} \Delta_{j+k-i}(\varphi^k)\Delta_i(g).$$

and

$$\Delta_j(P(\varphi)g) = \sum_{i=0}^{j} \Delta_{j-i}(P(\varphi))\Delta_i(g)$$

Since $\Delta_j(P(\varphi)) = \Delta_{k+j}(\varphi^k)$ for each $0 \leq j \leq n-1$, we conclude that

$$(\Delta_{n-1}, \ldots, \Delta_0)(P(\varphi)g) = (\Delta_{n+k-1}, \ldots, \Delta_k)(f).$$

If we define $S_+^k : \bigcap_{j=0}^{k-1} \ker \Delta_j \to \mathcal{C}(X_0, X_1)$ by $S_+^k(f) = S_+^k(\varphi^k g) = P(\varphi)g$, then this operator is linear and bounded: observe that for each $0 \leq j \leq n-1$ we have that there exist M_j such that $\|P_j g\|_{\mathcal{C}} \leq M_j \|g\|_{\mathcal{C}}$ and that $\|f\|_{\mathcal{C}} = \|\varphi^k g\|_{\mathcal{C}} = \|g\|_{\mathcal{C}}$, hence

$$\begin{aligned} \|S_+^k(f)\|_{\mathcal{C}} &= \Big\| \sum_{j=0}^{n-1} \Delta_{k+j}(\varphi^k)P_j(\varphi)g \Big\|_{\mathcal{C}} \leq \sum_{j=0}^{n-1} |\Delta_{j+k}(\varphi^k)| \|P_j g\|_{\mathcal{C}} \\ &\leq \sum_{j=0}^{n-1} |\Delta_{j+k}(\varphi^k)| M_j \|g\|_{\mathcal{C}} \leq C\|g\|_{\mathcal{C}} = C\|f\|_{\mathcal{C}}. \end{aligned}$$ □

Proposition 2.3.1 will be crucial in the following chapters to work with the truncated sequence of Taylor coefficients of $f \in \mathcal{C}(X_0, X_1)$. For any $k \in \mathbb{N}$, the shift maps S_+^k and S_-^k can be used to prove that translations on Taylor coefficients (2.19) and (2.20) are continuous (see Subsection 3.1).

2.4 Reiteration and duality

When studying the properties of the spaces X_θ obtained from a given Banach couple (X_0, X_1), in some cases one does not need to work with the whole scale $(X_\theta)_{0\le\theta\le1}$, but just with a suitable neighborhood of X_θ. The following result is known as the *Reiteration Theorem.*

Proposition 2.4.1. *Let (X_0, X_1) be a Banach couple pair and $0 \le \alpha < \beta \le 1$. Then for any $0 < \theta < 1$ the identity*

$$\Big((X_0, X_1)_\alpha, (X_0, X_1)_\beta\Big)_\theta = (X_0, X_1)_{(1-\theta)\alpha+\theta\beta} \tag{2.21}$$

holds with equality of norms.

The main point is that the linear map $C : \mathcal{C}(X_0, X_1) \to \mathcal{C}\Big((X_0, X_1)_\alpha, (X_0, X_1)_\beta\Big)$, defined by $C(f)(z) = f\big((1-z)\alpha + z\beta\big)$ is a contraction (see [14, 32.3]). A proof of Proposition 2.4.1 can be found in [40].

We discuss now duality issues involving interpolation spaces. A detailed account is given in Cwikel's notes [41]. Let (X_0, X_1) be a Banach couple and suppose that X is an intermediate space in such a way that both inclusions $X_0 \cap X_1 \subset X \subset X_0 + X_1$ have dense range. Then dualizing one obtains a chain

$$(X_0 + X_1)^* \subset X^* \subset (X_0 \cap X_1)^*.$$

of continous inclusions for the dual spaces. Using that $X_0 \cap X_1 \subset X$ is dense, one can identify the dual space of X by the action of functionals $f \in (X_0 \cap X_1)^*$ on $X_0 \cap X_1 \subset X$. Precisely, if we denote by $\langle \cdot, \cdot \rangle_{X_0\cap X_1}$ the duality between $X_0 \cap X_1$ and $(X_0 \cap X_1)^*$, then X^* coincides with the subspace of $y \in (X_0 \cap X_1)^*$ such that

$$\|y\| = \sup_{x\in X_0\cap X_1} |\langle x, y\rangle_{X_0\cap X_1}| < \infty.$$

For a general $x \in X$, the duality $\langle \cdot, \cdot \rangle_X : X \times X^* \to \mathbb{C}$ is given by

$$\langle x, x^*\rangle_X = \lim_{n\to\infty} \langle x_n, x^*\rangle_{X_0\cap X_1} \tag{2.22}$$

where $(x_n)_n \subset X_0 \cap X_1$ is any sequence converging to x for $\|\cdot\|_X$ (cf. [41, Fact 1.8]).

A natural setup where the previous density assumption holds is when $X_0 \cap X_1$ is dense in both endpoints X_0 and X_1. In such case, we will say following Cwikel [41, Def. 1.3] that (X_0, X_1) is a *regular* couple. Note that if (X_0, X_1) is a Banach couple that is not regular, by Lemma 2.1.4 we can always consider the regular couple (A_0, A_1) where A_j is the closure of $X_0 \cap X_1$ in X_j, which has the same interpolation spaces as (X_0, X_1).

If (X_0, X_1) is a regular couple we can apply the same reasoning as before to $X = X_0$ and $X = X_1$. Thus, for each $j = 0, 1$ the chain of inclusions $X_0 \cap X_1 \subset X_j \subset X_0 + X_1$ can be dualized to yield

$$(X_0 + X_1)^* \subset X_j^* \subset (X_0 \cap X_1)^*.$$

Using the analogous identifications of X_0^* and X_1^* indicated by the duality (2.22) one has the equalities

$$(X_0 + X_1)^* = X_0^* \cap X_1^* \quad \text{and} \quad (X_0 \cap X_1)^* = X_0^* + X_1^*,$$

and the corresponding dual norms $\|\cdot\|_{(X_0+X_1)^*}$ and $\|\cdot\|_{(X_0\cap X_1)^*}$ coincide, respectively, with the maximum and infimum norm associated to the Banach couple (X_0^*, X_1^*). Hence

$$X_0^* \cap X_1^* = (X_0 + X_1)^* \subset X_j^* \subset (X_0 \cap X_1)^* = X_0^* + X_1^*.$$

Moreover, given any interpolation space $(X_0, X_1)_\theta$ of a regular couple, then $(X_0, X_1)_\theta^*$ satisfies that

$$X_0^* \cap X_1^* \subset (X_0, X_1)_\theta^* \subset X_0^* + X_1^*.$$

Hence the dual space is an intermediate space for the dual Banach couple (X_0^*, X_1^*). In this context, Calderón proved in [14, 12.1] the remarkable fact that, under suitable hypothesis, this dual space is just the interpolation space of the dual couple.

Proposition 2.4.2 (Duality Theorem)**.** *Let (X_0, X_1) be a regular couple such that either X_0^* or X_1^* has the Radon-Nikodym property. Then for any $0 < \theta < 1$ we have*

$$(X_0, X_1)_\theta^* = (X_0^*, X_1^*)_\theta$$

with equality of norms. The duality is given by (2.22).

Recall that reflexive spaces or separable dual spaces have the Radon-Nikodym property [3, Prop. 5.5.6]. A proof of Proposition 2.4.2 is given in [80, Th. 8.37].

We examine now some results related to Proposition 2.4.2 that we shall use in forthcoming Section 3.5. Let (X_0, X_1) be a regular couple and (X_0^*, X_1^*) the dual couple. Given $f \in \mathcal{C}_{00}(X_0, X_1)$ and $g \in \mathcal{C}_{00}(X_0^*, X_1^*)$ the complex-valued function h given by

$$z \in \overline{\mathbb{S}} \mapsto h(z) = \langle f(z), g(z)\rangle_{X_0\cap X_1}$$

is continuous on $\overline{\mathbb{S}}$, analytic on $\mathbb{S}$ and bounded above by $\|f\|_{\mathcal{C}}\,\|g\|_{\mathcal{C}}$.
Indeed, by definition $f(z) = \sum_{i=1}^N \varphi_i(z)x_i$ and $g(z) = \sum_{j=1}^M \eta_j(z)x_j^*$ where $x_i \in X_0 \cap X_1$ and $x_i^* \in X_0^* \cap X_1^* \subset (X_0 \cap X_1)^*$. Then

$$h(z) = \langle f(z), g(z)\rangle_{X_0\cap X_1} = \Big\langle \sum_{i=1}^N \varphi_i(z)x_i, \sum_{j=1}^M \eta_j(z)x_j^* \Big\rangle_{X_0\cap X_1} = \sum_{j=1}^M\sum_{i=1}^N \eta_j(z)\varphi_i(z)\langle x_i, x_j^*\rangle_{X_0\cap X_1}.$$

Since $\varphi_i, \eta_j \in \mathcal{C}(\mathbb{C}, \mathbb{C})$ for all i, j (cf. Lemma 2.1.2), it follows that h is analytic on the open strip and continuous on $\overline{\mathbb{S}}$. Moreover, h is clearly bounded since it is a (finite) linear combination of bounded functions. In fact, since $f(j+it) \in X_0 \cap X_1 \subset X_j$ and $g(j+it) \in X_0^* \cap X_1^* \subset X_j^*$ for any $j = 0, 1$, one has for any $t \in \mathbb{R}$ that

$$\begin{aligned} |h(j+it)| &= \big|\big\langle f(j+it), g(j+it)\big\rangle_{X_0\cap X_1}\big| = \big|\big\langle f(j+it), g(j+it)\big\rangle_{X_j}\big| \\ &= \Big|\Big\langle \sum_{i=1}^N \varphi_i(j+it)x_i, \sum_{j=1}^M \eta_j(j+it)x_j^* \Big\rangle_{X_j}\Big| \\ &\le \Big\|\sum_{i=1}^N \varphi_i(j+it)x_i\Big\|_{X_j}\,\Big\|\sum_{j=1}^M \eta_j(j+it)x_j^*\Big\|_{X_j^*}, \end{aligned}$$

where we have used (2.22) for $X = X_j$ (see [41] around equation (1.3)).

Thus, for any $z \in \overline{\mathbb{S}}$ it follows that

$$\begin{aligned}|h(z)| &\leq \max_{j=0,1} \sup_t |h(j+it)| \leq \max_{j=0,1} \sup_t \left\{ \left\| \sum_{i=1}^{N} \varphi_i(j+it)x_i \right\|_{X_j} \left\| \sum_{j=1}^{M} \eta_j(j+it)x_j^* \right\|_{X_j^*} \right\} \\ &\leq \|f\|_{\mathcal{C}} \, \|g\|_{\mathcal{C}}.\end{aligned}$$

This implies the existence of a bounded bilinear map

$$\mathcal{C}_{00}(X_0, X_1) \times \mathcal{C}_{00}(X_0^*, X_1^*) \to \mathcal{C}_0(\mathbb{C}, \mathbb{C})$$

defined by $(f, g) \mapsto h \in \mathcal{C}_0(\mathbb{C}, \mathbb{C})$, where $h(z) = \langle f(z), g(z) \rangle_{X_0 \cap X_1}$. By Lemma 2.1.2 the subspace $\mathcal{C}_{00}(-,-)$ is dense in $\mathcal{C}_0(-,-)$, hence:

Lemma 2.4.1. *If (X_0, X_1) is a regular couple, then there is a bilinear contraction B : $\mathcal{C}_0(X_0, X_1) \times \mathcal{C}_0(X_0^*, X_1^*) \to \mathcal{C}_0(\mathbb{C}, \mathbb{C})$ given by*

$$B(f, g)(z) = \langle f(z), g(z) \rangle_{X_{Re(z)}} = \lim_{n\to\infty} \langle f_n(z), g_n(z) \rangle_{\Delta},$$

where $(f_n)_{n\in\mathbb{N}} \subset \mathcal{C}_{00}(X_0, X_1)$ and $(g_n)_{n\in\mathbb{N}} \subset \mathcal{C}_{00}(X_0^*, X_1^*)$ are converging sequences to f and g respectively.

2.5 Stein's interpolation principle

Let (M, μ) and (N, ν) be two σ-finite measure spaces and denote by $\mathcal{S}(\mu)$ and $\mathcal{S}(\nu)$ the corresponding spaces of simple functions and $L_0(\nu)$ the space of ν-measurable functions.

In 1956, some years prior the work of Calderón on complex interpolation, Elias Stein had proven in [87] an outstanding generalization of Riesz-Thorin Theorem: instead of using a fixed operator T one can consider a family of linear maps $(T_z)_{z\in\overline{\mathbb{S}}} \subset \mathcal{L}(S(\mu), L_0(\nu))$ which varies analytically on the strip and satisfies some regularity hypothesis.

Precisely, suppose that for each $z \in \overline{\mathbb{S}}$ one has defined a linear map $T_z : \mathcal{S}(\mu) \to L_0(\nu)$ such that

$$H(z) = \langle T_z(f), g \rangle = \int_N T_z(f)\, g d\nu < \infty \tag{2.23}$$

for all $f \in \mathcal{S}(\mu)$ and $g \in \mathcal{S}(\nu)$. Then the family $(T_z)_{z\in\overline{\mathbb{S}}}$ is called:

(i) *analytic* if, for $f \in \mathcal{S}(\mu)$ and $g \in \mathcal{S}(\nu)$, the map (2.23) is analytic on $\mathbb{S}$ and extends continuously to the closed strip $\overline{\mathbb{S}}$;

(ii) *admissible* if $f \in \mathcal{S}(\mu)$ and $g \in \mathcal{S}(\nu)$, the map (2.23) has *admissible growth* [90, pp. 205]:

$$\sup_{y\in\mathbb{R}} \left| e^{-a|y|} \log |H(x+iy)| \right| < \infty \quad \text{for all } 0 \leq x \leq 1 \text{ and some } a < \pi.$$

Under the previous considerations Stein obtains:

Theorem 2.5.1 (Stein Interpolation Theorem)**.**
Let $(T_z)_{z\in\overline{\mathbb{S}}} \subset \mathcal{L}(\mathcal{S}(\mu), L_0(\nu))$ be an admissible analytic family of operators. Assume that

$$\|T_{it} f\|_{q_0} \leq M_0(t)\|f\|_{p_0} \quad \textit{and} \quad \|T_{1+it}\|_{q_1} \leq M_1(t)\|f\|_{p_1},$$

where $f \in \mathcal{S}(\mu)$ and, for each $j = 0, 1$, $M_j : \mathbb{R} \to \mathbb{R}$ satisfies the bound

$$\sup_{t \in \mathbb{R}} \left| e^{-a|t|} \log M_j(t) \right| < \infty \quad \textit{for some } a < \pi.$$

Then for all $f \in \mathcal{S}(\mu)$ and any $0 < \theta < 1$ we have that

$$\|T_\theta f\|_{q_\theta} \leq e^{M(\theta)} \|f\|_{p_\theta},$$

where

$$M(\theta) = \int_{\partial_0} \log M_0(t) d\mu_\theta^0(it) + \int_{\partial_1} \log M_1(t) d\mu_\theta^1(1 + it) < \infty.$$

In particular, T_θ extends to a bounded operator $L_{p_\theta}(\mu) \to L_{q_\theta}(\nu)$.

In light of Stein's theorem, many authors were interested in obtaining a generalization of Theorem 2.5.1 to couples (X_0, X_1) of Banach spaces. Here we follow the description given by Cwikel and Janson in [42]. The setting of Stein's theorem assumes the existence of a suitable family of linear maps which are defined on a common dense subspace of $L_p(\mu)$ spaces, namely, the subspace of simple functions. In the case of a general couple, the natural substitute in view of Lemma 2.1.3 is the intersection space $X_0 \cap X_1$.

Thus, given a family of linear maps $(T_z)_{z \in \overline{\mathbb{S}}} \subset \mathcal{L}(X_0, \cap X_1, X_0 + X_1)$, in analogy with (2.23) we say that the family is *analytic* if, for each $x \in X_0 \cap X_1$ and $y \in (X_0 + X_1)^*$, the map

$$H(z) = \langle T_z(x), y \rangle \in \mathcal{C}(\mathbb{C}, \mathbb{C}), \tag{$\mathcal{A}$}$$

namely, H is analytic on $\mathbb{S}$ and bounded and continuous on $\overline{\mathbb{S}}$. Equivalently, for each $x \in X_0 \cap X_1$ the map $f_x(z) = T_z(x) : \overline{\mathbb{S}} \to X_0 + X_1$ is bounded, weakly continuous and analytic on $\mathbb{S}$.

The analytic family is called *uniformly bounded* if, for all $x \in X_0 \cap X_1$, it satisfies the boundary conditions

$$\|T_{it} x\|_{X_0} \leq M_0 \|x\|_{X_0} \quad \text{and} \quad \|T_{1+it} x\|_{X_1} \leq M_1 \|x\|_{X_1} \tag{$\mathcal{B}$}$$

for all $t \in \mathbb{R}$ and some constants $M_0, M_1 > 0$. Thus, the map $f_x(z) = T_z(x)$ verifies that $f|_{\partial_j} : \partial_j \to X_j$ boundedly, althought it is not necessarily continuous.

Despite that both conditions ($\mathcal{A}$) and ($\mathcal{B}$) are stronger than the ones appearing in Stein's interpolation Theorem, Cwikel and Janson showed [42, Th. 1] that, in general, such conditions are not enough to guarantee that $T_\theta : X_\theta \to X_\theta$ is bounded: $T_\theta(X_\theta)$ lies in a strictly bigger space that contains X_θ.

For this reason we assume a further condition that the family must satisfy:

$$\text{for each } x \in X_0 \cap X_1, \text{ the function } f_x(\cdot) = T_\cdot(x) : \overline{\mathbb{S}} \to X_0 + X_1 \in \mathcal{C}(X_0, X_1). \tag{$\mathcal{I}$}$$

Note that ($\mathcal{I}$) improves the boundary conditions of both ($\mathcal{A}$) and ($\mathcal{B}$) adding, for each $x \in X_0 \cap X_1$, that:

(i) the function $f_x(z) = T_z(x) : \partial\mathbb{S} \to X_0 + X_1$ is continuous;

(ii) for each $j = 0, 1$ the function

$$f_x(z) = T_z(x) : \partial_j \to X_j$$

is bounded *and continuous*.

Any analytic family $(T_z)_{z\in\overline{\mathbb{S}}}$ satisfying $(\mathcal{I})$ will be called *interpolating*. With this in hand, the following result is the desired generalization of Stein's theorem:

Theorem 2.5.2 (Stein interpolation principle)**.**
Let (X_0, X_1) be a compatible couple and $(T_z)_{z\in\overline{\mathbb{S}}} \subset \mathcal{L}(X_0\cap X_1, X_0+X_1)$ an interpolating *family of operators. Assume that the uniform bound $(\mathcal{B})$ holds for this family. Then there exist a bounded operator $T^{\mathcal{C}} : \mathcal{C}_{00}(X_0, X_1) \to \mathcal{C}(X_0, X_1)$ given by*

$$T^{\mathcal{C}}(f)(z) = T_z(f(z)).$$

Proof. Despite not being expressed in these terms, the proof is the same that appears in [42, Th. 1] (see also [80, Th. 8.28]).
Let $f \in \mathcal{C}_{00}(X_0, X_1)$ and consider the function $g(z) = T_z(f(z)) = T^{\mathcal{C}}(f)(z)$. Since $(T_z)_z$ is interpolating and f is of the form

$$f(z) = \sum_{i=1}^{N} \varphi_i(z)x_i,$$

where $x_i \in X_0 \cap X_1$ and $\varphi_i \in \mathcal{C}(\mathbb{C},\mathbb{C})$, we deduce that

$$g(z) = \sum_{i=1}^{N} \varphi_i(z)T_z(x_i) \in \mathcal{C}(X_0, X_1).$$

Moreover, using the bound for $(T_z)_z$ on the boundary of $\mathbb{S}$ we deduce that

$$\begin{aligned}\|g\|_{\mathcal{C}} = \max_{j=0,1}\{\sup_t \|T_{j+it}(f(j+it))\|_{X_j}\} &\le \max_{j=0,1}\{M_j \sup_t \|f(j+it)\|_{X_j}\}\\ &\le \max\{M_0, M_1\}\|f\|_{\mathcal{C}}.\end{aligned}$$

□

Corollary 2.5.1. *Assume that conditions of Theorem 2.5.2 hold. Then for any $0 < \theta < 1$ one has that*

$$\|T_\theta(x)\|_\theta \le \|T^{\mathcal{C}}\|\|x\|_\theta \quad \textit{for all } x \in X_0 \cap X_1.$$

In particular, T_θ extends to a bounded operator on X_θ.

Proof. Let $x \in X_0\cap X_1$ and use Lemma 2.1.3 to obtain $f \in \mathcal{C}_{00}(X_0, X_1)$ such that $f(\theta) = x$ and $\|f\|_{\mathcal{C}} \le (1+\varepsilon)\|x\|_\theta$. Then

$$\|T_\theta x\|_\theta = \|T_\theta(f(\theta))\|_\theta = \|T^{\mathcal{C}}(f)(\theta)\|_\theta \le \|T^{\mathcal{C}}(f)\|_{\mathcal{C}} \le \|T^{\mathcal{C}}\|\|f\|_{\mathcal{C}} \le \|T^{\mathcal{C}}\|(1+\varepsilon)\|x\|_\theta.$$

Taking the infimum over all possible f gives the result.
Since $X_0 \cap X_1$ is dense in X_θ (cf. Lemma 2.1.3) the second claim follows. □

Note the similarities with the proofs of Proposition 2.1.1 and Theorem 2.1.1. We also stress the fact that one could impose weaker growth conditions on $(\mathcal{B})$ analogous to that of Stein's Theorem (see [42, Th. 2]), but we shall not pursue this line here since Theorem 2.5.2 will be enough for our purposes.

The easiest example of interpolating family is given by the constant family, $T_z = T$ for every $z \in \overline{\mathbb{S}}$, where T is interpolating (cf. before Proposition 2.1.1). This case recovers Theorem 2.1.1. A non-trivial example is the following one studied in [25]:

Example 1. We consider the pair (ℓ_∞, ℓ_1) and a sequence $\mathfrak{u} = (u_n)_{n\in\mathbb{N}} \subset \ell_2$ of normalized block vectors in ℓ_2. Let us denote by

$$u_n = \sum_{j=p_{n-1}+1}^{p_n} u_n^j e_j$$

the coordinates of u_n. We define for each $x \in c_{00}$ the linear map $f_x(z) = T_z(x) : \overline{\mathbb{S}} \to \ell_\infty$ given by

$$T_z(x) = x \cdot |\mathfrak{u}|^{2z} = \sum_{n\in\mathbb{N}} x_n(sign(u_n)|u_n|^{2z}) = \sum_{n\in\mathbb{N}} x_n\Big[\sum_{j=p_{n-1}+1}^{p_n} sign(u_n^j)|u_n^j|^{2z} e_j\Big].$$

Here $|u_n|$ denotes the sequence whose coordinates are the modulus of those of u_n; the same applies to the sign sequence $sign(u_n)$. Thus, each $z \in \overline{\mathbb{S}}$ defines the multiplication operator

$$\begin{pmatrix}
sign(u_1^{p_0+1})|u_1^{p_0+1}|^{2z} & 0 & 0 \\
sign(u_1^{p_0+2})|u_1^{p_0+2}|^{2z} & 0 & 0 \\
\vdots & 0 & 0 \\
0 & sign(u_2^{p_1+1})|u_2^{p_1+1}|^{2z} & 0 \\
0 & sign(u_2^{p_1+2})|u_2^{p_1+2}|^{2z} & 0 \\
0 & \vdots & 0 \\
0 & 0 & sign(u_3^{p_2+1})|u_3^{p_2+1}|^{2z} \\
0 & 0 & sign(u_3^{p_2+2})|u_3^{p_2+2}|^{2z} \\
0 & 0 & \vdots
\end{pmatrix}$$

Then the family $(T_z)_{z\in\overline{\mathbb{S}}} \subset \mathcal{L}(c_{00}, \ell_\infty)$ defined as above satisfies that

$$\text{for each } x \in c_{00}, \text{ the function } f_x(\cdot) = T_\cdot(x) : \overline{\mathbb{S}} \to \ell_\infty \in \mathcal{C}(\ell_\infty, \ell_1).$$

Indeed:

(1) For each $x \in c_{00}$ we have that $T_z(x) : \overline{\mathbb{S}} \to \ell_\infty$ is $\|\cdot\|_\infty$-bounded. Indeed, note that $|u_n|^{2z} = |u_n|^{2Re(z)}|u_n|^{iIm(z)}$ and, since $||u_n|^{it}| = 1$ for any $t \in \mathbb{R}$, one has

$$\|sign(u_n)|u_n|^{2z}\|_\infty = \|sign(u_n)|u_n|^{2Re(z)}\|_\infty \quad \text{for all } z \in \overline{\mathbb{S}}.$$

Moreover, given $z \in [0,1]$ we have that $sign(u_n)|u_n|^{2z} \in \ell_{1/z} \subset \ell_\infty$. In fact,

$$\|sign(u_n)u_n^{2z}\|_{\ell_{1/z}} = \sum_{j=p_{n-1}+1}^{p_n} (|u_n^j|^{2z})^{1/z} = \sum_{j=p_{n-1}+1}^{p_n} |u_n^j|^2 = 1$$

since the sequence $(u_n)_n$ were normalized in ℓ_2. Thus $\|sign(u_n)|u_n|^{2z}\|_\infty \leq 1$ for each $n \in \mathbb{N}$, and hence

$$\|T_z(x)\|_\infty = \|\sum_n x_n(sign(u_n)|u_n|^{2z})\|_\infty \leq \sum_n |x_n| = \|x\|_{\ell_1} < \infty$$

for all $z \in \overline{\mathbb{S}}$.

(2) For each $x \in c_{00}$ we have that $T_z(x) : \overline{\mathbb{S}} \to \ell_\infty$ is $\|\cdot\|_\infty$-continuous. Let $z, w \in \overline{\mathbb{S}}$ and observe that

$$\begin{aligned}\|T_z(x) - T_w(x)\|_\infty &= \Big\| \sum_n x_n \big[sign(u_n)(|u_n|^{2z} - |u_n|^{2w})\big] \Big\|_\infty \\ &\leq \sum_n |x_n| \| |u_n|^{2z} - |u_n|^{2w} \|_\infty \\ &\leq \sum_n |x_n| \sup_j \{ \big| |u_n^j|^{2z} - |u_n^j|^{2w} \big| \}.\end{aligned}$$

Since x has finite support, the previous sum has finitely many vectors. Thus, since $z \in \overline{\mathbb{S}} \mapsto |u_n^j|^{2z}$ is continuous for each j, we may choose δ such that $\|T_z(x) - T_z(w)\|_\infty < \varepsilon$ if $|z-w| < \delta$.

(3) Applying the same argument as in (2) one gets that the family is analytic in $\mathbb{S}$: the derivative of the family at z_0 is given by

$$\begin{aligned}\frac{d}{dz} T_z(x)|_{z=z_0} &= x \cdot (2|\mathfrak{u}|^{2z_0} \log |\mathfrak{u}|) = \sum_n x_n \big(2 sign(u_n)|u_n|^{2z_0} \log |u_n|\big) \\ &= \sum_n x_n \Big(2 \sum_{j=p_{n-1}+1}^{p_n} sign(u_n^j)|u_n^j|^{2z_0} \log |u_n^j| e_j\Big).\end{aligned}$$

Then using the same ideas as in (2) it follows that

$$\Big\| \frac{T_{z_0}(x) - T_w(x)}{z_0 - w} - \frac{d}{dz} T_z(x)|_{z=z_0} \Big\|_\infty \leq \sum_n |x_n| \sup_j \Big\{ \Big| \frac{|u_n^j|^{2z_0} - |u_n^j|^{2w}}{z_0 - w} - 2|u_n^j|^{2z_0} \log |u_n^j| \Big| \Big\}.$$

Since for each j the function $|u_n^j|^{2z}$ is complex-valued analytic on $\mathbb{S}$, that supremum goes to zero as $w \to z_0$.

Moreover, for each $j = 0, 1$ we have that the map $t \in \mathbb{R} \mapsto T_{j+it}(x) \in \ell_{1/j}$ is bounded and continuous:

(i) For $j = 0$ and $t \in \mathbb{R}$ one has that

$$\|T_{it}(x)\|_\infty = \| \sum_n x_n (sign(u_n)|u_n|^{2it}) \|_\infty = \sup_j \{ |x_n| |sign(u_n^j)| | |u_n^j|^{2it} | \} = \|x\|_\infty.$$

The calculations made in (2) shows that $T_{it}(x) : \partial_1 \to \ell_\infty$ is also continuous.

(ii) Using that $\|u_n\|_{\ell_2} = 1$, for $j = 1$ and $t \in \mathbb{R}$ it follows that

$$\begin{aligned}\|T_{1+it}(x)\|_{\ell_1} = \Big\| \sum_n x_n (sign(u_n)|u_n|^{2+2it}) \Big\|_{\ell_1} &= \Big\| \sum_n x_n \sum_{j=p_{n-1}+1}^{p_n} sign(u_n^j)|u_n^j|^2 |u_n^j|^{2it} e_j \Big\|_{\ell_1} \\ &= \sum_n |x_n| \Big(\sum_{j=p_{n-1}+1}^{p_n} |sign(u_n^j)| |u_n^j|^2 \Big) \\ &= \sum_n |x_n| \|u_n\|_{\ell_2} = \|x\|_{\ell_1}.\end{aligned}$$

Moreover, for $t, t_0 \in \mathbb{R}$ it follows that

$$\|T_{1+it}(x) - T_{1+it_0}(x)\|_{\ell_1} = \sum_n |x_n| \sum_{j=p_{n-1}+1}^{p_n} \left||u_n^j|^2\right| \left||u_n^j|^{2it} - |u_n^j|^{2it_0}\right|.$$

Since x has finite support and the function $|u_n^j|^{2it}$ is continuous for all j, we conclude that $T_{1+it}(x) : \partial_0 \to \ell_1$ is $\|\cdot\|_{\ell_1}$-continuous.

If we consider the subspace

$$\mathcal{C}_{00}^{\Delta}(\ell_\infty, \ell_1) = \left\{f \in \mathcal{C}_{00}(\ell_\infty, \ell_1) \colon f = \sum_{i=1}^{N} \varphi_i(z)\, x_i,\, x_i \in c_{00}\right\},$$

then taking into account that c_{00} is dense in ℓ_1 it follows that $\mathcal{C}_{00}^{\Delta}(\ell_\infty, \ell_1)$ is dense in $\mathcal{C}_{00}(\ell_\infty, \ell_1)$; in particular, Lemma 2.1.3 holds replacing $\mathcal{C}_{00}(\ell_\infty, \ell_1)$ by $\mathcal{C}_{00}^{\Delta}(\ell_\infty, \ell_1)$. Moreover, by (i) and (ii) above the boundary operators $T_{j+it} : \ell_{1/j} \to \ell_{1/j}$ are into isometries for each $j = 0, 1$ and $t \in \mathbb{R}$. Then a direct modification of the proofs of Theorem 2.5.2 and Corollary 2.5.1 yields for any $0 < \theta < 1$ that

$$\|T_\theta(x)\|_{(\ell_\infty, \ell_1)_\theta} \leq \|x\|_{(\ell_\infty, \ell_1)_\theta} \quad \text{for all } x \in c_{00}.$$

Consequently, there exist for each $0 < \theta < 1$ a bounded contraction $T_\theta : \ell_{1/\theta} \to \ell_{1/\theta}$. In particular, for $\theta = 1/2$ we have

$$T_{1/2}(x) = \sum_n x_n(sign(u_n)|u_n|) = \sum_n x_n u_n,$$

which is the usual isometric multiplication operator on ℓ_2.

Despite its simplicity, this previous example will be extensively used in later sections. Is one of the main tools to study the properties of Rochberg spaces associated to the scale of ℓ_p spaces.

2.6 A primer on derived spaces

We close this chapter with a brief outline of the cycle of ideas developed in the papers [37, 36, 83]. The key idea, emphasized by the authors of [36], is that many operators studied in analysis can be regarded as the evaluations of a suitable analytic operator-valued function $(\Phi_z)_z$. In [37] Coifman, Rochberg and Weiss used this idea to obtain (see [53, 88] for background on the involved concepts):

Theorem 2.6.1 (Coifman-Rochberg-Weiss Commutator Theorem)**.** *Let $b \in BMO(\mathbb{R}^n)$ be a function of bounded mean oscillation and $T : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ a Calderón-Zygmund operator. Then for any $1 < p < \infty$ there exist a constant $C(p) > 0$ such that*

$$\|T(bf) - bT(f)\|_p \leq C(p)\|f\|_p, \quad \textit{for all } f \in L_p(\mathbb{R}^n). \tag{2.24}$$

Rochberg and Weiss figured out in [83] that Theorem 2.6.1 could be extended to the broader context of complex interpolation using the following observation: higher order derivatives $\Delta_k(f)$ of analytic maps $f \in \mathcal{C}(X_0, X_1)$ can give further information about the interpolation spaces X_θ. The simplest way to see this is employing the results of Section 2.3 regarding shift maps. In the particular case where $n = k = 1$, Proposition 2.3.1 yields:

Lemma 2.6.1. *For any $0 < \theta < 1$, the linear map $\Delta_1^\theta|_{\ker \delta_\theta} : \ker \delta_\theta \to X_\theta$ is bounded and onto.*

This was noted by Carro, Cerdà and Soria [17] in their study of abstract frameworks for interpolation methods; the main point is that althought $f'(\theta)$ need not to be in X_θ, it will as long as $f(\theta) = 0$. Thus, Lemma 2.6.1 can be sistematically used in the following way:

choose $f, g \in \mathcal{C}(X_0, X_1)$ such that $f(\theta) = g(\theta)$. Then $h = f - g \in \ker \delta_\theta$ satisfies that $h'(\theta) = f'(\theta) - g'(\theta) \in X_\theta$.

This idea has been termed *the cancellation principle* in some papers (see [43]). We briefly discuss the concepts of *derived space* and *differential map* of a Banach couple (X_0, X_1) introduced by Rochberg and Weiss in [83] and how using the cancellation principle one can obtain a commutator estimate analogous to (2.24) on any interpolation space $(X_0, X_1)_\theta$.

2.6.1 The derived space

Recall from Subsection 2.3 that given any couple (X_0, X_1), the maps Δ_0^θ and Δ_1^θ are bounded for any $0 < \theta < 1$. We fix θ and omit the superscript in Δ_j from the rest of this section. We observe that

$$\big(\Delta_1, \Delta_0\big) : \mathcal{C}(X_0, X_1) \to \big(X_0 + X_1\big)^2$$

given by $(\Delta_1, \Delta_0)(f) = (\Delta_1(f), \Delta_0(f)) = (f'(\theta), f(\theta))$ is a bounded operator. Then we can consider the range of (Δ_1, Δ_0),

$$(\Delta_1, \Delta_0)\Big(\mathcal{C}(X_0, X_1)\Big) = \{(f'(\theta), f(\theta)) \colon f \in \mathcal{C}(X_0, X_1)\} \tag{2.25}$$

endowed with the quotient norm

$$\|(x, y)\|_{d\theta} = \inf\{\|g\|_{\mathcal{C}} \colon (\Delta_1, \Delta_0)(f) = (x, y)\}.$$

The Banach space (2.25) is called the *derived space* and can be identified with the quotient space $\mathcal{C}(X_0, X_1)/(\ker \Delta_1 \cap \ker \Delta_0)$. We will denote it by $d(X_0, X_1)_\theta$ or simply dX_θ.

We discuss the close relationship that exists between X_θ and dX_θ. First note that there is a natural quotient map $\pi : dX_\theta \to X_\theta$ given by $\pi(x, y) = y$. Thus we can form the short exact sequence

$$0 \longrightarrow \ker \pi \longrightarrow dX_\theta \xrightarrow{\pi} X_\theta \longrightarrow 0$$

Since $\ker \pi = \{(x, 0) \colon (x, y) \in dX_\theta\}$ we can identify this subspace isomorphically with X_θ by Lemma 2.6.1. Thus, there is a short exact sequence

$$0 \longrightarrow X_\theta \xrightarrow{i} dX_\theta \xrightarrow{\pi} X_\theta \longrightarrow 0 \tag{2.26}$$

given by $i(x) = (x, 0)$ and $\pi\big(x, y\big) = y$.

Let us describe the quasilinear map defining (2.26). Choose an homogeneous bounded selection $B_\theta : X_\theta \to \mathcal{C}(X_0, X_1)$ for Δ_0. Then the map $\Omega_\theta : X_\theta \to X_0 + X_1$ defined by

$$\Omega_\theta(x) = \Delta_1(B_\theta(x))$$

is quasilinear on X_θ with ambient space $X_0 + X_1$. Indeed, Ω_θ is clearly homogeneous and satisfies for any $x, y \in X_\theta$ that

$$\|\Omega_\theta(x+y) - \Omega_\theta(x) - \Omega_\theta(y)\|_\theta = \big\|\Delta_1\big[B_\theta(x+y) - B_\theta(x) - B_\theta(y)\big]\big\|_\theta. \tag{2.27}$$

Since the function inside the brackets vanishes on θ, we deduce by Lemma 2.6.1 that

$$(2.27) \leq C'\big(\|B_\theta(x+y)\|_{\mathcal{C}} + \|B_\theta(x)\|_{\mathcal{C}} + \|B_\theta(y)\|_{\mathcal{C}}\big) \leq C(\|x\|_\theta + \|y\|_\theta)$$

for some positive constant C. Moreover, if we had picked a different selection V_θ for Δ_0, then $\mho_\theta = \Delta_1 V_\theta$ is equivalent to Ω_θ since for every $x \in \ell_2$ one has

$$\|\Omega_\theta(x) - \mho_\theta(x)\|_\theta = \|\Delta_1(B_\theta(x) - V_\theta(x))\|_\theta \leq C\|x\|_\theta,$$

where the last inequality follows from Lemma 2.6.1. It turns out that the twisted sum space $X_\theta \oplus_{\Omega_\theta} X_\theta = \{(x,y) \in (X_0 + X_1) \times X_\theta \colon x - \Omega_\theta(y)\}$ coincides as a set with dX_θ and the norm $\|\cdot\|_{d\theta}$ is equivalent to the canonical quasinorm

$$\|(x,y)\| = \|x - \Omega_\theta(y)\|_\theta + \|y\|_\theta$$

induced by Ω_θ. This last claim is consequence of Lemma 2.6.1 yet again; we delay the proof until next chapter since an analogous fact holds for general Rochberg spaces.

