

Geometrical and Topological Properties of Bumps and Starlike Bodies in Banach Spaces

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(Survey paper presented by J. Jaramillo)

AMS Subject Class. (2000): 46B20, 46G05, 57N20

Received January 21, 2002

1. INTRODUCTION

A bump b on a Banach space X is a (most often smooth, at least continuous) function with bounded nonempty support, $\text{supp}(b) = \overline{\{x \in X : b(x) \neq 0\}}$. The existence of smooth bump functions on a Banach space X is closely related in several ways to the (linear and nonlinear) structure of the space X , and has often important consequences on its geometrical properties (see [27]). In connection with bump functions there is the class of starlike bodies, which, perhaps, have not yet received the attention that they are worth.

A closed subset A of a Banach space X is said to be a starlike body if there exists a point x_0 in the interior of A such that every ray emanating from x_0 meets ∂A , the boundary of A , at most once. Up to a suitable translation, we can always assume (and we will do so) that $x_0 = 0$ is the origin of X . For a starlike body A , we define the characteristic cone of A as

$$ccA = \{x \in X : rx \in A \text{ for all } r > 0\},$$

and the Minkowski functional of A as

$$\mu_A(x) = \inf\{\lambda > 0 : \frac{1}{\lambda}x \in A\}$$

for all $x \in X$. It is easily seen that for every starlike body A its Minkowski functional μ_A is a continuous function which satisfies $\mu_A(rx) = r\mu_A(x)$ for every $r \geq 0$ and $x \in X$, and $\mu_A^{-1}(0) = ccA$. Moreover, $A = \{x \in X : \mu_A(x) \leq 1\}$,

and $\partial A = \{x \in X : \mu_A(x) = 1\}$. Conversely, if $\psi : X \rightarrow [0, \infty)$ is continuous and satisfies $\psi(\lambda x) = \lambda\psi(x)$ for all $\lambda \geq 0$, then $A_\psi = \{x \in X : \psi(x) \leq 1\}$ is a starlike body and $\mu_{A_\psi} = \psi$. More generally, for a continuous function $\psi : X \rightarrow [0, \infty)$ such that $\psi_x(\lambda) = \psi(\lambda x)$, $\lambda > 0$, is increasing and $\sup\{\psi_x(\lambda) : \lambda > 0\} > \varepsilon$ for every $x \in X \setminus \psi^{-1}(0)$, the set $\psi^{-1}([0, \varepsilon])$ is a starlike body whose characteristic cone is $\psi^{-1}(0)$.

A familiar important class of starlike bodies are *convex bodies*, that is, starlike bodies that are convex. For a convex body U , ccU is always a convex set, but in general the characteristic cone of a starlike body is not convex.

We will say that A is a C^p smooth starlike body provided its Minkowski functional μ_A is C^p smooth on the set $X \setminus ccA = X \setminus \mu_A^{-1}(0)$. A starlike body A is said to be Lipschitz provided its Minkowski functional μ_A is a Lipschitz function. Finally, two (smooth) starlike bodies A, B in a Banach space X are relatively homeomorphic (relatively diffeomorphic) whenever there is a self-homeomorphism (diffeomorphism) $g : X \rightarrow X$ so that $g(A) = B$.

Starlike bodies often appear in nonlinear functional analysis as natural substitutes of convex bodies or in connection with polynomials. Indeed, for every n -homogeneous polynomial $P : X \rightarrow \mathbb{R}$ the sets $A_c = \{x \in X : P(x) \leq c\}$, $c > 0$, are either starlike bodies or complements of starlike bodies; therefore the level sets of every n -homogeneous polynomials are boundaries of starlike bodies, and if one is interested in the geometrical behaviour of n -homogeneous polynomials then one should also pay some attention to the geometrical properties of starlike bodies. On the other hand, smooth bounded starlike bodies also arise in a natural way from smooth bump functions; indeed, for every Banach space $(X, \|\cdot\|)$ with a C^p smooth bump function there exist a functional ψ and constants $a, b > 0$ such that ψ is C^p smooth away from the origin, $\psi(\lambda x) = |\lambda|\psi(x)$ for every $x \in X$ and $\lambda \in \mathbb{R}$, and $a\|x\| \leq \psi(x) \leq b\|x\|$ for every $x \in X$ (see [27], proposition II.5.1) The function ψ has a useful conical shape and can sometimes take the role of a smooth norm in spaces which in general are not known to possess such norms. The level sets of this function are precisely the boundaries of the C^p smooth bounded starlike bodies $A_c = \{x \in X : \psi(x) \leq c\}$, $c \in \mathbb{R}^+$. Conversely, if a Banach space X has a C^p smooth bounded starlike body then it has a C^p smooth bump function as well.