The quasilinear map $\Omega_\theta : X_\theta \curvearrowright X_\theta$ receives the name of *differential map* of the scale at θ. A simple example is given by the Banach couple $(L_2(w_0), L_2(w_1))$ formed by two weighted Lebesgue spaces: by (2.14) an homogenoeus bounded selection $B : L_2(w_0^{1/2}w_1^{1/2}) \to \mathcal{C}(L_2(w_0), L_2(w_1))$ for Δ_0 is $B_w(x) = \left(\frac{w_0}{w_1}\right)^{\frac{(z-\theta)}{2}} x$. Since the selection is linear the differential map is also linear and is defined by $\Omega_\theta(x) = \frac{1}{2}x\log\left(\frac{w_0}{w_1}\right)$. In this case the derived space $dL_2(w_0^{1/2}w_1^{1/2})$ is isomorphic to $L_2(w_0^{1/2}w_1^{1/2}) \oplus L_2(w_0^{1/2}w_1^{1/2})$. Now we turn our attention to bounded operators on dX_θ. Let T be an interpolating operator for the couple (X_0, X_1) and consider the diagonal linear map

$$\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} : dX_\theta \longrightarrow dX_\theta \tag{2.28}$$

given by $(x,y) \in dX_\theta \mapsto (Tx, Ty)$. Then we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X_\theta \arrow[r, "i"] \arrow[d, "T"'] & dX_\theta \arrow[r, "\pi"] \arrow[d] & X_\theta \arrow[r] \arrow[d, "T"] & 0 \\
0 \arrow[r] & X_\theta \arrow[r, "i"] & dX_\theta \arrow[r, "\pi"] & X_\theta \arrow[r] & 0
\end{tikzcd}$$

where the middle arrow is the linear map (2.28). Recall what was discussed in Chapter 1 about operators on Z_2: once we have a twisted sum, to bound an operator on it translates into diverse conditions on the entries. In this diagonal case, the boundedness of (2.28) is equivalent (see Corollary 1.2.2) to

$$T \circ \Omega_\theta - \Omega_\theta \circ T : X_\theta \to X_\theta \quad \text{is bounded.}$$

This is precisely the commutator estimate

$$\|[T,\Omega_\theta](x)\|_\theta = \|T\big(\Omega_\theta(x)\big) - \Omega_\theta\big(T(x)\big)\|_\theta \leq C\|x\|_\theta.$$

To prove this last commutator estimate let us consider the diagram

$$\begin{tikzcd}
\mathcal{C}(X_0,X_1) \arrow[r, "\Delta_0"'] \arrow[d, "T_{\mathcal{C}}"'] & X_\theta \arrow[l, bend right, "B_\theta"'] \arrow[d, "T"] \\
\mathcal{C}(X_0,X_1) \arrow[r, "\Delta_0"] & X_\theta \arrow[l, bend left, "B_\theta"]
\end{tikzcd}$$

Then given any $x \in X_\theta$, we can define the function

$$h_x = B_\theta(T(x)) - T_{\mathcal{C}}(B_\theta(x)) \in \mathcal{C}(X_0,X_1),$$

which satisfies that $\|h_x\|_{\mathcal{C}} \leq 2\|B_\theta\|(\|T : X_\theta \to X_\theta\| + \|T_{\mathcal{C}}\|)\|x\|_\theta$. It follows that $h_x \in \ker \Delta_0$, and thus $\Delta_1(h_x) \in X_\theta$ by Lemma 2.6.1. Recall by (2.4) that $\Delta_1(T_{\mathcal{C}}(f)) = T(\Delta_1(f))$; hence for some constant $C > 0$ we have that

$$\|[T,\Omega_\theta](x)\|_\theta = \|\Delta_1(h_x)\|_\theta \leq \|h_x\|_{\mathcal{C}} \leq C\|x\|_\theta.$$

This gives the Rochberg-Weiss Commutator Theorem:

Theorem 2.6.2 (Rochberg-Weiss Commutator Theorem)**.** *Let (X_0,X_1) be a compatible couple and T an interpolating operator. Then for each $0 < \theta < 1$, the commutator $[T,\Omega_\theta] : X_\theta \to X_\theta$ is bounded.*

In this section we bound the diagonal operator (2.28) using commutator estimates. In the following chapter we will see it the other way around: we will show that once the theory has been stablished, to bound the operator (2.28) is rather simple and just an application of the interpolation principle for operators explained in Theorem 2.1.1.

2.6.2 The derived spaces generated by the pair (ℓ_∞, ℓ_1)

We end this chapter relating its ideas to the fundamental example of twisted Hilbert space studied in Chapter 1, the Kalton-Peck space. To this end, let us identify the derived space of the scale (ℓ_∞, ℓ_1). As we shown in (2.12), if $p^{-1} = \theta$ a bounded homogeneous selection for $\Delta_0 : \mathcal{C}(\ell_\infty, \ell_1) \to \ell_p$ is given, for any $x \in c_{00} \subset \ell_p$, by

$$B_p(x)(z) = x\Big(\frac{|x|}{\|x\|_p}\Big)^{pz-1}. \tag{2.29}$$

The differential map on a given $x \in c_{00}$ is the first Taylor coefficient of (2.29), i.e.,

$$\Omega_p(x) = \Delta_1(B_p(x)) = p\,x\log\Big(\frac{|x|}{\|x\|_p}\Big). \tag{2.30}$$

If we set $p = 2$ we deduce that the differential map $\Omega_2 : c_{00} \to \ell_\infty$ is just a *multiple* of the Kalton-Peck map (1.2). However, if we choose an arbitrary $x \in \ell_2$ then (2.29) does not necessarily belong to $\mathcal{C}(\ell_\infty, \ell_1)$, and thus $\Omega_p(x)$ is not defined as in (2.30). There are two ways of avoiding this problem:

- The first one is to consider a bigger Calderón space $\mathcal{G}(\ell_\infty, \ell_1)$ for which (2.29) belongs to if $x \in \ell_2$. Then one can work with the expression $\Omega_p(x)$ explicitely for every $x \in \ell_2$. A possible choice for $\mathcal{G}(\ell_\infty, \ell_1)$ is given in [12, pp. 298]: the space of analytic functions $f : \mathbb{S} \to \ell_\infty$ which extends to a $\sigma(\ell_\infty, \ell_1)$ continuous function $\overline{F} : \overline{\mathbb{S}} \to \ell_\infty$ and such that $\|\overline{F}\|_{\mathcal{G}} = \sup\{\|\overline{F}(j + it)\|_{\ell_{1/j}}, \, t \in \mathbb{R}\}$ is finite.
- The second approach is to observe that any quasilinear map $\underline{\Omega_2} : c_{00} \curvearrowright \ell_2$ can be extended to a unique (up to equivalence) quasilinear map $\widehat{\Omega_2} : \ell_2 \curvearrowright \ell_2$ (see [62, Theorem 3.1] and [10, Section 3.3]). In this case $d\ell_2$ can be regarded as the completion of $(c_{00} \oplus_{\mathsf{KP}} c_{00}, \| \cdot \|_{Z_2})$.

In both of the previous cases we can identify $d\ell_2$ with the Kalton-Peck space studied in Chapter 1. We will follow the second approach, which from the practical side means that all results obtained via the representation $d\ell_2 = Z_2$ are obtained first for $(c_{00} \oplus_{\mathsf{KP}} c_{00}, \|\cdot\|_{Z_2})$ and then extended by density to Z_2.

One of the main problems left over at the end of Chapter 1 was how to obtain specific bounded operators on Z_2; we posed the examples $C_2 = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$ and $H_2 = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}$, where H is the Hilbert matrix and C the Cesàro operator (see (1.13)). In order to show that C_2 and H_2 are bounded we have to bound the commutators $[\mathsf{KP}, H] : \ell_2 \to \ell_2$ and $[\mathsf{KP}, C] : \ell_2 \to \ell_2$ (see (1.14) and (1.15)).

Now that we have identified Z_2 as the derived space of the scale (ℓ_∞, ℓ_1) at $\theta = 1/2$, the commutator bounds (1.14) and (1.15) are almost inmediate by well known facts:

(i) The Hilbert matrix H is bounded on ℓ_p spaces for every $1 < p < \infty$ [54, Section 9.2] with norm

$$\|H\|_{\ell_p \to \ell_p} = \frac{\pi}{\sin(\frac{\pi}{p})}.$$

(ii) The Cesàro operator C is bounded on ℓ_p spaces for every $1 < p < \infty$ [54, 326] with norm

$$\|C\|_{\ell_p \to \ell_p} = \frac{p}{p-1}.$$

Now, since these operators are not bounded on the endpoints ℓ_∞ and ℓ_1, we can not apply directly Rochberg-Weiss Commutator Theorem to the couple (ℓ_∞, ℓ_1). However, we can apply a modification of the Reiteration Theorem (cf. Proposition 2.4.1) to this case: given any $1 < p < 2$, we can consider the couple (ℓ_p, ℓ_p^*) which also has ℓ_2 as interpolation space at $\theta = 1/2$. In this case the selection is given for $x \in c_{00}$ by (see (2.12))

$$\widehat{B}_2(x)(z) = x\left(\frac{|x|}{\|x\|_2}\right)^{\frac{2(1-z)}{p} + \frac{2z}{p^*} - 1},$$

and so the differential map is $\widehat{\Omega_2}(x) = \Delta_1\big(\widehat{B}_2(x)\big) = \big(\frac{2}{p^*} - \frac{2}{p}\big)x \log\left(\frac{|x|}{\|x\|_2}\right)$. Thus, if we apply Theorem 2.6.2 to the couple (ℓ_p, ℓ_p^*) we deduce that

$$[\widehat{\Omega_2}, H] : c_{00} \to \ell_2 \quad \text{and} \quad [\widehat{\Omega_2}, C] : c_{00} \to \ell_2$$

are bounded. Using that c_{00} is dense in ℓ_2 and that both the differential maps, the one appearing in (2.30) and $\widehat{\Omega_2}$, differ by a constant, we conclude that

$$[\mathsf{KP}, H] : \ell_2 \to \ell_2 \quad \text{and} \quad [\mathsf{KP}, C] : \ell_2 \to \ell_2$$

are bounded.

A further important estimate is (1.10), which was used in Section 1.2. Using interpolation methods, (1.10) is direct: any bounded sequence $a \in \ell_\infty$ defines a continuous operator on ℓ_p for all $1 \le p \le \infty$ given by pointwise multiplication $\tau_a(x) = ax = \sum_n a_n x_n$. Then a direct application of Theorem 2.6.2 provides

$$\|[\mathsf{KP}, \tau_a]\|_2 = \|\mathsf{KP}(\tau(x)) - \tau_a\big(\mathsf{KP}(x)\big)\|_2 = \|\mathsf{KP}(ax) - a\mathsf{KP}(x)\|_2 \le C\|x\|_2.$$

Chapter 3

Rochberg spaces

In this chapter we define Rochberg spaces and discuss the general properties they satisfy. Most results can be regarded as direct generalizations of the main theorems discussed in Chapter 2: reiteration, duality and interpolation principles.

3.1 Definition of Rochberg spaces

Given a couple (X_0, X_1), $0 < \theta < 1$ and $n \in \mathbb{N}$, the n-th *Rochberg space* at θ is defined as the space of first n Taylor coefficients of analytic functions $f \in \mathcal{C}(X_0, X_1)$, i.e.,

$$\mathfrak{R}_n(X_0, X_1)_\theta = \big\{(\Delta^\theta_{n-1}, \dots, \Delta^\theta_0)(f) \colon f \in \mathcal{C}(X_0, X_1)\big\} \tag{3.1}$$

endowed with the quotient norm

$$\|x\|_n = \inf\{\|f\|_{\mathcal{C}} \colon (\Delta^\theta_{n-1}, \dots, \Delta^\theta_0)(f) = x\}.$$

Thus $\mathfrak{R}_n(X_0, X_1)_\theta$ can be identified with the quotient space

$$\mathcal{C}(X_0, X_1) / \bigcap_{j=0}^{n-1} \ker \Delta^\theta_k. \tag{3.2}$$

Note that $X_\theta = \mathfrak{R}_1(X_0, X_1)_\theta$ and $dX_\theta = \mathfrak{R}_2(X_0, X_1)_\theta$. From now on, we will omit the superscript θ on the linear maps Δ^θ_k if there is no real necessity. Given any $1 \le k, l < n$ we can consider the following two natural operators:

the inclusion map $i_{l,n} : (X_0 + X_1)^l \to (X_0 + X_1)^n$ given by

$$i_{l,n}(x_{l-1}, \dots, x_0) = (x_{l-1}, \dots, x_0, 0, \overset{(n-l)}{\dots}, 0).$$

the projection map $\pi_{n,k} : (X_0 + X_1)^n \to (X_0 + X_1)^k$ given by

$$\pi_{n,k}(x_{n-1}, \dots, x_0) = (x_{k-1}, \dots, x_0).$$

The restriction of $\pi_{n,k}$ to $\mathfrak{R}_n(X_0, X_1)_\theta$ defines a quotient map

$$\pi_{n,k} : \mathfrak{R}_n(X_0, X_1)_\theta \to \mathfrak{R}_k(X_0, X_1)_\theta.$$

The fact that $i_{l,n} : \mathfrak{R}_l(X_0, X_1)_\theta \to \mathfrak{R}_n(X_0, X_1)_\theta$ also defines a bounded operator depends on the following property: given any string $(x_{l-1}, \dots, x_0)$ of l Taylor coefficients of some

$f \in \mathcal{C}(X_0, X_1)$, there exist a positive constant C and $g \in \mathcal{C}(X_0, X_1)$ whose n Taylor coefficients are

$$(x_{l-1}, \ldots, x_0, 0, \overset{(n-l)}{\ldots}, 0) \quad \text{and} \quad \|g\|_{\mathcal{C}} \leq C\|f\|_{\mathcal{C}}.$$

This is precisely what the Shift operator $S_-^{(n-l)}$ of Proposition 2.3.1 does: $g = S_-^{(n-l)}(f) \in \bigcap_{j=0}^{n-l-1} \ker \Delta_j$ satisfies that $(\Delta_{n-1}, \ldots, \Delta_{n-l})(S_-^{(n-l)}(f)) = (\Delta_{l-1}, \ldots, \Delta_0)(f)$ and that $\|g\|_{\mathcal{C}} \leq \|S_-^{(n-l)}\|\|f\|_{\mathcal{C}}$. Therefore, the inclusion map $i_{l,n} : \mathfrak{R}_l(X_0, X_1)_\theta \to \mathfrak{R}_n(X_0, X_1)_\theta$ is well defined and bounded.

Rochberg spaces can be naturally fitted in short exact sequences, something hinted at first in [82] and explictly proved in [12]:

Proposition 3.1.1. *For any $k, l \geq 1$ such that $n = k+l$ there exist a short exact sequence*

$$0 \longrightarrow \mathfrak{R}_l(X_0, X_1)_\theta \xrightarrow{i_{l,n}} \mathfrak{R}_n(X_0, X_1)_\theta \xrightarrow{\pi_{n,k}} \mathfrak{R}_k(X_0, X_1)_\theta \longrightarrow 0 \tag{3.3}$$

Proof. First note that $\pi_{n,k} \circ i_{l,n} = 0$, so $i_{l,n}\Big(\mathfrak{R}_l(X_0, X_1)_\theta\Big) \subset \ker \pi_{n,k}$. To see the other inclusion, we must show that given an analytic function $f \in \mathcal{C}(X_0, X_1)$ whose Taylor coefficients are $(\Delta_{n-1}, \ldots, \Delta_0)(f) = (x_{n-1}, \ldots, x_k, 0, \overset{(k)}{\ldots}, 0)$, there exist $g \in \mathcal{C}(X_0, X_1)$ with $\|g\|_{\mathcal{C}} \leq C\|f\|_{\mathcal{C}}$ and such that $(\Delta_{l-1}, \ldots, \Delta_0)(g) = (x_{n-1}, \ldots, x_k)$. Proposition 2.3.1 yields $S_+^k(f) = g$. $\square$

Using Proposition 3.1.1, Rochberg spaces can be intertwined forming the commutative diagrams

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathfrak{R}_j(X_0, X_1)_\theta \arrow[r, equal] \arrow[d, "i_{j,l}"'] & \mathfrak{R}_j(X_0, X_1)_\theta \arrow[d, "i_{j,l+k}"] & & \\
0 \arrow[r] & \mathfrak{R}_l(X_0, X_1)_\theta \arrow[r, "i_{l,l+k}"] \arrow[d, "\pi_{l,l-j}"'] & \mathfrak{R}_{l+k}(X_0, X_1)_\theta \arrow[r, "\pi_{l+k,k}"] \arrow[d, "\pi_{l+k,l+k-j}"'] & \mathfrak{R}_k(X_0, X_1)_\theta \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{R}_{l-j}(X_0, X_1)_\theta \arrow[r, "i_{l-j,l+k-j}"] \arrow[d] & \mathfrak{R}_{l+k-j}(X_0, X_1)_\theta \arrow[r, "\pi_{l+k-j,k}"] \arrow[d] & \mathfrak{R}_k(X_0, X_1)_\theta \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

for $l, k \geq 1$ and $1 \leq j < l$. Using this last diagram one can show that the sequence (3.3) is non-trivial as long as the case $l = k = 1$ is non-trivial. Precisely (see [12, Corollary 6]):

Proposition 3.1.2. *If the sequence*

$$0 \longrightarrow \mathfrak{R}_1(X_0, X_1)_\theta \xrightarrow{i_{1,2}} \mathfrak{R}_2(X_0, X_1)_\theta \xrightarrow{\pi_{2,1}} \mathfrak{R}_1(X_0, X_1)_\theta \longrightarrow 0$$

is non-trivial then (3.3) is non-trivial for all $k, l \geq 1$.

Proof. Given any $k \in \mathbb{N}$, consider the diagram (here we omit the arrows and the 0's to ease readability)

$$\begin{tikzcd}
\mathfrak{R}_1(X_0, X_1)_\theta \arrow[r, equal] \arrow[d] & \mathfrak{R}_1(X_0, X_1)_\theta \arrow[d] & \\
\mathfrak{R}_2(X_0, X_1)_\theta \arrow[r] \arrow[d] & \mathfrak{R}_k(X_0, X_1)_\theta \arrow[r] \arrow[d] & \mathfrak{R}_{k-2}(X_0, X_1)_\theta \arrow[d, equal] \\
\mathfrak{R}_1(X_0, X_1)_\theta \arrow[r] & \mathfrak{R}_{k-1}(X_0, X_1)_\theta \arrow[r] & \mathfrak{R}_{k-2}(X_0, X_1)_\theta
\end{tikzcd}$$

Note that if the vertical middle sequence is trivial then the left vertical sequence is trivial. Indeed, if $P : \mathfrak{R}_k(X_0, X_1)_\theta \to \mathfrak{R}_1(X_0, X_1)_\theta$ defines a retraction for $i_{1,k}$, then $Pi_{2,k}$ is a retraction for $i_{1,2}$. Thus, by hypothesis

$$0 \longrightarrow \mathfrak{R}_1(X_0, X_1)_\theta \xrightarrow{i_{1,k}} \mathfrak{R}_k(X_0, X_1)_\theta \xrightarrow{\pi_{k,k-1}} \mathfrak{R}_{k-1}(X_0, X_1)_\theta \longrightarrow 0$$

is non trivial for all $k \in \mathbb{N}$. Now consider the diagram

$$\begin{tikzcd}
\mathfrak{R}_{l-1}(X_0, X_1)_\theta \arrow[r, equal] \arrow[d] & \mathfrak{R}_{l-1}(X_0, X_1)_\theta \arrow[d] & \\
\mathfrak{R}_l(X_0, X_1)_\theta \arrow[r] \arrow[d] & \mathfrak{R}_{l+k}(X_0, X_1)_\theta \arrow[r] \arrow[d] & \mathfrak{R}_k(X_0, X_1)_\theta \arrow[d, equal] \\
\mathfrak{R}_1(X_0, X_1)_\theta \arrow[r] & \mathfrak{R}_{k+1}(X_0, X_1)_\theta \arrow[r] & \mathfrak{R}_k(X_0, X_1)_\theta
\end{tikzcd}$$

To end the proof observe that if the horizontal middle sequence is trivial, then the lower horizontal sequence is also trivial (note that $\pi_{l+k,k+1}S : \mathfrak{R}_k(X_0, X_1)_\theta \to \mathfrak{R}_{k+1}(X_0, X_1)_\theta$ is a linear selection if $S : \mathfrak{R}_k(X_0, X_1)_\theta \to \mathfrak{R}_{l+k}(X_0, X_1)_\theta$ is). □

3.1.1 Representation of $\mathfrak{R}_n(X_0, X_1)_\theta$ as a twisted sum

We focus now in identifying the quasilinear maps defining the short exact sequence (3.3). This is completely analogous to what we did for the case of derived spaces in Section 2.6. We choose a bounded homogeneous selection B^k_θ for the quotient map

$$(\Delta_{k-1}, \ldots, \Delta_0) : \mathcal{C}(X_0, X_1) \longrightarrow \mathfrak{R}_k(X_0, X_1)_\theta$$

and define the map $\Omega_{k,l} : \mathfrak{R}_k(X_0, X_1) \to (X_0 + X_1)^l$ given for $x \in \mathfrak{R}_k(X_0, X_1)_\theta$ by

$$\Omega_{k,l}(x) = \big(\Delta_{k+l-1}, \ldots, \Delta_k\big) B^k_\theta(x).$$

Proposition 2.3.1 ensures that $\Omega_{k,l} : \mathfrak{R}_k(X_0, X_1)_\theta \curvearrowright \mathfrak{R}_l(X_0, X_1)_\theta$ is quasilinear: note that for any $x, y \in \mathfrak{R}_k(X_0, X_1)_\theta$ we have

$$\Omega_{k,l}(x+y) - \Omega_{k,l}(x) - \Omega_{k,l}(y) = \big(\Delta_{k+l-1}, \ldots, \Delta_k\big)\big(B_\theta^k(x+y) - B_\theta^k(x) - B_\theta^k(y)\big).$$

By definition $f = B_\theta^k(x+y) - B_\theta^k(x) - B_\theta^k(y) \in \bigcap_{j=0}^{k-1} \ker \Delta_j$ and thus

$$\big(\Delta_{k+l-1}, \ldots, \Delta_k\big)(f) = \big(\Delta_{l-1}, \ldots, \Delta_0\big)(S_+^k(f))$$

by Proposition 2.3.1. It follows that $\Omega_{k,l}(x+y) - \Omega_{k,l}(x) - \Omega_{k,l}(y) \in \mathfrak{R}_l(X_0, X_1)_\theta$ and that

$$\begin{aligned}\|\Omega_{k,l}(x+y) - \Omega_{k,l}(x) - \Omega_{k,l}(y)\|_l &\leq \|S_+^k\| \|B_\theta^k(x+y) - B_\theta^k(x) - B_\theta^k(y)\|_{\mathcal{C}} \\ &\leq C\big(\|x\|_k + \|y\|_k\big).\end{aligned}$$

$\Omega_{k,l}$ depends on the selection B_θ^k. However, any other selection V_θ^k defines an equivalent quasilinear map since

$$\|(\Delta_{l+k-1}, \ldots, \Delta_k)\big(B_\theta^k(x) - V_\theta^k(x)\big)\|_l \leq C\|x\|_k$$

by Proposition 2.3.1 yet again. Cheating a bit, the maps $\Omega_{k,l}$ will also be called differentials. The quasilinear map $\Omega_{k,l}$ defines the twisted sum space

$$\mathfrak{R}_l(X_0, X_1)_\theta \oplus_{\Omega_{k,l}} \mathfrak{R}_k(X_0, X_1)_\theta = \{(x, y) \in (X_0+X_1)^l \times \mathfrak{R}_k(X_0, X_1)_\theta \colon x - \Omega_{k,l}(y) \in \mathfrak{R}_l(X_0, X_1)_\theta\}$$

endowed with the quasinorm

$$\|(x, y)\|_{l,k} = \|x - \Omega_{k,l}\|_l + \|y\|_k.$$

The fact that $\Omega_{k,l}$ is the quasilinear map defining the sequence (3.3) is contained in the following result (see [12, Prop. 7]):

Proposition 3.1.3. $\mathfrak{R}_{l+k}(X_0, X_1)_\theta = \mathfrak{R}_l(X_0, X_1)_\theta \oplus_{\Omega_{k,l}} \mathfrak{R}_k(X_0, X_1)_\theta$ *as sets and the quasinorms* $\|\cdot\|_{l+k}$ *and* $\|\cdot\|_{l,k}$ *are equivalent.*

Proof. Let $(x, y) = (x_{l-1}, \ldots, x_0, y_{k-1}, \ldots, y_0) \in \mathfrak{R}_{l+k}(X_0, X_1)_\theta$ be the string of first $l+k$ Taylor coefficients of some $f \in \mathcal{C}(X_0, X_1)$. Then $y \in \mathfrak{R}_k(X_0, X_1)_\theta$ and thus $(\Omega_{k,l}(y), y) \in \mathfrak{R}_{l+k}(X_0, X_1)_\theta$. Therefore, $(x - \Omega_{k,l}(y), 0) = (x, y) - (\Omega_{k,l}(y), y) \in \mathfrak{R}_{l+k}(X_0, X_1)$. Let $h \in \mathcal{C}(X_0, X_1)$ be a function with $(x - \Omega_{k,l}(y), 0)$ as Taylor coefficients and such that $\|f\|_{\mathcal{C}} \leq (1+\varepsilon)\|(x - \Omega_{k,l}(y), 0)\|_{l+k}$.
By Proposition 2.3.1 it follows that

$$x - \Omega_{k,l}(y) = (\Delta_{l-1}, \ldots, \Delta_0)(S_+^k(h)) = (\Delta_{l+k-1}, \ldots, \Delta_k)(h),$$

hence

$$\begin{aligned}\|x - \Omega_{k,l}(y)\|_l &\leq \|S_+^k(h)\|_{\mathcal{C}} \leq \|S_+^k\|(1+\varepsilon)\|(x - \Omega_{k,l}(y), 0)\|_{l+k} \\ &\leq (1+\varepsilon)\|S_+^k\|\big(\|(x, y)\|_{l+k} + \|(\Omega_{k,l}(y), y)\|_{l+k}\big) \\ &\leq (1+\varepsilon)\|S_+^k\|\big(\|(x, y)\|_{l+k} + \|y\|_k\big) \\ &\leq 2(1+\varepsilon)\|S_+^k\| \, \|(x, y)\|_{l+k}.\end{aligned}$$

We conclude that $\|(x, y)\|_{l,k} = \|x - \Omega_{k,l}(y)\|_l + \|y\|_k \leq 3\|S_+^k\|\|(x, y)\|_{l+k}$.

To see the other inequality, let $(x,y) \in \mathfrak{R}_l(X_0,X_1)_\theta \oplus_{\Omega_{k,l}} \mathfrak{R}_k(X_0,X_1)_\theta$, so that $x - \Omega_{k,l}(y) \in \mathfrak{R}_l(X_0,X_1)_\theta$ and $y \in \mathfrak{R}_k(X_0,X_1)_\theta$. Define the function

$$h = S_-^k\big(B_\theta^l(x - \Omega_{k,l}(y))\big) + B_\theta^k(y) \in \mathcal{C}(X_0,X_1),$$

where S_-^k is the shift of Proposition 2.3.1. Then

$$(\Delta_{l+k-1},\dots,\Delta_k,\Delta_{k-1},\dots,\Delta_0)(h) = (x,y) \in \mathfrak{R}_{l+k}(X_0,X_1)_\theta$$

and

$$\|(x,y)\|_{l+k} \leq \|h\|_{\mathcal{C}} \leq (1+\varepsilon)\|S_+^k\|\big(\|x - \Omega_{k,l}(y)\|_l + \|y\|_k\big). \quad \square$$

Proposition 3.1.3 shows that, for each $n \in \mathbb{N}$, there exist a family of Rochberg spaces $\{\mathfrak{R}_n(X_0,X_1)_\theta\}_{0<\theta<1}$ at "level n", and the differential maps are interlacing such levels:

$$\begin{tikzcd}
 & \{\mathfrak{R}_4(X_0,X_1)_\theta\}_{0<\theta<1} & \\
 & \{\mathfrak{R}_3(X_0,X_1)_\theta\}_{0<\theta<1} & \\
\mathcal{C}(X_0,X_1) \arrow[ruu, bend left, "{(\Delta_3,\Delta_2,\Delta_1,\Delta_0)}"] \arrow[ru, "{(\Delta_2,\Delta_1,\Delta_0)}"'] \arrow[rd, "{(\Delta_1,\Delta_0)}"] \arrow[rrdd, bend right, "\Delta_0"'] & & \\
 & \{\mathfrak{R}_2(X_0,X_1)_\theta\}_{0<\theta<1} & \\
 & & \{X_\theta\}_{0<\theta<1} \arrow[lu, "\Omega_{1,1}"] \arrow[luuu, bend right, "\Omega_{1,2}"'] \arrow[luuuu, bend right, "\Omega_{1,3}"']
\end{tikzcd}$$

Recall from Subsection A.1.2 that any quasilinear map Ω has associated an inverse quasi-linear map $\Omega^{-1} : \mathrm{Ran}(\Omega) \curvearrowright \mathrm{Dom}(\Omega)$. Thus, in the case of $\Omega_{l,k}$ we also have the short exact sequence

$$0 \longrightarrow \mathrm{Dom}(\Omega_{k,l}) \xrightarrow{i^{k,l+k}} \mathfrak{R}_{l+k}(X_0,X_1)_\theta \xrightarrow{\pi^{l+k,l}} \mathrm{Ran}(\Omega_{k,l}) \longrightarrow 0 \tag{3.4}$$

given by $i^{k,l+k}(x_{k-1},\dots,x_0) = (0,\dots,x_{k-1},\dots,x_0)$ and $\pi^{l+k,l}(x_{l+k-1},\dots,x_0) = (x_{l+k-1},\dots,x_k)$. By Proposition 3.1.3 we have

$$\mathrm{Dom}(\Omega_{k,l}) = \Big\{\big(\Delta_{k-1},\dots,\Delta_0\big)(f) \colon f \in \bigcap_{j=k}^{l+k-1} \ker \Delta_j\Big\} \tag{3.5}$$

with (quasi)-norm $\|x\|_{\mathrm{Dom}_{k,l}} = \|\Omega_{k,l}(x)\|_l + \|x\|_k$. Similarly, we have

$$\mathrm{Ran}(\Omega_{k,l}) = \big\{(\Delta_{l+k-1},\dots,\Delta_k)(f) \colon f \in \mathcal{C}(X_0,X_1)\big\}. \tag{3.6}$$

Using both (3.5) and (3.6) we can provide a description of $\Omega_{k,l}^{-1}$: pick a homogeneous bounded selection $B_\theta^{l+k-1,k}$ for the quotient map

$$(\Delta_{l+k-1},\dots,\Delta_k) : \mathcal{C}(X_0,X_1) \to \mathrm{Ran}(\Omega_{k,l})$$

and set $\Omega_{k,l}^{-1}(x) = \big(\Delta_{k-1}, \dots, \Delta_0\big) B_\theta^{l+k-1,k}(x)$ for $x \in \operatorname{Ran}(\Omega_{k,l})$. Observe that

$$B_\theta^{l+k-1,k}(x+y) - B_\theta^{l+k-1,k}(x) - B_\theta^{l+k-1,k}(y) \in \bigcap_{j=k}^{l+k-1} \ker \Delta_j,$$

and thus by (3.5) it follows that

$$\|\Omega_{k,l}^{-1}(x+y) - \Omega_{k,l}^{-1}(x) - \Omega_{k,l}^{-1}(y)\|_{\mathrm{Dom}_{k,l}} \leq C(\|x\|_{\mathrm{Ran}_{k,l}} + \|y\|_{\mathrm{Ran}_{k,l}})$$

for every $x, y \in \operatorname{Ran}(\Omega_{k,l})$. An analogous proof to that of Proposition 3.1.3 shows that $\Omega_{k,l}^{-1}$ does indeed define (3.4). Note also that we have a chain of continuous inclusions

$$\operatorname{Dom}(\Omega_{1,n}) \subset \cdots \subset \operatorname{Dom}(\Omega_{1,1}) \subset X_\theta \subset \operatorname{Ran}(\Omega_{1,1}) \subset \cdots \subset \operatorname{Ran}(\Omega_{n,1}) \tag{3.7}$$

The range inclusions follow by shifting Taylor coefficients using Proposition 2.3.1, while domain inclusion follow using repeatedly Proposition 3.1.3: if $i < j$ and $x = (0, \dots, 0, x) \in \operatorname{Dom}(\Omega_{1,j}) \subset \mathfrak{R}_{1+j}$, note that

$$\begin{aligned} \|x\|_{\mathrm{Dom}_{1,i}} &= \|(0, \dots, 0, x)\|_{1,i} \leq C_i \|(0, \dots, 0, x)\|_{1+i} \\ &\leq C_i \|(0, \dots, 0, x)\|_{i+1} + C_i \|\Omega_{1+i, j-i}(0, \dots, 0, x)\|_{j-i} \\ &\leq C_{i,j} \|(0, \dots, 0, x)\|_{1+j} \leq K_{i,j} \|(0, \dots, 0, x)\|_{1,j} = \|x\|_{\mathrm{Dom}_{1,j}}. \end{aligned}$$

3.1.2 Natural subspaces and quotients

Most of of the discussion about domain and range spaces can be generalized in the following way: fix $n \in \mathbb{N}$ and a subset $A \subset \{0, 1, \dots, n-1\}$. Then define:

$$X^A = \{\Delta_A(f) = (\Delta_i(f))_{i \in A} \colon f \in \mathcal{C}(X_0, X_1)\}$$

endowed with the quotient norm $\|x\|_A = \inf\{\|f\|_{\mathcal{C}} \colon \Delta_A(f) = x\}$ and

$$X_A = \{x = (x_{n-1}, \dots, x_0) \in \mathfrak{R}_n(X_0, X_1)_\theta \colon x_i = 0,\ i \in A\}$$

endowed with the (quasi)-norm $\|x\|_n$ of $\mathfrak{R}_n(X_0, X_1)_\theta$. If we denote by $[i, j] = \{k \in \mathbb{N} \colon i \leq k \leq j\} \subset \{0, \dots, n-1\}$, then

(a) $X^{[l-1,0]} = \mathfrak{R}_l(X_0, X_1)_\theta$ and $X_{[l-1,0]} = \mathfrak{R}_{n-l}(X_0, X_1)_\theta$.

(b) $X^{[n-1,l]} = \operatorname{Ran}(\Omega_{l,n-l})$ and $X_{[n-1,l]} = \operatorname{Dom}(\Omega_{l,n-l})$.

For any $A \subset \{0, \dots, n-1\}$ there is a short exact sequence

$$0 \longrightarrow X_A \longrightarrow \mathfrak{R}_n(X_0, X_1)_\theta \xrightarrow{\pi^A} X^A \longrightarrow 0 \tag{3.8}$$

given by $\pi^A(x_{n-1}, \dots, x_0) = (x_i)_{i \in A}$.

3.2 Basic results on Rochberg spaces

In Chapter 2 we used the alternative Calderón space $\mathcal{C}_0$ to define interpolation spaces since it was useful to apply density arguments. We can do the same for the general case of Rochberg spaces. To be precise, let us denote by $\mathfrak{R}_n^0(X_0, X_1)_\theta$ the Rochberg spaces obtained by replacing $\mathcal{C}(X_0, X_1)$ by $\mathcal{C}_0(X_0, X_1)$ in their definition (3.1). Propositions 3.1.1 and 3.1.3 hold for $\mathfrak{R}_n^0$. The following result was obtained in [11, Lemma 3.2] in a broader context:

Lemma 3.2.1. *$\mathfrak{R}_n^0(X_0, X_1)_\theta = \mathfrak{R}_n(X_0, X_1)_\theta$ up to equivalence of norms.*

Proof. Since $\mathcal{C}_0(X_0, X_1) \subset \mathcal{C}(X_0, X_1)$ one has continuous inclusions $\mathfrak{R}_n^0(X_0, X_1)_\theta \subset \mathfrak{R}_n(X_0, X_1)_\theta$ for every $n \in \mathbb{N}$. Thus we can consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{R}_1^0(X_0, X_1)_\theta \arrow[r, "i_{1,2}"] \arrow[d] & \mathfrak{R}_2^0(X_0, X_1)_\theta \arrow[r, "\pi_{2,1}"] \arrow[d] & \mathfrak{R}_1^0(X_0, X_1)_\theta \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{R}_1(X_0, X_1)_\theta \arrow[r, "i_{1,2}"] & \mathfrak{R}_2(X_0, X_1)_\theta \arrow[r, "\pi_{2,1}"] & \mathfrak{R}_1(X_0, X_1)_\theta \arrow[r] & 0
\end{tikzcd}$$

where vertical arrows are the formal inclusions. Since $\mathfrak{R}_1^0(X_0, X_1)_\theta = \mathfrak{R}_1(X_0, X_1)_\theta$ with equality of norms (see the comments before Lemma 2.1.2) we deduce that the middle arrow is an isomorphism by the 3-lemma [10, 2.1.3]. An induction argument on n and k provides a proof for the rest of the cases. $\square$

The following result generalizes Lemma 2.1.3. It is basically the same argument provided by Stafney [86, Lemma 2.5] and generalized to derived spaces in [39, Lemma 3.3.6].

Lemma 3.2.2. *For any $0 < \theta < 1$ the space $(X_0 \cap X_1)^n$ is dense in $\mathfrak{R}_n(X_0, X_1)_\theta$. Moreover, for any $x \in (X_0 \cap X_1)^n$ one has*

$$\|x\|_n = \inf\{\|f\|_{\mathcal{C}} \colon f \in \mathcal{C}_{00}(X_0, X_1),\ (\Delta_{n-1}, \dots, \Delta_0)(f) = x\}. \tag{3.9}$$

Proof. Let $x \in \mathfrak{R}_n(X_0, X_1)_\theta$ and $f \in \mathcal{C}_0(X_0, X_1)$ such that $(\Delta_{n-1}, \dots, \Delta_0)(f) = x$. Then by Lemma 2.1.2 there exist $g \in \mathcal{C}_{00}(X_0, X_1)$ such that $\|f - g\|_{\mathcal{C}} < \varepsilon$. Thus

$$\|x - (\Delta_{n-1}, \dots, \Delta_0)(g)\|_n \leq \|f - g\|_{\mathcal{C}} < \varepsilon.$$

To see the second statement, let $x = (x_{n-1}, \dots, x_0) \in (X_0 \cap X_1)^n$ and $f \in \mathcal{C}_{00}(X_0, X_1)$ such that $(\Delta_{n-1}, \dots, \Delta_0)(f) = x$. Such function can be defined in the following way: consider $f_0(z) = e^{(z-\theta)^2} x_0$ and define by recursion

$$f_1(z) = \frac{\varphi(z)}{\varphi'(\theta)} e^{(z-\theta)^2}(x_1 - \Delta_1(f_0)),\ \cdots,\ f_{n-1}(z) = \frac{\varphi(z)^{n-1} e^{(z-\theta)^2}}{(n-1)!\varphi'(\theta)^{n-1}}\Big(x_{n-1} - \sum_{j=0}^{n-2} \Delta_{n-1}(f_j)\Big).$$

Since $\Delta_k\Big(\frac{\varphi(z)^k}{k!\varphi'(\theta)^k} e^{(z-\theta)^2} x\Big) = x$ and $\Delta_k(f_i) = 0$ for all $k < i$, taking $f(z) = \sum_{j=0}^{n-1} f_j(z)$ we deduce that $\Delta_k(f) = x_k$ for all $k = 0, \dots, n-1$. Moreover, $f \in \mathcal{C}_{00}$ because each $f_j \in \mathcal{C}_{00}$.

Now, by the very definition of $\|\cdot\|_n$ as quotient norm, there exist $g \in \bigcap_{k=0}^{n-1} \ker \Delta_k \subset \mathcal{C}_0(X_0, X_1)$ such that $\|f-g\|_{\mathcal{C}} < \|x\|_n + \varepsilon$. Since $g \in \bigcap_{k=0}^{n-1} \ker \Delta_k$ we can apply the shift operator to obtain $S_-^n(g) \in \mathcal{C}_0(X_0, X_1)$. Note that we can use the shift $g \mapsto \frac{g}{\varphi^n}$. Taking into account that $S_-^n(g) \in \mathcal{C}_0$ and that $\mathcal{C}_{00}(X_0, X_1) \subset \mathcal{C}_0(X_0, X_1)$, there exist $h \in \mathcal{C}_{00}$ that approximates $S_-^n(g)$, namely, $\|S_-^n(g) - h\|_{\mathcal{C}} < \varepsilon$. Then $g - \varphi^n h = \varphi^n(S_-^n(g) - h) \in \bigcap_{k=0}^{n-1} \ker \Delta_k$, and using that $|\varphi(z)| \leq 1$ for all $z \in \overline{\mathbb{S}}$, we deduce that

$$\|g - \varphi^n h\|_{\mathcal{C}} = \|\varphi^n(S_-^n(g) - h)\|_{\mathcal{C}} = \|S_-^n(g) - h\|_{\mathcal{C}} < \varepsilon.$$

Thus, it follows that

$$\begin{aligned}\|x\|_n = \|(\Delta_{n-1}, \dots, \Delta_0)(f - \varphi^n h)\|_n \leq \|f - \varphi^n h\|_{\mathcal{C}} &\leq \|f - g\|_{\mathcal{C}} + \|g - \varphi^n h\|_{\mathcal{C}} \\ &< \|x\|_n + 2\varepsilon.\end{aligned}$$

Since $f - \varphi^n h \in \mathcal{C}_{00}(X_0, X_1)$, we conclude (3.9). □

3.2.1 Singular exact sequences of Rochberg spaces

Recall from the Appendix A.1.1 that a short exact sequence $0 \to Y \to X \to Z \to 0$ is singular if the quotient map $X \to Z$ is a strictly singular operator. One fundamental example of singular exact sequence is the Kalton-Peck sequence (1.3).

In the next chapter we will study the Rochberg spaces $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$. Since $Z_2 = \mathfrak{R}_2(\ell_\infty, \ell_1)_{1/2}$, we may ask whether the sequences

$$0 \longrightarrow \mathfrak{R}_l(\ell_\infty, \ell_1)_\theta \xrightarrow{i_{l,l+k}} \mathfrak{R}_{l+k}(\ell_\infty, \ell_1)_\theta \xrightarrow{\pi_{l+k,k}} \mathfrak{R}_k(\ell_\infty, \ell_1)_\theta \longrightarrow 0$$

are also singular for $l + k > 2$. The following result from [12, Prop. 9] gives an affirmative answer:

Proposition 3.2.1. *If $\pi_{2,1} : \mathfrak{R}_2(X_0, X_1)_\theta \to \mathfrak{R}_1(X_0, X_1)_\theta$ is singular then $\pi_{n,k} : \mathfrak{R}_n(X_0, X_1)_\theta \to \mathfrak{R}_k(X_0, X_1)_\theta$ is singular for any $n, k \in \mathbb{N}$.*

Proof. First note that $\pi_{n,1} = \pi_{2,1} \circ \pi_{3,2} \circ \cdots \circ \pi_{n,n-1}$ and thus $\pi_{n,1}$ is singular for every $n \in \mathbb{N}$ by hypothesis and the ideal property of $\mathcal{S}$. Moreover, since $\pi_{n,k} = \pi_{k+1,k} \circ \pi_{n,k+1}$, it suffices to show that $\pi_{k+1,k}$ is strictly singular for every $k \in \mathbb{N}$. We will proceed by induction on $k \in \mathbb{N}$. The case $k = 1$ holds by hypothesis, hence we can assume it true for $k \geq 1$. Now observe that we have a commutative diagram of the form

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathfrak{R}_1(X_0, X_1)_\theta \arrow[r, equal] \arrow[d] & \mathfrak{R}_1(X_0, X_1)_\theta \arrow[d] & & \\
0 \arrow[r] & \mathfrak{R}_{k-1}(X_0, X_1)_\theta \arrow[r] \arrow[d, "\pi_{k-1,k-2}"'] & \mathfrak{R}_k(X_0, X_1)_\theta \arrow[r, "\pi_{k,1}"] \arrow[d, "\pi_{k,k-1}"'] & \mathfrak{R}_1(X_0, X_1)_\theta \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{R}_{k-2}(X_0, X_1)_\theta \arrow[r] \arrow[d] & \mathfrak{R}_{k-1}(X_0, X_1)_\theta \arrow[r] \arrow[d] & \mathfrak{R}_1(X_0, X_1)_\theta \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $\pi_{k,1}$ and $\pi_{k-1,k-2}$ are strictly singular, we conclude that $\pi_{k,k-1}$ is singular by Proposition A.1.3. $\square$

3.3 Rochberg's Commutator Theorem

Let $T : X_0+X_1 \to X_0+X_1$ be an interpolating operator for the couple (X_0, X_1) and $T_{\mathcal{C}}$ the induced operator on $\mathcal{C}(X_0, X_1)$ given by $T_{\mathcal{C}}(f)(z) = T(f(z))$. The Interpolation Principle (cf. Theorem 2.1.1) implies that $T : X_\theta \to X_\theta$ is bounded. Rochberg noticed that this same principle could be used to define bounded operators on $\mathfrak{R}_n(X_0, X_1)_\theta$ for any $0 < \theta < 1$. Given $f \in \mathcal{C}(X_0, X_1)$, the sequence of Taylor coefficients $\Big(\Delta_k\big(T_{\mathcal{C}}(f)\big)\Big)_{k\in\mathbb{N}}$, already hinted in Proposition 2.1.1 (see (2.4)), is as follows: since T is linear and continuous

$$\Delta_k\big(T_{\mathcal{C}}(f)\big) = T\big(\Delta_k(f)\big) \quad \text{for any } f \in \mathcal{C}(X_0, X_1). \tag{3.10}$$

Thus

$$\Big(\Delta_k\big(T_{\mathcal{C}}(f)\big)\Big)_{k\in\mathbb{N}} = \begin{pmatrix} \ddots & & & & \\ & T & & & \\ & & T & & \\ & & & T & \\ & & & & T \end{pmatrix} \begin{pmatrix} \vdots \\ \Delta_3(f) \\ \Delta_2(f) \\ \Delta_1(f) \\ \Delta_0(f) \end{pmatrix} \tag{3.11}$$

It follows that $T_{\mathcal{C}}$ preserves the kernels $\bigcap_{j=0}^{n-1} \ker \Delta_j$ for any $n \in \mathbb{N}$ and we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \bigcap_{j=0}^{n-1} \ker \Delta_j \arrow[r] \arrow[d, "T_{\mathcal{C}}"'] & \mathcal{C}(X_0, X_1) \arrow[r] \arrow[d, "T_{\mathcal{C}}"] & \mathfrak{R}_n(X_0, X_1)_\theta \arrow[r] \arrow[d, dotted, "T_n"] & 0 \\ 0 \arrow[r] & \bigcap_{j=0}^{n-1} \ker \Delta_j \arrow[r] & \mathcal{C}(X_0, X_1) \arrow[r] & \mathfrak{R}_n(X_0, X_1)_\theta \arrow[r] & 0 \end{tikzcd}$$

The operator $T_{\mathcal{C}}$ induces a bounded operator T_n on $\mathfrak{R}_n(X_0, X_1)$ given by

$$T_n(x) = \begin{pmatrix} T & & & & \\ & \ddots & & & \\ & & T & & \\ & & & T & \\ & & & & T \end{pmatrix} \begin{pmatrix} x_{n-1} \\ \vdots \\ x_2 \\ x_1 \\ x_0 \end{pmatrix} = (Tx_{n-1}, \ldots, Tx_0), \tag{3.12}$$

and whose norm is bounded above by $\|T_{\mathcal{C}}\|$. This essentially proves [82, Th. 5.1]:

Theorem 3.3.1 (Rochberg Commutator Theorem). *Let (X_0, X_1) be a Banach couple and T an interpolating operator. Then:*

(1) The diagonal operator $T_n : \mathfrak{R}_n(X_0, X_1)_\theta \to \mathfrak{R}_n(X_0, X_1)_\theta$ is bounded.

(2) For any $n = k + l$ one has the commutator estimate

$$\big\|\Omega_{k,l}\big(T_k(x)\big) - T_l\big(\Omega_{k,l}(x)\big)\big\|_l \leq C\|x\|_k,$$

for some constant C depending only on T.

Proof. (1) Given $x \in \mathfrak{R}_n(X_0, X_1)_\theta$ take $f \in \mathcal{C}(X_0, X_1)$ such that $\|f\|_{\mathcal{C}} \leq (1+\varepsilon)\|x\|_{\mathfrak{R}_n}$. Then by (3.10) we deduce that

$$\|T_n(x)\|_n = \|(\Delta_{n-1}, \dots, \Delta_0)(T_{\mathcal{C}}(f))\|_n \leq \|T_{\mathcal{C}}(f)\|_{\mathcal{C}} \leq \|T_{\mathcal{C}}\|(1+\varepsilon)\|x\|_n.$$

To prove (2) observe the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{R}_l(X_0,X_1)_\theta \arrow[r, "i_{l,n}"] \arrow[d, "T_l"'] & \mathfrak{R}_n(X_0,X_1)_\theta \arrow[r, "\pi_{n,k}"] \arrow[d, "T_n"] & \mathfrak{R}_k(X_0,X_1)_\theta \arrow[r] \arrow[d, "T_k"] \arrow[ll, bend right, "\Omega_{k,l}"'] & 0 \\
0 \arrow[r] & \mathfrak{R}_l(X_0,X_1)_\theta \arrow[r, "i_{l,n}"] & \mathfrak{R}_n(X_0,X_1)_\theta \arrow[r, "\pi_{n,k}"] & \mathfrak{R}_k(X_0,X_1)_\theta \arrow[r] \arrow[ll, bend left, "\Omega_{k,l}"] & 0
\end{tikzcd}$$

By (1) all involved operators are bounded. Then using Corollary 1.2.5 the conclusion follows. □

Corollary 3.3.1. *Let $n \in \mathbb{N}$ and $A \subset \{0, \dots, n-1\}$. Assume that T is interpolating for (X_0, X_1). Then:*

(i) $T_n : X_A \to X_A$ is bounded.

(ii) $T_{|A|} : X^A \to X^A$ is bounded.

Proof. (i) is inmediate since X_A is a subspace of $\mathfrak{R}_n(X_0, X_1)_\theta$ and T_n acts diagonally on $\mathfrak{R}_n(X_0, X_1)_\theta$. To see (ii), note by (i) and (3.8) that there exist a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X_A \arrow[r] \arrow[d, "T_n"'] & \mathfrak{R}_n(X_0,X_1)_\theta \arrow[r] \arrow[d, "T_n"] & X^A \arrow[r] \arrow[d, dotted, "T_{|A|}"] & 0 \\
0 \arrow[r] & X_A \arrow[r] & \mathfrak{R}_n(X_0,X_1)_\theta \arrow[r] & X^A \arrow[r] & 0
\end{tikzcd}$$

□

Corollary 3.3.2. *If T is interpolating for (X_0, X_1), then*

(1) $T_k : \mathrm{Dom}(\Omega_{k,l}) \to \mathrm{Dom}(\Omega_{k,l})$ is bounded.

(2) $T_l : \mathrm{Ran}(\Omega_{k,l}) \to \mathrm{Ran}(\Omega_{k,l})$ is bounded.

(3) For any $x \in \mathrm{Ran}(\Omega_{k,l})$ we have that

$$\|\Omega_{k,l}^{-1}\big(T_k(x)\big) - T_l\big(\Omega_{k,l}^{-1}(x)\big)\|_{\mathrm{Dom}_{k,l}} \leq C\|x\|_{\mathrm{Ran}_{k,l}}.$$

Proof. The first two statements follow by Corollary 3.3.1 while (3) is a consequence of (1) and (2) using the inverse representation (3.4). □

3.4 Reiteration of Rochberg spaces

Let us agree for the remainder of this section that given a Banach couple (X_0, X_1) and $0 < \alpha < \theta < \beta < 1$, we denote by:

- $\mathfrak{R}_n(0,1)$ the Rochberg spaces associated to (X_0, X_1) at $(1-\theta)\alpha + \theta\beta$.
- $\mathfrak{R}_n(\alpha,\beta)$ the Rochberg spaces associated to (X_α, X_β) at θ.
- Ω_θ the differential map associated to $(X_0, X_1)_\theta$ and $\widehat{\Omega}_\theta$ the differential map associated to $(X_\alpha, X_\beta)_\theta$.

Proposition 2.4.1 yields that $\mathfrak{R}_1(0,1)$ coincides with $\mathfrak{R}_1(\alpha,\beta)$ with equality of norms. We shall see that $\mathfrak{R}_n(0,1)$ and $\mathfrak{R}_n(\alpha,\beta)$ are isomorphic. First we will need the following reiteration result for differential maps obtained in [26, Proposition 3.7] (see also [22, Theorem 4.15]):

Proposition 3.4.1. *Let (X_0, X_1) be a Banach couple and $0 < \alpha < \theta < \beta < 1$. Then*

$$\widehat{\Omega}_\theta = (\beta - \alpha)\Omega_{(1-\theta)\alpha+\theta\beta}.$$

Proof. Note that given $f \in \mathcal{C}(X_0, X_1)$, the operator $C : \mathcal{C}(X_0, X_1) \to \mathcal{C}(X_\alpha, X_\beta)$ given by

$$C(f)(z) = f\big((1-z)\alpha + z\beta\big)$$

is bounded with $\|C(f)\| \le \|f\|$ (see Section 2.4). Consider now a homogeneous bounded selection $B : X_{(1-\theta)\alpha+\beta\theta} \to \mathcal{C}(X_0, X_1)$ for the evaluation operator $\delta_{(1-\theta)\alpha+\beta\theta} : \mathcal{C}(X_0, X_1) \to X_{(1-\theta)\alpha+\beta\alpha}$. Taking into account the reiteration identity $X_{(1-\theta)\alpha+\beta\theta} = (X_\alpha, X_\beta)_\theta$, we conclude that the map $CB : (X_\alpha, X_\beta)_\theta \to \mathcal{C}(X_\alpha, X_\beta)$, defined for each $x \in X_{(1-\theta)\alpha+\beta\theta}$ by

$$CB(x)(z) = B(x)\big((1-z)\alpha + \beta z\big),$$

is a homogeneous bounded selection for the evaluation map $\widehat{\delta}_\theta : \mathcal{C}(X_\alpha, X_\beta) \to (X_\alpha, X_\beta)_\theta$. Therefore, the differential map $\widehat{\Omega}_\theta$ is defined, for each $x \in (X_\alpha, X_\beta)_\theta$, by:

$$\begin{aligned}\widehat{\Omega}_\theta(x) &= \widehat{\delta}'_\theta\big(CB(x)\big) = B'(x)\big((1-z)\alpha + \beta z\big)\,(-\alpha+\beta)|_{z=\theta}\\ &= (\beta-\alpha)\delta'_{((1-\theta)\alpha+\beta\theta)}B(x)\\ &= (\beta-\alpha)\Omega_{(1-\theta)\alpha+\beta\theta}.\end{aligned}$$

□

We show now the reiteration result for Rochberg spaces:

Proposition 3.4.2. *The spaces $\mathfrak{R}_n(0,1)$ and $\mathfrak{R}_n(\alpha,\beta)$ are isomorphic.*

Proof. Denote by $c = \beta - \alpha$; we will show that the map $(c^{n-1}, \dots, c, 1) : \mathfrak{R}_n(0,1) \to \mathfrak{R}_n(\alpha,\beta)$ given by

$$(c^{n-1}, \dots, c, 1)\big((x_{n-1}, \dots, x_0)\big) = (c^{n-1}x_{n-1}, \dots, cx_1, x)$$

is an isomorphism. Let us proceed by induction: $(c,1) : \mathfrak{R}_2(0,1) \to \mathfrak{R}_2(\alpha,\beta)$ is bounded: by Proposition 3.4.1 it follows that

$$\begin{aligned}\|(c,1)(x,y)\|_{\mathfrak{R}_2(\alpha,\beta)} &= \|cx - \Omega(\alpha,\beta)y\|_{\mathfrak{R}_1(\alpha,\beta)} + \|y\|_{\mathfrak{R}_1(\alpha,\beta)}\\ &= \|cx - c\Omega(0,1)y\|_{\mathfrak{R}_1(0,1)} + \|y\|_{\mathfrak{R}_1(0,1)}\\ &= c\|x - \Omega(0,1)y\|_{\mathfrak{R}_1(0,1)} + \|y\|_{\mathfrak{R}_1(0,1)}\\ &\le \max\{c,1\}\|(x,y)\|_{\mathfrak{R}_2(0,1)}.\end{aligned}$$

Since we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{R}_1(0,1) \arrow[r] \arrow[d, "c"'] & \mathfrak{R}_2(0,1) \arrow[r] \arrow[d, "(c,1)"'] & \mathfrak{R}_1(0,1) \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & \mathfrak{R}_1(\alpha,\beta) \arrow[r] & \mathfrak{R}_2(\alpha,\beta) \arrow[r] & \mathfrak{R}_1(\alpha,\beta) \arrow[r] & 0
\end{tikzcd}$$

where all the vertical arrows are bounded and $c : \mathfrak{R}_1(0,1) \to \mathfrak{R}_1(\alpha,\beta)$ is an isomorphism, we conclude by the 3-lemma that $(c,1) : \mathfrak{R}_2(0,1) \to \mathfrak{R}_2(\alpha,\beta)$ is an isomorphism.

Passing to the next level we prove that $(c^2,c,1) : \mathfrak{R}_3(0,1) \to \mathfrak{R}_3(\alpha,\beta)$ is bounded: using Proposition 3.4.1 and the definition of $\Omega_{1,2}$ we deduce that

$$\begin{aligned}
\|(c^2,c,1)(x,y,z)\|_{\mathfrak{R}_3(\alpha,\beta)} &= \|(c^2x,cy) - \Omega_{1,2}(\alpha,\beta)z\|_{\mathfrak{R}_2(\alpha,\beta)} + \|z\|_{\mathfrak{R}_1(\alpha,\beta)} \\
&= \|(c^2,c)\big((x,y) - \Omega_{1,2}(0,1)z\big)\|_{\mathfrak{R}_2(\alpha,\beta)} + \|z\|_{\mathfrak{R}_1(0,1)} \\
&= |c|\|(c,1)\big((x,y) - \Omega_{1,2}(0,1)z\big)\|_{\mathfrak{R}_2(\alpha,\beta)} + \|z\|_{\mathfrak{R}_1(0,1)} \\
&\le |c|\|(c.1)\|\|(x,y) - \Omega_{1,2}(0,1)z\|_{\mathfrak{R}_2(0,1)} + \|z\|_{\mathfrak{R}_1(0,1)} \\
&\le K\|(x,y,z)\|_{\mathfrak{R}_3(0,1)}.
\end{aligned}$$

We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{R}_1(0,1) \arrow[r] \arrow[d, "c^2"'] & \mathfrak{R}_3(0,1) \arrow[r] \arrow[d, "(c^2,c,1)"'] & \mathfrak{R}_2(0,1) \arrow[r] \arrow[d, "(c,1)"] & 0 \\
0 \arrow[r] & \mathfrak{R}_1(\alpha,\beta) \arrow[r] & \mathfrak{R}_3(\alpha,\beta) \arrow[r] & \mathfrak{R}_2(\alpha,\beta) \arrow[r] & 0
\end{tikzcd}$$

where all arrows are bounded and c^2 and $(c,1)$ are isomorphisms. Hence, $(c^2,c,1)$: $\mathfrak{R}_3(0,1) \to \mathfrak{R}_3(\alpha,\beta)$ is an isomorphism. Proceeding in this way inductively we finish the proof. □

A quite important conclusion is that $T_n : \mathfrak{R}_n(0,1) \to \mathfrak{R}_n(0,1)$ is bounded if and only if $T_n : \mathfrak{R}_n(\alpha,\beta) \to \mathfrak{R}_n(\alpha,\beta)$ is bounded. This suggests that boundness of T_n at $\mathfrak{R}_n(X_0,X_1)_\theta$ only depends on the behaviour of T on a neighborhood of the interpolation scale at θ.

Corollary 3.4.1. *Suppose that T is interpolating for the couple (X_α, X_β). Then T_n : $\mathfrak{R}_n(0,1) \to \mathfrak{R}_n(0,1)$ is bounded.*

Proof. It follows from Rochberg Commutator Theorem that T_n is bounded on $\mathfrak{R}_n(\alpha,\beta)$. By Proposition 3.4.2 there exist an isomorphism of the form $(c^{n-1},\dots,c,1) : \mathfrak{R}_n(0,1) \to \mathfrak{R}_n(\alpha,\beta)$. Then given any $(x_{n-1},\dots,x_0) \in \mathfrak{R}_n(0,1)$ we deduce that

$$\begin{aligned}
\|T_n(x_{n-1},\dots,x_0)\|_{\mathfrak{R}_n(0,1)} &= \|(Tx_{n-1},\dots,Tx_0)\|_{\mathfrak{R}_n(0,1)} \\
&= \Big\|\Big(\frac{1}{c^{n-1}}c^{n-1}Tx_{n-1},\dots,\frac{1}{c}cTx_1,Tx_0\Big)\Big\|_{\mathfrak{R}_n(0,1)} \\
&= \Big\|\big(\frac{1}{c^{n-1}},\dots,\frac{1}{c},1\big)(Tc^{n-1}x_{n-1},\dots,Tcx_1,Tx_0)\Big\|_{\mathfrak{R}_n(0,1)} \\
&\le C\|(Tc^{n-1}x_{n-1},\dots,Tcx_1,Tx_0)\|_{\mathfrak{R}_n(\alpha,\beta)} \\
&= C\|T_n(c^{n-1}x_{n-1},\dots,cx_1,x_0)\|_{\mathfrak{R}_n(\alpha,\beta)} \\
&\le K\|(c^{n-1}x_{n-1},\dots,cx_1,x_0)\|_{\mathfrak{R}_n(\alpha,\beta)} \\
&= K\|(c^{n-1},\dots,c,1)(x_{n-1},\dots,x_0)\|_{\mathfrak{R}_n(\alpha,\beta)} \\
&\le M\|(x_{n-1},\dots,x_0)\|_{\mathfrak{R}_n(0,1)}.
\end{aligned}$$

□

Corollary 3.4.2. *Given $1 < p < \infty$ one has that $\mathfrak{R}_n(H_p(\mathbb{T}), H_{p^*}(\mathbb{T}))_\theta$ is isomorphic to $\mathfrak{R}_n(L_1(\mathbb{T}), L_\infty(\mathbb{T}))_\theta$.*

Proof. By Proposition 2.2.1, the Boas isomorphism $\mathcal{B}$ is interpolating for the couple $(L_p(\mathbb{T}), L_{p^*}(\mathbb{T}))$. Thus $\mathcal{B}_n : \mathfrak{R}_n(L_p(\mathbb{T}), L_{p^*}(\mathbb{T}))_\theta \to \mathfrak{R}_n(H_p(\mathbb{T}).H_{p^*}(\mathbb{T}))_\theta$ is an isomorphism for any $0 < \theta < 1$ by Rochberg's Commutator Theorem. A direct applicaction of Proposition 3.4.2 finishes the proof. □

3.5 Duality for Rochberg spaces

Given a regular couple (X_0, X_1) that is $(X_0 + X_1, i_0, i_1)$-compatible, the dual couple (X_0^*, X_1^*) is $\big((X_0 \cap X_1)^*, i_0^*, i_1^*\big)$-compatible (see Subsection 2.4), and we can consider the Rochberg spaces $\mathfrak{R}_n(X_0^*, X_1^*)_\theta$ and the associated short exact sequences

$$0 \longrightarrow \mathfrak{R}_l(X_0^*, X_1^*)_\theta \xrightarrow{i_{l,n}} \mathfrak{R}_n(X_0^*, X_1^*)_\theta \xrightarrow{\pi_{n,k}} \mathfrak{R}_k(X_0^*, X_1^*)_\theta \longrightarrow 0 \tag{3.13}$$

At the same time, dualizing the short exact sequence (cf. Appendix A.1.3)

$$0 \longrightarrow \mathfrak{R}_k(X_0, X_1)_\theta \xrightarrow{i_{k,n}} \mathfrak{R}_n(X_0, X_1)_\theta \xrightarrow{\pi_{n,l}} \mathfrak{R}_l(X_0, X_1)_\theta \longrightarrow 0 \tag{3.14}$$

we obtain the short exact sequence

$$0 \longrightarrow \mathfrak{R}_l(X_0, X_1)_\theta^* \xrightarrow{\pi_{n,l}^*} \mathfrak{R}_n(X_0, X_1)_\theta^* \xrightarrow{i_{k.n}^*} \mathfrak{R}_k(X_0, X_1)_\theta^* \longrightarrow 0 \tag{3.15}$$

The following result was first obtained for finite dimensional spaces by Rochberg himself [82, Th. 4.1] and later extended to the infinite dimensional case by Cabello, Castillo and Corrêa in [11, Section 5]. It shows that (3.13) and (3.15) are (in the terminology of [9]) isomorphically equivalent sequences:

Theorem 3.5.1. *[Duality for Rochberg spaces] Let (X_0, X_1) be a regular couple such that either X_0^* or X_1^* has the Radon-Nikodym property. Then $\mathfrak{R}_n(X_0, X_1)_\theta^*$ is isomorphic to $\mathfrak{R}_n(X_0^*, X_1^*)_\theta$ via the map $D_n : \mathfrak{R}_n(X_0^*, X_1^*)_\theta \to \mathfrak{R}_n(X_0, X_1)_\theta^*$ defined for $(x_{n-1}, \dots, x_0) \in (X_0 \cap X_1)^n$ by*

$$D_n(x_{n-1}^*, \dots, x_0^*)(x_{n-1}, \dots, x_0) = \sum_{i=0}^{n-1} \langle x_i, x_{n-j-1}^* \rangle_{X_0 \cap X_1}.$$

Proof. Fix $x^* = (x_{n-1}^*, \dots, x_0^*) \in (X_0^* \cap X_1^*)^n$ and $x = (x_{n-1}, \dots, x_0) \in (X_0 \cap X_1)^n$. By Lemma 3.2.2 we can pick $f \in \mathcal{C}_{00}(X_0, X_1)$ and $g \in \mathcal{C}_{00}(X_0^*, X_1^*)$ such that

- $(\Delta_{n-1}, \dots, \Delta_0)(f) = x$ and $\|f\|_{\mathcal{C}} \leq (1 + \varepsilon)\|x\|_n$;
- $(\Delta_{n-1}, \dots, \Delta_0)(g) = x^*$ and $\|g\|_{\mathcal{C}} \leq (1 + \varepsilon)\|x^*\|_n$.

By Lemma 2.4.1 and (2.16) the function $h(z) = \langle f(z), g(z) \rangle_{X_0 \cap X_1}$ belongs to $\mathcal{C}_{00}(\mathbb{C}, \mathbb{C})$ and satisfies

$$\Delta_n(h) = \sum_{k=0}^{n} \langle \Delta_k(f), \Delta_{n-k}(g) \rangle_{X_0 \cap X_1}$$

and that

$$|\Delta_n(h)| \leq \frac{\|f\|_{\mathcal{C}} \, \|g\|_{\mathcal{C}}}{\operatorname{dist}(\theta, \partial\mathbb{S})^{n-1}}.$$

Now note that $|D_n(x^*)(x)| = |\Delta_n(h)| \leq C(\theta,n)\|x\|_n \,\|x^*\|_n$; thus $D_n(x^*)(\cdot)$ defines a linear functional on $(X_0 \cap X_1)^n \subset \mathfrak{R}_n(X_0,X_1)_\theta$. Since $(X_0 \cap X_1)^n$ is dense in $\mathfrak{R}_n(X_0,X_1)_\theta$ by Lemma 3.2.2, we can extend $D_n(x^*)(\cdot)$ to a linear functional on the whole Rochberg space. Now, since $(X_0^* \cap X_1^*)^n$ is dense in $\mathfrak{R}_n(X_0^*,X_1^*)_\theta$, the operator D_n can be extended to a bounded operator on $\mathfrak{R}_n(X_0^*,X_1^*)_\theta$. To conclude the proof we just have to show that D_n is an isomorphism onto. This reduces to an induction and chasing argument on the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{R}_l(X_0^*,X_1^*)_\theta & \xrightarrow{i_{l,n}} & \mathfrak{R}_n(X_0^*,X_1^*)_\theta & \xrightarrow{\pi_{n,k}} & \mathfrak{R}_k(X_0^*,X_1^*)_\theta & \longrightarrow & 0 \\ & & {\scriptstyle D_l}\big\downarrow & & {\scriptstyle D_n}\big\downarrow & & {\scriptstyle D_k}\big\downarrow & & \\ 0 & \longrightarrow & \mathfrak{R}_l(X_0,X_1)_\theta^* & \xrightarrow{\pi_{n,l}^*} & \mathfrak{R}_n(X_0,X_1)_\theta^* & \xrightarrow{i_{k,n}^*} & \mathfrak{R}_k(X_0,X_1)_\theta^* & \longrightarrow & 0 \end{array}$$

Indeed, the diagram is commutative:

- Beginning at the upper left corner, given $(x_{l-1}^*,\dots,x_0^*) \in \mathfrak{R}_l(X_0^*,X_1^*)_\theta$ then

  $$D_n(i_{l.n}(x_{l-1}^*,\dots,x_0^*))(x_{n-1},\dots,x_0) = D_n\big((x_{l-1}^*,\dots,x_0^*,0,\overset{(k)}{\dots},0)\big)(x_{n-1},\dots,x_0).$$

  Going the other way around gives the same

  $$D_l(x_{l-1}^*,\dots,x_0^*)\big(\pi_{n,l}(x_{n-1},\dots,x_0)\big) = D_l(x_{l-1}^*,\dots,x_0^*)(x_{l-1},\dots,x_0).$$

- Starting now at the upper middle space, given $(x_{n-1}^*,\dots,x_0^*) \in \mathfrak{R}_n(X_0^*,X_1^*)_\theta$ one has

  $$D_k\big(\pi_{n,k}(x_{n-1}^*,\dots,x_0^*)\big)(x_{k-1},\dots,x_0) = D_k(x_{k-1}^*,\dots,x_0^*)(x_{k-1},\dots,x_0),$$

  while

  $$D_n(x_{n-1}^*,\dots,x_0^*)\big(i_{k,n}(x_{k-1},\dots,x_0)\big) = D_n(x_{n-1}^*,\dots,x_0^*)(x_{k-1},\dots,x_0,0,\overset{(l)}{\dots},0).$$

If $l=k=1$ then D_1 is just the isomorphism induced by the Duality Theorem (cf. Proposition 2.4.2), hence D_2 is injective and surjective by the 3-lemma, and thus an isomorphism. This implies by induction on l and k that D_n is an isomorphism for all $n \in \mathbb{N}$. $\square$

If we denote by $\Omega_{k,l}^\star$ the differential map of the dual couple and $\Omega_{k,l}^*$ the dual of $\Omega_{k,l}$ (i.e. the quasilinear map defining (3.15)) then the final part of the proof yields the commutator bound

$$\Big\|D_l\big(\Omega_{k,l}^\star(x_{k-1}^*,\dots,x_0^*)\big) - \Omega_{k,l}^*\big(D_k(x_{k-1}^*,\dots,x_0^*)\big)\Big\|_l \leq \|(x_{k-1}^*,\dots,x_0^*)\|_k.$$

We end this section noting that the operator D_n induces a bilinear pairing

$$\omega_n : \mathfrak{R}_n(X_0^*,X_1^*)_\theta \times \mathfrak{R}_n(X_0,X_1)_\theta \to \mathbb{C}$$

given by $\omega_n(x,y) = D_n(x)(y)$. We shall make extensive use of this pairing in the next chapter, so we include here a visual depiction of how it acts. If

x_{n-1}^*	x_{n-2}^*	$\cdots$	x_2^*	x_1^*	x_0^*

x_{n-1}	x_{n-2}	$\cdots$	x_2	x_1	x_0

represents the components of two vectors $x^* = (x^*_{n-1}, \dots, x^*_0) \in \mathfrak{R}_n(X_0^*, X_1^*)_\theta$ and $x = (x_{n-1}, \dots, x_0) \in \mathfrak{R}_n(X_0, X_1)_\theta$, then $\omega_n(x, y)$ is obtained by flipping the componentes of x to reach

$$\boxed{\begin{array}{cccccc} x^*_{n-1} & x^*_{n-2} & \cdots & x^*_2 & x^*_1 & x^*_0 \end{array}}$$
$$\boxed{\begin{array}{cccccc} x_0 & x_1 & \cdots & x_{n-3} & x_{n-2} & x_{n-1} \end{array}}$$

and operating coordinatewise and summing over all terms:

$$\boxed{\begin{array}{c} x^*_{n-1} \\ x_0 \end{array}} \qquad \boxed{\begin{array}{c} x^*_{n-2} \\ x_1 \end{array}} \quad \begin{array}{c} \cdots \\ \cdots \end{array} \quad \boxed{\begin{array}{c} x^*_2 \\ x_{n-3} \end{array}} \qquad \boxed{\begin{array}{c} x^*_1 \\ x_{n-2} \end{array}} \qquad \boxed{\begin{array}{c} x^*_0 \\ x_{n-1} \end{array}}$$
$$\boxed{\langle x_0, x^*_{n-1}\rangle \quad + \quad \langle x_1, x^*_{n-2}\rangle \quad \cdots \quad \langle x_{n-3}, x^*_2\rangle \quad + \quad \langle x_{n-2}, x^*_1\rangle \quad + \quad \langle x_{n-1}, x^*_0\rangle}$$

3.6 Stein's principle for Rochberg spaces

Recall from Section 3.3 that if T is interpolating, the Interpolation Principle gives that $T_n : \mathfrak{R}_n(X_0, X_1)_\theta \to \mathfrak{R}_n(X_0, X_1)_\theta$ is bounded. As we noted in (3.12), the induced operator T_n on $\mathfrak{R}_n(X_0, X_1)_\theta$ is just the initial $n \times n$ lower submatrix of (3.11). Keeping this idea, if we use an interpolating family of operators $(T_z)_z \subset \mathcal{L}(X_0 \cap X_1, X_0 + X_1)$, Stein's Interpolation Principle (cf. Theorem 2.5.2) gives that $T^{\mathcal{C}} : \mathcal{C}_{00}(X_0, X_1) \to \mathcal{C}(X_0, X_1)$ is bounded, where $T^{\mathcal{C}}(f)(z) = T_z(f(z))$.
In this case the family $(T_z)_z$ will also induce a bounded operator on $\mathfrak{R}_n(X_0, X_1)_\theta$. To see how it is defined we have to obtain the sequence of Taylor coefficients

$$\Big(\Delta_k\big(T^{\mathcal{C}}(f)\big)\Big)_{k\in\mathbb{N}} \quad \text{for all } k \in \mathbb{N} \text{ and } f \in \mathcal{C}_{00}(X_0, X_1). \tag{3.16}$$

Unlike the previous case, the matrix that gives (3.16) from the sequence $\big(\Delta_k(f)\big)_{k\in\mathbb{N}}$ is not diagonal, but *upper triangular*. Let us check this: since $(T_z)_z$ is analytic, it satisfies that $\langle T_z(x), y\rangle$ is complex valued analytic on $\mathbb{S}$ for all $x \in X_0 \cap X_1$ and $y \in (X_0 + X_1)^*$; this is equivalent to the fact that the map $z \in \mathbb{S} \mapsto T_z \in \mathcal{L}(X_0 \cap X_1, X_0 + X_1)$ is itself analytic (see [93, Chapter V]). Therefore, the derivatives $\frac{d^n}{dz^n} T_z$ exist and belong to $\mathcal{L}(X_0 \cap X_1, X_0 + X_1)$. These can be related to T_z via the Cauchy Integral Formula and by the expression (see [38, Chapter 4, §4]):

$$\Big\langle \frac{d^n}{dz^n} T_z(x), y \Big\rangle = \frac{d^n}{dz^n} \langle T_z(x), y\rangle \quad \text{for all } x \in X_0 \cap X_1,\ y \in (X_0 + X_1)^*.$$

Now, if $T_z(f) \in \mathcal{C}(X_0, X_1)$ is considered for a given $f \in \mathcal{C}_{00}(X_0, X_1)$, we can take its derivative at θ like any other function on Calderón space:

$$\Delta_1\big(T_z(f)\big) = \frac{dT_z}{dz}_{|\theta}(f(\theta)) + T_\theta(f'(\theta)) \in X_0 + X_1,$$

the equality being a consequence of the chain rule. Note that this is well defined since $f \in \mathcal{C}_{00}(X_0, X_1)$, and thus $f(\theta), f'(\theta) \in X_0 \cap X_1$. Hence, if we apply succesively the chain rule to $T_z(f)$ we obtain

$$\frac{d^k}{dz^k}\Big(T_z(f)\Big) = \sum_{i=0}^{k} \binom{k}{i} \frac{d^i}{dz^i} T_z\big(f^{(k-i)}\big).$$

Thus

$$\Delta_k\big(T_z(f)\big) = \sum_{i=0}^{k} \Big[\frac{1}{i!}\frac{d^i\,T_z}{dz^i}\Big]_{|\theta} \big(\Delta_{k-i}(f)\big). \tag{3.17}$$

We deduce

$$\Big(\Delta_k(T^{\mathcal{C}}(f))\Big)_{k\in\mathbb{N}} = \begin{pmatrix} \ddots & & \ddots & \ddots & & & \vdots \\ \ddots & T_\theta & \frac{1}{1!}\frac{dT_z}{dz}{}_{|\theta} & \frac{1}{2!}\frac{d^2\,T_z}{dz^2}{}_{|\theta} & \cdots & \cdots & \frac{1}{k!}\frac{d^k\,T_z}{dz^k}{}_{|\theta} \\ & 0 & T_\theta & \frac{1}{1!}\frac{dT_z}{dz}{}_{|\theta} & \frac{1}{2!}\frac{d^2\,T_z}{dz^2}{}_{|\theta} & \cdots & \frac{1}{(k-1)!}\frac{d^{k-1}\,T_z}{dz^{k-1}}{}_{|\theta} \\ & & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & \cdots & 0 & T_\theta & \frac{1}{1!}\frac{dT_z}{dz}{}_{|\theta} & \frac{1}{2!}\frac{d^2\,T_z}{dz^2}{}_{|\theta} \\ & & & \cdots & 0 & T_\theta & \frac{1}{1!}\frac{dT_z}{dz}{}_{|\theta} \\ & & & & \cdots & 0 & T_\theta \end{pmatrix} \begin{pmatrix} \vdots \\ \Delta_5(f) \\ \Delta_4(f) \\ \Delta_3(f) \\ \Delta_2(f) \\ \Delta_1(f) \\ \Delta_0(f) \end{pmatrix}$$

We denote by L_n^θ the corresponding $n \times n$ lower submatrix

$$L_n^\theta = \begin{pmatrix} T_\theta & \frac{1}{1!}\frac{dT_z}{dz}{}_{|\theta} & \frac{1}{2!}\frac{d^2\,T_z}{dz^2}{}_{|\theta} & \cdots & \frac{1}{(k-1)!}\frac{d^{k-1}\,T_z}{dz^{k-1}}{}_{|\theta} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & T_\theta & \frac{1}{1!}\frac{dT_z}{dz}{}_{|\theta} & \frac{1}{2!}\frac{d^2\,T_z}{dz^2}{}_{|\theta} \\ 0 & 0 & 0 & T_\theta & \frac{1}{1!}\frac{dT_z}{dz}{}_{|\theta} \\ 0 & 0 & 0 & 0 & T_\theta \end{pmatrix}$$

We are ready to state Stein's Interpolation Theorem for Rochberg spaces. This result was hinted first by Rochberg [82, pp. 257] and studied by Carro in [16] and by Castillo and Ferenczi in [25].

Theorem 3.6.1. *Let (X_0, X_1) be a Banach couple and $(T_z)_{z\in\overline{\mathbb{S}}} \subset \mathcal{L}\big(X_0 \cap X_1, X_0 + X_1\big)$ an interpolating family of operators. Then for any $0 < \theta < 1$ the linear map L_n^θ : $(X_0 \cap X_1)^n \to \mathfrak{R}_n(X_0, X_1)_\theta$ is bounded, and thus it can be extended to a bounded operator*

$$L_n^\theta : \mathfrak{R}_n(X_0, X_1)_\theta \to \mathfrak{R}_n(X_0, X_1)_\theta.$$

Proof. Given $x = (x_{n-1}, \dots, x_0) \in (X_0 \cap X_1)^n \subset \mathfrak{R}_n(X_0, X_1)_\theta$, there exist by Lemma 3.2.2 a function $f \in \mathcal{C}_{00}(X_0, X_1)$ such that $(\Delta_{n-1}, \dots, \Delta_0)(f) = x$ and $\|f\|_{\mathcal{C}} \leq (1+\varepsilon)\|x\|_n$. Then by Stein's Interpolation Principle (cf. Theorem 2.5.2) we have that $T^{\mathcal{C}}(f) \in \mathcal{C}(X_0, X_1)$. Thus by (3.17) we conclude that

$$\|L_n^\theta(x)\|_n = \|(\Delta_{n-1}, \dots, \Delta_0)(T^{\mathcal{C}}(f))\|_n \leq \|T^{\mathcal{C}}\|\|f\|_{\mathcal{C}} \leq (1+\varepsilon)\|T^{\mathcal{C}}\|\|x\|_n.$$

Since $(X_0 \cap X_1)^n$ is dense in $\mathfrak{R}_n(X_0, X_1)_\theta$ by Lemma 3.2.2, the result follows. $\square$

We apply Theorem 3.6.1 to Example 1 of Chapter 2. In this case, the family $(T_z)_{z \in \overline{\mathbb{S}}} \subset \mathcal{L}\big(c_{00}, \ell_\infty\big)$ given for each $x \in c_{00}$ by

$$T_z(x) = x \cdot |\mathfrak{u}|^{2z} = \sum_n x_n(sign(u_n)|u_n|^{2z})$$

satisfied that $f_x(\cdot) = T_\cdot(x) : \overline{\mathbb{S}} \to \ell_\infty \in \mathcal{C}(\ell_\infty, \ell_1)$ for every $x \in c_{00}$, i.e., $(T_z)_z$ satisfies the interpolating condition $(\mathfrak{I})$ for the couple (ℓ_∞, ℓ_1) on a dense subspace of ℓ_1, which is the intersection subspace for the couple (ℓ_∞, ℓ_1). This is enough to apply Theorem 3.6.1 (see the case $n = 1$ at the end of Section 2.5) and deduce that the corresponding triangular operator L_n^θ is bounded on $\mathfrak{R}_n(\ell_\infty, \ell_1)_\theta$. To obtain L_n^θ note that

$$\frac{d^k T_z}{dz^k}_{|\theta}(x) = 2^k\, x \cdot \Big(|\mathfrak{u}|^{2\theta} \log^{k-1}|\mathfrak{u}|\Big) = 2^k \sum_n x_n(sign(u_n)|u_n|^{2\theta} \log^{k-1}|u_n|).$$

Setting $\theta = 1/2$ one has that

$$\frac{1}{k!}\frac{d^k T_z}{dz^k}_{|z=1/2}(x) = \frac{2^k}{k!}\, x \cdot \mathfrak{u} \log^{k-1}|\mathfrak{u}| = \frac{2^k}{k!} \sum_n x_n u_n \log^{k-1}|u_n|.$$

Thus, it follows that the operator

$$T_n^{\mathfrak{u}} = \begin{pmatrix} \mathfrak{u} & \frac{1}{1!}\mathfrak{u}\log|\mathfrak{u}| & \frac{1}{2!}\mathfrak{u}\log^2|\mathfrak{u}| & \cdots & \frac{1}{(n-1)!}\mathfrak{u}\log^{n-1}|\mathfrak{u}| \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \mathfrak{u} & \frac{1}{1!}\mathfrak{u}\log|\mathfrak{u}| & \frac{1}{2!}\mathfrak{u}\log^2|\mathfrak{u}| \\ 0 & 0 & 0 & \mathfrak{u} & \frac{1}{1!}\mathfrak{u}\log|\mathfrak{u}| \\ 0 & 0 & 0 & 0 & \mathfrak{u} \end{pmatrix}$$

is bounded on $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$. We will refer to $T_n^{\mathfrak{u}}$ as a *block operator*. Moreover, since there exist a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{R}_l(X_0, X_1)_\theta & \longrightarrow & \mathfrak{R}_{l+k}(X_0, X_1)_\theta & \longrightarrow & \mathfrak{R}_k(X_0, X_1)_\theta & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle T_l^{\mathfrak{u}}} & & \Big\downarrow{\scriptstyle T_{l+k}^{\mathfrak{u}}} & & \Big\downarrow{\scriptstyle T_k^{\mathfrak{u}}} & & \\ 0 & \longrightarrow & \mathfrak{R}_l(X_0, X_1)_\theta & \longrightarrow & \mathfrak{R}_{l+k}(X_0, X_1)_\theta & \longrightarrow & \mathfrak{R}_k(X_0, X_1)_\theta & \longrightarrow & 0 \end{array}$$

and $T_1^{\mathfrak{u}} : \ell_2 \to \ell_2$ is an isometric embedding, we deduce that $T_n^{\mathfrak{u}}$ is an isomorphic embedding for all $n \in \mathbb{N}$.