It is therefore reasonable to ask to what extent the geometrical properties of convex bodies are shared with the more general class of starlike bodies. Surprisingly enough, very little work concerning smooth starlike bodies and their geometrical properties has been attempted until very recently.

This work is mainly a compilation of some recent research about the topological and geometrical properties of starlike bodies and bump functions that has been carried out by Manuel Cepedello, Robert Deville, Tadeusz Dobrowolski, Marian Fabian and the present authors during the last few years. Our aim here is to organize some of the results obtained from that research in a coherent way, stressing the interplay between infinite-dimensional differential topology and nonlinear functional analysis. In particular we relate some questions about topological and geometrical properties of starlike bodies to other interesting problems on nonlinear analysis, such as the failure of Rolle's theorem in infinite dimensions and other ways of characterizing the smoothness properties of a Banach space. As said above, starlike bodies and bump functions are tightly related, so it is no surprise that looking at the geometrical properties of one of these classes of objects can help us to learn more about the nature of the other.

We will avoid the most technical proofs, trying to focus on the ideas behind them rather than overwhelming the reader with cumbersome details. However, we believe that some of the new tools developed in the proofs (such as the twisted tube method of Section 4, or the construction of mappings whose derivatives are surjections of Section 6) might have some applications beyond the problems considered herein. In such cases we will try to be more accurate in our account.

The structure of this essay is as follows.

1. Introduction
2. Classifying starlike bodies
3. Smooth Lipschitz contractibility of boundaries of starlike bodies in infinite dimensions
4. The failure of Rolle's and Brouwer's theorems in infinite dimensions
5. How small can the range of the gradient of a bump be?
6. How large can the range of a derivative be?
7. What does the range of a derivative look like?
8. Geometrical properties of starlike bodies. The failure of James' theorem for starlike bodies.

Sections 2 and 3 concern some (smooth) topological properties of starlike bodies; the results of Section 2 are part of [10], while those of Section 3 constitute the main theorems from [4]. Sections 4, 5, 6 and 7 are devoted to a study of the geometrical properties of bump functions; more specifically, we ask and answer questions about the size of the sets of gradients of smooth

bump functions. This study enables us to answer to some natural questions about the geometrical properties of starlike bodies (such as the topological size of the cones of tangent hyperplanes to smooth starlike bodies), which we consider in Section 8, and in particular we deduce that James' theorem on the characterization of reflexivity cannot be extended to the class of starlike bodies. Most of the material of Sections 4, 5, 6 and 8 can be found in [6], [7], [13], [14]. Finally, the material of Section 7 can be found in [23], [24] and [8].

2. CLASSIFYING STARLIKE BODIES

It was V.L. Klee [39] that first gave a topological classification of the convex bodies of a Hilbert space. This result was generalized for every Banach space with the help of Bessaga's non-complete norm technique (see the book by Bessaga and Pelczynski [22]). To get a better insight in the history of the topological classification of convex bodies the reader should have a look at the papers by Stocker [51], Corson and Klee [25], Bessaga and Klee [20], [21], and Dobrowolski [30]. These results have recently been sharpened to get a full classification of the C^p smooth convex bodies of every Banach space [9]. In its most general form the result on a classification of (smooth) convex bodies reads as follows (see [9]); here, as in the whole section, $p = 0, 1, 2, \dots, \infty$, and " C^0 diffeomorphic" means just "homeomorphic".

THEOREM 2.1. *Let U be a C^p convex body in a Banach space X .*

- (a) *If ccU is a linear subspace of finite codimension (say $X = ccU \oplus Z$, with Z finite-dimensional), then U is C^p relatively diffeomorphic to $ccU + B_Z$, where B_Z is an Euclidean ball in Z .*
- (b) *If ccU is not a linear subspace or ccU is a linear subspace such that the quotient space X/ccU is infinite-dimensional, then U is C^p relatively diffeomorphic to a closed half-space (that is, $\{x \in X : x^*(x) \geq 0\}$, for some $x^* \in X^*$).*

Our aim here is to discuss to what extent this result can be generalized for (smooth) starlike bodies. The following example shows that part (b) of Theorem 2.1 is not true for starlike bodies whose characteristic cones are not convex sets.

EXAMPLE 2.2. Let $A = \{(x, y) \in \mathbb{R}^2 : |xy| \leq 1\}$. It is plain that A is a starlike body in the plane \mathbb{R}^2 , and its characteristic cone is the pair of lines

defined by the equation $xy = 0$. Then A cannot be relatively diffeomorphic (not even relatively homeomorphic) to a half-plane of \mathbb{R}^2 . Indeed, ∂A is not connected, while the boundary of a closed half-plane (that is to say, a line) is always connected. Similar examples show that for every $n \in \mathbb{N}$ there exists a starlike body A_n in the plane \mathbb{R}^2 such that ∂A_n has exactly n connected components. Hence A_n is not relatively homeomorphic to A_m whenever $n \neq m$.