Chapter 4

Rochberg spaces generated by the pair (ℓ_∞, ℓ_1)

In this chapter we focus in studying the Rochberg spaces defined by the Banach couple (ℓ_∞, ℓ_1). We obtained most of the results in [33] and [30], but our presentation here closely follows the structure of [31]. We will denote by $\mathfrak{R}_n$ the Rochberg space $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$. Thus $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are respectively the Hilbert space ℓ_2 and the Kalton-Peck space Z_2. If $n = l+k$ we have the short exact sequence

$$0 \longrightarrow \mathfrak{R}_l \xrightarrow{i_{l,n}} \mathfrak{R}_n \xrightarrow{\pi_{n,k}} \mathfrak{R}_k \longrightarrow 0 \tag{4.1}$$

generated by the differential map $\Omega_{k,l} : \mathfrak{R}_k \to \ell_\infty^l$. We are interested in the case $l = n-1$ and $k = 1$ since in this case the differential map $\Omega_{1,n-1} : \ell_2 \to \ell_\infty^{n-1}$ can be explicitly identified on a dense subspace of ℓ_2. Recall that $\Omega_{1,n-1}$ is defined for each $x \in \ell_2$ by

$$\Omega_{1,n-1}(x) = \big(\Delta_{n-1}, \dots, \Delta_1\big)(B_{1/2}(x)),$$

where $B_{1/2} : \ell_2 \to \mathcal{C}(\ell_\infty, \ell_1)$ is an homogeneous bounded selector for the quotient map Δ_0. As we explained in Section 2.2.1, the map (2.12) provides an example of selector for finitely supported vectors, and thus for $x \in c_{00}$ one has

$$\mathsf{KP}^k(x) := \Delta_k(B_{1/2}(x)) = \frac{2^k}{k!} x \log^k \frac{|x|}{\|x\|_2} = \Big(\frac{2^k}{k!} x_n \log^k \frac{|x_n|}{\|x\|_2}\Big)_{n\in\mathbb{N}}.$$

Hence the differential map $\Omega_{1,n-1} : c_{00} \to \ell_\infty$ is

$$\mathsf{KP}_{1,n-1}(x) = \Big(\mathsf{KP}^{n-1}(x), \dots, \mathsf{KP}^2(x), \mathsf{KP}(x)\Big) = \Big(\frac{2^k}{k!} x \log^k \frac{|x|}{\|x\|_2}\Big)_{k=0}^{n-1}.$$

Note that Kalton-Peck map is precisely $\mathsf{KP}_{1,1}$. Therefore, $\mathfrak{R}_n$ can be described as the completion of

$$\mathfrak{R}_n = \{(x_{n-1}, \dots, x_0) \in \ell_\infty^{n-1} \times c_{00} \colon (x_{n-1}, \dots, x_1) - \mathsf{KP}_{1,n-1}(x_0) \in \mathfrak{R}_{n-1}\}$$

endowed with the quasinorm

$$\|(x_{n-1}, \dots, x_0)\|_{1,n-1} = \|(x_{n-1}, \dots, x_0) - \mathsf{KP}_{1,n-1}(x_0)\|_{n-1} + \|x_0\|_{\ell_2}.$$

As we noted in Section 2.6.2 for the case of $Z_2 = \mathfrak{R}_2$, all results in this chapter that involve an explicit expression for $\mathsf{KP}_{1,n-1}$ must be proven first for $(c_{00}^{n-1} \oplus_{\mathsf{KP}_{1.n-1}} c_{00}, \|\cdot\|_n)$ and then extended by density to $\mathfrak{R}_n$.

Now we note a couple of basic facts concerning $\mathfrak{R}_n$:

(1) $\mathfrak{R}_n$ has all 3SP-properties (see Subsection A.2 for the definition) that ℓ_2 has. This is inmediate using (4.1) and induction n. In particular, $\mathfrak{R}_n$ is superreflexive and ℓ_2-saturated for every $n \in \mathbb{N}$ [27, 4.5 and 3.2.d].

(2) Let $(e_n)_{n\in\mathbb{N}}$ be the canonical basis of ℓ_2. Then the sequence $(u_l)_{l\in\mathbb{N}}$ defined by

$$u_{nm+k} = (0, \dots, \overset{(k)}{e_{m+1}}, \dots, 0) \quad \text{for every } 0 \leq k \leq n-1 \text{ and } m \geq 0, \tag{4.2}$$

is a basis for $\mathfrak{R}_n$. Indeed, we claim that the sequence of closed subspaces $(X_m)_{m\in\mathbb{N}}$ given by

$$X_m = \operatorname{span}\big\{(e_m, 0, \dots, 0), (0, e_m, 0, \dots, 0), \dots, (0, \dots, 0, e_m)\big\} \tag{4.3}$$

defines a symmetric finite dimensional decomposition of $\mathfrak{R}_n$. Since $\dim(X_m) = n$ for every $n \in \mathbb{N}$, we can select the basis $(u_{nm+k})_{k=0}^{n-1}$ for each X_m and then the spliced sequence $\big((u_{nm+k})_{k=0}^{n-1}\big)_{m\geq 0}$ forms a basis for $\mathfrak{R}_n$ [18, Prop. 6.5].

To prove the previous claim we have to show that, given any permutation $\sigma : \mathbb{N} \to \mathbb{N}$ and a sequence of signs $\varepsilon = (\varepsilon_i)_{i\in\mathbb{N}}$, the natural projection

$$\big(P_{\sigma,\varepsilon}\big)_n\big(x_{n-1}, \dots, x_0\big) = \Big(\sum_{k=0}^{\infty} \varepsilon_k x_{n-1}^k e_{\sigma(k)}, \dots, \sum_{k=0}^{\infty} \varepsilon_k x_0^k e_{\sigma(k)}\Big) \tag{4.4}$$

is uniformly bounded on $\mathfrak{R}_n$. This follows by the Commutator Theorem: since $(e_n)_n$ is a basis for ℓ_p with $1 \leq p < \infty$, we have that the natural projections

$$P_m\Big(\sum_{k=1}^{\infty} x_k e_k\Big) = \sum_{k=0}^{m} x_k e_k$$

associated to $(e_n)_n$ are uniformly bounded on ℓ_p for every $1 \leq p < \infty$. Then P_m is interpolating for the couple (ℓ_p^*, ℓ_p) where $p > 1$, and thus by Corollary 3.4.1 we conclude that

$$(P_m)_n\Big(\sum_{k=0}^{\infty} x_{n-1}^k e_k, \dots, \sum_{k=0}^{\infty} x_0^k e_k\Big) = \Big(\sum_{k=0}^{m} x_{n-1}^k e_k, \dots, \sum_{k=0}^{m} x_0^k e_k\Big)$$

is bounded on $\mathfrak{R}_n$. To show that (4.4) is bounded, we use the operators

$$P_{\sigma,\varepsilon}\Big(\sum_{k=0}^{\infty} x_n e_n\Big) = \sum_{k=0}^{\infty} \varepsilon_k x_k e_{\sigma(k)},$$

which are uniformly bounded on ℓ_p for all $1 \leq p < \infty$ since $(e_n)_n$ is symmetric.

For future use, we remark that the canonical sequence $(e_n)_{n\in\mathbb{N}}$ is a symmetric basis for $\operatorname{Dom}(\mathsf{KP}_{1,n-1})$. The proof is a consequence of (4.4) being bounded, since $\operatorname{Dom}(\mathsf{KP}_{1,n-1})$ is a natural subspace for $\mathfrak{R}_n$ (see also Corollary 3.3.2).

4.1 Basic sequences in $\mathfrak{R}_n$

It is a classical result that Hilbert spaces only admit one type of *seminormalized* basic sequence, meaning that any seminormalized basic sequence has a subsequence equivalent to the canonical basis of ℓ_2. Kalton and Peck [62, Th. 5.4] showed that every normalized basic sequence in Z_2 has a subsequence equivalent to either the canonical basis of ℓ_2 or of ℓ_f. This fact has deep implications on the structure of the space:

- It is the basic ingredient to show that Z_2 has no complemented subspace with unconditional basis.
- It provides a simple and automatic proof for the singularity of KP, which plays a key role in the proof of Proposition 1.1.2 (see also the Section 4.3).

This situation was pushed forward in [23], where the analogous result is proved for the third Rochberg space $\mathfrak{R}_3$. The same result is true in general: the n-th Rochberg space $\mathfrak{R}_n$ has exactly n types of basic sequences that coincide with the canonical bases of domain spaces $\mathrm{Dom}(\mathsf{KP}_{1,j})$ for $0 \le j \le n-1$. Note that we have a chain of continuous inclusions (see (3.7))

$$\mathrm{Dom}(\mathsf{KP}_{1,n-1}) \subset \mathrm{Dom}(\mathsf{KP}_{1,n-2}) \subset \cdots \subset \mathrm{Dom}(\mathsf{KP}_{1,1}) \subset \mathrm{Dom}(\mathsf{KP}_{1,0}) = \ell_2 \tag{4.5}$$

From here, the delicate part of the proof relies in two technical results, Lemma 4.1.1 and Proposition 4.1.1, that identify $\mathrm{Dom}(\mathsf{KP}_{1,j})$ as the Orlicz space generated by the function $f_j(t) = t^2 \log^{2j} t$. This is done by studying the asymptotic behaviour of $\|\sum x_n e_n\|_{\mathrm{Dom}(\mathsf{KP}_{1,j})} = \|(0,\dots,0,\sum x_n e_n)\|_{\mathfrak{R}_{j+1}}$.

Lemma 4.1.1. *Let $(u_n)_{n\in\mathbb{N}}$ be a normalized block basic sequence in ℓ_2. Then the sequence $w_n = (\mathsf{KP}_{1,n-1}(u_n), u_n)$ is equivalent to the usual basis of* $\mathrm{Dom}(\mathsf{KP}_{1,n-1})$.

Proof. We shall prove that if $(x_n)_{n\in\mathbb{N}}$ is a sequence of scalars in ℓ_2, i.e., $\sum_n x_n e_n \in \ell_2$, then

$$\sum_n x_n w_n \in \mathfrak{R}_n \quad \text{if and only if} \quad \sum_n x_n e_n \in \mathrm{Dom}\Omega_{1,n-1}.$$

Consider the block operator $T_n^{\mathfrak{u}}$ associated to the sequence $\mathfrak{u} = (u_n)_n$. Recall by Subsection 3.6 that this is an isomorphic embbeding defined by

$$T_n^{\mathfrak{u}} = \begin{pmatrix} \mathfrak{u} & \mathsf{KP}^1(\mathfrak{u}) & \mathsf{KP}^2(\mathfrak{u}) & \cdots & \mathsf{KP}^{n-1}(\mathfrak{u}) \\ 0 & \mathfrak{u} & \mathsf{KP}^1(\mathfrak{u}) & \mathsf{KP}^2(\mathfrak{u}) & \cdots \\ 0 & 0 & \mathfrak{u} & \mathsf{KP}^1(\mathfrak{u}) & \mathsf{KP}^2(\mathfrak{u}) \\ 0 & 0 & 0 & \mathfrak{u} & \mathsf{KP}^1(\mathfrak{u}) \\ 0 & 0 & 0 & 0 & \mathfrak{u} \end{pmatrix} \tag{4.6}$$

where $\mathsf{KP}^k(\mathfrak{u})$ denotes the linear map defined by $\mathsf{KP}^k(\mathfrak{u})(e_n) = \frac{2^k}{k!} u_k \log^k |u_k|$. In particular, for $k = 0$ this is the usual multiplication operator $\mathfrak{u}(e_n) = u_n$ on ℓ_2. Then note that

$$\sum_n x_n w_n = (\sum_n x_n \mathsf{KP}_{1,n-1}(u_n), \sum_n x_n u_n) = T_n^{\mathfrak{u}}(0\dots,0,\sum_n x_n e_n).$$

Indeed:

$$\begin{aligned}
T_U(0,\dots,0,\sum_n x_n e_n) &= \Big(\mathsf{KP}^{n-1}(\mathfrak{u})(\sum_n x_n e_n),\dots,\mathsf{KP}^1(\mathfrak{u})(\sum_n x_n e_n),\mathfrak{u}(\sum_n x_n e_n)\Big)\\
&= \Big(\sum_n x_n \mathsf{KP}^{n-1}(\mathfrak{u})(e_n),\dots,\sum_n x_n\mathsf{KP}^1(\mathfrak{u})(e_n),\sum_n x_n\mathfrak{u}(e_n)\Big)\\
&= \Big(\sum_n x_n \mathsf{KP}^{n-1}(u_n),\dots,\sum_n x_n\mathsf{KP}^1(u_n),\sum_n x_n u_n\Big)\\
&= \Big(\sum_n x_n \big(\mathsf{KP}^{n-1}(u_n),\dots,\mathsf{KP}^1(u_n)\big),\sum_n x_n u_n\Big)\\
&= \Big(\sum_n x_n \mathsf{KP}_{1,n-1}(u_n),\sum_n x_n u_n\Big).
\end{aligned}$$

Thus, taking into account that block operators are isomorphic embeddings we deduce that

$$\|\sum_n x_n w_n\| = \|T_U(0,\dots,0,\sum_n x_n e_n)\| \sim \|(0,\dots,0,\sum_n x_n e_n)\|_{\mathfrak{R}_n}.$$

This means that $\sum_n x_n w_n$ converges if and only if $(0,\dots,0,\sum_n x_n e_n)$ converges in $\mathfrak{R}_n$; equivalently, if and only if $\sum_n x_n e_n$ converges in $\mathrm{Dom}(\mathsf{KP}_{1,n-1})$. □

If, in the statement of Lemma 4.1.1, we consider that $(u_n)_n$ is just a seminormalized block basic sequence, then the associated block operator $T_n^{\mathfrak{u}}$ defined as in (4.6) is still an isomorphic embedding. Indeed, since $(u_n)_n$ is seminormalized, the operator

$$\tau_{\|u\|}(x) = \sum_n \|u_n\| x_n$$

is invertible and interpolating for the couple (ℓ_∞,ℓ_1). By the Commutator Theorem $(\tau_{\|u\|})_n : \mathfrak{R}_n \to \mathfrak{R}_n$ is bounded. Moreover, if we denote by $\mathfrak{v}$ the normalized sequence $v_n = \frac{u_n}{\|u_n\|}$, then $T_n^{\mathfrak{u}} = T_n^{\mathfrak{v}} \circ (\tau_{\|u\|})_n$. Thus $T_n^{\mathfrak{u}}$ is an isomorphic embedding, being the composition of two of them. This gives that the sequence $(\mathsf{KP}_{1,n-1}(u_n),u_n)_m$ is also equivalent to the canonical basis of $\mathrm{Dom}(\mathsf{KP}_{1,n-1})$.

We now show that the domain spaces $\mathrm{Dom}(\mathsf{KP}_{1,j})$, for $0 \le j \le n-1$, are Orlicz sequence spaces.

Proposition 4.1.1. *For $0 \le j \le n-1$ the space $\mathrm{Dom}(\mathsf{KP}_{1,j})$ is isomorphic to the Orlicz space ℓ_{f_j} generated by the Orlicz function*

$$f_j(t) = t^2 \log^{2j} t$$

Proof. Let us fix some notation and ideas to ease the process:

- $\mathsf{KP}_{1,n-1}(x) = x\left[\frac{2^j}{j!}\log^j x\right]_{j=1}^{n-1}$ for normalized x.
- To estimate $\|(0,\dots,0,x)\|_n$ we need to perform $n-1$ steps until arriving to $\|\cdot\|_{\ell_2}$. We will simplify the later norm to plain $\|\cdot\|$.

Step 1.

$$\bigstar = \|u\mathsf{KP}_{1,n-1}(x) - \mathsf{KP}_{1,n-1}(xu)\|_{n-1} = \left\| xu\left[\frac{2^j}{j!}\log^j x\right]_{j=1}^{n-1} - xu\left[\frac{2^j}{j!}\log^j xu\right]_{j=1}^{n-1}\right\|_{n-1}$$

Step 2.

$$\left\|x\left[\frac{2^j}{j!}\log^j x\right]_{j=1}^{n-1}\right\|_{n-1} = \left\|x\left[\frac{2^j}{j!}\log^j x\right]_{j=2}^{n-1} - \mathsf{KP}_{1,n-2}(\alpha_1 x\log x)\right\|_{n-2}$$

where α_1 is the first coefficient of $\left[\frac{2^j}{j!}\log^j x\right]_{j=1}^{n-1}$ (namely, 2). Thus, adding and subtracting $\alpha_1 \log x \mathsf{KP}_{1,n-2}(x)$ one gets

$$\begin{aligned} \bigstar &\leq \left\|x\left[\frac{2^j}{j!}\log^j x\right]_{j=2}^{n-1} - \alpha_1 \log x \mathsf{KP}_{1,n-2}(x)\right\|_{n-2} \\ &+ \|\alpha_1 \log x \mathsf{KP}_{1,n-2}(x) - \mathsf{KP}_{1,n-2}(\alpha_1 x\log x)\|_{n-2} \\ &\leq \left\|x\left[\frac{2^j}{j!}\log^j x\right]_{j=2}^{n-1} - \alpha_1 \log x \mathsf{KP}_{1,n-2}(x)\right\|_{n-2} + \alpha_1 \|\log x\|_\infty \|x\|. \end{aligned}$$

but also

$$\bigstar \geq \left\|x\left[\frac{2^j}{j!}\log^j x\right]_{j=2}^{n-1} - \alpha_1 \log x \mathsf{KP}_{1,n-2}(x)\right\|_{n-2} - \alpha_1 \|\log x\|_\infty \|x\|.$$

In other words, up to an order $\|\log x\|$ term,

$$\bigstar \sim \left\|x\left[\frac{2^j}{j!}\log^j x\right]_{j=2}^{n-1} - \alpha_1 \log x \mathsf{KP}_{1,n-2}(x)\right\|_{n-2}$$

Now

$$\begin{aligned} \bigstar &= \left\|x\left[\frac{2^j}{j!}\log^j x\right]_{j=2}^{n-1} - x\alpha_1 \log x\left[\frac{2^j}{j!}\log^j x\right]_{j=1}^{n-2}\right\|_{n-2} \\ &= \left\|x \quad \left[\frac{2^j}{j!}\log^j x\right]_{j=2}^{n-1} - \alpha_1\left[\frac{2^j}{j!}\log^{j+1} x\right]_{j=1}^{n-2}\right\|_{n-2} \\ &= \left\|x \quad \left[\frac{2^{j+1}}{j+1!}\log^{j+1} x\right]_{j=1}^{n-2} - \alpha_1\left[\frac{2^j}{j!}\log^{j+1} x\right]_{j=1}^{n-2}\right\|_{n-2} \\ &= \left\|x \quad \left[\left(\frac{2^{j+1}}{j+1!} - \alpha_1\frac{2^j}{j!}\right)\log^{j+1} x\right]_{j=1}^{n-2}\right\|_{n-2} \end{aligned}$$

with first coefficient $\alpha_2 = \frac{2^2}{2!} - \alpha_1 \frac{2}{1!}$.

Step 3. Keeping up with the same ideas

$$\begin{aligned} \bigstar &= \left\|x \quad \left[\left(\frac{2^{j+1}}{j+1!} - \alpha_1\frac{2^j}{j!}\right)\log^{j+1} x\right]_{j=1}^{n-2}\right\|_{n-2} \\ &\sim \left\|x \quad \left[\left(\frac{2^{j+1}}{j+1!} - \alpha_1\frac{2^j}{j!}\right)\log^{j+1} x\right]_{j=2}^{n-2} - \alpha_2 \log^2 x\left[\frac{2^j}{j!}\log^j x\right]_{j=1}^{n-3}\right\|_{n-3} \end{aligned}$$

up to a $\alpha_2\|\log^2 x\|_\infty$ term. Thus

$$\begin{aligned}
\bigstar &= \left\| x \quad \left[\left(\frac{2^{j+1}}{j+1!} - \alpha_1 \frac{2^j}{j!}\right) \log^{j+1} x\right]_{j=2}^{n-2} - \alpha_2 \left[\frac{2^j}{j!} \log^{j+2} x\right]_{j=1}^{n-3} \right\|_{n-3} \\
&= \left\| x \quad \left[\left(\frac{2^{j+2}}{j+2!} - \alpha_1 \frac{2^{j+1}}{j+1!}\right) \log^{j+2} x - \alpha_2 \frac{2^j}{j!} \log^{j+2} x\right]_{j=1}^{n-3} \right\|_{n-3} \\
&= \left\| x \quad \left[\left(\frac{2^{j+2}}{j+2!} - \alpha_1 \frac{2^{j+1}}{j+1!} - \alpha_2 \frac{2^j}{j!}\right) \log^{j+2} x\right]_{j=1}^{n-3} \right\|_{n-3}
\end{aligned}$$

So the pattern seems clear now: after $n-1$ steps,

$$\bigstar = \left\| x \quad \left(\frac{2^{n-1}}{n-1!} - \alpha_1 \frac{2^{n-2}}{n-2!} - \cdots - \alpha_{n-1} \frac{2}{1!}\right) \log^{n-1} x \right\|$$

and therefore $x \log^{n-1} x \in \ell_2$ means $\sum x^2 \log^{2(n-1)} x$ finite, namely, the element x belongs to the Orlicz space $\ell_{f_{n-1}}$ defined by the Orlicz function f_{n-1}. $\square$

The combination of Lemma 4.1.1 and Proposition 4.1.1 yields, by an induction argument, that there exist n types of (normalized) basic sequences in $\mathfrak{R}_n$.

Theorem 4.1.1. *Every normalized basic sequence in $\mathfrak{R}_n$ admits a subsequence equivalent to the basis of one of the spaces ℓ_{f_j}, $1 \le j \le n$.*

Proof. Let $w_k = (x_{n-1}^k, \ldots, x_0^k)$ be a normalized basic sequence in $\mathfrak{R}_n$. If $\|x_0^k\| \longrightarrow 0$ then

$$\|(x_{n-1}^k, \ldots, x_0^k) - \big((x_{n-1}^k, \ldots, x_1^k) - \mathsf{KP}_{1,n-1}(x_0^k), 0\big)\|_n = \|(\mathsf{KP}_{1,n-1}(x_0^k), x_0^k)\|_n \to 0.$$

Thus, passing to a subsequence and using a standard perturbation of bases argument, we can assume that w_k is a basic sequence in $\mathfrak{R}_{n-1}$ that therefore admits a subsequence equivalent to the basis of one of the spaces ℓ_{f_j}, $0 \le j \le n-2$ by induction hypothesis and [62, Theorem 5.4].

If $\|x_0^n\| \ge \varepsilon > 0$ then we can assume after perturbation that there is a block basic sequence $(u_n)_{n\in\mathbb{N}}$ in ℓ_2 so that $\sum \|x_0^k - u_k\| < \infty$. Since

$$\begin{aligned}
(x_{n-1}^k, \ldots, x_0^k) &= (x_{n-1}^k, \ldots, x_0^k) - (\mathsf{KP}_{1,n-1}(u_k), u_k) + (\mathsf{KP}_{1,n-1}(u_k), u_k) \\
&= \big((x_{n-1}^k, \ldots, x_1^k) - \mathsf{KP}_{1,n-1}(u_k), x_0^k - u_k\big) + (\mathsf{KP}_{1,n-1}(u_k), u_k).
\end{aligned}$$

and $x_0^k - u_k \to 0$, we can assume (by the first part of the proof) that $\big((x_{n-1}^k, \ldots, x_1^k) - \mathsf{KP}_{1,n-1}(u_k), x_0^k - u_k\big)_k$ admits a subsequence equivalent to the canonical basis of ℓ_{f_j} for some $0 \le j \le n-2$. On the other hand, by Lemma 4.1.1 and Proposition 4.1.1 the sequence $\big(\mathsf{KP}_{1,n-1}(u_k), u_k\big)_k$ is equivalent to the canonical basis of $\ell_{f_{n-1}}$.

We conclude that, up to a subsequence, $(w_k)_k$ is equivalent to the canonical basis of $\ell_{f_{n-1}}$. Indeed, if $\sum t_k\big(\mathsf{KP}_{1,n-1}(u_k), u_k\big)$ converges, then $\sum t_k\big((x_{n-1}^k, \ldots, x_1^k) - \mathsf{KP}_{1,n-1}(u_k), x_0^k - u_k\big)$ converges because $\ell_{f_{n-1}} \subset \ell_{f_{n-2}} \subset \cdots \subset \ell_{f_1} \subset \ell_2$. Thus, passing to a subsequence if necessary, the sum

$$\sum t_k(x_{n-1}^k, \ldots, x_0^k) = \sum t_k\big((x_{n-1}^k, \ldots, x_1^k) - \mathsf{KP}_{1,n-1}(u_k), x_0^k - u_k\big) + \sum t_k\big(\mathsf{KP}_{1,n-1}(u_k), u_k\big)$$

converges. Conversely, if $\sum t_k w_k$ converges, then $\sum t_k\big(\mathsf{KP}_{1,n-1}(u_k), u_k\big)$ converges. If not, then

$$\begin{aligned}\Big\|\sum t_k w_k\Big\|_{\mathfrak{R}_n} &= \Big\|\sum t_k\big((x^k_{n-1},\dots,x^k_1) - \mathsf{KP}_{1,n-1}(u_k), x^k_0 - u_k\big) + \sum t_k\big(\mathsf{KP}_{1,n-1}(u_k), u_k\big)\Big\|_{\mathfrak{R}_n}\\ &\geq \Big\|\sum t_k\big(\mathsf{KP}_{1,n-1}(u_k), u_k\big)\Big\|_{\mathfrak{R}_n} - \Big\|\sum t_k\big((x^k_{n-1},\dots,x^k_1) - \mathsf{KP}_{1,n-1}(u_k), x^k_0 - u_k\big)\Big\|_{\mathfrak{R}_n}\\ &\sim \Big\|\sum t_k e_k\Big\|_{\ell_{f_{n-1}}} - \Big\|\sum t_k e_k\Big\|_{\ell_{f_j}},\end{aligned}$$

and taking into account that f_{n-1} and f_j are not equivalent Orlicz functions, it follows that $\sum_k t_k w_k$ does not converge, a contradiction. □

Observe that the proof also works if we assume that the basic sequence $(w_k)_{k\in\mathbb{N}}$ is seminormalized instead of normalized. Just note that the case $n = 2$ proved by Kalton and Peck in [62, Th. 5.4] is also true for seminormalized sequences (cf. [63, Lemma 3]), since every seminormalized basic sequence in ℓ_2 is equivalent to the canonical basis [3, Remark 2.1.2]. Up from here we would use Lemma 4.1.1 for the case in which $(u_n)_n$ is seminormalized.

The existence of block operators $T^{\mathfrak{u}}_n$ is the key to prove the preceding result. If X is a twisted Hilbert space generated by a Banach couple of Orlicz sequence spaces and $(u_k)_k \subset \ell_2$ is a sequence of normalized blocks, we know no method to show that the sequence $\big(\Omega_{1,1}(u_k), u_k\big)_k$ is equivalent to the usual basis of $\mathrm{Dom}(\Omega_{1,1})$ out of the case of ℓ_p spaces.

Corollary 4.1.1. *Let $(x_n)_n \subset \mathfrak{R}_n$ be a seminormalized basic sequence in $\mathfrak{R}_n$ such that $\|\pi_{n,1}x_n\| \geq \varepsilon > 0$. Then $(x_n)_n$ admits a subsequence equivalent to the canonical basis of $\ell_{f_{n-1}}$.*

Proof. The proof of Theorem 4.1.1 gives that $(x_n)_n$ is equivalent, up to a subsequence, to some basic sequence of the form $(\mathsf{KP}_{1,n-1}(u_n), u_n)_n$, where $(u_n)_n$ is a seminormalized block basic sequence in ℓ_2. Now Lemma 4.1.1 and Proposition 4.1.1 finish the proof. □

Corollary 4.1.2. *Suppose that $(z_m)_m \subset \mathfrak{R}_n$ is a seminormalized basic sequence equivalent to the canonical basis of ℓ_2. Then $\|\pi_{n.n-1}z_m\| \to 0$.*

Proof. We are going to show that if the thesis does not hold, then $(z_m)_m$ admits a subsequence equivalent to some basic sequence $(v_1,\dots,v_k)$ in $\mathfrak{R}_k$ where $\|v_k\| \geq \varepsilon > 0$. This implies by Corollary 4.1.1 that, up to a subsequence, it is equivalent to the canonical basis of $\ell_{f_{k-1}}$. This is absurd, since ℓ_2 is not ismorphic to ℓ_{f_j} for any $j \in \mathbb{N}$.

To that end, let $(z_m)_m = (z_1, z_2, \dots, z_n)_m$ be a seminormalized basic sequence in $\mathfrak{R}_n$ equivalent to the canonical basis of ℓ_2 such that

$$\|\pi_{n,n-1}z_m\| = \|(z_2,\dots,z_n)_m\| \geq \varepsilon > 0.$$

Since $\|(z_2,\dots,z_n)_m\|_{n-1} = \|(z_2,\dots,z_{n-1}) - \mathsf{KP}_{1,n-2}(z_n)\|_{n-2} + \|z_n\|$, there are two possibilities:

- If $\|z_n\| \geq \varepsilon_0 > 0$, then $\|\pi_{n,1}z_m\| > 0$ and by Corollary 4.1.1 we deduce that $(z_n)_n$ has a subsequence equivalent to the canonical basis of $\ell_{f_{n-1}}$, which is absurd since ℓ_2 is not isomorphic to $\ell_{f_{n-1}}$.

- If $\|z_n\| \to 0$ then

$$\|(z_1,\dots,z_n)_m - \big((z_1,\dots,z_{n-1})_m - \mathsf{KP}_{1,n-1}(z_n), 0\big)\| = \|(\mathsf{KP}_{1,n-1}(z_n), z_n)\| = \|z_n\| \to 0.$$

Thus, using a perturbation argument and passing to a subsequence we can assume that $(z_1,\dots,z_n)_m$ is a basic sequence in $\mathfrak{R}_{n-1}$. More precisely, there exist a basic sequence $(v_m)_m = (v_1,\dots,v_{n-1})_m \subset \mathfrak{R}_{n-1}$ such that

$$\|(z_1,\dots,z_n)_m - (v_1,\dots,v_{n-1},0)_m\| \le 2^{-m}$$

and both sequences are equivalent.

Now, we have that

$$\begin{aligned}\|\pi_{n,n-1}(v_1,\dots,v_{n-1},0)_m\| &\ge \|\pi_{n,n-1}(z_1,\dots,z_n)_m\| - \|\pi_{n,n-1}\big((z_1,\dots,z_n)_m - (v_1,\dots,v_{n-1},0)_m\big)\|\\ &\ge \|\pi_{n,n-1}(z_1,\dots,z_n)_m\| - \|(z_1,\dots,z_n)_m - (v_1,\dots,v_{n-1},0)_m\|\\ &\ge \varepsilon - 2^{-m}.\end{aligned}$$

For m sufficiently big, we deduce (passing to a subsequence) that

$$\|\pi_{n,n-1}(v_1,\dots,v_{n-1},0)_m\|_{n-1} \ge \varepsilon_1 > 0.$$

This is the same to say as $\|\pi_{n-1,n-2}(v_1,\dots,v_{n-1})\|_{n-1} \ge \varepsilon_1 > 0$. Hence we can apply the same argument inductively to reach, in the worst case scenario to a seminormalized basic sequence $(u_m)_m = (u_1,u_2)_m$ in Z_2, equivalent to a subsequence of $(z_m)_m$, such that $\|u_2\| > 0$. Thus, it admits a subsequence equivalent to the canonical basis of ℓ_f, which is impossible since ℓ_2 is not isomorphic to ℓ_f. □

Corollary 4.1.3. *For any $n \in \mathbb{N}$, the natural quotient map $\pi_{n,n-1} : \mathfrak{R}_n \to \mathfrak{R}_{n-1}$ is strictly singular.*

Proof. Assume that $\pi_{n,n-1}$ is an isomorphism on some infinite dimensional subspace $X \subset \mathfrak{R}_n$. Since $\mathfrak{R}_n$ is ℓ_2-saturared, X contains an infinite dimensional subspace $Y \subset X$ isomorphic to ℓ_2, and generated by a normalized basis $(y_n)_n \subset Y$ equivalent to the canonical basis of ℓ_2. Since $\pi_{n,n-1}|_Y$ is an isomorphism, we have that $\|\pi_{n,n-1} y_n\| \ge \varepsilon > 0$. This contradicts Corollary 4.1.2. □

Since $\pi_{n,k} = \pi_{k+1,k} \circ \cdots \circ \pi_{n-1,n-2} \circ \pi_{n,n-1}$, by the ideal property of $\mathcal{SS}$ and Corollary 4.1.3 we deduce that:

Corollary 4.1.4. *The natural quotient map $\pi_{n,k} : \mathfrak{R}_n \to \mathfrak{R}_k$ is strictly singular for every $n,k \in \mathbb{N}$ with $n > k$. Equivalently, $\mathsf{KP}_{k,l}$ is singular for every $k,l \in \mathbb{N}$.*

This would also follow by the combination of Proposition 1.1.1 and Proposition 3.2.1. Here we provide a different proof which is closer in spirit to that of Kalton and Peck [62, Th. 6.4]. Using Corollary 4.1.4 and Proposition A.1.4 we deduce:

Corollary 4.1.5. *Given any Banach space X and $1 \le k \le n-1$, a bounded operator $T : \mathfrak{R}_n \to X$ is strictly singular if and only if $T \circ i_{k,n} : \mathfrak{R}_k \to \mathfrak{R}_n$ is stricltly singular.*

We include here an important consequence of Theorem 4.1.1:

Proposition 4.1.2. *$\mathfrak{R}_n$ does not embed in $\mathfrak{R}_m$ whenever $n > m$.*

Proof. Assume that $n > m$ and that $T : \mathfrak{R}_n \hookrightarrow \mathfrak{R}_m$ defines an isomorphic embedding. If $(u_k^j)_{k\in\mathbb{N}}$ denotes the canonical basis of ℓ_{f_j} then $(Tu_k^{n-1})_k$ is a seminormalized basic sequence in $\mathfrak{R}_m$ equivalent to the canonical basis of $\ell_{f_{n-1}}$. From Theorem 4.1.1 we deduce that $(Tu_k^{n-1})_k$ admits a subsequence equivalent to the canonical basis ℓ_{f_j} for some $0 \leq j \leq m-1 < n$. Since $(u_k^{n-1})_k$ is symmetric, it implies that both $\ell_{f_{n-1}}$ and ℓ_{f_j} are isomorphic, which is a contradiction. □

Proposition 4.1.2 was proved in [11, Section 4] estimating the type-2 constants in the form $a_{m,2}(\mathfrak{R}_n) \sim \log^{n-1} m$. Our proof is conceptually similar, since its main point is that one cannot embbed the domain space $\ell_{f_{n-1}}$ into $\mathfrak{R}_j$, because that forces it to coincide, up to equivalence of norms, with ℓ_{f_j} for some $j < n-1$, which can not occur since the assymptotic growth of their Orlicz functions is different.

4.2 Duality for $\mathfrak{R}_n$

It is a direct consequence of Lemma 2.1.4 that $\mathcal{C}_0(\ell_\infty, \ell_1) = \mathcal{C}_0(c_0, \ell_1)$. Hence, given any $0 < \theta < 1$ one has $\mathfrak{R}_n^0(\ell_\infty, \ell_1)_\theta = \mathfrak{R}_n^0(c_0, \ell_1)_\theta$ isometrically. By Lemma 3.2.1 it follows that $\mathfrak{R}_n^0(c_0, \ell_1)_\theta = \mathfrak{R}_n(c_0, \ell_1)_\theta$ and $\mathfrak{R}_n^0(\ell_\infty, \ell_1) = \mathfrak{R}_n(\ell_\infty, \ell_1)$ with equivalent norms. By Theorem 3.5.1 we deduce that the linear map $\widehat{D_n} : \mathfrak{R}_n(\ell_1, \ell_\infty)_{1/2} \to \mathfrak{R}_n(\ell_\infty, \ell_1)^*_{1/2}$ given for any $(y_{n-1}, \dots, y_0) \in (\ell_1)^n$ and $(x_{n-1}, \dots, x_0) \in \mathfrak{R}_n(\ell_1, \ell_\infty)_{1/2} \subset (\ell_\infty)^n$ by

$$\widehat{D_n}(x_{n-1}, \dots, x_0)(y_{n-1}, \dots, y_0) = \sum_{i=0}^{n-1} \langle x_i, y_{n-i-1} \rangle_{\ell_1}$$

is an isomorphism. Now note that $\mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2}$ is isometrically isomorphic to $\mathfrak{R}_n(\ell_1, \ell_\infty)_{1/2}$ via the map $u_n : \mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2} \to \mathfrak{R}_n(\ell_1, \ell_\infty)_{1/2}$ given by

$$u_n\big((x_{n-1}, \dots, x_0)\big) = \big((-1)^{n-1} x_{n-1}, \dots, -x_1, x_0\big)$$

as a consequence of the fact that the map $f \in \mathcal{C}(X_0, X_1) \mapsto \overline{f} \in \mathcal{C}(X_0, X_1)$ given by $\overline{f}(z) = f(1-z)$ defines an isometric isomorphism for any Banach couple (X_0, X_1) (see [80, pp. 305]).

If we denote by $D_n = \widehat{D_n} u_n : \mathfrak{R}_n \to \mathfrak{R}_n^*$ we deduce that $\mathfrak{R}_n$ is isomorphic to its dual:

Theorem 4.2.1. *The map $D_n : \mathfrak{R}_n \to \mathfrak{R}_n^*$ given for any $(x_{n-1}, \dots, x_0), (y_{n-1}, \dots, y_0) \in \mathfrak{R}_n$ by*

$$D_n(x_{n-1}, \dots, x_0)(y_{n-1}, \dots, y_0) = \sum_{i=0}^{n-1} (-1)^i \langle x_i, y_{n-i-1} \rangle. \tag{4.7}$$

defines an isomorphism between $\mathfrak{R}_n$ and its dual.