However, it seems natural to think that every two (smooth) starlike bodies with the same characteristic cone should be diffeomorphic. This is indeed true and it is a fact that, though elementary, will help us to unravel the tangle of starlike bodies and get a first generalization of Theorem 2.1.

PROPOSITION 2.3. *Let X be a Banach space, and let A_1, A_2 be C^p smooth starlike bodies such that $ccA_1 = ccA_2$. Then there exists a C^p diffeomorphism $g : X \rightarrow X$ such that $g(A_1) = A_2$, $g(\partial A_1) = \partial A_2$, and $g(0) = 0$. Moreover, $g(x) = \eta(x)x$, where $\eta : X \rightarrow [0, \infty)$, and hence g preserves the rays emanating from the origin.*

Proof. First let us see that the statement is true if we make the additional assumption that $A_1 \subseteq A_2$. So, let us suppose that A and B are starlike bodies such that the origin is an interior point of both A and B , $ccA = ccB$, and $A \subseteq B$ (so that $\mu_B(x) \leq \mu_A(x)$ for every x , where μ_A and μ_B are the Minkowski functionals of A and B respectively), and see that there exists a C^p diffeomorphism $g : X \rightarrow X$ such that $g(A) = B$, $g(0) = 0$, and $g(\partial A) = \partial B$.

Let $\lambda(t)$ be a non-decreasing real function of class C^∞ defined for $t > 0$, such that $\lambda(t) = 0$ for $t \leq 1/2$ and $\lambda(t) = 1$ for $t \geq 1$. Let

$$g(x) = \left[\lambda(\mu_A(x)) \frac{\mu_A(x)}{\mu_B(x)} + 1 - \lambda(\mu_A(x)) \right] x$$

for $x \notin ccA$, and $g(x) = x$ whenever $\mu_B(x) = 0$. It is clear that g is a C^p smooth mapping. With the help of the implicit function theorem it is not difficult to see that g^{-1} is C^p smooth as well.

Now let us consider the general case. Let

$$A = \{x \in X : \mu_{A_1}(x) + \mu_{A_2}(x) \leq 1\},$$

which is a C^p smooth convex body satisfying $ccA = ccA_j$ and $A \subseteq A_j$, for $j = 1, 2$. We already know that there exist self-diffeomorphisms of X , g_1 and g_2 , such that $g_j(A) = A_j$ and $g_j(\partial A) = \partial A_j$, $j = 1, 2$. Then, if we put $g = g_2 \circ g_1^{-1}$, we get a self-diffeomorphism of X transforming A_1 onto A_2 and ∂A_1 onto ∂A_2 . ■

As said above, one cannot dream of extending part (b) of Theorem 2.1 to the class of general starlike bodies. The complexity of the characteristic cones of (unbounded) starlike bodies really makes a difference that forces us to devise a new classification scheme that suits all starlike bodies, whatever their characteristic cones may be. If one wants to stick to the Bessaga-Klee classification scheme then the best result one can get is that Theorem 2.1 still holds for the class of starlike bodies whose characteristic cones are *convex* sets.

We will next state and prove such a result. To this end we need to use the following result (see [11] for the proof), which implies that every closed convex cone in a separable Banach space can be regarded as the characteristic cone of some C^∞ smooth convex body. We say that a nonempty subset C of a Banach space X is a *cone* (resp., a cone over a set K) provided $[0, \infty)C = C$ (resp., $C = [0, \infty)K$). The cone C is proper if $C \neq X$.

THEOREM 2.4. *For every proper closed convex set C in a separable Banach space X there exists a C^∞ smooth convex function $f : X \rightarrow [0, \infty)$ so that $f^{-1}(0) = C$. Moreover, when C is a cone, $U = f^{-1}([0, 1])$ is a C^∞ smooth convex body in X so that $ccU = C$.*

Now we have arrived at the following generalization of Theorem 2.1.

THEOREM 2.5. *Let A be a C^p starlike body in a separable Banach space X . Assume that ccA is a convex subset of X .*

- (a) *If ccA is a linear subspace of finite codimension (say $X = ccA \oplus Z$, with Z finite-dimensional), then A is C^p relatively diffeomorphic to $ccA + B_Z$, where B_Z is an Euclidean ball in Z .*
- (b) *If ccA is either not a linear subspace or else ccA is a linear subspace such that the quotient space X/ccA is infinite-dimensional, then A is C^p relatively diffeomorphic to a closed half-space.*

Moreover, in the case $p = 0$ this is true for all Banach spaces X .

Proof. To obtain (a) it is enough to apply proposition 2.3 for $A_1 = A$ and $A_2 = ccA + B_Z$.

To obtain (b), write $C = ccA$, which is a closed convex cone of X . By proposition 2.4 there exists a C^∞ smooth convex body U so that $ccU = C = ccA$. Then, by proposition 2.3 the starlike bodies U and A are C^p relatively diffeomorphic. On the other hand, by the assumption, $ccU = C$ is either not a linear subspace or else is a linear subspace such that $\dim(X/C) = \infty$. Now,

part (b) of Theorem 2.1 tells us that U is C^p relatively diffeomorphic to a closed half-space, and hence so is A .

Finally, in the case $p = 0$, it is easy to see that, for every closed convex cone $C \subset X$, the set $U = \overline{C + B}$, where B is the unit ball of X , is a closed convex body so that $C = ccU$. Hence, the above argument applies. ■

In particular, for an infinite-dimensional separable Banach space X , the boundary of every smooth bounded starlike body $A \subset X$ is C^p diffeomorphic to a hyperplane. We now apply the above result to get smooth negligibility of starlike bodies.

COROLLARY 2.6. *Let X be a separable Banach space, and let A be a C^p smooth starlike body such that its characteristic cone is a linear subspace of infinite codimension in X . Then there exists a C^p diffeomorphism from X onto $X \setminus A$.*

Proof. According to Theorem 2.5, there exists a C^p self-diffeomorphism of X mapping A onto a closed half-space. Therefore $X \setminus A$ is C^p diffeomorphic to an open half-space. Since an open half-space is obviously C^∞ diffeomorphic to the whole space, we may conclude that $X \setminus A$ and X are C^p diffeomorphic.

As said above, examples like 2.2 show that the classification scheme used in Theorem 2.5 is useless when one wants to cover such cases as those of starlike bodies with nonconvex characteristic cones. Let us have a closer look at those examples. In the case of the bodies A_n whose construction is hinted in Example 2.2, and whose boundary has n connected components, one could wonder whether every starlike body in \mathbb{R}^k whose boundary has exactly n connected components must be relatively homeomorphic to A_n .

More generally, it is natural to ask whether for every couple of starlike bodies A and B in a Banach space X with homeomorphic boundaries ∂A and ∂B it happens that A and B are relatively homeomorphic.

Surprisingly enough, the answers to these questions are all negative in the finite-dimensional setting, as we will show later on (see Examples 2.13, 2.14 and 2.15 below).

However, in infinite dimensions things turn less complicated, topologically speaking. The following theorem answers the above question in the affirmative, providing a full classification of starlike bodies in terms of the homotopy type of their boundaries in infinite-dimensional Banach spaces.

THEOREM 2.7. *Let X be an infinite-dimensional Banach space and let A, B be starlike bodies in X , with boundaries ∂A and ∂B . The following statements are equivalent:*

- (1) ∂A has the same homotopy type as ∂B
- (2) ∂A and ∂B are homeomorphic
- (3) A and B are relatively homeomorphic.

Proof. The proof involves infinite-dimensional topology, see [22]. The bodies A and B , and their boundaries ∂A and ∂B are manifolds modelled on the separable Hilbert space (in the sequel those manifolds will be called Hilbert manifolds). A fundamental theorem of infinite-dimensional topology states that two Hilbert manifolds are homeomorphic provided they have the same homotopy type. Since A and B are contractible, in fact, they are homeomorphic to X . Moreover, ∂A and ∂B are instances of the so-called Z -sets in A and B , respectively. The fact that ∂A and ∂B have the same homotopy type implies that they actually are homeomorphic. Then, by the homeomorphism extension theorem for Z -sets, any homeomorphism $h : \partial A \rightarrow \partial B$ extends to a homeomorphism H of A onto B . Finally, it is easy to extend H to a self-homeomorphism of X . We refer the reader to [10] for the details. ■

The starlike bodies of a Banach space X are, in some sense, in one-to-one correspondence with the closed subsets K (resp. the open subsets U) of the unit sphere S of X . Let A be a starlike body in X . Let $r : X \setminus \{0\} \rightarrow S$ be the radial retraction. Clearly, $S(A) = r(ccA \setminus \{0\})$ is a closed subset of S such that $ccA = [0, \infty)S(A)$, the cone over $S(A)$, while $r(\partial A) = S \setminus S(A)$ is an open subset of S . As it is easily seen below, a closed subset K of S gives rise to a starlike body whose characteristic cone is the cone over K .