Since $(y_{n-1}, \dots, y_0) \in \mathfrak{R}_n$, it follows that $y_{n-i-1} \in \operatorname{Ran}(\mathsf{KP}_{n-i-1,1})$, and thus $\langle x_i, y_{n-i-1} \rangle$ has to be understood in the sense of (2.22): as the limit $\lim_k \langle x_i, y_{n-i-1}^k \rangle_{\ell_1}$ where $(y_{n-i-1}^k)_k \subset \ell_1$ converges to y_{n-i-1} in $\|\cdot\|_{\operatorname{Ran}\mathsf{KP}_{n-i-1,i}}$. From now on, we will use this convention without further mention. We have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{R}_l(\ell_\infty, \ell_1)_{1/2} & \xrightarrow{i_{l,n}} & \mathfrak{R}_n(\ell_\infty, \ell_1)_{1/2} & \xrightarrow{\pi_{n,k}} & \mathfrak{R}_k(\ell_\infty, \ell_1)_{1/2} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle (-1)^k u_l} & & \Big\downarrow{\scriptstyle u_n} & & \Big\downarrow{\scriptstyle u_k} & & \\
0 & \longrightarrow & \mathfrak{R}_l(\ell_1, \ell_\infty)_{1/2} & \xrightarrow{i_{l,n}} & \mathfrak{R}_n(\ell_1, \ell_\infty)_{1/2} & \xrightarrow{\pi_{n,k}} & \mathfrak{R}_k(\ell_1, \ell_\infty)_{1/2} & \longrightarrow & 0
\end{array}$$

Combining this last diagram with the one appearing during the proof of Theorem 3.5.1 we obtain:

Theorem 4.2.2. *For any $l + k = n$ there exist a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{R}_l \arrow[r, "i_{l,n}"] \arrow[d, "(-1)^k D_l"'] & \mathfrak{R}_n \arrow[r, "\pi_{n,k}"] \arrow[d, "D_n"'] & \mathfrak{R}_k \arrow[r] \arrow[d, "D_k"'] & 0 \\ 0 \arrow[r] & \mathfrak{R}_l^* \arrow[r, "\pi_{n,l}^*"] & \mathfrak{R}_n^* \arrow[r, "i_{k,n}^*"] & \mathfrak{R}_k^* \arrow[r] & 0 \end{tikzcd}$$

Theorem 4.2.2 shows that

$$(-1)^k D_l \mathsf{KP}_{k,l} \equiv \mathsf{KP}_{l,k}^* D_k. \tag{4.8}$$

A consequence of this is:

Corollary 4.2.1. *For any $k, l \in \mathbb{N}$ we have that*

(i) The spaces $\mathrm{Dom}(\mathsf{KP}_{l,k})^*$ *and* $\mathrm{Ran}(\mathsf{KP}_{k,l})$ *are isomorphic.*

(ii) The spaces $\mathrm{Ran}(\mathsf{KP}_{l,k})^*$ *and* $\mathrm{Dom}(\mathsf{KP}_{k,l})$ *are isomorphic.*

Proof. First note that $(X_0 \cap X_1)^k \subset \mathrm{Dom}(\Omega_{k,l})$ for every $k \in \mathbb{N}$. Indeed, given $(x_1, \ldots, x_k) \in (X_0 \cap X_1)^k$ we deduce using constant functions that $x_i \in \mathrm{Dom}(\Omega_{1,i})$. Equivalently, that $(0, \ldots, 0, x_i) \in \mathfrak{R}_{l+i}$. Then using the embeddings $i_{l+i,l+k}$ it follows that

$$y_i = (0, \overset{(l+i-1)}{\ldots}, 0, x_i, 0, \overset{(k-i)}{\ldots}, 0) \in \mathfrak{R}_{l+k}.$$

Hence $\sum_{i=1}^k y_i = (0, \ldots, 0, x_1, x_2, \ldots, x_k) \in \mathfrak{R}_{l+k}$, i.e., $(x_1, \ldots, x_k) \in \mathrm{Dom}(\Omega_{k,l})$.
By Lemma 3.2.2 we deduce that $\mathrm{Dom}(\Omega_{k,l})$ is dense in $\mathfrak{R}_k$, and thus $\Omega_{k,l} : \mathfrak{R}_k \to (\ell_\infty)^l$ is a quasilinear map with dense domain. By Proposition A.1.5 we have that

$$\mathrm{Ran}(\mathsf{KP}_{l,k}^*) = \mathrm{Dom}(\mathsf{KP}_{l,k})^* \quad \text{and} \quad \mathrm{Dom}(\mathsf{KP}_{l,k}^*) = \mathrm{Ran}(\mathsf{KP}_{l,k})^*. \tag{4.9}$$

On the other hand, equation (4.8) implies by Lemma A.1.1 that

$$\mathrm{Dom}((-1)^k D_l \mathsf{KP}_{k,l}) = \mathrm{Dom}(\mathsf{KP}_{l,k}^* D_k) \quad \text{and} \quad \mathrm{Ran}((-1)^k D_l \mathsf{KP}_{k,l}) = \mathrm{Ran}(\mathsf{KP}_{l,k}^* D_k). \tag{4.10}$$

Since D_k and $(-1)^k D_l$ are both isomorphisms, it follows that:

(a_1) $\mathrm{Dom}((-1)^k D_l \mathsf{KP}_{k,l})$ and $\mathrm{Dom}(\mathsf{KP}_{k,l})$ coincide up to equivalence of norms;

(a_2) $\mathrm{Ran}((-1)^k D_l \mathsf{KP}_{k,l})$ and $\mathrm{Ran}(\mathsf{KP}_{k,l})$ are isomorphic;

(a_3) $\mathrm{Ran}(\mathsf{KP}_{l,k}^* D_k)$ and $\mathrm{Ran}(\mathsf{KP}_{l,k}^*)$ coincide up to equivalence of norms;

(a_4) $\mathrm{Dom}(\mathsf{KP}_{l,k}^* D_k)$ and $\mathrm{Dom}(\mathsf{KP}_{l,k}^*)$ are isomorphic.

Using (4.9), (a_3), (4.10) and (a_2) we deduce (i). The claim (ii) follows by (4.9), (a_4), (4.10) and (a_1). □

In the particular case $l = 1$, Corollary 4.2.1 and Proposition 4.1.1 imply that $\mathrm{Ran}(\mathsf{KP}_{i,1})$ is isomorphic to $\mathrm{Dom}(\mathsf{KP}_{1,i})^* = \ell_{f_i}^*$. Thus, the range space $\mathrm{Ran}(\mathsf{KP}_{i,1})$ can be identified with the Orlicz space ℓ_{g_i} generated by the dual Orlicz function to f_i, which in this case is given by $g_i(t) = t^2 \log^{-2i} t$ (see [71, Example 4.c.1]).

Corollary 4.2.2. *For any $n > l$ the operator $i_{l,n} : \mathfrak{R}_l \to \mathfrak{R}_n$ is strictly cosingular.*

Proof. By Theorem 4.2.2 we have that $i_{l,n} = D_n^{-1} \circ \pi_{n,l}^* \circ \big((-1)^k D_l\big)$. Since $\pi_{n,l}$ is strictly singular by Corollary 4.1.4, we deduce by Proposition B.0.1 that $\pi_{n,l}^*$ is strictly cosingular. The result follows because $\mathcal{SC}$ is an operator ideal [78, 1.10]. □

4.2.1 Involution in $\mathcal{L}(\mathfrak{R}_n)$

Theorem 4.2.1 enables us to define an involution on the space of bounded operators $\mathcal{L}(\mathfrak{R}_n)$ analogous to the Hilbert space adjoint. Precisely, consider the bilinear mapping $\omega_n : \mathfrak{R}_n \times \mathfrak{R}_n \to \mathbb{C}$ induced by (4.7), i.e., the map given for any $(x, y) = \big((x_{n-1}, \dots, x_0), (y_{n-1}, \dots, y_0)\big) \in \mathfrak{R}_n \times \mathfrak{R}_n$ by

$$\omega_n(x, y) = D_n(x)(y) = \sum_{i=0}^{n-1} (-1)^i \langle x_i, y_{n-i-1} \rangle.$$

Given any operator $T : \mathfrak{R}_n \to \mathfrak{R}_n$ and $x \in \mathfrak{R}_n$, the map $\omega_n(x, T(\cdot)) : \mathfrak{R}_n \to \mathbb{C}$ defines a bounded linear functional in $\mathfrak{R}_n$ by Theorem 4.2.1. Then there exist $x' \in \mathfrak{R}_n$ such that

$$\omega_n(x', y) = \omega_n(x, Ty) \quad \text{for all } y \in \mathfrak{R}_n.$$

Since ω_n is bilinear, this defines a map $T^+ : \mathfrak{R}_n \to \mathfrak{R}_n$ given by $T^+ x = x'$ satisfying

$$\omega_n(x, Ty) = \omega_n(T^+ x, y) \quad \text{for all } x, y \in \mathfrak{R}_n. \tag{4.11}$$

The preceding comments just show that, for any $T \in \mathcal{L}(\mathfrak{R}_n)$, the following diagram is commutative:

$$\begin{array}{ccc}
\mathfrak{R}_n & \xrightarrow{D_n} & \mathfrak{R}_n^* \\
{\scriptstyle T^+}\downarrow & & \downarrow{\scriptstyle T^*} \\
\mathfrak{R}_n & \xrightarrow{D_n} & \mathfrak{R}_n^*
\end{array}$$

and thus

$$T^+ = D_n^{-1} T^* D_n. \tag{4.12}$$

If we assume the convention that the duality $\langle x, y\rangle$ is sesquilinear on x and linear on y, then the map $D_n : \mathfrak{R}_n \to \mathfrak{R}_n^*$ is an antilinear isomorphism (meaning that it is linear and conjugate homogeneous) in the same way as the natural isomorphism $H \to H^*$ between a Hilbert space ant its dual. By (4.12), T^+ is linear being the composition of two antilinear maps. Note that for any $T, W \in \mathcal{L}(\mathfrak{R}_n)$ and $\alpha \in \mathbb{C}$ we have that

$$(T + W)^+ = T^+ + W^+ \quad \text{and} \quad (\alpha T)^+ = \overline{\alpha} T^+.$$

It is also clear that $(WT)^+ = T^+ W^+$. Moreover, since $\omega_n(x, y) = (-1)^{n+1}\overline{\omega_n(y, x)}$we deduce that the map $T \mapsto T^+$ defines an involution on $\mathcal{L}(\mathfrak{R}_n)$: for any $x, y \in \mathfrak{R}_n$ we have

$$\omega_n(T^{++}x, y) = \omega_n(x, T^+ y) = (-1)^{n+1}\overline{\omega_n(T^+ y, x)} = (-1)^{n+1}\overline{\omega_n(y, Tx)} = \omega_n(Tx, y).$$

Hence $\omega_n\big((T^{++} - T)x, y\big) = 0$ for all $x, y \in \mathfrak{R}_n$. Since D_n is an isomorphism we deduce that $(T^{++} - T)x = 0$ for all $x \in \mathfrak{R}_n$, namely, that $T^{++} = T$.

Clearly T^+ is bounded if and only if T is. Moreover, T^+ inherits most properties of the dual operator T^*. For instance:

- T^+ has closed range if and only if T has.
- T^+ is an isomotphic embedding if and only if T is onto.
- $T^+ \in \Phi_+$ if and only if $T \in \Phi_-$. In the case $T^+ = T$, $T \in \Phi_+$ implies $T \in \Phi$.

To grasp some ideas about how the involution $+$ works let us see how the matrix representation of an operator on $\mathfrak{R}_n$ changes through $+$. For the sake of clarity, let us restrict ourselves to the space $\mathfrak{R}_n^n$ defined as the span of the first n^2 vectors $u_{nm+l} = (0, \dots, \overset{(l)}{e_m}, \dots, 0)$ of the canonical basis. Equivalentely,

$$\mathfrak{R}_n^n = \{x = (x_{n-1}, \dots, x_0) \in \mathbb{C}^{n^2} : \|(x_{n-1}, \dots, x_1) - \mathsf{KP}_{1,n-1}(x_0)\|_{\mathfrak{R}_{n-1}^n} + \|x_0\|_{\ell_2^n} < \infty\}$$

where $x_l = (x_l^i)_{i=1}^n$ and

$$\mathsf{KP}_{1,n-1}(x_0) = \Big(\big(x_0^l \log^{n-1} \frac{|x_0^l|}{\|x_0\|}\big)_{l=1}^n, \dots, \big(x_0^l \log \frac{|x_0^l|}{\|x_0\|}\big)_{l=1}^n\Big) \in \mathbb{C}^{(n-1)n}.$$

An operator $T : \mathfrak{R}_n^n \to \mathfrak{R}_n^n$ is given by an $n \times n$ matrix of the form

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{nn} \end{pmatrix} \tag{4.13}$$

where each entry a_{ij} represents also a $n \times n$ matrix on $\mathbb{C}^n$. If we consider the duality $\langle \cdot, \cdot \rangle$ on ℓ_2^n, then given any $(x_{n-1}, \dots, x_0), (y_{n-1}, \dots, y_0) \in \mathfrak{R}_n^n$, we may consider on $\mathfrak{R}_n^n$ the same duality map as in the infinite dimensional case

$$\omega_n\big((x_{n-1}, \dots, x_0), (y_{n-1}, \dots, y_0)\big) = \sum_{i=0}^{n-1} (-1)^i \langle x_i, y_{n-i-1} \rangle.$$

We show now the matrix representation of T^+. Note that

$$\begin{aligned}
\omega_n\big((x_{n-1}, \dots, x_0), T(y_{n-1}, \dots, y_0)\big) &= \omega_n\big((x_{n-1}, \dots, x_0), (\sum_{j=0}^{n-1} a_{1,j+1} y_{n-j-1}, \dots, \sum_{j=0}^{n-1} a_{n,j+1} y_{n-j-1})\big) \\
&= \sum_{i=0}^{n-1} (-1)^i \langle x_i, \sum_{j=0}^{n-1} a_{i+1,j+1} y_{n-j-1} \rangle \\
&= \sum_{i=0}^{n-1} (-1)^i \sum_{j=0}^{n-1} \langle x_i, a_{i+1,j+1} y_{n-j-1} \rangle \\
&= \sum_{i=0}^{n-1} (-1)^i \sum_{j=0}^{n-1} \langle a_{i+1,j+1}^* x_i, y_{n-j-1} \rangle \\
&= \sum_{j=0}^{n-1} \Big[\sum_{i=0}^{n-1} (-1)^i \langle a_{i+1,j+1}^* x_i, y_{n-j-1} \rangle \Big] \\
&= \sum_{j=0}^{n-1} \Big[\langle \sum_{i=0}^{n-1} (-1)^i a_{i+1,j+1}^* x_i, y_{n-j-1} \rangle \Big] \\
&= \sum_{j=0}^{n-1} (-1)^j (-1)^j \Big[\langle \sum_{i=0}^{n-1} (-1)^i a_{i+1,j+1}^* x_i, y_{n-j+1} \rangle \Big] \\
&= \sum_{j=0}^{n-1} (-1)^j \Big[\langle \sum_{i=0}^{n-1} (-1)^{i+j} a_{i+1,j+1}^* x_i, y_{n-j+1} \rangle \Big] \\
&= \omega_n\big(T^+(x_{n-1}, \dots, x_0), (y_{n-1}, \dots, y_0)\big),
\end{aligned}$$

where $T^+ : \mathfrak{R}_n^n \to \mathfrak{R}_n^n$ is the linear map defined by the matrix

$$\begin{pmatrix} a^*_{n,n} & -a^*_{n-1,n} & a^*_{n-2,n} & \cdots & (-1)^n a^*_{2,n} & (-1)^{n+1} a^*_{1,n} \\ -a^*_{n,n-1} & a^*_{n-1,n-1} & -a^*_{n-2,n-1} & a^*_{n-3,n-1} & \cdots & (-1)^n a^*_{1,n-1} \\ a^*_{n,n-2} & -a^*_{n-1,n-2} & a^*_{n-2,n-2} & \ddots & \ddots & \cdots \\ \vdots & a^*_{n-1,n-3} & \ddots & a^*_{3,3} & -a^*_{2,3} & a^*_{1,3} \\ (-1)^n a^*_{n,2} & \vdots & \ddots & -a^*_{3,2} & a^*_{2,2} & -a^*_{1,2} \\ (-1)^{n+1} a^*_{n,1} & (-1)^n a^*_{n-1,1} & \cdots & a^*_{3,1} & -a^*_{2,1} & a^*_{1,1} \end{pmatrix} \tag{4.14}$$

Thus, if $T : \mathfrak{R}_n^n \to \mathfrak{R}_n^n$ comes defined by a matrix with coefficients $(a_{i,j})$ then the coefficientes of the matrix defining T^+ are $((-1)^{i+j} a^*_{n-j+1,n-i+1})$. This means that if we restrict ourselves to vectors formed by any diagonal entries of the matrix (4.13), say $a = (a_{i,i+c})_{i=1}^{n-|c|}$ where $c \in \{-n+1, \ldots, n-1\}$, then duality just reverse the ordering of $(-1)^c a$.

A similar argument can be applied to settle the infinite dimensional case (see [30]): every operator in $T : \mathfrak{R}_n \to \mathfrak{R}_n$ admits a representation of the form

$$T = \begin{pmatrix} \alpha_1 & \beta_1 & \delta_1 & \cdots & \varepsilon \\ \gamma_1 & \alpha_2 & \beta_2 & \delta_2 & \cdots \\ \vdots & \gamma_2 & \alpha_3 & \beta_3 & \delta_{n-2} \\ \pi_1 & \ddots & \gamma_3 & \alpha_4 & \beta_{n-1} \\ \varphi & \pi_2 & \cdots & \gamma_{n-1} & \alpha_n \end{pmatrix} \tag{4.15}$$

where all entries are linear maps between suitable sequence spaces (see the beginning of Section 1.2 for a discussion in the case $n = 2$). If we represent T with respect to the canonical basis $(u_l)_{l\in\mathbb{N}}$ of $\mathfrak{R}_n$, then we can apply to (4.15) the preceding calculations shown for (4.13) and (4.14). Thus, it follows that the operator T^+ is

$$\begin{pmatrix} \alpha^*_n & -\beta^*_{n-1} & \delta^*_{n-2} & \cdots & (-1)^{n+1}\varepsilon^* \\ -\gamma^*_{n-1} & \alpha^*_{n-1} & -\beta^*_{n-2} & \delta^*_{n-3} & \cdots \\ \vdots & -\gamma^*_{n-2} & \alpha^*_{n-2} & -\beta^*_{n-3} & \delta^*_1 \\ (-1)^n \pi^*_2 & \ddots & -\gamma^*_{n-3} & \alpha^*_{n-3} & -\beta^*_1 \\ (-1)^{n+1}\varphi^* & (-1)^n \pi^*_1 & \cdots & -\gamma^*_1 & \alpha^*_1 \end{pmatrix} \tag{4.16}$$

$(\mathcal{L}(\mathfrak{R}_n), +)$ is not a C^*-algebra because it does not satisfy the identity

$$\|T^+T\| = \|T^+\|\|T\| = \|T\|^2 \quad \text{for any } T \in \mathcal{L}(\mathfrak{R}_n). \tag{4.17}$$

Indeed, the operator $T = i_{1,n}\pi_{n,1} : \mathfrak{R}_n \to \mathfrak{R}_n$ is given by $T(x_{n-1}, , \ldots, x_0) = (x_0, 0, \ldots, 0)$. It is inmediate that if I_{ℓ_2} denotes the identity map on ℓ_2, then T has the matrix representation

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & I_{\ell_2} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & 0 & 0 & 0 & \vdots \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Therefore, either by a direct calculation or by (4.16), we deduce that $T^+ = -T$. Since $T^2 = 0$ we conclude that (4.17) fails. We will prove in Section 4.3 that $\mathcal{SS}(\mathfrak{R}_n)$ is a closed $+$-subalgebra of $\mathcal{L}(\mathfrak{R}_n)$, and thus $(\mathcal{L}(\mathfrak{R}_n)/\mathcal{SS}(\mathfrak{R}_n), +)$ is an algebra. However, it is still unsettled whether such quotient algebra is a C^*-algebra.

Let us see an application of the previous duality results:

Proposition 4.2.1. *Suppose that $T : \mathfrak{R}_n \to \mathfrak{R}_n$ is the linear map*

$$T = \begin{pmatrix} \alpha_1 & \beta_1 & \delta_1 & \cdots & \varepsilon \\ \vdots & \alpha_2 & \beta_2 & \delta_2 & \cdots \\ \gamma_1 & \ddots & \alpha_3 & \beta_3 & \delta_{n-2} \\ \pi_1 & \gamma_2 & \ddots & \alpha_4 & \beta_{n-1} \\ \varphi & \pi_2 & \gamma_3 & \cdots & \alpha_n \end{pmatrix} \tag{4.18}$$

where all entries are scalar multiples of the identity map. Then T is bounded if and only if is of the form

$$\begin{pmatrix} \alpha & \beta & \delta & \cdots & \varepsilon \\ 0 & \alpha & \beta & \delta & \cdots \\ \vdots & 0 & \alpha & \beta & \delta \\ 0 & \ddots & 0 & \alpha & \beta \\ 0 & 0 & \cdots & 0 & \alpha \end{pmatrix} \tag{4.19}$$

Proof. To see that (4.19) is bounded note that we can decompose (4.19) as a sum $\alpha I + \sum_{i=1}^{n-1} \eta_i A_i$ where η_i are scalars and

$$A_i = \begin{pmatrix} 0 & \cdots & I & \cdots & 0 \\ 0 & 0 & \cdots & I & \cdots \\ \vdots & 0 & 0 & \cdots & I \\ 0 & \ddots & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

is the linear map having all null entries except at the i-th superdiagonal (defined by elements $a_{k,l}$ such that $|k-l| = i-1$). We have that $A_i = i_{n-i,n} \circ \pi_{n,n-i}$ for each $1 \le i \le n-1$, hence (4.19) is bounded being a sum of bounded operators.

We will prove the converse by induction on n; the case $n = 1$ has nothing to prove, while the case $n = 2$ is the Johnson-Lindenstrauss-Schechtman Theorem (cf. Corollary 1.2.3). Thus, we assume that the result is true for $\mathfrak{R}_{n-1}$. Consider the bounded homogenous map $B : \ell_2 \to \mathfrak{R}_n$ given by $B(x) = (\mathsf{KP}_{1,n-1}(x), x) = (\mathsf{KP}^{n-1}(x), \dots, \mathsf{KP}(x), x)$. If (4.18) is bounded then

$$\varphi\,\mathsf{KP} + \pi_2 = \pi_{n,1} \circ T \circ i_{2,n} \circ \pi_{n,2} \circ B : \ell_2 \to \ell_2$$

is bounded; hence, $\varphi = 0$. By the same reason, the map

$$\pi_2\,\mathsf{KP} + \gamma_3 = \pi_{n,2} \circ T \circ i_{3,n} \circ \pi_{n,3} \circ B : \ell_2 \to \ell_2$$

is bounded. Thus $\pi_2 = 0$. We can work inductively showing that all entries of the last row are null with the exception of α_n. This implies that the restriction of T to $\mathfrak{R}_{n-1}$ defines a $(n-1)\times(n-1)$ scalar bounded matrix on $\mathfrak{R}_{n-1}$, and thus we can apply induction on the upper-left $(n-1)\times(n-1)$ submatrix to conclude that T has the form

$$\begin{pmatrix} \alpha & \beta & \cdots & \theta_1 & \varepsilon \\ 0 & \alpha & \beta & \cdots & \theta_2 \\ \vdots & 0 & \alpha & \beta & \cdots \\ 0 & \ddots & 0 & \alpha & \beta_{n-1} \\ 0 & 0 & \cdots & 0 & \alpha_n \end{pmatrix} \tag{4.20}$$

Now we can consider the +-adjoint of (4.20) given by

$$\begin{pmatrix} \overline{\alpha_n} & -\overline{\beta_{n-1}} & \cdots & (-1)^n\overline{\theta_2} & \overline{\varepsilon} \\ 0 & \overline{\alpha} & -\overline{\beta} & \cdots & (-1)^n\overline{\theta_1} \\ \vdots & 0 & \overline{\alpha} & -\overline{\beta} & \cdots \\ 0 & \ddots & 0 & \overline{\alpha} & -\overline{\beta} \\ 0 & 0 & \cdots & 0 & \overline{\alpha} \end{pmatrix}$$

Applying induction once again we deduce that T is equal to

$$\begin{pmatrix} \alpha & \beta & \cdots & \theta_1 & \varepsilon \\ 0 & \alpha & \beta & \cdots & \theta_2 \\ \vdots & 0 & \alpha & \beta & \cdots \\ 0 & \ddots & 0 & \alpha & \beta \\ 0 & 0 & \cdots & 0 & \alpha \end{pmatrix} \tag{4.21}$$

Using the same decomposition argument as in the initial part of the proof we deduce that (4.21) is bounded if and only if

$$V = \begin{pmatrix} 0 & 0 & \cdots & \theta_1 & 0 \\ 0 & 0 & 0 & \cdots & \theta_2 \\ \vdots & 0 & 0 & 0 & \cdots \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{4.22}$$

is bounded, which means that for any $x \in \ell_2$ we have

$$\begin{aligned} \|(\theta_1-\theta_2)\mathsf{KP}(x)\|_{\ell_2} \le \|(\theta_1\mathsf{KP}(x),\theta_2 x)\|_{Z_2} &\le C_n\|(\theta_1\mathsf{KP}(x),\theta_2 x,0,\dots,0)\|_n \\ &= C_n\|V\circ B(x)\|_n \le K_n\|x\|_{\ell_2}. \end{aligned}$$

and this finally forces $\theta_1=\theta_2$. □

Proposition 4.2.1 is a natural generalization of the Johnson-Lindenstrauss-Schechtman Theorem (Corollary 1.2.3) to $\mathfrak{R}_n$. Note also that the proof essentially depends on two facts: KP is unbounded and the duality identity (4.16).

4.2.2 Symplectic operators

Let $T : \mathfrak{R}_n \to \mathfrak{R}_n$ be a bounded operator. We say that T is a *symplectic operator* if preserves ω_n:

$$\omega_n\big(Tx, Ty\big) = \omega_n(x, y), \quad \text{for all } x, y \in \mathfrak{R}_n. \tag{4.23}$$

The meaning of the name will be clarified in forthcoming Section 4.4. Taking into account that ω_n separates duals and that

$$\omega_n(Tx, Ty) = \omega_n(T^+Tx, y), \quad \text{for all } x, y \in \mathfrak{R}_n,$$

it follows that operator T is symplectic if and only if $T^+T = I$. Moreover, any symplectic operator has complemented range: the map $P = TT^+$ defines a bounded projection onto $T(\mathfrak{R}_n)$. In the paper [30] we studied symplectic operators on Banach spaces. A simple example of symplectic operator is the following one: take a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ and a sequence of signs $(\varepsilon_i)_i \subset \{-1, 1\}^{\mathbb{N}}$. Then the isometry $\tau : \ell_\infty \to \ell_\infty$ given by $\tau(x) = \big(\varepsilon_i x_{\sigma(i)}\big)_{i\in\mathbb{N}}$ is interpolating for (ℓ_∞, ℓ_1). Then the induced operator

$$\tau_n = \begin{pmatrix} \tau & & & & \\ & \ddots & & & \\ & & \tau & & \\ & & & \tau & \\ & & & & \tau \end{pmatrix}$$

is symplectic by (4.16). More elaborated and crucially important examples are the block operators:

Theorem 4.2.3. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of normalized blocks in ℓ_2. Then the block operator $T_n^{\mathfrak{u}}$ is symplectic, i.e.,*

$$\omega_n(T_n^{\mathfrak{u}}x, T_n^{\mathfrak{u}}y) = \omega_n(x, y) \quad \textit{for all } x, y \in \mathfrak{R}_n.$$

In particular, block operators have complemented range.

Proof. It is enough to prove that $T_n^{\mathfrak{u}}$ preserves ω_n on the basis vectors (4.2). To avoid confussion with notation, let us denote by $x_{i,k}$ the vector of $\mathfrak{R}_n$ having e_i at the k^{th} position and zeroes in the other coordinates:

$$x_{i,k} = \big(0, \dots, 0, \overset{(k)}{e_i}, 0, \dots, 0\big), \quad 0 \le k \le n-1,\, i \in \mathbb{N}.$$

Thus, we shall prove that

$$\omega_n(T_n^{\mathfrak{u}}(x_{i,k}), T_n^{\mathfrak{u}}(x_{j,l})) = \omega_n(x_{i,k}, x_{j,l}). \tag{4.24}$$

First, suppose that $k + l = n - 1$. By definition

$$\begin{aligned}
&\omega_n(T_n^{\mathfrak{u}}(x_{i,k}), T_n^{\mathfrak{u}}(x_{j,l})) \\
&\qquad = \omega_n\Big(\big(\frac{2^{k-1}}{(k-1)!}u_i \log^{k-1}|u_i|, \dots, u_i, \dots, 0\big), \big(\frac{2^{l-1}}{(l-1)!}u_j \log^{l-1}|u_j|, \dots, u_j, \dots, 0\big)\Big) \\
&\qquad = (-1)^k\langle u_i, u_j\rangle = \omega_n(x_{i,k}, x_{j,l}).
\end{aligned}$$

If $k+l<n-1$ then (4.24) cancels out as we are multiplying by zeroes. If $k+l>n-1$ then $\omega_n(x_{i,k},x_{j,l})=0$ and (4.24) becomes, after setting $m=k+l-(n-1)$,

$$\begin{aligned}
&\omega_n(T_n^{\mathfrak{u}}(x_{i,k}),T_n^{\mathfrak{u}}(x_{j,l})) \qquad (4.25)\\
&\quad=(-1)^k\Big\langle u_i,\frac{2^m}{m!}u_j\log^m|u_j|\Big\rangle+(-1)^{k+1}\Big\langle\frac{2^1}{1!}u_i\log|u_i|,\frac{2^{m-1}}{(m-1)!}u_j\log^{m-1}|u_j|\Big\rangle\\
&\quad+\cdots+(-1)^{k+m}\Big\langle\frac{2^m}{m!}u_i\log^m|u_i|,u_j\Big\rangle\\
&\quad=(-1)^k\sum_{p=0}^m(-1)^p\Big\langle\frac{2^p}{p!}u_i\log^p|u_i|,\frac{2^{m-p}}{(m-p)!}u_j\log^{m-p}|u_j|\Big\rangle.
\end{aligned}$$

If $i\neq j$ then all summands in (4.25) are null because u_i and u_j have disjoint support. In particular $\langle u_i,u_j\rangle=0$ and the result follows. If $i=j$ then note that

$$\Big\langle\frac{2^p}{p!}u_i\log^p|u_i|,\frac{2^{m-p}}{(m-p)!}u_i\log^{m-p}|u_i|\Big\rangle=\frac{2^m}{p!(m-p)!}\sum_{\nu=0}^N\overline{u_i^\nu}u_i^\nu\log^m|u_i^\nu|.$$

Thus, for some constant $C(u_i)$ depending on u_i, (4.25) becomes

$$\begin{aligned}
(-1)^kC(u_i)\Big[\sum_{p=0}^m(-1)^p\frac{2^m}{p!(m-p)!}\Big]&=(-1)^kC(u_i)\Big[\sum_{p=0}^m(-1)^p\frac{2^m}{m!}\binom{m}{p}\Big]\\
&=(-1)^kC(u_i)\,\frac{2^m}{m!}\left[\sum_{p=0}^m(-1)^p\binom{m}{p}\right].
\end{aligned}$$

Now, the Binomial Theorem $0=(1-1)^m=\sum_{k=0}^m\binom{m}{k}1^{m-k}(-1)^k$ cancels out all terms. $\square$

4.3 Operators on $\mathfrak{R}_n$

Operators on ℓ_p spaces behave as follows: given a Banach space X and $1\le p<\infty$, any operator $T:\ell_p\to X$ is either strictly singular or invertible on some complemented copy of ℓ_p. The proof reduces to classical facts about basic sequences in ℓ_p (see [3, Section 2.1] for the following claims): if T is not strictly singular, then we can assume that it is an isomorphism on some infinite dimensional subspace $E\subset\ell_p$ generated by a normalized block basic sequence of the canonical basis $(e_n)_n$. But any such space E is complemented and isometrically isomorphic to ℓ_p. Kalton proved in [59] that operators on Z_2 satisfy the same property (see Proposition 1.1.2). One of the main results of this section shows that the same is true for $\mathfrak{R}_n$: every operator $T:\mathfrak{R}_n\to X$ is strictly singular or invertible on some complemented copy of $\mathfrak{R}_n$.

Reduced to its bare bones, the proof essentially depends on two facts:

(1) The quotient map $\pi_{n,n-1}:\mathfrak{R}_n\to\mathfrak{R}_{n-1}$ is strictly singular (cf. Corollary 4.1.4).

(2) The existence of a large family of isomorphic embeddings with complemented range formed by the so called block operators $T_n^{\mathfrak{v}}$.

Fact (1) is a global to local principle: as a consequence, $T : \mathfrak{R}_n \to X$ is strictly singular if and only if $T_{|_{\ell_2}} : \ell_2 \to X$ is strictly singular. Thus, one has that T is invertible on some subspace $W \subset \ell_2$ generated by some normalized block basic sequence $\mathfrak{w}$. On the other hand, fact (2) will be used as a local to global principle: once we have the subspace $W \subset \ell_2$ on which T is invertible, we can consider the associated block operator $T_n^{\mathfrak{w}} \in \mathcal{L}(\mathfrak{R}_n)$ and obtain that $TT_n^{\mathfrak{w}}$ is invertible on the whole $\ell_2 \subset \mathfrak{R}_n$. And this is strong enough to force the desired property. Using the results about operators we will obtain several consequences on the structure of $\mathfrak{R}_n$ which we study in further subsections: unconditional structure, operator ideals, Fredholm operators and a generalized version to $\mathfrak{R}_n$ of Proposition 1.2.5 and the hyperplane problem for Z_2.

We begin with an important result that will be used in this chapter. It was noted by Kalton [59] for Z_2:

Proposition 4.3.1. *If $T : \mathfrak{R}_n \to \mathfrak{R}_n$ is not strictly singular then there exists $\alpha \neq 0$ and block operators $T_n^{\mathfrak{w}}$ and $T_n^{\mathfrak{v}}$ such that $TT_n^{\mathfrak{w}} - \alpha T_n^{\mathfrak{v}}$ is strictly singular.*

Proof. Since $T : \mathfrak{R}_n \to \mathfrak{R}_n$ is not strictly singular, by Corollary 4.1.5 we deduce that T is not strictly singular in ℓ_2. Thus, there exist some infinite dimensional subspace $Y \subset \ell_2$ such that $T|_Y$ is an isomorphism. Using the Bessaga-Pełczyński Selection Principle [3, Prop. 1.3.10], we can assume that Y is generated by a normalized block basic sequence $(w_m)_m$. If we denote by $z_n = (w_n, 0, \ldots, 0) \in i_{1,n}(\ell_2) \subset \mathfrak{R}_n$, it follows that the sequence $(Tz_m)_m \subset \mathfrak{R}_n$ is a seminormalized basic sequence equivalent to the canonical basis of ℓ_2. By Corollary 4.1.2 we deduce that $\|\pi_{n,n-1}Tz_m\| \to 0$. Hence, there exist a sequence $(x_m)_m \subset i_{1,n}(\ell_2) \subset \mathfrak{R}_n$ such that $\|x_m - Tz_m\| \to 0$. Thus, passing again to subsequences and using a perturbation argument we have that

$$\|x_m - Tz_m\| \leq 2^{-m},$$

where $(x_m)_m$ is basic and seminormalized (since $(Tz_n)_n$ is) in $i_{1,n}(\ell_2)$. By Bessaga-Pełczyński selection principle and passing to subsequences yet again, we deduce that there exist some seminormalized block basic sequence $(u_m)_m$ of $(e_n)_n \subset i_{1,n}(\ell_2)$ such that

$$\|Tz_m - (u_m, 0, \ldots, 0)\| \leq 2^{-m}.$$

Now observe that the sequence $(\|u_n\|)_n$ is bounded on $\mathbb{R}$ and such that

$$0 < \alpha = \inf \|u_n\| \leq \sup \|u_n\| \leq \beta < \infty.$$

Thus, there exist a converging subsequence $(\|u_{n_k}\|)_k$; we may assume that it converges to α. Passing to a subsequence we can ssume that $|\|u_n\| - \alpha| \leq 2^{-n}$. It follows then

$$\begin{aligned}\Big\|Tz_n - \alpha\Big(\frac{u_n}{\|u_n\|}, 0 \ldots, 0\Big)\Big\| &= \Big\|Tz_n - (u_n, 0, \ldots, 0) + (u_n, 0, \ldots, 0) - \alpha\Big(\frac{u_n}{\|u_n\|}, 0, \ldots, 0\Big)\Big\| \\ &\leq \|Tz_n - (u_n, 0, \ldots, 0)\| \\ &+ \Big\|\|u_n\|\Big(\frac{u_n}{\|u_n\|}, 0, \ldots, 0\Big) - \alpha\Big(\frac{u_n}{\|u_n\|}, 0, \ldots, 0\Big)\Big\| \\ &\leq 2^{-n} + |\|u_n\| - \alpha| \leq 2^{-n+1}.\end{aligned}$$

Summing all up, passing to subsequences if necessary, we deduce that there exist normalized block basic sequences $(w_m)_m$ and $(v_m)_m = \big(\frac{u_m}{\|u_m\|}\big)_m$ of $(e_n)_n \subset \ell_2$, and $\alpha \neq 0$ such that

$$\|T(w_m, 0, \ldots, 0) - \alpha(v_m, 0, \ldots, 0)\| \leq 2^{-m}. \tag{4.26}$$

Define $T_n^{\mathfrak{w}}$ and $T_n^{\mathfrak{v}}$ as the block operators associated to the sequences $(w_m)_m$ and (v_m). Then

$$\|TT_n^{\mathfrak{w}}(e_n, 0, \ldots, 0) - \alpha T_n^{\mathfrak{v}}(e_n, 0, \ldots, 0)\| = \|T(w_m, 0, \ldots, 0) - \alpha(v_m, 0, \ldots, 0)\| \leq 2^{-m}.$$

Define $t_n = (TT_n^{\mathfrak{w}} - \alpha T_n^{\mathfrak{v}})(e_n, 0, \ldots, 0) \in \mathfrak{R}_n$ and consider the following operator $\widehat{K}$ given by

$$\widehat{K}(x, \ldots, 0) = \sum_{n=1}^{\infty} (e_n, \ldots, 0)^*\big((x, \ldots, 0)\big)t_n, \quad \text{for each } (x, \ldots, 0) \in \ell_2 \subset \mathfrak{R}_n.$$

Since $\sum_{n=1}^{\infty} \|(e_n, \ldots, 0)^*\|\|t_n\| = \sum_{n=1}^{\infty} \|t_n\| < \infty$ we have that $\widehat{K}$ is nuclear. If we denote by $K : \mathfrak{R}_n \to \mathfrak{R}_n$ the natural nuclear extension to $\mathfrak{R}_n$, then K is compact and $(TT_n^{\mathfrak{w}} - \alpha T_n^{\mathfrak{v}} - K)(\ell_2) = 0$, so $TT_n^{\mathfrak{w}} - \alpha T_n^{\mathfrak{v}} - K$ is stricty singular by Corollary 4.1.5. Since K is compact, it follows that $TT_n^{\mathfrak{w}} - \alpha T_n^{\mathfrak{v}}$ is strictly singular. $\square$

We will use several times the following classical result about complemented subspaces. See [92, 1.1] or [2, 2.2] for a proof:

Lemma 4.3.1. *Let $T : X \to Y$ be a bounded operator between two Banach spaces. If $M \subset X$ is a closed subspace such that $T|_M$ is an isomorphism and $Y = T(M) \oplus N$, then $X = M \oplus T^{-1}(N)$.*

We prove now the main result of this section:

Theorem 4.3.1.