PROPOSITION 2.8. *Let K be a closed subset of S , there exists a starlike body $A = A_K$ such that $S(A) = K$. If X is separable and C^p smooth, then we may require that the body A is C^p smooth as well.*

Proof. Take any continuous function $\lambda : S \rightarrow [0, 1]$ with $\lambda^{-1}(0) = K$. Define $\psi(x) = \|x\|\lambda(\frac{x}{\|x\|})$ for $x \neq 0$ and $\psi(0) = 0$. We see that $\psi : X \rightarrow [0, \infty)$ is a positively homogeneous continuous function with $\psi^{-1}(0) = [0, \infty)K$. It is enough to set $A = \psi^{-1}([0, 1])$. In the smooth case, if X is C^p smooth, there exists a bounded C^p smooth starlike body whose characteristic cone is $\{0\}$ [27]. Let μ stand for the Minkowski functional of this body. Using the fact that X admits C^p smooth partitions of unity, one can find a continuous function

$\lambda : X \rightarrow [0, 1]$ which is C^p smooth off $\lambda^{-1}(0) = [0, \infty)K$. Define $\psi(x) = \mu(x)\lambda(\frac{x}{\mu(x)})$ for $x \neq 0$ and $\psi(0) = 0$. Clearly, $\psi : X \rightarrow [0, \infty)$ is a positively homogeneous continuous function which is C^p smooth off $\psi^{-1}(0) = [0, \infty)K$. Set $A = \psi^{-1}([0, 1])$. ■

Remark 2.9. The smooth assertion holds true if one replaces the separability assumption by the existence of C^p smooth partitions of unity.

For a fixed closed set $K \subset S$, all (smooth) starlike bodies of the form A_K are relatively (diffeomorphic) homeomorphic. In the infinite-dimensional setting, as a consequence of Theorem 2.7, we also have:

COROLLARY 2.10. *For two closed sets $K_1, K_2 \subset S$ in an infinite-dimensional Banach space X , the starlike bodies A_{K_1} and A_{K_2} are relatively homeomorphic if and only if the complements $S \setminus K_1$ and $S \setminus K_2$ have the same homotopy type.*

Proof. This is a consequence of Theorem 2.7 because the boundary of A_{K_i} is homeomorphic to $S \setminus K_i$, $i = 1, 2$. ■

We do not know what necessary and sufficient conditions for K_i , $i = 1, 2$ one has to impose in order their complements in S have the same homotopy type. If K is a Z -set in S (e.g., K is compact), then the complement of K is homeomorphic to S ; hence, in such a case A_K is relatively homeomorphic to the unit ball. If K_1 is a one-point set and K_2 is a small closed ball intersected with S , then K_1 is a Z -set, while B_2 is not a Z -set, but the complements of K_1 and K_2 have the same homotopy type (they are contractible), and therefore A_{K_1} and A_{K_2} are relatively homeomorphic (with the unit ball). The following example shows that the contractibility of K_1 and K_2 does not suffice to obtain the same homotopy type of their complements.

EXAMPLE 2.11. Let $K_1 \subset S$ be a one point set and $K_2 = S \cap X_0$, where X_0 is a codimension 1 vector subspace of X . Then, K_1 and K_2 are contractible, but the complement of K_2 is disconnected, while the complement of K_1 is contractible (even homeomorphic to X). We see that A_{K_1} is relatively homeomorphic to the unit ball in X , while $ccA_{K_2} = X_0$ and, consequently, A_{K_2} is relatively homeomorphic to $X_0 \times [-1, 1]$, which, in turn, (having disconnected boundary in $X_0 \times \mathbb{R}$) is not homomorphic to the unit ball in X .

THE FINITE-DIMENSIONAL CASE

Below we provide several examples showing that Corollary 2.10 and Theorem 2.7 cannot be extended in any reasonable way for a finite-dimensional space X .

EXAMPLE 2.12. Let $S = S^1$ and B be the unit sphere and the unit ball in $X = \mathbb{R}^2$, respectively. Consider two compacta K_1 and K_2 in S ; K_1 is a copy of an infinite convergent sequence space and K_2 is a copy of the Cantor set. Then, the bodies A_{K_1} and A_{K_2} (having their boundaries homeomorphic) are not homeomorphic.

To see this it suffices to notice that each A_{K_i} is homeomorphic to $B \setminus K_i$. It is then clear that any nonisolated point of K_1 has a basis of neighborhoods (in A_{K_1}) that can be chosen to be topologically different from any neighborhood of any point of K_2 . We can obviously make those starlike bodies to be real-analytic, so an improvement in smoothness is not any help.

In higher dimensions, one can provide more regular examples.

EXAMPLE 2.13. Let $S = S^2$ be the unit sphere in $X = \mathbb{R}^3$. Consider $C_1 = U_1 \cup U_2 \cup U_3$, where $U_1 = \{(x, y, z) \in S \mid |z| < 1/8\}$, $U_2 = \{(x, y, z) \in S \mid |z - 1| < 1/8\}$, and $U_3 = -U_2$, and $C_2 = U_1 \cup U_2 \cup U'_3$, where $U'_3 = \{(x, y, z) \in S \mid |z - 1/2| < 1/8, y > 0\}$. Letting $K_i = S \setminus C_i$, $i = 1, 2$, we see that the boundaries of the starlike bodies A_{K_i} (being homeomorphic to C_i) are homeomorphic. However, there is no homeomorphism of A_{K_1} onto A_{K_2} .

In \mathbb{R}^4 , we have the following.

EXAMPLE 2.14. Let $S = S^3$ be the unit sphere in $X = \mathbb{R}^4$. Let K be the (doubled) Fox-Artin arc in S , that is, K is a topological arc whose complement is a contractible 3-manifold which is not homeomorphic to \mathbb{R}^3 , see [49, p. 68]. Then, for a starlike body $A = A_K$, ccA is a cone over an arc, therefore, it is contractible. Moreover, A_K is not homeomorphic to a half-space in \mathbb{R}^4 though both bodies have contractible boundaries.

In general, for every $n \geq 4$, the sphere $S = S^{n-1}$ in $X = \mathbb{R}^n$ contains an open contractible $(n-1)$ -manifold U that is not homeomorphic to \mathbb{R}^{n-1} . One can take U to be the so-called Whitehead manifold. In each dimension, there are continuum many pairwise non-homeomorphic such objects. While the complement $S^3 \setminus U$ is a continuum that is not contractible, for $n > 4$, always

one can pick U so that $S^{n-1} \setminus U$ is a contractible $(n-1)$ -manifold. To see this, let M be a contractible $(n-1)$ -manifold with non-simply connected boundary; the existence of M is due to N.H.A. Newman for $n > 5$ (see [35]), and due to B. Mazur and V. Poenaru for $n = 5$. Gluing together two copies of M along their boundaries we obtain the double space N , which is a topological copy of S^{n-1} (cf. [1], p. 2, items (4) and (9)). The complement of one copy of M in N is just the interior of the other copy, which yields a requested manifold U . Since U is not simply connected at infinity, U is not homeomorphic to \mathbb{R}^{n-1} ; moreover, the manifold U , being the interior of a contractible manifold, is itself contractible.

EXAMPLE 2.15. Write $K = S \setminus U$. Any starlike body A_K in \mathbb{R}^n , $n > 4$, has both ccA_K and ∂A_K contractible. However, A_K is not homeomorphic to a half-space.

3. SMOOTH LIPSCHITZ CONTRACTIBILITY OF BOUNDARIES OF STARLIKE BODIES IN INFINITE DIMENSIONS

The well known Brouwer's fixed point theorem states that every continuous self-map of the unit ball of a finite-dimensional Banach space admits a fixed point. This is equivalent to saying that there is no continuous retraction from the unit ball onto the unit sphere, or that the unit sphere is not contractible (the identity map on the sphere is not homotopic to a constant map). This result is no longer true in infinite dimensions (see [22]). In [47] B. Nowak showed that for several infinite-dimensional Banach spaces Brouwer's theorem fails even for Lipschitz mappings, and in [17] Y. Benyamini and Y. Sternfeld generalized Nowak's result for all infinite-dimensional normed spaces, establishing that for every infinite-dimensional space $(X, \|\cdot\|)$ there exists a Lipschitz retraction from the unit ball $B_X = \{x \in X : \|x\| \leq 1\}$ onto the sphere $S_X = \{x \in X : \|x\| = 1\}$, and that S_X is Lipschitz contractible. In recent years a lot of work has been done on smoothness and Lipschitz properties in Banach spaces (see [27, 16]). Following this trend it is natural to ask whether Nowak-Benyamini-Sternfeld's results can be sharpened so as to get C^p smooth Lipschitz retractions of the unit ball onto the sphere of every infinite-dimensional Banach space whose norm is C^p smooth.

The main result of this section tells us that this is indeed possible. In fact we generalize those results in two ways. Not only do they hold for the smooth category but also for a wider class of objects than balls and spheres, namely, that of bounded starlike bodies and their boundaries. Indeed, for

every infinite-dimensional Banach space with a C^p Lipschitz bounded starlike body A (where $p = 0, 1, 2, \dots, \infty$), there is a C^p Lipschitz retraction of A onto its boundary ∂A , and ∂A is also C^p Lipschitz contractible.