- *Every operator $T : \mathfrak{R}_n \to X$ into any Banach space X is either strictly singular, or an isomorphism on a complemented copy of $\mathfrak{R}_n$.*

- *Every operator $T : \mathfrak{R}_n \to \mathfrak{R}_n$ is strictly singular or an isomorphism on a complemented copy E of $\mathfrak{R}_n$ such that $T(E)$ is also complemented.*

Proof. For the first part, let us assume that $T : \mathfrak{R}_n \to X$ is not strictly singular. Then it follows by Corollary 4.1.5 that $T \circ i_{1,n} : \ell_2 \to X$ is not strictly singular. We distinguish two cases: asume first that $T \circ i_{1,n}$ is an isomorphism and consider the following diagram:

$$\begin{tikzcd}
 & \ell_2 \arrow[r, equal] \arrow[d, "i_{1,n}"'] & \ell_2 \arrow[d, "\overline{i_{1,n}}"] & & \\
0 \arrow[r] & \mathfrak{R}_n \arrow[r, "{(T,1)}"] \arrow[d, "\pi_{n,n-1}"'] & X \oplus \mathfrak{R}_n \arrow[r] \arrow[d, "Q"] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{R}_{n-1} \arrow[r] & \mathrm{PO} \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

where $\overline{i_{1,n}}(x) = (Tx, i_{1,n}x)$. Since the diagram is commutative and $\pi_{n,n-1}$ is strictly singular, we deduce that $Q(T,1)$ is strictly singular. Now, by definition we have that $Q(T,1) = Q(T,0) + Q(0,1)$ and $Q(0,1)$ is an embedding. Indeed, we can assume after normalization that $\|(T \circ i_{1,n})^{-1}\| \le 1$. From this it follows

$$\begin{aligned}
\|Q((0,x))\| &= \inf_{y\in\ell_2} \|(0,x) - \overline{i_{1,n}}(y)\| = \inf_{y\in\ell_2} \|(0,x) - (Ty,y)\| \\
&= \inf_{y\in\ell_2} \|(-Ty, x-y)\| = \inf_{y\in\ell_2} \{\|Ty\| + \|x-y\|\} \\
&\ge \inf_{y\in\ell_2} \{\|T^{-1}\|^{-1}\,\|y\| + \|x\| - \|y\|\} \ge \inf_{y\in\ell_2} \{\|x\| + (\|T^{-1}\|^{-1} - 1)\|y\|\} \\
&\ge \|x\|.
\end{aligned}$$

In particular, $Q(0,1)$ is an upper semi-Fredholm operator, hence $Q(T,0) = Q(T,1) - Q(0,1)$ is also an upper semi-Fredhom operator by Proposition B.1.2. Thus, $Q(T,0)$ it is an isomorphism on some finite codimensional subspace of $\mathfrak{R}_n$. Hence, the same is true for T, and since $\mathfrak{R}_n$ is isomorphic to its kn-codimensional subspaces, we conclude that T is an isomorphism on a complemented subspace isomorphic to $\mathfrak{R}_n$. On the other hand, if $T \circ i_{1,n}$ is not an isomorphism on the whole ℓ_2, then there exist a infinite dimensional subspace $W \subset \ell_2$ generated by a normalized block basic sequence $(w_n)_n$ of $(e_n)_n$ such that $T|_W$ is an isomorphism. Consider the operator $TT_n^{\mathfrak{w}} : \mathfrak{R}_n \to X$, where $T_n^{\mathfrak{w}}$ is the block operator associated to $(w_n)_n$. Then $TT_n^{\mathfrak{w}} \circ i_{1,n} = T|_W$ is an isomorphism on ℓ_2, and by the first part of the proof it is an isomorphism on a n-codimensional subspace of $E \subset \mathfrak{R}_n$. Hence T is an isomorphism on $T_n^{\mathfrak{w}}(E)$, which is a complemented isomorphic copy of $\mathfrak{R}_n$.

To prove the second part we assume again that $T : \mathfrak{R}_n \to \mathfrak{R}_n$ is not strictly singular. Then by Proposition 4.3.1 there exist block operators $T_n^{\mathfrak{w}}$ and $T_n^{\mathfrak{v}}$, and $\alpha \neq 0$ such that $TT_n^{\mathfrak{w}} - \alpha T_n^{\mathfrak{v}}$ is strictly singular. Then by Theorem 4.2.3 we have that $(T_n^{\mathfrak{v}})^+ T_n^{\mathfrak{v}} = I$, hence

$$\alpha^{-1}(T_n^{\mathfrak{v}})^+ TT_n^{\mathfrak{w}} = I + S,$$

where S is strictly singular. By Proposition B.1.2 we deduce that $\alpha^{-1}(T_n^{\mathfrak{v}})^+ TT_n^{\mathfrak{w}}$ is Fredholm of index 0. Then there exist decompositions

$$\mathfrak{R}_n = F \oplus M \quad \text{and} \quad \mathfrak{R}_n = (T_n^{\mathfrak{v}})^+ TT_n^{\mathfrak{v}}(F) \oplus N,$$

where M and N are both of the same finite dimension and $(T_n^{\mathfrak{v}})^+ TT_n^{\mathfrak{w}}$ is an isomoprhism on F. Then $TT_n^{\mathfrak{w}}$ is an isomorphism on F, hence T is an isomorphism on $T_n^{\mathfrak{w}}(F) = E$. Now observe that both E and $T(E)$ are complemented: since $\mathfrak{R}_n = (T_n^{\mathfrak{v}})^+ TT_n^{\mathfrak{w}}(F) \oplus N$ and $(T_n^{\mathfrak{v}})^+$ is an isomorphism on $TT_n^{\mathfrak{w}}(F)$, it follows by Lemma 4.3.1 that:

$$\mathfrak{R}_n = TT_n^{\mathfrak{w}}(F) \oplus \big((T_n^{\mathfrak{v}})^+\big)^{-1}(N) = T(E) \oplus \big((T_n^{\mathfrak{v}})^+\big)^{-1}(N). \tag{4.27}$$

Similarly, since T is an isomorphism on E, it follows again by Lemma 4.3.1 and (4.27) that

$$\mathfrak{R}_n = E \oplus \big((T_n^{\mathfrak{v}})^+ T\big)^{-1}(N).$$

To end the proof just note that, since F is finite codimensional in $\mathfrak{R}_n$ and $\mathfrak{R}_n$ is isomorphic to its mn-codimensional subspaces, F contains a complemented subspace L isomorphic to $\mathfrak{R}_n$. Then T is an isomorphism on $E' = T_n^{\mathfrak{w}}(L) \subset T_n^{\mathfrak{w}}(F) = E$ and both $T(E')$ and E' are complemented because E and $T(E)$ were. □

Corollary 4.3.1. *Every operator $T : \mathfrak{R}_n \to \mathfrak{R}_k$ is strictly singular for every $k < n$.*

Proof. This is a direct consequence of Theorem 4.3.1 and Proposition 4.1.2. □

Corollary 4.3.1 is blatantly false for general Rochberg spaces. The case of weighted Hilbert spaces is a simple counterexample since the differential map $\Omega_{1,n-1} : \mathfrak{R}_1(\ell_2(w_0), \ell_2(w_1))_{1/2} \curvearrowright \mathfrak{R}_{n-1}(\ell_2(w_0), \ell_2(w_1))_{1/2}$ is linear, and thus all Rochberg spaces are isomorphic to each other by induction on n (see [31, Section 5]).

Corollary 4.3.2. *Every infinite dimensional complemented subspace of $\mathfrak{R}_n$ contains a further complemented subspace isomorphic to $\mathfrak{R}_n$.*

An inmediate consequence of Corollary 4.3.2 is:

Corollary 4.3.3. *$\mathfrak{R}_n$ has no complemented copies of $\mathfrak{R}_k$ for any $k < n$.*

Corollary 4.3.4. *If $\mathfrak{R}_n = \mathfrak{R}_n \oplus F$ with F infinite dimensional, then F is isomorphic to $\mathfrak{R}_n$.*

Proof. It follows from Pełczyński's decomposition argument [3, Th. 2.2.3] that if F is a complemented subspace of $\mathfrak{R}_n$ and $F \oplus F \simeq F$ then F is isomorphic to $\mathfrak{R}_n$. By Corollary 4.3.2 one has that $F \simeq \mathfrak{R}_n \oplus N$, and therefore by hypothesis

$$F \oplus F \simeq F \oplus \mathfrak{R}_n \oplus N \simeq \mathfrak{R}_n \oplus N \simeq F. \quad \square$$

4.3.1 Unconditional structure of $\mathfrak{R}_n$

We describe the unconditional structure of $\mathfrak{R}_n$. The following result was obtained by Kalton and Peck in [62, Corollary 6.9] for $n = 2$ and in [23, Prop. 7.10] for $n = 3$.

Proposition 4.3.2. *$\mathfrak{R}_n$ has no unconditional basis.*

Proof. If $(x_n)_n$ were an unconditional basis, then $(y_n)_n = \left(\frac{x_n}{\|x_n\|}\right)_n$ is also a (non-necessarily) equivalent unconditional basis. Then by Theorem 4.1.1 the sequence $(y_n)_n$ admits a subsequence equivalent to the canonical basis of ℓ_2 or some ℓ_{f_j} for $1 \leq j \leq n-1$. Since $(y_n)_n$ is unconditional, this subspace spanned by the subsequence is complemented in $\mathfrak{R}_n$. Now note that every Orlicz space ℓ_{f_j} contains a complemented copy of ℓ_2; indeed, if $(u_n)_n$ is a normalized block basic sequence in ℓ_{f_j} such that $\lim_{n\to\infty} \|u_n\|_\infty = 0$ then $(u_n)_n$ is equivalent to the canonical basis of ℓ_2 [69, Lemma 2] and its closed linear span is complemented [69, Lemma 5]. It follows that $\mathfrak{R}_n$ contains a complemented copy of ℓ_2, a contradiction with Corollary 4.3.3. $\square$

A stronger result can be obtained using the following result of Casazza and Kalton [19, Th. 3.8]: let X be a Banach space and $(V_n)_{n\in\mathbb{N}}$ an unconditional finite dimensional decomposition of X such that $\sup_{n\in\mathbb{N}} \dim V_n < \infty$. If X has l.u.st then X has an unconditional basis.

Theorem 4.3.2. *$\mathfrak{R}_n$ has no l.u.st. In particular, $\mathfrak{R}_n$ is not complemented in a Banach lattice.*

Proof. As we noted in (4.3), $\mathfrak{R}_n$ has an UFDD $(X_m)_{m\in\mathbb{N}}$ where

$$X_m = \operatorname{span}\{(e_m, 0, \dots, 0), \dots, (0, \dots, 0, e_m)\}$$

and $\sup_m \dim X_m = n < \infty$. Thus, by the aforementioned result by Casazza and Kalton, we deduce that $\mathfrak{R}_n$ has l.u.st if and only if it has an unconditional basis. By Proposition 4.3.2 it follows that $\mathfrak{R}_n$ does not have l.u.st. $\square$

This last result was proved for $n = 2$ by Johnson, Lindenstrauss and Schechteman in [57, Th. 4]. Their approach can be generalized to $\mathfrak{R}_n$ using Proposition 4.2.1 (see also [31, Th. 6.8]), which provides an alternative proof of Theorem 4.3.2.

Taking into account that a complemented subspace of Banach space with l.u.st also have l.u.st, combining Corollary 4.3.2 with Theorem 4.3.2 yields:

Corollary 4.3.5. *$\mathfrak{R}_n$ has no complemented copies of Banach spaces with l.u.st. In particular, $\mathfrak{R}_n$ contains no complemented Banach lattice.*

4.3.2 Operator ideals in $\mathcal{L}(\mathfrak{R}_n)$

In this subsection we will see that the lattice of closed ideals in $\mathcal{L}(\mathfrak{R}_n)$ is quite simple due to Theorem 4.3.1. In particular, using that $\mathfrak{R}_n$ is isomorphic to its dual we shall prove that strictly singular and strictly cosingular operators on $\mathfrak{R}_n$ coincide. The following result is the basic ingredient for the proof; the argument is a modification of [99, Th. 6.2].

Lemma 4.3.2. *Assume that $T : \mathfrak{R}_n \to \mathfrak{R}_n$ is not strictly singular. Then there exist operators $A, B \in \mathcal{L}(\mathfrak{R}_n)$ such that $I_{\mathfrak{R}_n} = ATB$.*

Proof. By the second part of Theorem 4.3.1 we have that if T is not strictly singular, then it is an isomorphism on some complemented subspace E isomorphic to $\mathfrak{R}_n$ and such that $T(E)$ is also complemented. Let $W : \mathfrak{R}_n \to T(E)$ be any isomorphism and i denote the corresponding inclusion. Then we have a commutative diagram of the form

$$\begin{array}{ccccc}
\mathfrak{R}_n & \xrightarrow{T} & \mathfrak{R}_n & & \\
{\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} & & \\
E & \xrightarrow{T|_E} & T(E) & \xleftarrow{W} & \mathfrak{R}_n
\end{array}$$

Then $I_{\mathfrak{R}_n} = W^{-1} \circ T \circ \big((T|_E)^{-1} \circ W\big)$. Now observe that defining $A = W^{-1}$ and $B = (T|_E)^{-1} \circ W$ ends the proof. Indeed, we have that $W^{-1} : T(E) \to \mathfrak{R}_n$, and since $T(E)$ is complemented we can extend it to an operator (which we still denote by) W^{-1} on $\mathfrak{R}_n$. On the other hand $(T|_E)^{-1} \circ W : \mathfrak{R}_n \to E$ and $E \subset \mathfrak{R}_n$, so $(T|_E)^{-1} \circ W \in \mathcal{L}(\mathfrak{R}_n)$. $\square$

Recall that a proper ideal $\mathcal{A}(X)$ is the unique maximal ideal if it contains every proper ideal of X.

Theorem 4.3.3.

(1) $\mathcal{SS}(\mathfrak{R}_n) = \mathcal{SC}(\mathfrak{R}_n)$ is the unique maximal ideal of $\mathcal{L}(\mathfrak{R}_n)$.

(2) $\mathcal{K}(\mathfrak{R}_n) \subsetneq \mathcal{SS}(\mathfrak{R}_n)$.

Proof. (1) If $T \in \mathcal{SC}(\mathfrak{R}_n)$ then by Theorem 4.3.1 it follows that $T \in \mathcal{SS}(\mathfrak{R}_n)$; indeed, if T is not strictly singular then by Lemma 4.3.2 we deduce that the identity $I = ATB$ is strictly cosingular by the ideal property of $\mathcal{SC}$, which is impossible. The other inclusion is also direct applying duality: if $T \in \mathcal{SS}(\mathfrak{R}_n)$ then $T^* \in \mathcal{SC}(\mathfrak{R}_n^*)$. Since $T^+ = D_n^{-1} T^* D_n$ we deduce that $T^+ \in \mathcal{SC}(\mathfrak{R}_n)$ by the ideal property of $\mathcal{SC}$. Then by the first part of the proof we conclude that $T^+ \in \mathcal{SS}(\mathfrak{R}_n)$. Then applying duality once again we deduce that $T \in \mathcal{SC}(\mathfrak{R}_n)$. The fact that $\mathcal{SS}(\mathfrak{R}_n)$ is the only maximal operator ideal follows by Lemma 4.3.2: any ideal $\mathcal{A}(\mathfrak{R}_n)$ which contains a non-strictly singular operator must contain the identity. To see (2) note that the operator $i_{1,n} \circ \pi_{n,1} : \mathfrak{R}_n \to \mathfrak{R}_n$ is strictly singular by Corollary 4.1.4 but it is not compact (consider the sequence $(0, \dots, 0, e_n) \in \mathfrak{R}_n$). $\square$

A useful consequence of Theorem 4.3.3 is that $T \in \mathcal{SS}(\mathfrak{R}_n)$ if and only if $T^+ \in \mathcal{SS}(\mathfrak{R}_n)$. The following corollary is key to understand the properties of the involution $+ : \mathcal{L}(\mathfrak{R}_n) \to \mathcal{L}(\mathfrak{R}_n)$.

Corollary 4.3.6. *Let $T : \mathfrak{R}_n \to \mathfrak{R}_n$ be any operator. If T^+T is strictly singular then T is strictly singular.*

Proof. If T is not strictly singular, by Proposition 4.3.1 there exists $\alpha \neq 0$ and block operators $T_n^{\mathfrak{w}}, T_n^{\mathfrak{v}}$ such that $TT_n^{\mathfrak{w}} = \alpha T_n^{\mathfrak{v}} + S$ with $S \in \mathcal{S}(\mathfrak{R}_n)$. Since S^+ is also strictly singular, by Theorem 4.2.3 we deduce that

$$(T_n^{\mathfrak{w}})^+ T^+ T T_n^{\mathfrak{w}} = (TT_n^{\mathfrak{w}})^+ T T_n^{\mathfrak{w}} = (\alpha (T_n^{\mathfrak{v}})^+ + S^+)(\alpha T_n^{\mathfrak{v}} + S) = \alpha' (T_n^{\mathfrak{v}})^+ T_n^{\mathfrak{v}} + S' = \alpha' I + S',$$

where $S' \in \mathcal{S}(\mathfrak{R}_n)$. Since T^+T is strictly singular, this implies that the identity I is strictly singular, a contradiction. $\square$

As we explained in Appendix B.1, both operator ideals $\mathcal{SS}$ and $\mathcal{SC}$ are contained in the operator ideal $\mathcal{I}n$ of inessential operators, which coincides with the perturbation class of Fredholm operators. Thus, using Theorem 4.3.3 we can give a complete answer to the Perturbation Class Problem for the Rochberg spaces $\mathfrak{R}_n$ (see the diagram in Section B.1):

Corollary 4.3.7. *$P\Phi(\mathfrak{R}_n) = P\Phi_+(\mathfrak{R}_n) = P\Phi_-(\mathfrak{R}_n) = \mathcal{SS}(\mathfrak{R}_n)$.*

We stress the fact that $\mathfrak{R}_n$ does not satisfy the usual hypothesis (see [51, 49]) that imply the thesis of Corollary 4.3.7.

To end this subsection we study how strictly singular relate to compact operators on $\mathfrak{R}_n$. More precisely, note that the example $T = i_{1,n} \circ \pi_{n,1}$ of non compact strictly singular operator satisfies that $T^2 = 0$. In fact, the operator $S = i_{n-1,n} \circ \pi_{n,n-1}$ is also strictly singular, not compact, and such that $S^{n-1} = T$. Thus, if we compose S with itself n times, the operator is 0 (in particular compact). This last fact generalizes to: the composition of n strictly singular operators on $\mathfrak{R}_n$ is compact. To show it, we introduce some notation:

Definition 1. *Let us say that a seminormalized basic sequence $(x_n)_n \subset \mathfrak{R}_n$ has type f_j, for $0 \leq j \leq n-1$, if it is equivalent to the canonical basis of ℓ_{f_j}, where $\ell_{f_0} = \ell_2$.*

The following result shows the behaviour of type f_j sequences under the action of strictly singular operators. What we obtain is that any strictly singular operator acts like a *decreasing shift* on the type of a seminormalized basic sequence, meaning that $(Sx_n)_n$ must be norm null or have a lower type than that of $(x_n)_n$ whenever $S \in \mathcal{SS}(\mathfrak{R}_n)$.

Lemma 4.3.3. *Let $(x_n)_n$ be a seminormalized basic sequence sequence of type f_k and $S \in \mathcal{SS}(\mathfrak{R}_n)$. Then $(Sx_n)_n$ has a norm null subsequence or a seminormalized basic subsequence of type f_j where $j < k$.*

Proof. Let $(x_n)_n$ be of type f_k for some $0 \leq k \leq n-1$ and assume that $(Sx_n)_n$ has no norm-null subsequence. Then passing to a subsequence que have that $\|Sx_n\| \geq \varepsilon > 0$. Thus, $(Sx_n)_n$ is seminormalized and weakly null since S is bounded and $(x_n)_n$ is basic in a reflexive space, hence shrinking. Using the Bessaga-Pełczyński selection principle we can assume that $(Sx_n)_n$ is seminormalized and basic. By Theorem 4.1.1 we have that $(Sx_n)_n$ has a (seminormalized basic) subsequence $(Sx_{n_l})_l$ of type f_j for $0 \leq j \leq k-1$. Indeed,

$(Sx_{n_l})_l$ can not be of type f_k because S is strictly singular. Moreover, we have the chain of continuous and proper inclusions

$$\ell_{f_{n-1}} \subset \ell_{f_{n-2}} \subset \cdots \subset \ell_{f_1} \subset \ell_2 \tag{4.28}$$

and those spaces are not isomorphic. Then, if $(Sx_{n_l})_l$ were of type f_j por some $j > k$ then, up to a subsequence,

$$\|\sum a_l e_l\|_{\ell_{f_j}} \sim \|\sum a_l S x_{n_l}\| \leq \|S\| \|\sum a_l x_{n_l}\| \sim \|\sum a_l e_l\|_{\ell_{f_k}}.$$

Since by (4.28) we have the inequality $\|x\|_{\ell_{f_k}} \leq \|x\|_{\ell_{f_j}}$, we conclude that the bases of ℓ_{f_k} and ℓ_{f_j} are equivalent, a contradiction. $\square$

Compare the following theorem with known results [48] on L_p spaces:

Theorem 4.3.4. *If $S_1, \ldots, S_n \in \mathcal{SS}(\mathfrak{R}_n)$ then $S_1 \cdots S_n \in \mathcal{K}(\mathfrak{R}_n)$.*

Proof. Recall that an operator T on a reflexive Banach space is compact if and only if for every normalized weakly null sequence $(x_n)_n$, we have that $(Tx_n)_n$ has a norm-null subsequence.

Let $(x_n)_n$ be a normalized weakly null sequence. Then passing to a subsequence and using the Bessaga-Pełczyński selection principle we can assume by Theorem 4.1.1 that $(x_n)_n$ is of type f_k for some $0 \leq k \leq n-1$. Consider now the sequence $(S_1 x_n)_n$. By Lemma 4.3.3 we have two possibilities:

(1) $\|S_1 x_{n_k}\| \to 0$ for some subsequence;

(2) $(S_1 x_n)_n$ has a seminormalized basic subsequence of type f_j for some $j < k$.

In the first case S_1 is compact an thus the product $S_1 \cdots S_n$ is also compact. In the second case we can consider the sequence $(S_2 S_1 x_n)_n$ and use Lemma 4.3.3 again to reach either to a norm-null subsequence or to a further subsequence of $(S_2 S_1 x_n)_n$ that is of type f_m for $m < j < k$. It is clear that this process can be repeated at most n times, the number of types of basic sequences, until reaching to a norm null subsequence. Thus $(S_1 \cdots S_n x_n)_n$ has a norm-null subsequence, hence $S_1 \cdots S_n \in \mathcal{K}(\mathfrak{R}_n)$. $\square$

4.3.3 Semi-Fredholm operators on $\mathfrak{R}_n$

We focus now in studying specific properties of semi-Fredholm operators on $\mathfrak{R}_n$, a topic we considered in [33]. We begin with a result in the spirit of Proposition A.1.4 for Fredholm operators: the upper semi-Fredholm character of an operator $T : X \to W$ on singular twisted sums only depends on its behaviour on the kernel of the quotient map.

Proposition 4.3.3. *Suppose that $0 \to Y \xrightarrow{i} X \xrightarrow{q} Z \to 0$ is a singular sequence. Then $T : X \to W$ is upper semi-Fredholm if and only if $Ti : Y \to W$ is upper semi-Fredholm.*

Proof. Since $i : Y \to X$ is an embedding, the direct implication is inmediate by standard properties of upper semi-Fredholm operators (cf. Proposition B.1.1). Conversely, suppose that $T \notin \Phi_+$. Then there exist an infinite dimensional closed subspace $M \subset X$ so that $T|_M$ is compact (see Remark 1 in Appendix B.1). Since $q : X \to Z$ is strictly singular, Proposition A.1.2 guarantees the existence of an infinite dimensional subspace $N \subset ker(q) = Y$ and a compact operator $K : N \to M$ such that $(I + K) : N \to M$ is an isomorphic embedding. It follows that $Ti|_N$ is compact, hence $Ti \notin \Phi_+$. Indeed, if $(y_n)_n \subset N$ is a bounded sequence then $T(I+K)(y_n)$ has a convergent subsequence since $T|_M$ is compact and $(I+K)(N) \subset M$. Taking into account that $T(I+K)(y_n) = Ty_n + TKy_n$ and that TK is compact, it follows that Ty_n has a convergent subsequence. $\square$

The following proposition is an analogue of Corollary 4.3.6 for Fredholm operators:

Proposition 4.3.4. *If $T \in \Phi_+(\mathfrak{R}_n)$ then $T^+T \in \Phi(\mathfrak{R}_n)$.*

Proof. Since $(T^+T)^+ = T^+T$, it is enough to show that $T^+T \in \Phi_+(\mathfrak{R}_n)$. Suppose that $T^+T \notin \Phi_+(\mathfrak{R}_n)$. Then by Proposition 4.3.3 and Corollary 4.1.4 it follows that $T^+T\imath_{1,n} \notin \Phi_+(\ell_2, \mathfrak{R}_n)$. Thus, there exists an infinite dimensional closed subspace $M \subset X$ so that $T^+Ti_{1,n}|_M$ is compact. We can assume that M is generated by a normalized block basic sequence $(w_n)_n$ in ℓ_2. Therefore, if we denote by $i_w : \ell_2 \to \ell_2$ the isometric embedding defined by $i_w e_n = w_n$, then $T^+T\imath_{1,n}i_w$ is compact. Let $T_n^{\mathfrak{w}}$ be the block operator associated to the sequence $(w_n)_n$. Since $T^+TT_n^{\mathfrak{w}}\imath_{1,n}e_n = T^+Ti_{1,n}i_we_n$ for each $n \in \mathbb{N}$, we deduce that $T^+TT_n^{\mathfrak{w}}i_{1,n}$ is compact; hence $T^+TT_n^{\mathfrak{w}}$ is strictly singular by Proposition A.1.4. Thus $(T_n^{\mathfrak{w}})^+T^+TT_n^{\mathfrak{w}} \in \mathcal{SS}(\mathfrak{R}_n)$, hence by Corollary 4.3.6 it follows that $TT_n^{\mathfrak{w}} \in \mathcal{SS}(\mathfrak{R}_n)$, and so $T \notin \Phi_+(\mathfrak{R}_n)$. $\square$

As an application we obtain that all copies of $\mathfrak{R}_n$ in $\mathfrak{R}_n$ are complemented:

Theorem 4.3.5. *Every subspace of $\mathfrak{R}_n$ isomorphic to $\mathfrak{R}_n$ is complemented.*

Proof. Let $T \in \mathfrak{L}(\mathfrak{R}_n)$ an isomorphism into with $R(T) = M$. Then $T^+T \in \Phi(\mathfrak{R}_n)$ by Proposition 4.3.4 and thus, the subspace $N(T^+) \cap R(T) \subset R(T)$ is finite dimensional. If we denote by N the complement

$$R(T) = N \oplus \big(N(T^+) \cap R(T)\big)$$

then N is finite codimensional on $R(T) = M$ and T^+ is an isomorphism on N. Then $T^+(N)$ is finite codimensional on $\mathfrak{R}_n$ since T^+T is Fredholm; therefore, $T^+(N)$ is complemented on $\mathfrak{R}_n$ and so is N by Lemma 4.3.1. As N is finite codimensional in M, it follows that M is complemented in $\mathfrak{R}_n$. $\square$

In fact, not only the range of isomorphic embeddings $\mathfrak{R}_n \to \mathfrak{R}_n$ are complemented, but also those of any semi-Fredhom operator:

Theorem 4.3.6. *Every semi-Fredholm operator on $\mathfrak{R}_n$ has complemented kernel and range.*

Proof. If $T \in \Phi_+(\mathfrak{R}_n)$ then the kernel is finite dimensional, and the range $R(T)$ is complemented using the arguments of Theorem 4.3.5. If $T \in \Phi_-(\mathfrak{R}_n)$ then $R(T)$ is finite codimensional and $T^* \in \Phi_+(\mathfrak{R}_n^*)$. Taking into account that $\mathfrak{R}_n$ is isomorphic to $\mathfrak{R}_n^*$ we conclude that $R(T^*)$ is complemented using the first part of the theorem, hence $N(T) = {}^\perp R(T^*)$ is also complemented. $\square$

4.3.4 The hyperplane problem on $\mathfrak{R}_n$

Recall Proposition 1.2.5 from Section 1.2.4: if one has a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \ell_2 \arrow[r] \arrow[d, "T"'] & Z_2 \arrow[r] \arrow[d] & \ell_2 \arrow[r] \arrow[d, "S"] & 0 \\ 0 \arrow[r] & \ell_2 \arrow[r] & Z_2 \arrow[r] & \ell_2 \arrow[r] & 0 \end{tikzcd}$$

then $T-S$ is compact. Equivalently, the difference between the diagonal entries of any upper triangular operator

$$\begin{pmatrix} T & \square \\ 0 & S \end{pmatrix} : Z_2 \to Z_2 \tag{4.29}$$

is strictly singular. As we explained in Section 1.2.6, the result is closely related to Johnson-Lindenstrauss-Schechteman Conjecture and the hyperplane problem; its main point is that any Fredholm operator of the form (4.29) has even index.

We are interested in generalizing to $\mathfrak{R}_n$ both the JLS-conjecture and the hyperplane problem. Since $\mathfrak{R}_n$ is isomorphic to its n-codimensional subspaces, the natural conjecture that extends the hyperplane problem to this setting is:

Problem 4 (Generalized Hyperplane Problem)**.** *Is $\mathfrak{R}_n$ isomorphic to its k-codimensional subspaces for $1 \le k \le n-1$? Equivalently, does there exist a Fredholm operator $T \in \mathcal{L}(\mathfrak{R}_n)$ that has index km for $1 \le k \le n-1$ and $m \in \mathbb{N}$?*

Obtaining a formulation of JLS-Conjecture in $\mathfrak{R}_n$ requires some work. Note that given an upper triangular operator on $\mathfrak{R}_n$,

$$R = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ 0 & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & a_{34} & a_{35} & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{n-1n-1} & a_{n-1n} \\ 0 & 0 & \cdots & 0 & 0 & a_{nn} \end{pmatrix} \in \mathcal{L}(\mathfrak{R}_n)$$

then, for any $k < n$, the operator R defines two upper triangular operators on lower Rochberg spaces: its restriction to $\mathfrak{R}_k$

$$R_k = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,k-1} & a_{1k} \\ 0 & a_{22} & a_{23} & a_{24} & \cdots & a_{2k} \\ 0 & 0 & a_{33} & a_{34} & a_{35} & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{k-1k-1} & a_{k-1k} \\ 0 & 0 & \cdots & 0 & 0 & a_{kk} \end{pmatrix} \in \mathcal{L}(\mathfrak{R}_k)$$

and the corresponding induced operator on the quotient space

$$R^{n-k} = \begin{pmatrix} a_{k+1k+1} & a_{k+1k+2} & a_{k+1k+3} & \cdots & a_{k+1,n-1} & a_{k+1n} \\ 0 & a_{k+2k+2} & a_{k+2k+3} & a_{k+2k+4} & \cdots & a_{k+2n} \\ 0 & 0 & a_{k+3k+3} & a_{k+3k+4} & a_{k+3k+5} & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{n-1n-1} & a_{n-1n} \\ 0 & 0 & \cdots & 0 & 0 & a_{nn} \end{pmatrix} \in \mathcal{L}(\mathfrak{R}_{n-k})$$

Thus, there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{R}_k \arrow[r] \arrow[d, "R_k"'] & \mathfrak{R}_n \arrow[r] \arrow[d, "R"] & \mathfrak{R}_{n-k} \arrow[r] \arrow[d, "R^{n-k}"] \arrow[ll, bend right, "\mathsf{KP}_{n-k,k}"'] & 0 . \\
0 \arrow[r] & \mathfrak{R}_k \arrow[r] & \mathfrak{R}_n \arrow[r] & \mathfrak{R}_{n-k} \arrow[r] \arrow[ll, bend left, "\mathsf{KP}_{n-k,k}"] & 0
\end{tikzcd}$$

We obtain now a generalization of Proposition 1.2.5:

Theorem 4.3.7. *If $R \in \mathcal{L}(\mathfrak{R}_n)$ then $R_k - R^k \in \mathcal{SS}(\mathfrak{R}_k)$ for every $k < n$.*

Proof. We work inductively. The case $n = 2$ is just Proposition 1.2.5. The case $n = 3$ will help us to explain the strategy: an upper triangular operator

$$R = \begin{pmatrix} \alpha & \beta & \varepsilon \\ 0 & \gamma & \delta \\ 0 & 0 & \eta \end{pmatrix} \in \mathcal{L}(\mathfrak{R}_3)$$

generates the two commutative diagrams:

$$\begin{tikzcd}
0 \arrow[r] & Z_2 \arrow[r] \arrow[d, "R_2"'] & \mathfrak{R}_3 \arrow[r] \arrow[d, "R"] & \ell_2 \arrow[r] \arrow[d, "R^1"] & 0 \\
0 \arrow[r] & Z_2 \arrow[r] & \mathfrak{R}_3 \arrow[r] & \ell_2 \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & \ell_2 \arrow[r] \arrow[d, "R_1"'] & \mathfrak{R}_3 \arrow[r] \arrow[d, "R"] & Z_2 \arrow[r] \arrow[d, "R^2"] & 0 \\
0 \arrow[r] & \ell_2 \arrow[r] & \mathfrak{R}_3 \arrow[r] & Z_2 \arrow[r] & 0
\end{tikzcd}$$

Since $R_2 \in \mathcal{L}(Z_2)$, $\alpha - \gamma$ is compact by induction; and since $R^2 \in \mathcal{L}(Z_2)$, $(R_2)_1 - (R_2)^1 = \gamma - \eta$ is compact as well. Therefore $\alpha - \gamma$ is compact too. Since $\alpha - \gamma$ is the restriction of the operator

$$R_2 - R^2 = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} - \begin{pmatrix} \gamma & \delta \\ 0 & \eta \end{pmatrix} = \begin{pmatrix} \alpha - \gamma & \beta - \delta \\ 0 & \gamma - \eta \end{pmatrix}$$

to ℓ_2, it follows that $R_2 - R^2$ must be strictly singular.
Assume the result has been proved for $n-1$ and pick $R \in \mathfrak{L}(\mathfrak{R}_n)$. Then using induction on R_{n-1} and on R^{n-1} we deduce, respectively, that

$$a_{ii} - a_{jj} \in \mathcal{K}(\ell_2) \quad \text{for all} \quad 1 \le i, j \le n-1$$

and

$$a_{ii} - a_{jj} \in \mathcal{K}(\ell_2) \quad \text{for all} \quad 2 \le i, j \le n.$$

Hence $a_{ii} - a_{jj}$ is compact in ℓ_2 for any $1 \le i, j \le n$. Given any $1 \le k \le n-1$, we have that $R_k - R^k$ equals

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k} \\ 0 & a_{22} & a_{23} & \cdots & a_{2k} \\ 0 & 0 & a_{33} & \cdots & a_{3k} \\ 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{kk} \end{pmatrix} - \begin{pmatrix} a_{n-k+1\, n-k+1} & a_{n-k+1\, n-k+2} & \cdots & \cdots & a_{n-k+1\, n} \\ 0 & a_{n-k+2\, n-k+2} & a_{n-k+2\, n-k+3} & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{n-1 n-1} & a_{n-1 n} \\ 0 & \cdots & 0 & 0 & a_{nn} \end{pmatrix}$$

and thus, $R_k - R^k$ is strictly singular in $\mathfrak{R}_k$ since its restriction to ℓ_2 is strictly singular. $\square$

Corollary 4.3.8. *Given a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{R}_n \arrow[r] \arrow[d, "T"'] & \mathfrak{R}_{2n} \arrow[r] \arrow[d] & \mathfrak{R}_n \arrow[r] \arrow[d, "S"] & 0 \\ 0 \arrow[r] & \mathfrak{R}_n \arrow[r] & \mathfrak{R}_{2n} \arrow[r] & \mathfrak{R}_n \arrow[r] & 0 \end{tikzcd}$$

in which T and S are upper triangular operators then $T - S$ is strictly singular.

Corollary 4.3.9. *If an upper triangular operator*

$$T = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ 0 & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & a_{34} & a_{35} & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{n-1n-1} & a_{n-1n} \\ 0 & 0 & \cdots & 0 & 0 & a_{nn} \end{pmatrix} \in \mathcal{L}(\mathfrak{R}_n)$$

is a Fredholm operator, then $ind(T)$ is a multiple of n.

Proof. Since there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \ell_2 \arrow[r] \arrow[d, "a_{11}"'] & \mathfrak{R}_n \arrow[r] \arrow[d, "T"] & \mathfrak{R}_{n-1} \arrow[r] \arrow[d, "R^{n-1}"] & 0 \\ 0 \arrow[r] & \ell_2 \arrow[r] & \mathfrak{R}_n \arrow[r] & \mathfrak{R}_{n-1} \arrow[r] & 0 \end{tikzcd}$$

we have that T is Fredholm if and only if a_{11} and R^{n-1} are Fredholm, in which case $ind(T) = ind(a_{11}) + ind(R^{n-1})$. We can reason inductively on R^{n-1} to reach that

$$ind(T) = \sum_{i=1}^{n} ind(a_{ii}).$$

Since the differences $a_{ii} - a_{jj}$ are compact for every $1 \leq i, j \leq n$, the invariance of the index under compact perturbations (cf. Subsection B.1) implies that $ind(a_{ii}) = ind(a_{jj})$ and the result follows. □

Taking into account Corollary 4.3.9, any Fredholm operator $T \in \mathcal{L}(\mathfrak{R}_n)$ of the form $T = S + U$, where $S \in \mathcal{SS}(\mathfrak{R}_n)$ and $U \in \mathcal{L}(\mathfrak{R}_n)$ is upper triangular, has index equal to a multiple of n. Hence we can formulate a general version of Johnson-Lindenstrauss-Schechteman as follows:

Conjecture 2 (Generalized JLS-conjecture)**.** *Any bounded operator $T : \mathfrak{R}_n \to \mathfrak{R}_n$ is a strictly singular perturbation of an upper triangular operator on $\mathfrak{R}_n$.*

A positive answer to this last problem solves the Generalized Hyperplane Problem discussed above.

4.4 $\mathfrak{R}_n$ is a non trivial symplectic Banach space

In this section we will work with $\mathfrak{R}_n^{\mathbb{R}}$, the real version of $\mathfrak{R}_n$, namely, the subspace of $\mathfrak{R}_n$ formed by real sequences $(x_{n-1}, \ldots, x_0) \in \ell_\infty^{\mathbb{R}}$ which belong to $\mathfrak{R}_n$. Since KP preserves real sequences, this subspace is closed (just use that $\ell_2^{\mathbb{R}}$ is closed in $\ell_2^{\mathbb{C}}$ and induction on $\|\cdot\|_n$) and defines an analogous short exact sequence

$$0 \longrightarrow \mathfrak{R}_l^{\mathbb{R}} \longrightarrow \mathfrak{R}_n^{\mathbb{R}} \longrightarrow \mathfrak{R}_k^{\mathbb{R}} \longrightarrow 0 \tag{4.30}$$

to its complex counterpart. Most results proven for $\mathfrak{R}_n$ also work for its real version. In particular, if $(u_n)_n$ is a normalized block basic sequence in $\ell_2^{\mathbb{R}} \subset \ell_2^{\mathbb{C}}$, then the corresponding block operator $T_n^{\mathfrak{u}} : \mathfrak{R}_n^{\mathbb{R}} \to \mathfrak{R}_n^{\mathbb{R}}$ is bounded on $\mathfrak{R}_n^{\mathbb{R}}$. Hence the quotient map of (4.30) is singular using the analogous result to Corollary 4.1.4. Moreover, the bilinear map $\omega_n : \mathfrak{R}_n \times \mathfrak{R}_n \to \mathbb{C}$ restricts to real values: if $x, y \in \mathfrak{R}_n^{\mathbb{R}}$ then

$$\omega_n^{\mathbb{R}}(x,y) = \omega_n(x,y) = \sum_{i=0}^{n-1} (-1)^i \langle x_i, y_{n-i-1} \rangle \in \mathbb{R}. \tag{4.31}$$

By the complex case it follows that $D_n^{\mathbb{R}} : \mathfrak{R}_n^{\mathbb{R}} \to (\mathfrak{R}_n^{\mathbb{R}})^*$ is injective and bounded. Using that $D_1 : \ell_2^{\mathbb{R}} \to (\ell_2^{\mathbb{R}})^*$ is an isomorphism, an induction and chasing argument over the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{R}_l^{\mathbb{R}} \arrow[r] \arrow[d, "(-1)^k D_l"'] & \mathfrak{R}_n^{\mathbb{R}} \arrow[r] \arrow[d, "D_n"'] & \mathfrak{R}_k^{\mathbb{R}} \arrow[r] \arrow[d, "D_k"] & 0 \\
0 \arrow[r] & (\mathfrak{R}_l^{\mathbb{R}})^* \arrow[r] & (\mathfrak{R}_n^{\mathbb{R}})^* \arrow[r] & (\mathfrak{R}_k^{\mathbb{R}})^* \arrow[r] & 0
\end{tikzcd}$$

implies that $D_n^{\mathbb{R}} : \mathfrak{R}_n^{\mathbb{R}} \to (\mathfrak{R}_n^{\mathbb{R}})^*$ is bijective as in the complex setting.

A bilinear map $\omega : X \times X \to \mathbb{R}$ on a real Banach space X is said to be *a linear symplectic form* if

(1) ω is continuous, i.e., $|\omega(x,y)| \leq K\|x\|\,\|y\|$ for all $x, y \in X$.

(2) ω is alternating: $\omega(x,y) = -\omega(y,x)$ for all $x, y \in X$.

(3) the induced map $D_\omega : X \to X^*$ given by $D_\omega(x)(y) = \omega(x,y)$ is an isomorphism onto.