Before stating this result let us recall a few topological definitions. Let M, N be closed subsets of a Banach space X . We will say that two maps $f, g : M \rightarrow N$ are C^p Lipschitz homotopic provided there exist an open subset U of X containing M , an $\varepsilon > 0$, and a C^p smooth mapping $H : (-\varepsilon, 1 + \varepsilon) \times U \rightarrow X$ such that the restriction of H to $[0, 1] \times M$ is a Lipschitz homotopy joining f to g , that is, $H : [0, 1] \times M \rightarrow N$ is Lipschitz continuous and satisfies $H(0, x) = f(x)$ and $H(1, x) = g(x)$ for all $x \in M$. Moreover we will demand that $H(t, x) = f(x)$ for $t \leq 0, x \in M$, and $H(t, x) = g(x)$ for $t \geq 1, x \in M$. With this definition, ‘being C^p Lipschitz homotopic’ endows the set of C^p Lipschitz mappings from M into N with an equivalence relationship (one can join C^p smooth homotopies without losing smoothness or Lipschitzness). A closed subset M of X is said to be C^p Lipschitz contractible if the identity map on M is C^p Lipschitz homotopic to a constant map on M . For instance, it is easy to check that every C^p Lipschitz starlike body A is C^p Lipschitz contractible. It is also easy to see that every two maps on a (C^p Lipschitz) contractible set are always (C^p Lipschitz) homotopic (they are both homotopic to a constant). Finally, we will say that $r : A \rightarrow \partial A$ is a C^p smooth Lipschitz retraction from the starlike body A onto its boundary provided there exist an open subset U of X containing A and a C^p smooth mapping $R : U \rightarrow X$ such that R fixes all the points of ∂A , and the restriction of R to A is Lipschitz continuous and coincides with r .

THEOREM 3.1. *Let X be an infinite-dimensional Banach space and let A be a C^p Lipschitz bounded starlike body. Then:*

- (1) *The boundary ∂A is C^p Lipschitz contractible.*
- (2) *There is a C^p Lipschitz retraction from A onto ∂A .*
- (3) *There is a C^p Lipschitz map $T : A \rightarrow A$ with no approximate fixed points, that is, $\inf\{\|x - T(x)\| : x \in A\} > 0$.*

As a corollary we obtain the following generalization of Benyamini-Sternfeld’s theorem:

COROLLARY 3.2. *Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space with an equivalent norm $\|\cdot\|$ which is C^p smooth, and let B_X and S_X be its unit ball and unit sphere respectively. Then*

- (1) S_X is C^p Lipschitz contractible.
- (2) There is a C^p Lipschitz retraction of B_X onto S_X .
- (3) There is a C^p Lipschitz map $T : B_X \rightarrow B_X$ with no approximate fixed points.

If one is not interested in the Lipschitz property, it is a trivial consequence of the main result in [2] that the sphere S_X is C^p contractible and there are C^p smooth retractions from B_X onto S_X . Unfortunately, the deleting diffeomorphisms obtained in [2], [9] are not Lipschitz, and Corollary 3.2 cannot be deduced by using those results. As a matter of fact, Corollary 3.2 provides a new result even in the case $X = \ell_2$ with the usual hilbertian norm.

The general scheme of the proof of Theorem 3.1 follows that of [17], which is in turn a generalization with some modifications of Nowak's approach [47]. The proofs in [17], [47] are already involved in themselves and in our case they are even more complicated with the difficulties peculiar to smooth maps and starlike bodies. That is why we omit the proof of Theorem 3.1 in the Lipschitz case; we refer the interested reader to [4]. However, if one drops the Lipschitz condition, a simpler proof is available, as we will see in the following section.

4. THE FAILURE OF ROLLE'S AND BROUWER'S THEOREMS IN INFINITE-DIMENSIONS

Rolle's theorem in finite-dimensional spaces states that, for every bounded open subset U of \mathbb{R}^n and for every continuous function $f : \bar{U} \rightarrow \mathbb{R}$ such that f is differentiable in U and constant on the boundary ∂U , there exists a point $x \in U$ such that $f'(x) = 0$. Unfortunately, Rolle's theorem does not remain valid in infinite dimensions. It was S. A. Shkarin [50] that first showed the failure of Rolle's theorem in superreflexive infinite-dimensional spaces and in non-reflexive spaces which have smooth norms. The class of spaces for which Rolle's theorem fails was substantially enlarged in [12], where it was also shown that an approximate version of Rolle's theorem remains nevertheless true in all Banach spaces. In fact, as a consequence of the existence of diffeomorphisms deleting points in infinite-dimensional spaces (see [2], [9]), it is easy to see that Rolle's theorem fails in all infinite-dimensional Banach spaces which have smooth norms [3].