In this case, (X, ω) is called a *symplectic Banach space*.

Any symplectic Banach space is necessarily isomorphic to its dual and reflexive by the Hahn-Banach Theorem. In fact, one has the following result (see [30, Lemma 2.2] or [59, pp. 98-99] for an explicit proof):

Lemma 4.4.1. *A continuous alternating bilinear map ω on real Banach space X is symplectic if and only if X is reflexive and $D_\omega : X \to X^*$ is isomorphism into.*

There is a direct method to construct symplectic Banach spaces: take any reflexive Banach space Y and consider $E = Y \oplus Y^*$ under the alternating form $\Omega_Y : E \oplus E \to \mathbb{R}$ given by

$$\Omega_Y[(z, z^*), (w, w^*)] = w^*(z) - z^*(w).$$

Then $(Y \oplus Y^*, \Omega_Y)$ is a symplectic Banach space. Two symplectic Banach spaces (X_1, ω_1) and (X_2, ω_2) are *equivalent* if there exist a linear isomorphism $T : X_1 \to X_2$ so that

$\omega_2(Tx, Ty) = \omega_1(x, y)$ for all $x, y \in X$. Observe that this is the same as saying that the following diagram is commutative:

$$\begin{tikzcd} X_1 \arrow[r, "T"] \arrow[d, "L_{\omega_1}"'] & X_2 \arrow[d, "L_{\omega_2}"] \\ X_1^* & X_2^* \arrow[l, "T^*"] \end{tikzcd}$$

A symplectic Banach space (X, ω) is said *trivial* if it is equivalent to some symplectic Banach space of the form $(Y \oplus Y^*, \Omega_Y)$.

In the late seventies there were some interest in obtaining specific examples of symplectic Banach spaces due to their connections with the theory of Banach manifolds (see [63, 98, 30] for a detailed account). Up to that moment, the only known symplectic Banach spaces were:

- Even finite dimensional Banach spaces.
- The infinite dimensional Hilbert spaces.
- The (trivial) symplectic spaces of the form $(Y \oplus Y^*, \Omega_Y)$ given previously.

Moreover, Weinstein showed in [98, Corollary 5.2] that all symplectic structures on a given Hilbert space are equivalent, thus trivial. For this reason, Weinstein asked if non-trivial symplectic Banach space existed at all. According to him, an equivalent way of checking triviality of symplectic Banach spaces is the following: given a symplectic Banach space (X, ω), call a subspace $F \subset X$ *isotropic* if

$$\omega(x, y) = 0 \quad \text{for all} \quad x, y \in F.$$

A subspace $F \subset X$ is a *Langrangian subspace* if F is isotropic, complemented in X and its complement is also isotropic. Now, if F is Lagrangian and X is decomposed as $X = F \oplus G$, then G can be identified with F^* via the isomorphism $\varphi : G \to F^*$ defined by

$$\varphi(x)(y) = \omega(x, y), \quad \text{for any } x \in G \text{ and } y \in F. \tag{4.32}$$

Here $\varphi(x)$ is obtained restricting the induced map $L_\omega : X \to X^*$ to G and composing with the projection $X^* \to F^*$ (see [98, pp. 336]). It follows that (X, ω) is equivalent to the trivial symplectic space $(F \oplus F^*, \Omega_F)$. Conversely, if (X, ω) is equivalent to the trivial symplectic structure $(Y \oplus Y^*, \Omega_Y)$ via an isomorphism $T : X \to Y \oplus Y^*$, then $T^{-1}(Y \times \{0\})$ is a Lagrangian subspace of X.

Using this approach, Kalton and Swanson [63] showed that $(Z_2^{\mathbb{R}}, \omega_2^{\mathbb{R}})$ is a symplectic Banach space such that any isotropic subspace is finite dimensional, thus proving that Z_2 has no Lagrangian subspace, solving in the negative the question raised by Weinstein.

We studied in [30] symplectic Banach spaces and, in particular, the Rochberg spaces $\mathfrak{R}_n^{\mathbb{R}}$ were added to the list of non-trivial symplectic spaces. To see this observe first that $\mathfrak{R}_n^{\mathbb{R}}$ is symplectic for all $n \geq 1$:

Theorem 4.4.1. *All Rochberg spaces $\mathfrak{R}_n^{\mathbb{R}}$ are symplectic.*

Proof. Observe that $\omega_n^{\mathbb{R}}$ is alternated if and only if n is even; so, the result holds for even n by the comments at the beginning of this section. For n odd, consider the isomorphism $\sigma : \ell_2 \to \ell_2$ given by $\sigma(x) = (-x_2, x_1, -x_4, x_3, \ldots)$. Then σ is interpolating for (ℓ_∞, ℓ_1). Since σ preserves real sequences we deduce that the induced diagonal operator σ_n is bounded on $\mathfrak{R}_n^{\mathbb{R}}$. We define the bilinear map $\overline{\omega_n} : \mathfrak{R}_n^{\mathbb{R}} \times \mathfrak{R}_n^{\mathbb{R}} \to \mathbb{R}$ given by

$$\begin{aligned}\overline{\omega_n}\big((x_{n-1}, \ldots, x_0), (y_{n-1}, \ldots, y_0)\big) &= \omega_n^{\mathbb{R}}\big((x_{n-1}, \ldots, x_0), \sigma_n(y_{n-1}, \ldots, y_0)\big) \\ &= \sum_{i=0}^{n-1} (-1)^i \langle x_i, \sigma y_{n-i-1} \rangle.\end{aligned}$$

This map is now alternated due to the fact that $\sigma^* = -\sigma$. Indeed,

$$\begin{aligned}\overline{\omega_n}\big((x_{n-1}, \ldots, x_0), (y_{n-1}, \ldots, y_0)\big) &= \sum_{i=0}^{n-1} (-1)^i \langle x_i, \sigma y_{n-i-1} \rangle \\ &= \sum_{i=0}^{n-1} (-1)^i \langle \sigma^* x_i, y_{n-i-1} \rangle \\ &= \sum_{i=0}^{n-1} (-1)^i \, (-1) \langle \sigma x_i, y_{n-i-1} \rangle \\ &= (-1) \sum_{i=0}^{n-1} (-1)^i \langle y_{n-i-1}, \sigma x_i \rangle \\ &= (-1) \sum_{j=0}^{n-1} (-1)^{n-j-1} \, \langle y_j, \sigma x_{n-j-1} \rangle \\ &= (-1) \sum_{j=0}^{n-1} (-1)^j \langle y_j, \sigma x_{n-j-1} \rangle \\ &= -\overline{\omega_n}\big((y_{n-1}, \ldots, y_0), (x_{n-1}, \ldots, x_0)\big).\end{aligned}$$

Boundedness follows from the boundness of ω_n and σ_n:

$$\big|\overline{\omega_n}(x, y)\big| = \big|\omega_n^{\mathbb{R}}(x, \sigma_n y)\big| \leq K \, \|x\| \, \|\sigma_n y\| \leq C \|x\| \, \|y\|.$$

To obtain that $(\mathfrak{R}_n^{\mathbb{R}}, \overline{\omega_n})$ is symplectic it suffices by Lemma 4.4.1 to show that the induced linear map $L_{\overline{\omega_n}} : \mathfrak{R}_n^{\mathbb{R}} \to (\mathfrak{R}_n^{\mathbb{R}})^*$ is an isomorphism into. Assume that there exists $x \in \mathfrak{R}_n^{\mathbb{R}}$ such that $L_{\overline{\omega_n}}(x)(y) = 0$ for all $y \in \mathfrak{R}_n^{\mathbb{R}}$. Thus $L_{\omega_n}(x)(\sigma_n y) = 0$ for all $y \in \mathfrak{R}_n^{\mathbb{R}}$. Taking into account that σ_n is invertible in $\mathfrak{R}_n^{\mathbb{R}}$, it follows that $L_{\omega_n}(x)(y) = 0$ for all $y \in \mathfrak{R}_n^{\mathbb{R}}$, so that $x = 0$. Moreover, as σ_n is an isomorphism, its clear that $\overline{\omega_n}$ has closed range because $\omega_n^{\mathbb{R}}$ has closed range. □

Note that even Rochberg spaces are symplectic spaces in a natural way. The case of odd Rochberg spaces required to "twist" the duality $\omega_n^{\mathbb{R}}$ with a complex structure on $\mathfrak{R}_n^{\mathbb{R}}$, i.e., a bounded operator $T : \mathfrak{R}_n^{\mathbb{R}} \to \mathfrak{R}_n^{\mathbb{R}}$ such that $T^2 = -I$. If n is even and $T \in \mathcal{L}(\mathfrak{R}_n^{\mathbb{R}})$ then we can define the adjoint operator $T^+ \in \mathcal{L}(\mathfrak{R}_n^{\mathbb{R}})$ in the same way as we did in Subsection 4.2.2:

$$\omega_n^{\mathbb{R}}(T^+ x, y) = \omega_n^{\mathbb{R}}(x, Ty) \quad \text{ for all } x, y \in \mathfrak{R}_n^{\mathbb{R}}.$$

It was proved in Theorem 4.2.3 that block operators preserve ω_n; thus if $T_n^{\mathfrak{u}}$ is a block operator defined by some normalized block basic sequence $(u_n)_n \subset \ell_2^{\mathbb{R}}$, then $T_n^{\mathfrak{u}}$ preserves the symplectic map $\omega_n^{\mathbb{R}}$, i.e., block operators preserve the symplectic structure.

For odd n, we have a different duality given by $\overline{\omega_n}$, and so a new involution $\sharp$ on $\mathcal{L}(\mathfrak{R}_n^{\mathbb{R}})$ defined as:

$$\overline{\omega_n}(T^\sharp x, y) = \overline{\omega_n}(x, Ty) \quad \text{ for all } x, y \in \mathfrak{R}_n^{\mathbb{R}}.$$

To prove that $(\mathfrak{R}_{2n}, \omega_{2n}^{\mathbb{R}})$ and $(\mathfrak{R}_{2n-1}, \overline{\omega_{2n-1}})$ are also non-trivial symplectic spaces we will show, like Kalton and Swanson did for $Z_2^{\mathbb{R}}$, that any isotropic subspace $F \subset \mathfrak{R}_n^{\mathbb{R}}$ must be finite dimensional. To see this, note that if $P : \mathfrak{R}_n^{\mathbb{R}} \to \mathfrak{R}_n^{\mathbb{R}}$ is a bounded projection onto some isotropic subspace F, then by (4.32) we have $\omega_n^{\mathbb{R}}(Px, Py) = 0$ for all $x, y \in \mathfrak{R}_n^{\mathbb{R}}$. It follows that:

(i) If n is even, $P^+P = 0$;

(ii) If n is odd, $P^\sharp P = 0$.

Using (i) we can prove:

Corollary 4.4.1. $(\mathfrak{R}_{2n}^{\mathbb{R}}, \omega_{2n}^{\mathbb{R}})$ *has no Lagrangian subspace.*

Proof. By Corollary 4.3.6 it follows that P is strictly singular. Since P is a projection, we conclude that P has finite range. Thus, any isotropic complemented subspace of $\mathfrak{R}_{2n}^{\mathbb{R}}$ is finite dimensional. □

To obtain the same result for odd Rochberg spaces we need an analogue of Corollary 4.3.6 for the involution $\sharp$. First note that if $T \in \mathcal{L}(\mathfrak{R}_{2n-1}^{\mathbb{R}})$ then

$$\overline{\omega_{2n-1}}(T^\sharp x, y) = \overline{\omega_{2n-1}}(x, Ty) = \omega_{2n-1}^{\mathbb{R}}(x, \sigma_n Ty) \quad \text{for all } x, y \in \mathfrak{R}_{2n-1}^{\mathbb{R}}.$$

Thus $T^\sharp = \big(\sigma_n T\big)^+ = T^+\sigma_n^+ = -T^+\sigma_n$. Using this last identity we can obtain:

Lemma 4.4.2. *If* $T \in \mathcal{L}(\mathfrak{R}_{2n-1}^{\mathbb{R}})$ *and* $T^\sharp T$ *is strictly singular, then* T *is strictly singular.*

Proof. Assume that T is not strictly singular. By Proposition 4.3.1 there exists $\alpha \neq 0$ and block operators $T_{2n-1}^{\mathfrak{u}}, T_{2n-1}^{\mathfrak{v}}$ such that $TT_{2n-1}^{\mathfrak{u}} = \alpha T_{2n-1}^{\mathfrak{v}} - S$ with S strictly singular. Therefore

$$\begin{aligned}(T_{2n-1}^{\mathfrak{u}})^\sharp T^\sharp T T_{2n-1}^{\mathfrak{u}} &= (TT_{2n-1}^{\mathfrak{u}})^\sharp TT_{2n-1}^{\mathfrak{u}} = \big((\alpha T_{2n-1}^{\mathfrak{v}})^\sharp - S^\sharp\big)(\alpha T_{2n-1}^{\mathfrak{v}} - S)\\ &= \alpha'(T_{2n-1}^{\mathfrak{v}})^\sharp T_{2n-1}^{\mathfrak{v}} + S' = -\alpha'(T_{2n-1}^{\mathfrak{v}})^+\sigma_n T_{2n-1}^{\mathfrak{v}} + S',\end{aligned}$$

where S' is strictly singular. This means that if $T^\sharp T$ is strictly singular, then $(T_{2n-1}^{\mathfrak{v}})^+\sigma_n T_{2n-1}^{\mathfrak{v}}$ must be strictly singular, but since $\sigma_n T_{2n-1}^{\mathfrak{v}}$ is invertible, $(T_{2n-1}^{\mathfrak{v}})^+$ must be strictly singular, as well as $T_{2n-1}^{\mathfrak{v}}$, which is a contradiction since block operators are never strictly singular. □

Chapter 5

Rochberg spaces for other interpolation scales

In this chapter we will study the Rochberg spaces associated to the couple formed by the 2-convexified Tsirelson space $\mathcal{T}_2$ and its dual. The main objective is to prove that they are weak Hilbert spaces (see Section 5.1 below for the definition). Suárez had already proved in [91] that the second Rochberg space $d\mathcal{T}_2$ is weak Hilbert. His proof depends on quite technical estimates about the sequence of local type 2 constants $a_{m,2}(d\mathcal{T}_2)$. In the general case we need to obtain an analogous estimate for the local type 2 constants $a_{m,2}\big(\mathfrak{R}_n(\mathcal{T}_2, \mathcal{T}_2^*)_{1/2}\big)$.
We prove first that for general Banach couples the constant $a_{m,2}\big(\mathfrak{R}_n(X_0, X_1)_\theta\big)$ does essentially depend only on $a_{m,2}(X_j)$ for $j = 0, 1$. This result is a generalization of the estimate obtained by Suárez in [91, Prop. 3] for $n = 2$. We will use the norm (see (2.10))

$$\|f\|_{\mathcal{C}_\theta} = \int_{\partial_0} \|f(it)\|_{X_0} d\mu_\theta^0(it) + \int_{\partial_1} \|f(1+it)\|_{X_1} d\mu_\theta^1(1+it).$$

to ease some calculations.

Proposition 5.0.1. *If $g \in \mathcal{C}(X_0, X_1)$ then $\|(\Delta_{n-1}, \ldots, \Delta_0)(g)\|_n \le C_{n-1}\|g\|_{\mathcal{C}_\theta}$.*

Proof. We will work by induction on n; the case $n = 1$ is (2.10). Let $(x_{n-1}, \ldots, x_0) = x \in \mathfrak{R}_n(X_0, X_1)_\theta$ and let $f \in \mathcal{C}(X_0, X_1)$ such that $(\Delta_{n-1}, \ldots, \Delta_0)(f) = x$ and $\|f\|_{\mathcal{C}} \le (1+\varepsilon)\|x\|_n$. If we denote $x = (x_{n-1}, x^{n-2})$ then we can decompose

$$x = (x_{n-1}, \ldots, x_0) = (x_{n-1} - \Omega_{n-1,1}(x^{n-2}), 0, \overset{(n-1)}{\ldots}, 0) + (\Omega_{n-1,1}(x^{n-2}), x^{n-2}).$$

Both elements belong to $\mathfrak{R}_n(X_0, X_1)_\theta$:

(1) $(\Delta_{n-1}, \ldots, \Delta_0)(f - B_\theta^{n-1}(x^{n-2})) = (x_{n-1} - \Omega_{n-1,1}(x^{n-2}), 0, \overset{(n-1)}{\ldots}, 0)$

(2) $(\Delta_{n-1}, \ldots, \Delta_0)(B_\theta^{n-1}(x^{n-2})) = (\Omega_{n-1,1}(x^{n-2}), x_{n-2}, \ldots, x_0)$

where $B_\theta^{n-1} : \mathfrak{R}_{n-1}(X_0, X_1)_\theta \to \mathcal{C}(X_0, X_1)$ is an homogeneous bounded selector for $(\Delta_{n-2}, \ldots, \Delta_0)$. Thus it follows that

$$\|(x_{n-1}, \ldots, x_0)\|_n \le \|(x_{n-1} - \Omega_{n-1,1}(x^{n-2}), 0, \ldots, 0)\|_n + \|(\Omega_{n-1,1}(x^{n-2}), x_{n-2}, \ldots, x_0)\|_n. \tag{5.1}$$

Now observe the bound on the second summand:

$$\begin{aligned}\|(\Omega_{n-1,1}(x^{n-2}), x^{n-2})\|_n &= \|(\Delta_{n-1}, \dots, \Delta_0)(B_\theta^{n-1}(x^{n-2}))\|_n \\ &\leq \|B_\theta^{n-1}(x^{n-2})\|_{\mathcal{C}} \leq C\|(x_{n-2}, \dots, x_0)\|_{n-1} \qquad (5.2)\\ &= \|(\Delta_{n-2}, \dots, \Delta_0)(f)\|_{n-1} \leq K_{n-2}\|f\|_{\mathcal{C}_\theta}, \qquad (5.3)\end{aligned}$$

where in the last inequality we used induction on $n-1$.
To bound the first summand of (5.1) we note that since $h = f - B_\theta^{n-1}(x^{n-2}) \in \bigcap_{j=0}^{n-2} \ker \Delta_j$, then $h = \varphi_\theta^{n-1} g$ for some $g \in \mathcal{C}(X_0, X_1)$ (see [22, Lemma 3.8-(2)] or [12, Lemma 3]) and $\Delta_{n-1}(h) = (n-1)!\varphi'(\theta)^{n-1}\Delta_0(g)$. Then

$$G(z) = (n-1)!\varphi'(\theta)^{n-1} g(z) = (n-1)!\varphi'(\theta)^{n-1} \frac{h(z)}{\varphi_\theta^{n-1}(z)} \in \mathcal{C}(X_0, X_1).$$

satisfies $\Delta_{n-1}(h) = \Delta_0(G)$. Moreover:

$$\begin{aligned}\|G\|_{\mathcal{C}_\theta} &= (n-1)!|\varphi'(\theta)^{n-1}| \int_{\partial_0} \frac{1}{|\varphi(it)|^{n-1}} \|h(it)\|_{X_0} d\mu_0^\theta(it) \\ &+ (n-1)!|\varphi'(\theta)^{n-1}| \int_{\partial_1} \frac{1}{|\varphi(1+it)|^{n-1}} \|h(1+it)\|_{X_1} d\mu_1^\theta(1+it) \\ &= (n-1)!|\varphi'(\theta)^{n-1}|\, \|h\|_{\mathcal{C}_\theta} \leq C_{n-1}(\|f\|_{\mathcal{C}_\theta} + \|B_\theta^{n-1}(x^{n-2})\|_{\mathcal{C}_\theta}).\end{aligned}$$

By (5.2) we deduce that there exist a constant C_{n-1} such that

$$\|(x_{n-1} - \Omega_{n-1,1}(x^{n-2}), 0, \dots, 0)\|_n \leq C_{n-1}\|f\|_{\mathcal{C}_\theta}. \qquad (5.4)$$

Combine (5.3) and (5.4) in (5.1) to end the proof. □

Corollary 5.0.1. *Suppose that $f \in \ker \delta_\theta$. Then $\|(\Delta_n, \dots, \Delta_1)(f)\|_n \leq K_n\|f\|_{\mathcal{C}_\theta}$.*

Proof. If $f \in \ker \delta_\theta$, then by Proposition 5.0.1 we have that

$$\|(\Delta_n, \dots, \Delta_1)(f)\|_n \sim \|(\Delta_n, \dots, \Delta_0)(f)\|_{n+1} \leq C_n\|f\|_{\mathcal{C}_\theta}. \quad \square$$

The proof of the following proposition depends on several lemmata involving probabilistic estimates about sums of functions of the Calderón space, estimates that we will prove later. To ease readability we explicitly state the notation that we will use:

- Let us write $\mathfrak{R}_n = \mathfrak{R}_n(X_0, X_1)_\theta$.
- B will be a bounded homogeneous selection for $\Delta_0 : \mathcal{C}(X_0, X_1) \to X_\theta$.
- $K(z) = \frac{1}{a_{m,2}(X_0)^{1-z}\, a_{m,2}(X_1)^z}$.
- $B_\varepsilon(z) = \sum_{j=1}^m \varepsilon_j B(x_j)(z)$.
- $F_\varepsilon \in \mathcal{C}(X_0, X_1)$ will be the function

$$F_\varepsilon(z) = K(z) \sum_{j=1}^m \varepsilon_j B(x_j)(z) = K(z)\, B_\varepsilon(z).$$

- $\Sigma_{n-1} = \sum_{k=1}^{n-1} \frac{1}{k!} \log^k \left(\frac{a_{m,2}(X_0)}{a_{m,2}(X_1)}\right)\Big((\Delta_{n-1-k}, \dots, \Delta_0)(B(z)), \overset{(k-1)}{\dots}, 0\Big)$.

- $\nabla_\varepsilon^{(n-1)} = \Big\| \sum_{j=1}^m \varepsilon_j \Omega_{1,n-1}(x_j) - \Omega_{1,n-1}\big(\sum_{j=1}^m \varepsilon_j x_j\big) \Big\|_{n-1}$.
- $\square_\varepsilon^{n-1} = \Big\| \Omega_{1,n-1}(\sum_{j=1}^m \varepsilon_j x_j) - \sum_{j=1}^m \varepsilon_j \Omega_{1,n-1}(x_j) - \Sigma_{n-1} \Big\|_{n-1}$.

Proposition 5.0.2. *If (X_0, X_1) is a couple, then for any $0 < \theta < 1$ there exists a function f_n such that*

$$a_{m,2}(\mathfrak{R}_n(X_0,X_1)_\theta) \leq f_n(a_{m,2}(X_0), a_{m,2}(X_1), a_{m,2}(X_\theta)).$$

Proof. If $\xi_j = (x_j^n, \ldots, x_j^2, x_j^1) \in \mathfrak{R}_n$ then

$$\begin{aligned}
\mathbb{E}\Big\| \sum_{j=1}^m \varepsilon_j \xi_j \Big\|_n &= \mathbb{E}\Big\| \sum_{j=1}^m \varepsilon_j (x_j^n, \ldots, x_j^2) - \Omega_{1,n-1}(\sum_{j=1}^m \varepsilon_j x_j^1) \Big\|_{n-1} + \mathbb{E}\Big\| \sum_{j=1}^m \varepsilon_j x_j^1 \Big\|_\theta \\
&\leq \mathbb{E}\Big\| \sum_{j=1}^m \varepsilon_j \big((x_j^n, \ldots, x_j^2) - \Omega_{1,n-1}(x_j^1)\big) \Big\|_{n-1} \\
&\quad + \mathbb{E}\Big\| \sum_{j=1}^m \varepsilon_j x_j^1 \Big\|_\theta + \mathbb{E}\Big\| \Omega_{1,n-1}\big(\sum_{j=1}^m \varepsilon_j x_j^1\big) - \sum_{j=1}^m \varepsilon_j \Omega_{1,n-1}(x_j^1) \Big\|_{n-1} \\
&\leq a_{m,2}(\mathfrak{R}_{n-1}) \Big(\sum_{j=1}^m \|(x_j^n, \ldots, x_j^2) - \Omega_{1,n-1}(x_j^1)\|_{n-1}^2 \Big)^{1/2} \\
&\quad + a_{m,2}(X_\theta) \Big(\sum_{j=1}^m \|x_j^1\|_\theta^2 \Big)^{1/2} + \mathbb{E}\nabla_\varepsilon^{(n-1)} \\
&\leq 2\Big(a_{m,2}(\mathfrak{R}_{n-1}) + a_{m,2}(X_\theta) \Big) \Big(\sum_{j=1}^m \|\xi_j\|_n^2 \Big)^{1/2} + \mathbb{E}\nabla_\varepsilon^{(n-1)}.
\end{aligned}$$

The estimate of forthcoming Lemma 5.0.4 yields

$$\begin{aligned}
\mathbb{E}\nabla_\varepsilon^{(n-1)} &\leq \mathbb{E}\square_\varepsilon^{(n-1)} + \mathbb{E}\|\Sigma_{n-1}\|_{n-1} \\
&\leq \gamma \Big(\sum_{j=1}^m \|\xi_j\|_n^2 \Big)^{1/2} + \mathbb{E}\|\Sigma_{n-1}\|_{n-1},
\end{aligned}$$

where $\gamma = C_n \|B\| \Big(a_{m,2}(X_\theta) + a_{m,2}(X_0)^{1-\theta}\, a_{m,2}(X_1)^\theta \Big)$. By Lemma 5.0.2 it follows:

$$\mathbb{E}\nabla_\varepsilon^{(n-1)} \leq \big(\gamma + \eta\big) \Big(\sum_{j=1}^m \|\xi_j\|_n^2 \Big)^{1/2},$$

where

$$\eta = \Big[\sum_{k=1}^{n-2} \frac{1}{k!} \Big| \log^k \Big(\frac{a_{m,2}(X_0)}{a_{m,2}(X_1)} \Big) \Big| C_{n-k} \|B\| \max\{a_{m,2}(X_0), a_{m,2}(X_1)\} + \frac{1}{(n-1)!} \Big| \log^{n-1} \Big(\frac{a_{m.2}(X_0)}{a_{m,2}(X_1)} \Big) \Big| a_{m,2}(X_\theta) \Big].$$

Summing all up, we have:

$$\begin{aligned}
\mathbb{E}\Big\| \sum_{j=1}^m \varepsilon_j \xi_j \Big\|_n &\leq 2\Big(a_{m,2}(\mathfrak{R}_{n-1}) + a_{m,2}(X_\theta) \Big) \Big(\sum_{j=1}^m \|\xi_j\|_n^2 \Big)^{1/2} \\
&\quad + \big(\gamma + \eta\big) \Big(\sum_{j=1}^m \|\xi_j\|_n^2 \Big)^{1/2} \\
&= \Big(2\big(a_{m,2}(\mathfrak{R}_{n-1}) + a_{m,2}(X_\theta) \big) + \gamma + \eta \Big) \Big(\sum_{j=1}^m \|\xi_j\|_n^2 \Big)^{1/2}.
\end{aligned}$$

A straightforward induction argument is enough to conclude with the proof. $\square$

We prove now the anounced technical lemmata needed in the proof of Proposition 5.0.2.

Lemma 5.0.1.

$$a_{m,2}(X_0)^{1-\theta}\, a_{m,2}(X_1)^{\theta}(\Delta_{n-1},\dots,\Delta_1)(F_\varepsilon)=\sum_{j=1}^{m}\varepsilon_j\Omega_{1,n-1}(x_j)+\Sigma_{n-1}.$$

Proof. Using the generalized Leibniz rule for derivatives we have that

$$(\Delta_{n-1},\dots,\Delta_1)(F_\varepsilon)=\Big(\frac{1}{(n-1)!}\sum_{k=0}^{n-1}\binom{n-1}{k}B_\varepsilon^{(n-1-k)}(\theta)K^{(k)}(\theta),\dots,B'_\varepsilon(\theta)K(\theta)+B_\varepsilon(\theta)K'(\theta)\Big).$$

If we isolate the terms that multiply the highest derivative of the function $B_\varepsilon(z)$ in each component we obtain the summands:

$$\Big(\frac{1}{(n-1)!}B_\varepsilon^{n-1}(\theta)K(\theta),\dots,B'_\varepsilon(\theta)K(\theta)\Big)=K(\theta)\sum_{j=1}^{m}\Omega_{1,n-1}(x_j)$$

and

$$\Big(\frac{1}{(n-1)!}\sum_{k=1}^{n-1}\binom{n-1}{k}B_\varepsilon^{(n-1-k)}(\theta)K^{(k)}(\theta),\dots,B_\varepsilon(\theta)K'(\theta)\Big).\tag{5.5}$$

Now rewrite (5.5) as

$$\Big(\sum_{k=1}^{n-1}\frac{1}{k!(n-1-k)!}B_\varepsilon^{(n-1-k)}(\theta)K^{(k)}(\theta),\dots,B_\varepsilon(\theta)K'(\theta)\Big).$$

Now we repeat the same process with this last summand: we isolate the terms that multiply the highest derivative of $B_\varepsilon(z)$ in each component to reach

$$\begin{aligned}(5.5)=&\Big(\frac{1}{(n-2)!}B_\varepsilon^{(n-2)}(\theta)K'(\theta),\dots,B_\varepsilon(\theta)K'(\theta)\Big)\\&+\Big(\sum_{k=2}^{n-1}\frac{1}{k!(n-1-k)!}B_\varepsilon^{(n-1-k)}(\theta)K^{(k)}(\theta),\dots,0\Big).\end{aligned}$$

The first summand is $K(\theta)\log\frac{a_{m.2}(X_0)}{a_{m,2}(X_1)}(\Delta_{n-2},\dots,\Delta_0)(B_\varepsilon)$ while the second is

$$\Big(\sum_{k=2}^{n-1}\frac{1}{k!(n-1-k)!}B_\varepsilon^{(n-1-k)}(\theta)K^{(k)}(\theta),\sum_{k=2}^{n-2}\frac{1}{k!(n-2-k)!}B_\varepsilon^{(n-2-k)}(\theta)K^{(k)}(\theta),\dots,0\Big).\tag{5.6}$$

Isolating the highest derivatives of B_ε in (5.6) yet again we deduce that

$$\begin{aligned}(5.6)=&\frac{1}{2!}K(\theta)\log^2\Big(\frac{a_{m,2}(X_0)}{a_{m,2}(X_1)}\Big)\big((\Delta_{n-3},\dots,\Delta_0)(B_\varepsilon(z)),0\big)\\&+\Big(\sum_{k=3}^{n-1}\frac{1}{k!(n-1-k)!}B_\varepsilon^{(n-1-k)}(\theta)K^{(k)}(\theta),\sum_{k=3}^{n-2}\frac{1}{k!(n-2-k)!}B_\varepsilon^{(n-2-k)}(\theta)K^{(k)}(\theta),\dots,0\Big).\end{aligned}$$

Working inductively it follows that

$$a_{m,2}(X_0)^{1-\theta}\, a_{m,2}(X_1)^{\theta}(\Delta_{n-1},\dots,\Delta_1)(F_\varepsilon) = \sum_{j=1}^{m} \varepsilon_j \Omega_{1,n-1}(x_j)$$
$$+ \sum_{k=1}^{n-1} \frac{1}{k!} \log^k \Big(\frac{a_{m,2}(X_0)}{a_{m,2}(X_1)}\Big) \Big((\Delta_{n-1-k},\dots,\Delta_0)(B_\varepsilon(z)), \overset{(k-1)}{\dots}, 0\Big) \;\square$$

Lemma 5.0.2. *If*

$$\eta = \Big[\sum_{k=1}^{n-2} \frac{1}{k!}\Big|\log^k \Big(\frac{a_{m,2}(X_0)}{a_{m,2}(X_1)}\Big)\Big| C_{n-k}\|B\| \max\{a_{m,2}(X_0), a_{m,2}(X_1)\} + \frac{1}{(n-1)!}\Big|\log^{n-1}\Big(\frac{a_{m.2}(X_0)}{a_{m,2}(X_1)}\Big)\Big| a_{m,2}(X_\theta)\Big]$$

then

$$\mathbb{E}\|\Sigma_{n-1}\|_{n-1} \le \eta \Big(\sum_{j=1}^{m} \|\xi_j\|_n^2\Big)^{1/2},$$

Proof. First note that obviously $(\Delta_{n-2},\dots,\Delta_0)(B_\varepsilon) \in \mathfrak{R}_{n-1}$ by definition. Observe that the elements $\Big((\Delta_{n-1-k},\dots,\Delta_0)(B_\varepsilon), \overset{(k-1)}{\dots}, 0\Big)$ are in $\mathfrak{R}_{n-1}$ for each $k = 2,\dots,n-1$: they are the image of $(\Delta_{n-2},\dots,\Delta_0)(B_\varepsilon)$ by the operator $i_{n-k,n-1} \circ \pi_{n-1,n-k} : \mathfrak{R}_{n-1} \to \mathfrak{R}_{n-1}$. Therefore

$$\Big\|\sum_{k=1}^{n-1} \frac{1}{k!} \log^k \Big(\frac{a_{m,2}(X_0)}{a_{m,2}(X_1)}\Big)\Big((\Delta_{n-1-k},\dots,\Delta_0)(B_\varepsilon), \overset{(k-1)}{\dots}, 0\Big)\Big\|_{n-1}$$
$$\le \sum_{k=1}^{n-1} \frac{1}{k!}\Big|\log^k \Big(\frac{a_{m,2}(X_0)}{a_{m,2}(X_1)}\Big)\Big| \big\|\big((\Delta_{n-1-k},\dots,\Delta_0)(B_\varepsilon), \overset{(k-1)}{\dots}, 0\big)\big\|_{n-1} = (\clubsuit)$$

Corollary 5.0.1 shows that there exist a constant C_k depending only on k such that

$$\big\|\big((\Delta_{n-1-k},\dots,\Delta_0)(B_\varepsilon), \overset{(k-1)}{\dots}, 0\big)\big\|_{n-1} \sim \|(\Delta_k,\dots,\Delta_0)(B_\varepsilon)\|_{k+1} \le C_k \|B_\varepsilon\|_{\mathcal{C}_\theta}.$$

Therefore

$$(\clubsuit) \le \sum_{k=1}^{n-2} \frac{1}{k!}\Big|\log^k \Big(\frac{a_{m,2}(X_0)}{a_{m,2}(X_1)}\Big)\Big| C_{n-k}\|B_\varepsilon\|_{\mathcal{C}_\theta} + \frac{1}{(n-1)!}\Big|\log^{n-1}\Big(\frac{a_{m.2}(X_0)}{a_{m,2}(X_1)}\Big)\Big| \Big\|\sum_{j=1}^{m} \varepsilon_j x_j\Big\|_\theta.$$

Averaging at both sides in last inequality and using forthcoming Lemma 5.0.3 we conclude that

$$\mathbb{E}(\clubsuit) \le \Big[\sum_{k=1}^{n-2} \frac{1}{k!}\Big|\log^k \Big(\frac{a_{m,2}(X_0)}{a_{m,2}(X_1)}\Big)\Big| C_{n-k}\|B\| \max\{a_{m,2}(X_0), a_{m,2}(X_1)\}$$
$$+ \frac{1}{(n-1)!}\Big|\log^{n-1}\Big(\frac{a_{m.2}(X_0)}{a_{m,2}(X_1)}\Big)\Big| a_{m,2}(X_\theta)\Big]\Big(\sum_{j=1}^{m}\|x_j\|_\theta^2\Big)^{1/2}$$
$$\le \Big[\sum_{k=1}^{n-2} \frac{1}{k!}\Big|\log^k \Big(\frac{a_{m,2}(X_0)}{a_{m,2}(X_1)}\Big)\Big| C_{n-k}\|B\| \max\{a_{m,2}(X_0), a_{m,2}(X_1)\}$$
$$+ \frac{1}{(n-1)!}\Big|\log^{n-1}\Big(\frac{a_{m.2}(X_0)}{a_{m,2}(X_1)}\Big)\Big| a_{m,2}(X_\theta)\Big]\Big(\sum_{j=1}^{m}\|\xi_j\|_n^2\Big)^{1/2}. \qquad \square$$

Lemma 5.0.3. $\mathbb{E}\|B_\varepsilon\|_{\mathcal{C}_\theta} \le \|B\| \max\{a_{m,2}(X_0), a_{m,2}(X_1)\}\Big(\sum_{j=1}^m \|x_j\|_\theta^2\Big)^{1/2}$.

Proof. Note that

$$\|B_\varepsilon\|_{\mathcal{C}_\theta} = \int_{\partial_0} \Big\|\sum_{j=1}^m \varepsilon_j B(x_j)(it)\Big\|_{X_0} d\mu_\theta^0(it) + \int_{\partial_1} \Big\|\sum_{j=1}^m \varepsilon_j B(x_j)(1+it)\Big\|_{X_1} d\mu_\theta^1(1+it).$$

Averaging at both sides of last equality we deduce that

$$\begin{aligned}
\mathbb{E}\|B_\varepsilon\|_{\mathcal{C}_\theta} &= \int_{\partial_0} \mathbb{E}\Big\|\sum_{j=1}^m \varepsilon_j B(x_j)(it)\Big\|_{X_0} d\mu_\theta^0(it) + \int_{\partial_1} \mathbb{E}\Big\|\sum_{j=1}^m \varepsilon_j B(x_j)(1+it)\Big\|_{X_1} d\mu_\theta^1(1+it)\\
&\le a_{m.2}(X_0)\int_{\partial_0}\Big(\sum_{j=1}^m \|B(x_j)(it)\|_{X_0}^2\Big)^{1/2} d\mu_\theta^0(it)\\
&+ a_{m,2}(X_1)\int_{\partial_1}\Big(\sum_{j=1}^m \|B(x_j)(1+it)\|_{X_1}^2\Big)^{1/2} d\mu_\theta^1(1+it)\\
&\le a_{m.2}(X_0)\int_{\partial_0}\Big(\sum_{j=1}^m \|B(x_j)\|_{\mathcal{C}}^2\Big)^{1/2} d\mu_\theta^0(it)\\
&+ a_{m,2}(X_1)\int_{\partial_1}\Big(\sum_{j=1}^m \|B(x_j)\|_{\mathcal{C}}^2\Big)^{1/2} d\mu_\theta^1(1+it)\\
&\le \|B\| \max\{a_{m,2}(X_0), a_{m,2}(X_1)\}\Big(\sum_{j=1}^m \|x_j\|_\theta^2\Big)^{1/2}
\end{aligned}$$

□

Lemma 5.0.4. $\mathbb{E}\Box_\varepsilon \le C_n\|B\|\Big(a_{m,2}(X_\theta) + a_{m,2}(X_0)^{1-\theta}\, a_{m,2}(X_1)^\theta\Big)\Big(\sum_{j=1}^m \|x_j\|_\theta^2\Big)^{1/2}$.

Proof. By Lemma 5.0.1

$$\begin{aligned}
\Box_\varepsilon &= \Big\|\Omega_{1,n-1}\Big(\sum_{j=1}^m \varepsilon_j x_j\Big) - \sum_{j=1}^m \varepsilon_j \Omega_{1,n-1}(x_j) - \Sigma_{n-1}\Big\|_{\mathfrak{R}_{n-1}}\\
&= \Big\|(\Delta_{n-1},\ldots,\Delta_1)\big[B\big(\sum_{j=1}^m \varepsilon_j x_j\big) - a_{m,2}(X_0)^{1-\theta}\, a_{m,2}(X_1)^\theta F_\varepsilon\big]\Big\|_{\mathfrak{R}_{n-1}}.
\end{aligned}$$

Taking into account that the function inside the brackets vanishes on θ, it follows by Corollary 5.0.1 that

$$\Box_\varepsilon \le C_n\Big(\|B\big(\sum_{j=1}^m \varepsilon_j x_j\big)\|_{\mathcal{C}_\theta} + a_{m,2}(X_0)^{1-\theta}\, a_{m,2}(X_1)^\theta \|F_\varepsilon\|_{\mathcal{C}_\theta}\Big). \tag{5.7}$$

Since $\mathbb{E}\big\|B\big(\sum_{j=1}^m \varepsilon_j x_j\big)\big\|_{\mathcal{C}} \le \|B\| a_{m,2}(X_\theta)\Big(\sum_{j=1}^m \|x_j\|_\theta^2\Big)^{1/2}$ and by [91, Lemma 1] we have

$$\mathbb{E}\|F_\varepsilon\|_{\mathcal{C}_\theta} \le \|B\|\Big(\sum_{j=1}^m \|x_j\|_\theta^2\Big)^{1/2};$$

averaging in (5.7) we conclude

$$\mathbb{E}\Box_\varepsilon \le C_n\|B\|\Big(a_{m,2}(X_\theta) + a_{m,2}(X_0)^{1-\theta}\, a_{m,2}(X_1)^\theta\Big)\Big(\sum_{j=1}^m \|x_j\|_\theta^2\Big)^{1/2}. \quad \Box$$

5.1 The space $\mathcal{T}_2$

A finite dimensional subspace E of a Banach space X is said to be M-complemented if there exist a projection $P : X \to X$ onto E such that $\|P\| \le M$. The subspace E is C-euclidean if $d(E, \ell_2^{\dim(E)}) \le C$.

A Banach space X is weak Hilbert if there exist $0 < \delta < 1$ and a constant C such that every n-dimensional subspace $E \subset X$ contains a C-euclidean and C-complemented subspace F in X with $\dim(F) \ge \delta \dim(E)$. This is not the original definition, but it is equivalent [79, Th. 12.2]. The fundamental example of weak Hilbert space which is not a Hilbert space is the 2-convexification of Tsirelson space.

Let us recall first the definition of Tsirelson space following [20]: given finite subsets $E, F \subset \mathbb{N}$ denote by $E < F$ that $\max E < \min F$, and define $Ex = \sum_{n \in E} x_n$ for any $x = \sum_n x_n \in c_{00}$. The norm of Tsirelson space is defined inductively: one fixes $x \in c_{00}$ and consider the following sequence of norms $(\|\cdot\|_m)_{m=0}^{\infty}$

$$\begin{cases} \|x\|_0 = \|x\|_{c_0} \\ \\ \|x\|_{m+1} = \max\left\{\|x\|_m, \frac{1}{2}\max\left[\sum_{j=1}^k \|E_j x\|_m\right]\right\}, \quad \text{for } m \ge 0, \end{cases}$$

where the inner maximum is defined over all possible choices of finite subsets $E_1, \dots, E_k$ such that $k \le E_1 < E_2 < \cdots < E_k$.

One has for every fixed $x \in c_{00}$ and every $m \in \mathbb{N}$ that $\|x\|_m \le \|x\|_{m+1} \le \|x\|_{\ell_1}$. Thus, the limit $\|x\|_{\mathcal{T}} := \lim_{m\to\infty} \|x\|_m$ exist for every $x \in c_{00}$ and defines a norm on c_{00}. The Tsirelson space $\mathcal{T}$ is the completion of c_{00} with the norm $\|\cdot\|_{\mathcal{T}}$.

The 2-convexified Tsirelson space $\mathcal{T}_2$ is defined as the completion of c_{00} under the norm

$$\left\|\sum x_n e_n\right\|_{\mathcal{T}_2} = \left\|\sum |x_n|^2 e_j\right\|_{\mathcal{T}}^{1/2}.$$

The basic facts concerning $\mathcal{T}_2$ are:

(1) Both $\mathcal{T}_2$ and its dual $\mathcal{T}_2^*$ are weak Hilbert spaces [79, Chapter 13]. In particular, they are both reflexive [79, Th. 14.1].

(2) The canonical basis $(e_n)_n \subset \mathcal{T}_2$ and the dual basis $(e_n^*)_n \subset \mathcal{T}_2^*$ are both unconditional [20, Th. X.e.3].

Let us denote by $E_m \subset \mathcal{T}_2$ and by $E_m^* \subset \mathcal{T}_2^*$ the complemented subspaces

$$E_m = \operatorname{span}\{e_j \colon j \ge m\} \quad \text{and} \quad E_m^* = \operatorname{span}\{e_j^* \colon j \ge m\}.$$

The following result highlights the importance of the subspaces E_m and E_m^* (see [20, Prop. A.b.2]):

Lemma 5.1.1. *Given any $m \in \mathbb{N}$, there exists a constant $C > 0$ such that every 5^{5^m}-dimensional subspace of $E_m \subset \mathcal{T}_2$ is C-euclidean.*

Lemma 5.1.1 also holds for $\mathcal{T}_2^*$ if one replaces E_m by the subspace E_m^* (see the comments in [20, pp. 130]). Using both facts it follows that for any $m \in \mathbb{N}$ one has

$$\max\{a_{5^{5^m},2}(E_m), a_{5^{5^m},2}(E_m^*)\} \le C. \tag{5.8}$$

Recall that given a Banach space X and subspaces $Y \subset X$ and $Z \subset X^*$, Y is called λ-*norming* over Z if

$$\|z\| \leq \lambda \sup\{|z(y)| \colon \|y\| \leq 1\}$$

for every $z \in Z$. We will need the following result of Johnson [55, Lemma 1.6]:

Proposition 5.1.1. *Let $Y \subset X$ and $Z \subset X^*$ be closed subspaces such that Y is λ-norming over Z. If every 5^m-dimensional subspace of Y is K-euclidean then every subspace $E \subset Z$ with $\dim(E) \leq m$ is $3\lambda K$-euclidean and $3\lambda K$-complemented in X^*.*

5.2 The Rochberg spaces $\mathfrak{R}_n\big(\mathcal{T}_2, \mathcal{T}_2^*\big)$

There exist a continuous injective operator $\ell_2 \to \mathcal{T}_2$ with dense range, and thus by [35, Corollary 4.3] (see also [97, Corollary 1] or [80, Prop. 8.86]) we have the interpolation identity

$$\big(\mathcal{T}_2, \overline{\mathcal{T}_2^*}\big)_{1/2} = \ell_2.$$

Here $\overline{\mathcal{T}_2^*}$ denotes antidual of $\mathcal{T}_2$, namely, the Banach space $\mathcal{T}_2^*$ with the scalar multiplication $\alpha \odot x = \overline{\alpha} x$. We will call $\mathfrak{R}_n^{\mathcal{T}}$ the Rochberg space $\mathfrak{R}_n\big(\mathcal{T}_2, \overline{\mathcal{T}_2^*}\big)_{1/2}$. This space satisfies the following properties:

(i) Since $\mathcal{T}_2$ is reflexive and (see the proof of [80, Prop. 8.86])

$$\Big(\mathcal{T}_2^*, \big(\overline{\mathcal{T}_2^*}\big)^*\Big)_{1/2} = \big(\mathcal{T}_2^*, \overline{\mathcal{T}_2^{**}}\big)_{1/2} = \big(\mathcal{T}_2^*, \overline{\mathcal{T}_2}\big)_{1/2} = \big(\overline{\mathcal{T}_2}, \mathcal{T}_2^*\big)_{1/2} = \big(\mathcal{T}_2, \overline{\mathcal{T}_2^*}\big)_{1/2},$$

we deduce from Theorem 3.5.1 that $\mathfrak{R}_n^{\mathcal{T}}$ is isomorphic to its dual by the map $D_n^{\mathcal{T}} \colon \mathfrak{R}_n^{\mathcal{T}} \to (\mathfrak{R}_n^{\mathcal{T}})^*$ given, for $(x_{n-1}, \ldots, x_0) \in \mathfrak{R}_n^{\mathcal{T}}$ and $(y_{n-1}, \ldots, y_0) \in (X \cap X^*)^n$, by

$$D_n^{\mathcal{T}}(x_{n-1}, \ldots, x_0)(y_{n-1}, \ldots, y_0) = \sum_{i=0}^{n-1} (-1)^i \langle x_i, y_{n-i-1}\rangle. \tag{5.9}$$

(ii) Since both bases $(e_n)_n$ and $(e_n^*)_n$ are unconditional, the sequence of closed subspaces

$$X_m = \operatorname{span}\{(0, \ldots, 0, \overset{(l)}{e_m}, 0, \ldots, 0) \colon 0 \leq l \leq n-1\}$$

define an UFDD of $\mathfrak{R}_n^{\mathcal{T}}$. Thus, the sequence $(u_l)_{l \geq 1}$ defined in (4.2) is a basis for $\mathfrak{R}_n^{\mathcal{T}}$.

If we consider the complemented subspaces

$$V_m = \operatorname{span}\{u_l \colon l \geq mn - (n-1)\}$$

then $V_m = \mathfrak{R}_n(E_m, E_m^*)_{1/2}$. This follow by applying the projection

$$P_{k \geq m}\big((x_n)_n\big) = (x_n)_{n \geq m}$$

to the corresponding endpoints of the couple $\big(\mathcal{T}_2, \mathcal{T}_2^*\big)$.

Once we have Proposition 5.0.2, the proof that $\mathfrak{R}_n^{\mathcal{T}}$ is weak Hilbert closely follows the arguments of Suárez [91], and thus we will just give a sketch of a proof:

Theorem 5.2.1. *$\mathfrak{R}_n^{\mathcal{T}}$ is weak Hilbert for all $n \in \mathbb{N}$.*

Proof.

(i) Use (5.8) and Proposition 5.0.2 applied to the couple (E_m, E_m^*) to obtain that for every 5^{5^m}-dimensional subspace $E \subset V_m = \mathfrak{R}_n(E_m, E_m^*)_{1/2}$ one has that $a_{5^{5^m},2}(E) \leq C < \infty$ for some positive constant C not depending on m.

Since $\mathfrak{R}_n^{\mathcal{T}}$ is isomorphic to its dual by the duality (5.9), it follows by the definition of $D_n^{\mathcal{T}}$ that this map restricts to an isomorphism $\widehat{D_n^{\mathcal{T}}} : V_m \to V_m^*$. Thus $a_{5^{5^m},2}(V_m^*) \leq M a_{5^{5^m},2}(V_m)$ for some constant $M > 0$. Hence for every 5^{5^m}-dimensional subspace of V_m it follows that

$$c_{5^{5^m},2}(E) \leq a_{5^{5^m},2}(E^*) \leq a_{5^{5^m},2}(V_m^*) \leq M a_{5^{5^m},2}(V_m) \leq M' < \infty.$$

We deduce from Kwapień bound (C.2) that the 5^{5^m}-dimensional subspaces of V_m are K-Hilbertian for some constant K.

(ii) Using that V_m is reflexive and applying two times Proposition 5.1.1, first to V_m and later to V_m^*, it follows by (i) that every m-dimensional subspace $E \subset V_m$ is K-euclidean and $9K$-complemented in V_m; since V_m is complemented in $\mathfrak{R}_n^{\mathcal{T}}$, we conclude furthermore that E is L-complemented in $\mathfrak{R}_n^{\mathcal{T}}$ for some constant L.

(iii) First note that if E is a $2mn$-dimensional subspace of $\mathfrak{R}_n^{\mathcal{T}}$ then

$$F = E \cap \operatorname{span}\{u_l : l \geq mn - (n-1)\} = E \cap V_m$$

is a subspace of V_m such that

$$\dim(F) \geq 2mn - (mn - n) = mn + n \geq m.$$

Thus, setting $C = \max\{9K, L\}$ we deduce by (ii) that F contains a C-euclidean and C-complemented subspace of $\mathfrak{R}_n^{\mathcal{T}}$.

Therefore, given an arbitrary N-dimensional $E \subset \mathfrak{R}_n^{\mathcal{T}}$ where $N = 2mn + k$ with $0 \leq k \leq 2mn - 1$, it follows that E contains a C-euclidean and C-complemented subspace F such that

$$\dim(F) \geq m \geq \frac{1}{4n}(2mn + k) = \frac{1}{4n}\dim(E). \quad \square$$

There is a further problem concerning $\mathfrak{R}_n^{\mathcal{T}}$ that we have been unable to solve: to decide if $\mathfrak{R}_n^{\mathcal{T}}$ and $\mathfrak{R}_m^{\mathcal{T}}$ are isomorphic when $n \neq m$. We know by a deep result of Kalton [60, Th. 7.6] that the sequence $0 \longrightarrow \mathfrak{R}_1^{\mathcal{T}} \longrightarrow \mathfrak{R}_2^{\mathcal{T}} \longrightarrow \mathfrak{R}_1^{\mathcal{T}} \longrightarrow 0$ is not trivial. This implies by Proposition 3.1.2 that $0 \longrightarrow \mathfrak{R}_l^{\mathcal{T}} \longrightarrow \mathfrak{R}_{l+k}^{\mathcal{T}} \longrightarrow \mathfrak{R}_k^{\mathcal{T}} \longrightarrow 0$ is not trivial for any $l, k \in \mathbb{N}$. In particular, $\mathfrak{R}_n^{\mathcal{T}}$ and $\mathfrak{R}_1^{\mathcal{T}} = \ell_2$ are not isomorphic for $n \geq 2$. However, it could happen that all Rochberg spaces $\mathfrak{R}_n^{\mathcal{T}}$ are isomorphic for $n \geq 2$. We conjecture that the Rochberg spaces $\mathfrak{R}_n^{\mathcal{T}}$ are not isomorphic.

Appendices

Appendix A

Short exact sequences and quasilinear maps

This appendix deals with the theory of short exact sequences of Quasi-Banach spaces, for which our basic references are the books [10, 27] and the recent paper [28].

A *short exact sequences* of Quasi-Banach spaces is a diagrams composed by Quasi-Banach spaces and operators of the form

$$0 \longrightarrow Y \xrightarrow{\ i\ } X \xrightarrow{\ q\ } Z \longrightarrow 0 \tag{A.1}$$

where the kernel of each arrow coincides with the image of the preceding one. Note that the first and last arrow tell us, respectively, that i is injective and that q is surjective. There are two direct ways to obtain sequences of the form (A.1): either with a quotient map $q : X \to Z$, in which case one completes the diagram with the kernel

$$0 \longrightarrow \ker q \longrightarrow X \xrightarrow{\ q\ } Z \longrightarrow 0$$

or either with an isomorphic embedding $i : Y \to X$, in which case the completion is obtained by the cokernel

$$0 \longrightarrow Y \xrightarrow{\ \imath\ } X \longrightarrow X/i(Y) \longrightarrow 0$$

All sequences (A.1) are particular instances of the previous types: since the sequence is exact, q is a quotient map by the Open Mapping Theorem; hence $\ker q = i(Y)$ is a closed subspace of X and Z is isomorphic to the quotient space $X/i(Y)$. Therefore, short exact sequences provide an unified way of representing a Quasi-Banach space X having a subspace Y with an associated quotient Z.

Given any two Quasi-Banach spaces Y, Z, we can always form the short exact sequence

$$0 \longrightarrow Y \xrightarrow{\ j\ } Y \oplus Z \xrightarrow{\ \pi\ } Z \longrightarrow 0$$

where $j(y) = (y, 0)$ and $\pi(y, z) = z$, referred to as the *trivial* sequence. We say that two short exact sequences $0 \to Y \to X_1 \to Z \to 0$ and $0 \to Y \to X_2 \to Z \to 0$ are *equivalent* if there exist an isomorphism $T : X_1 \to X_2$ making the following diagram conmmute

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle T} & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & X_2 & \longrightarrow & Z & \longrightarrow & 0
\end{array}$$

A short exact sequence is called *trivial* if it is equivalent to the trivial sequence. Equivalently (see [10, 27] for a proof)

- There exist a retraction operator $r : X \to Y$ for i.
- There exist a section operator $s : Z \to X$ for q.

A.1 Quasilinear maps

The key discovery of Kalton [58] (althought a forerunner concept appears in [47, 81]) is that exact sequences of Quasi-Banach spaces correspond to certain (in general not linear nor bounded) maps $F : Z \to Y$ called quasilinear maps. In this work we will be using quasilinear maps which do not necessarily take values in Y, but in a strictly bigger superspace. This point of view was treated in [28]. Precisely, let X, Y be Quasi-Banach spaces and suppose that Y is a subspace of some Banach space Σ. Then an homogeneous map $F : X \to \Sigma$ is *quasilinear* from X to Y with ambient space Σ, something we denote by $F : X \curvearrowright Y$, if for all $x, y \in X$ one has that $F(x+y) - F(x) - F(y) \in Y$ and there exist a constant $M > 0$ such that

$$\|F(x+y) - F(x) - F(y)\|_Y \leq C\big(\|x\|_X + \|y\|_X\big). \tag{A.2}$$

The homogeneity property plus the estimate (A.2) imply that the set

$$Y \oplus_F X = \big\{(y, x) \in \Sigma \times X : y - F(z) \in Y\big\}$$

is a vector space and a Quasi-Banach space when endowed with the quasinorm

$$\|(y, x)\| = \|y - F(x)\|_Y + \|x\|_X. \tag{A.3}$$

Hence, any quasilinear map $F : X \curvearrowright Y$ induces a short exact sequence

$$0 \longrightarrow Y \xrightarrow{\ i\ } Y \oplus_F X \xrightarrow{\ q\ } X \longrightarrow 0$$

Note that the quasinorm of $Y \oplus_F X$ is just the norm of the direct sum that has been "twisted" by means of a quasilinear map. Thus, it is customary referring to $Y \oplus_F X$ as a *twisted sum of* Y *and* X.

The previous discussion reduces the study of short exact sequences to that of quasilinear maps. The next step is to decide when two given quasilinear maps define equivalent sequences. Let $F, G : X \curvearrowright Y$ be two quasilinear maps with the same ambient space Σ. We say that F and G are equivalent, and we denote it by $F \equiv G$, if they define equivalent short exact sequences. The following result was provided by Kalton and Peck [62, Th. 2.4] (see also [28, Section 2]):

Proposition A.1.1. *Two quasilinear* $F, G : X \curvearrowright Y$ *with the same ambient space* Σ *are* equivalent *if and only if there exist a linear map* $L : X \to \Sigma$ *such that* $(F - G - L) : X \to Y$ *is bounded in the sense that*

$$\|Fx - Gx - Lx\|_Y \leq C\|x\|_X \quad \textit{for all } x \in X.$$

Since the zero map $0 : X \curvearrowright Y$ defines a trivial exact sequence, a simple consequence of Proposition A.1.1 is that a quasilinear F defines the trivial exact sequence if and only if there exist a linear map $L : X \to \Sigma$ such that $F - L : X \to Y$ is bounded. In this case we say that F is a *trivial quasilinear map*.

An important fact concerning twisted sums is that (A.3) is just a quasinorm, and thus, in general, $Y \oplus_F X$ need not to be isomorphic to a Banach space. However, if X has type $p > 1$ (for instance, if X is superreflexive), then (A.3) is equivalent to a norm [58, Th. 2.6] and hence $Y \oplus_F X$ is isomorphic to a Banach space. In fact, all twisted sums studied in this work will be isomorphic to Banach spaces.

A.1.1 Singular quasilinear maps

A quasilinear map $F : X \curvearrowright Y$ is *singular* if the quotient map of the associated short exact sequence

$$0 \longrightarrow Y \longrightarrow Y \oplus_F X \longrightarrow X \longrightarrow 0 \tag{A.4}$$

is strictly singular; in this case we will also say that $Y \oplus_F X$ is a singular twisted sum and that the sequence (A.4) is singular. It can be proven [10, Prop. 9.1.2] that $F : X \curvearrowright Y$ is singular if and only if its restriction to any infnite dimensional subspace of X is not trivial. Recall the following characterization of strictly singular quotient maps:

Proposition A.1.2. *A quotient map $Q : Z \to Z/Y$ is strictly singular if and only if for every infinite dimensional subspace $Z' \subset Z$ there exist an infinite dimensional subspace $Y' \subset Y$ and a nuclear operator $K : Y' \to Z'$ such that $i + K : Y' \to Z'$ defines an isomorphic embedding, where $i : Y \to Z$.*

A proof can be found in [34, Prop. 3.2]. We will need the following criteria for strictly singular maps on commutative diagrams proved in [12, Lemma 8]:

Proposition A.1.3. *Suppose we have a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\imath"] \arrow[d, "\tau"'] & B \arrow[r, "\rho"] \arrow[d, "T"'] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

where both ρ and τ are strictly singular. Then T is strictly singular.

The strictly singular character of any operator defined on a singular twisted sum only depends on its behaviour on the kernel of the quotient map. The proof appears in [29, Prop. 11] along with many related results (see also [10, Chapter 9]).

Proposition A.1.4. *Let W be any Banach space and $0 \to X \xrightarrow{i} Y \xrightarrow{\rho} Z \to 0$ a singular exact sequence. Then any operator $\tau : Y \to W$ is strictly singular if and only if $\tau|_X : X \to W$ is strictly singular.*

A.1.2 Domain and range spaces. Inverse representation

Given a quasilinear map $\Omega : X \curvearrowright Y$ with ambient space Σ we consider the following two spaces:

- The *domain space* $\mathrm{Dom}(\Omega) = \{x \in X : \Omega(x) \in Y\}$ endowed with the quasinorm $\|x\|_D = \|\Omega(x)\|_Y + \|x\|_X$.
- The *range space* $\mathrm{Ran}(\Omega) = \{\omega \in \Sigma : (\omega, x) \in Y \oplus_\Omega X\} = \{\omega \in \Sigma : \text{exist } x \in X \text{ such that } \omega - \Omega(x) \in X\}$ endowed with the quotient quasinorm $\|\omega\|_R = \inf\{\|(\omega, x)\|_\Omega : (\omega, x) \in Y \oplus_\Omega X\}$.

Thus, $\mathrm{Dom}(\Omega)$ can be identified with the closed subspace $\{x \in X : (0, x) \in Y \oplus_\Omega X\}$ and $\mathrm{Ran}(\Omega)$ can be identified with the quotient $(Y \oplus_\Omega X)/\mathrm{Dom}(\Omega)$. One clearly has that $\Omega : \mathrm{Dom}(\Omega) \to X$ and $\Omega : X \to \mathrm{Ran}(\Omega)$ are bounded. In particular, $\Omega : X \to Y$ is bounded if and only if $\mathrm{Dom}(\Omega) = X$ and $\mathrm{Ran}(\Omega) = Y$. Moreover, the domain and range spaces only depend on the equivalence class of Ω:

Lemma A.1.1. *If* Ψ *is equivalent to* Ω *then* $\mathrm{Dom}(\Psi) = \mathrm{Dom}(\Omega)$ *and* $\mathrm{Ran}(\Psi) = \mathrm{Ran}(\Omega)$.

By their very definition, the domain space and range space fit into a short exact sequence

$$0 \longrightarrow \mathrm{Dom}(\Omega) \xrightarrow{\ j\ } Y \oplus_\Omega X \xrightarrow{\ p\ } \mathrm{Ran}(\Omega) \longrightarrow 0 \tag{A.5}$$

given by $j(x) = (0, x)$ and $p(\omega, x) = \omega$. It was shown in [28, Section 3] that (A.5) is generated by a quasilinear map $\Omega^{-1} : \mathrm{Ran}(\Omega) \curvearrowright \mathrm{Dom}(\Omega)$ with ambient space X such that:

(1) The quasinorm $\|\cdot\|_\Omega$ is equivalent to the quasinorm induced by (A.5), in the sense that there exist positive constants m, M such that

$$m\|(\omega, x)\|_\Omega \leq \|x - \Omega^{-1}(\omega)\|_D + \|\omega\|_R \leq M\|(\omega, x)\|_\Omega,$$

for every $(\beta, x) \in Y \oplus_\Omega X$.

(2) $\mathrm{Dom}(\Omega^{-1}) = Y$ and $\mathrm{Ran}(\Omega^{-1}) = X$.

By (2) it follows that

$$\Omega^{-1} \circ \Omega : X \to X \quad \text{and} \quad \Omega \circ \Omega^{-1} : \mathrm{Ran}(\Omega) \to \mathrm{Ran}(\Omega)$$

and

$$\Omega \circ \Omega^{-1} : Y \to Y \quad \text{and} \quad \Omega^{-1} \circ \Omega : \mathrm{Dom}(\Omega) \to \mathrm{Dom}(\Omega)$$

are bounded; by this reason Ω^{-1} is called *the inverse* of Ω. If we combine (2) with (1) then we obtain a symmetric situation represented by the diagram

$$\begin{tikzcd}
 & \mathrm{Ran}(\Omega) \arrow[dr, bend left, "\Omega^{-1}"] & \\
Y \arrow[r] \arrow[dr, bend right, "\Omega^{-1}"'] & Y \oplus_\Omega X \arrow[r] \arrow[u] & X \arrow[ul, bend left=10, "\Omega"'] \\
 & \mathrm{Dom}(\Omega) \arrow[u] \arrow[ul, bend right=10, "\Omega"'] &
\end{tikzcd}$$

A.1.3 Duality of twisted sums

We briefly comment some results about duality of twisted sums. Suppose we have a quasilinear map $\Omega : X \curvearrowright Y$ with ambient space Σ defining a short exact sequence of Banach spaces

$$0 \longrightarrow Y \xrightarrow{\ i\ } Y \oplus_\Omega X \xrightarrow{\ q\ } X \longrightarrow 0$$

Then the Hahn-Banach Theorem implies that the dual sequence

$$0 \longrightarrow X^* \xrightarrow{\ q^*\ } (Y \oplus_\Omega X)^* \xrightarrow{\ i^*\ } Y^* \longrightarrow 0 \tag{A.6}$$

is also exact [27, 2.2.d]. If $\mathrm{Dom}(\Omega)$ is dense in X then the operator $J : \mathrm{Dom}(\Omega) \times Y \to Y \oplus_\Omega X$ given by $J(x, y) = (y, x)$ is injective and has dense range. Hence the dual operator

$$J^* : (Y \oplus_\Omega X)^* \to (\mathrm{Dom}(\Omega))^* \times Y^*$$

is also injective with dense range; moreover, note that X^* can be regarded as a subspace of $(\mathrm{Dom}(\Omega))^*$. Given a bounded homogenous selection B for the map i^* in (A.6), one can define a map $\Omega^* : Y^* \to (\mathrm{Dom}(\Omega))^*$ by the expression

$$B(y) = J^*\big(\Omega^* y, y\big), \quad \text{for all } y \in Y^*.$$

In [28, Section 4] it was proved that $\Omega^* : Y^* \curvearrowright X^*$ is quasilinear with ambient space $\mathrm{Dom}(\Omega)^*$ and such that $X^* \oplus_{\Omega^*} Y^*$ is isomorphic to $(Y \oplus_\Omega X)^*$. The map Ω^* is called *the dual* of Ω and defines (A.6). The next result shows the relationship between the domain and range spaces of Ω and that of the dual map Ω^* (see [28, Prop. 4.7]):

Proposition A.1.5. *Let* $\Omega : X \curvearrowright Y$ *a quasilinear map such that* $\mathrm{Dom}(\Omega)$ *is dense in* X*. Then:*

(i) $\mathrm{Dom}(\Omega^*) = \mathrm{Ran}(\Omega)^*$.

(ii) $\mathrm{Ran}(\Omega^*) = \mathrm{Dom}(\Omega)^*$.

A.2 3-space properties and twisted Hilbert spaces

A 3-*space problem* ask to decide whether for every short exact sequence

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$$

where both Y and Z have property P, the twisted sum X has P. When a 3-space problem has an afirmative answer, the property P is labeled as a 3-*space property*. Examples include the properties of *being reflexive*, *separable* or *isomorphic to* c_0. The clasical source that studies 3-space problems is [27].

Enflo, Lindenstruss and Pisier showed in [47] that "being isomorphic to a Hilbert space" is not a 3-space property. Precisely, there is a short exact sequence

$$0 \longrightarrow \ell_2 \longrightarrow X \longrightarrow \ell_2 \longrightarrow 0$$

where X is not isomorphic to a Hilbert space. Any such space X is called a nontrivial *twisted Hilbert space*. The Kalton-Peck space Z_2 studied in Chapter 1 is a fundamental example of twisted Hilbert space.

Appendix B

Operator ideals

Denote by $\mathcal{L}$ the class of all operators acting between Banach spaces. Following Pietsch [78] we say that a subclass of operators $\mathcal{A} \subset \mathcal{L}$ is an *operator ideal* if, for every pair of Banach spaces X, Y, the *component* $\mathcal{A}(X,Y) := \mathcal{A} \cap \mathcal{L}(X,Y)$ satisfies that:

(1) Finite range operators belong to $\mathcal{A}(X,Y)$;

(2) $T + U \in \mathcal{A}(X,Y)$ for all $T, U \in \mathcal{A}(X,Y)$;

(3) If X_0 and Y_0 are Banach spaces, then $WTU \in \mathcal{A}(X_0, Y_0)$ whenever $W \in \mathcal{L}(Y, Y_0)$, $T \in \mathcal{A}(X,Y)$ and $U \in \mathcal{L}(X_0, X)$.

Conditions (2) and (3) imply that, for each Banach spaces X, Y, the component $\mathcal{A}(X,Y)$ is a linear subspace of $\mathcal{L}(X,Y)$. When $X = Y$ we shall denote the component of an operator ideal by $\mathcal{A}(X)$ or $\mathcal{A}$ if no confusion arises. Given an operator ideal $\mathcal{A}$, there is a rather general way to produce new operator ideals out of $\mathcal{A}$. Pietsch list in [78, I, Chapter 4] eight natural ways of doing so, which he calls *procedures*. The most relevant for us are the closure and the dual procedures:

- The closure of $\mathcal{A}$, denoted by $\overline{\mathcal{A}}$, is defined by its components as the closures $\overline{\mathcal{A}(X,Y)}$ in $\mathcal{L}(X,Y)$.

- The dual operator ideal is defined on its components by

$$\mathcal{A}^d(X,Y) = \{T \in \mathcal{L}(X,Y) \colon T^* \in \mathcal{A}(Y^*, X^*)\}.$$

An operator ideal $\mathcal{A}$ is closed if $\mathcal{A} = \overline{\mathcal{A}}$ and symmetric when $\mathcal{A} = \mathcal{A}^d$.

Plenty of specific examples have appeared since the inception of the theory. We refer the reader to [78, 45, 44, 95] for a comprehensive list of examples and applications to Operator Theory and Banach space Theory. Here we are mainly concerned with three classical operator ideals: compact operators, strictly singular operators and strictly cosingular operators. Recall that an operator $T : X \to Y$ is compact if it takes bounded sets into relative compact sets. Equivalently, if for every bounded sequence (x_n) in X, the sequence (Tx_n) has a converging subsequence. The class of compact operators forms a closed and symmetric ideal which we denote by $\mathcal{K}$.

On the other hand, a bounded operator $T : X \to Y$ is:

- *strictly singular* if its restriction $T|_M$ to any infinite dimensional subspace $M \subset X$ is never an isomorphism.

- *strictly cosingular* if for every infinite codimensional subspace $N \subset Y$, the composition $Q_N T : X \to Y/N$ is not surjective.

We denote by $\mathcal{SS}$ and $\mathcal{SC}$ the classes of strictly singular and strictly cosingular operators, respectively. Both classes are closed operator ideals [78, 1.9 and 1.10] that, in general, strictly contain $\mathcal{K}$.

We recall now an important result due to Pełczyński [74], which relate both ideals by duality:

Proposition B.0.1.

$$\mathcal{SS}^d \subset \mathcal{SC} \quad \text{and} \quad \mathcal{SC}^d \subset \mathcal{SS}.$$

Hence, in the particular case of reflexive spaces, an operator $T : X \to Y$ is strictly singular (cosingular) if and only if $T^* : Y^* \to X^*$ is strictly cosingular (singular).

The following key result concerning strictly singular operators was obtained by Kato [64] (see also [71, 2.c.4] or [78, 1.9.3] for a proof):

Proposition B.0.2. *An operator $T \in \mathcal{L}(X, Y)$ is strictly singular if and only if for every $\varepsilon > 0$ there exist a further infinite dimensional subspace $M \subset X$ such that $T|_M$ is compact and $\|T|_M\| \leq \varepsilon$.*

B.1 Fredholm operators and perturbation classes

We shall describe now some properties of Fredholm operators. Here we will follow the description given in [50].

Definition 2. *Let T be an operator between Banach spaces. Then T is said to be*

- upper semi-Fredholm *if T has finite dimensional kernel and closed range. We shall denote this by $T \in \Phi_+(X, Y)$.*

- lower semi-Fredholm *if T has finite codimensional range. We denote it by $T \in \Phi_-(X, Y)$.*

- Fredholm *if is both upper and lower semi-Fredholm.*

We shall denote these three classes, repectively, by $\Phi_+(X, Y)$, $\Phi_-(X, Y)$ and $\Phi(X, Y)$. When $X = Y$ we abbreviate it to $\Phi_\pm(X)$. The following proposition appears in [50, A.1.5] and summarizes the elemental properties of semi-Fredholm operators:

Proposition B.1.1. *Let $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$ be any operators. Then*

- *If $T \in \Phi_+(X, Y)$ and $S \in \Phi_+(Y, Z)$ then $ST \in \Phi_+(X, Z)$.*

- *If $ST \in \Phi_+$ then $T \in \Phi_+$.*

- *If $T \in \Phi_-(X, Y)$ and $S \in \Phi_-(Y, Z)$ then $ST \in \Phi_-(X, Z)$.*

- *If $ST \in \Phi_-$ then $T \in \Phi_-$.*

- *$T \in \Phi_+(X, Y)$ if and only if $T^* \in \Phi_-(Y^*, X^*)$.*

Remark 1. A simple fact that will be used later is the following well-known observation: an operator $T \in \Phi_+(X,Y)$ if and only if $T|_M$ is not compact for any infinite dimensional subspace $M \subset X$:

- Suppose that $T \notin \Phi_+$. If $T(X)$ is not closed then T is not an isomorphism on any finite codimensional subspace of X (if it were on, say $X_0 \subset X$, then $T(X_0)$ is closed in Y and finite codimensional in $T(X)$, thus $T(X)$ is closed). Then Proposition B.0.2 yields that $T|_M$ is compact on some infinite dimensional subspace.

  If, on the other hand, $T(X)$ is closed but $T \notin \Phi_+$, then $T|_{ker(T)} = 0$ is obviously compact.

- Conversely, assume that $T \in \Phi_+(X,Y)$. If N is a closed complement of $\ker T$ then T is an isomorphism on N, and for every infinite dimensional subspace M of X the intersection $N \cap M$ is infinite dimensional. Hence $T|_M$ is not compact.

One can associate to any semi-Fredholm operator $T \in \Phi(X,Y)$ a value in $\mathbb{Z} \cup \{-\infty, +\infty\}$ called the *index of* T defined by

$$ind(T) = \dim(\ker T) - \dim\big(Y/T(X)\big).$$

A classical result of Fredholm theory states that this index remains invariant under compact perturbations of T [65, IV, Th. 5.26]. One can replace the class of Fredholm operators by Φ_+ and the operator ideal $\mathcal{K}$ by $\mathcal{SS}$ to obtain the analogous stability property:

Proposition B.1.2. *If $T \in \Phi_+(X,Y)$ then $T+S \in \Phi_+(X,Y)$ and $ind(T+S) = ind(T)$ for all $S \in \mathcal{SS}(X,Y)$.*

This result is due to Kato [64, 5. Th. 2] (see also [71, 2.c.10]). Shortly thereafter Vladimirskii showed [96, Corollary 1] the dual result that lower semi-Fredholm are stable under strictly cosingular perturbations (see also [84, C.V. Th. 3.4] or [78, Section 26.6.8]):

Proposition B.1.3. *If $T \in \Phi_-(X,Y)$ then $T+L \in \Phi_-(X,Y)$ and $ind(T+L) = ind(T)$ for all $L \in \mathcal{SC}(X,Y)$.*

On the other hand, an operator $T \in \mathcal{L}(X,Y)$ is called *inessential*, denoted by $\mathcal{I}n(X,Y)$, whenever $I_X - LT \in \Phi(X)$ for all $L \in \mathcal{L}(Y,X)$ (equivalently, such that $I_Y - TS \in \Phi(Y)$ for all $S \in \mathcal{L}(Y,X)$). Inessential operators were introduced by Kleinecke [66], who proved that $\mathcal{I}_n$ form a closed operator ideal containing both $\mathcal{SS}$ and $\mathcal{SC}$ (cf. also [78, Section 26.7]).

The preceding discussion about perturbation theory of Fredholm operators can be stated in a slightly more general way: given a class $\mathcal{A} \subset \mathcal{L}$ of bounded operators, its *perturbation class* $P\mathcal{A}$ is defined by its components when $\mathcal{A}(X,Y) \neq \emptyset$ as

$$P\mathcal{A}(X,Y) = \{T \in \mathcal{L}(X,Y) \colon T+L \in \mathcal{A}(X,Y), \text{ for all } L \in \mathcal{A}(X,Y)\}.$$

This definition is due to Lebow and Schechter, who proved in [68, Th. 2.7] that inessential operators $\mathcal{I}n$ are precisely the perturbation class of Fredholm operators, i.e., $\mathcal{I}n = P\Phi$. The aforementioned Propositons B.1.2 and B.1.3 imply that $\mathcal{SS} \subset P\Phi_+$ and $\mathcal{SC} \subset P\Phi_-$. Moreover, the stability of the index for semi-Fredholm operators implies that $P\Phi_+ \cup$

$P\Phi_- \subset \mathcal{I}n$ (see [15, 5.6.9] or [51, Proposition 3.3]). The following diagram resumes the situation (here the arrows denote a formal inclusion):

$$\begin{tikzcd}
 & & \mathcal{SS} \arrow[r] & P\Phi_+ \arrow[dr, bend left] & & \\
\mathcal{K} \arrow[r] & \mathcal{SS} \cap \mathcal{SC} \arrow[ur] \arrow[dr] & & & P\Phi_+ \cup P\Phi_- \arrow[r] & \mathcal{I}n = P\Phi \\
 & & \mathcal{SC} \arrow[r] & P\Phi_- \arrow[ur, bend right] & &
\end{tikzcd}$$

The *perturbation class problem* [49, 51] asked whether

$$\mathcal{SS}(X,Y) = P\Phi_+(X,Y) \quad \text{and} \quad \mathcal{SC}(X,Y) = P\Phi_-(X,Y) \tag{B.1}$$

for all Banach spaces X, Y. Note that when the perturbation class problem has an affirmative answer, the perturbation class provide an intrisic description of the classes $P\Phi_+$ and $P\Phi_-$, something which were asked by Gohberg, Markus and Feldman in [48].

It is known [49] that the identities (B.1) only hold for concrete examples (see also [85] and [51, Section 5] and the references therein for a detailed account).

Appendix C

Some local properties of a Banach space

Given a finite sequence $(x_i)_{i=1}^n$ of vectors in X, we denote by

$$\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\|_X$$

the average over all choices of signs $(\varepsilon_i)_{i=1}^n \subset \{-1,1\}^n$. If $1 \le p \le 2$ then we define $a_{n,p}(X)$ as the infimum of the constants $a > 0$ such that for every $(x_i)_{i=1}^n \subset X$ we have the bound

$$\mathbb{E}\Big\|\sum_{i=0}^n \varepsilon_i x_i\Big\|_X \le a\Big(\sum_{i=0}^n \|x\|_X^p\Big)^{1/p}.$$

A Banach space X has *type* p if $a_p(X) = \sup_{n\in\mathbb{N}} a_{n,p}(X) < \infty$. Similarly, for $2 \le q \le \infty$ we define $c_{n,q}(X)$ as the infimum of the constants $c > 0$ such that for every $(x_i)_{i=1}^n$ we have that

$$\Big(\sum_{i=0}^n \|x\|_X^q\Big)^{1/q} \le c\,\mathbb{E}\Big\|\sum_{i=0}^n \varepsilon_i x_i\Big\|_X.$$

A Banach space X has *cotype* q of $c_q(X) = \sup_{n\in\mathbb{N}} c_{n,q}(X) < \infty$. There exist several relations between the local type/cotype constants of a given Banach space. Here we note that $c_{n,2}(X^*) \le a_{n,2}(X)$ for every $n \in \mathbb{N}$ [44, Prop. 11.10].

Given two isomorphic Banach spaces X and Y, the Banach-Mazur distance between X and Y is defined as

$$d(X,Y) = \inf\{\|T\|\,\|T^{-1}\| \colon T : X \to Y \text{ is an isomorphism}\}. \tag{C.1}$$

Let X be a n-dimensional subspace. Then $d(X,\ell_2^n)$ is finite and the infimum in (C.1) is attained [3, Lemma 7.4.6]. Therefore, taking into account that

$$\mathbb{E}\,\Big\|\sum_{i=1}^m \varepsilon_i x_i\Big\|_X \le \|T^{-1}\|\,\mathbb{E}\,\Big\|\sum_{i=1}^m \varepsilon_i T(x_i)\Big\|_{\ell_2^n} \le d(X,\ell_2^n)\Big(\sum_{i=1}^m \|x_i\|_X^2\Big)^{1/2}$$

it follows that $a_{n,2}(X) \le d(X,\ell_2^n)$. An important result due to Kwapień is (see [3, Theorem 7.4.7] for a proof):

Proposition C.0.1. *If X has type 2 and cotype 2 then there exist a Hilbert space H such that*

$$d(X,H) \le a_2(X)\, c_2(X).$$

In particular, X is isomorphic to a Hilbert space.

If X is n-dimensional, then a result of Tomczak-Jaegermann [94, Th. 2] combined with Kwapień's result yields a universal constant M such that

$$d(X,\ell_2^n) \le a_2(X)\, C_2(X) \le M a_{n,2}(X)\, c_{n,2}(X). \tag{C.2}$$

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