Of course, Rolle's theorem is trivially true in the Banach spaces which do not have any smooth bumps (if X is such a space then every function on X satisfying the hypothesis of Rolle's theorem must be a constant). Thus,

in many infinite-dimensional Banach spaces, Rolle's theorem is either false or trivial, depending on the smoothness properties of the spaces considered. In this setting, it does not seem too risky to conjecture, as it was done in [12], that Rolle's theorem should fail in an infinite-dimensional Banach space if and only if our space has a C^1 smooth bump function. However, none of the results quoted above allows to completely characterize the spaces for which Rolle's theorem fails. What makes the problem difficult is that the spaces are not assumed to be separable, nor even to have smooth norms. As shown by R. Haydon [38], there are (nonseparable) Banach spaces with smooth bump functions which possess no equivalent smooth norms. Besides, it is natural to demand that the smooth bumps which do not satisfy Rolle's theorem be Lipschitz whenever smooth Lipschitz bumps are available in the space considered, and this requirement makes the problem even more delicate.

In spite of those difficulties, the above conjecture has recently proved to be right [13], thus providing an interesting characterization of smoothness in Banach spaces.

THEOREM 4.1. *Let X be an infinite-dimensional Banach space which has a C^p smooth (Lipschitz) bump function. Then there exists another C^p smooth (Lipschitz) bump function $f : X \rightarrow [0, 1]$ with the property that $f'(x) \neq 0$ for every $x \in \text{int}(\text{supp } f)$.*

Here, as in the whole section, $1 \leq p \leq \infty$, and $\text{supp } f$ denotes the support of f , that is, $\text{supp } f = \overline{\{x \in X : f(x) \neq 0\}}$. Let us recall that $b : X \rightarrow \mathbb{R}$ is said to be a bump function on X provided b is not constantly zero and b has a bounded support.

From this result it is easily deduced the following

COROLLARY 4.2. *Let X be an infinite-dimensional Banach space. The following statements are equivalent.*

- (1) X has a C^p smooth (and Lipschitz) bump function.
- (2) There exist a bounded contractible open subset U of X and a continuous function $f : \overline{U} \rightarrow \mathbb{R}$ such that f is C^p smooth (and Lipschitz) in U , $f = 0$ on ∂U , and yet $f'(x) \neq 0$ for all $x \in U$, that is, Rolle's theorem fails in X .
- (3) There exist a C^p smooth (and Lipschitz) function $f : X \rightarrow [0, 1]$ and a bounded contractible open subset U of X such that $f = 0$ precisely on $X \setminus U$ and yet $f'(x) \neq 0$ for all $x \in U$.

To complete the picture of Rolle's theorem in infinite-dimensional Banach spaces, let us quote the positive result from [12] on an approximate substitute of Rolle's theorem, which guarantee the existence of arbitrarily small derivatives (instead of vanishing ones) for every function satisfying (in an approximate manner) the conditions of the classical Rolle's theorem. Here, Baire category arguments can make up for the lack of local compactness, but one has to pay an ε , as is usual in such cases.

THEOREM 4.3. *Let U be a bounded connected open subset of a Banach space X . Let $f : \bar{U} \rightarrow \mathbb{R}$ be a bounded continuous function which is (Gâteaux) differentiable in U . Let $R > 0$ and $x_0 \in U$ be such that $\text{dist}(x_0, \partial U) = R$. Suppose that $f(\partial U) \subseteq [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$. Then there exists some $x_\varepsilon \in U$ such that $\|f'(x_\varepsilon)\| \leq \frac{\varepsilon}{R}$.*

The “twisted tube” method that we developed in order to prove Theorem 4.1 is interesting in itself and, with little more work, provides a useful characterization of C^p smoothness in infinite-dimensional Banach spaces related to the existence of a certain kind of *deleting diffeomorphisms*. Namely, we have the following

THEOREM 4.4. *Let X be an infinite-dimensional Banach space. The following assertions are equivalent.*

- (1) *X has a C^p smooth bump function.*
- (2) *There exists a nonempty contractible closed subset D of the unit ball B_X and a C^p diffeomorphism $f : X \rightarrow X \setminus D$ so that f restricts to the identity outside B_X .*

When X has a (not necessarily equivalent) C^p smooth norm this result was already known [2], [9], [3] and, moreover, one can take for D a single point, or a small ball. Theorem 4.4 provides a new result in the case when X possesses a C^p smooth bump but has no C^p smooth norm. Unfortunately, it is still unknown whether Theorem 4.4 is true in full generality when D is a single point. The proof we will give here does not clarify this question (in our proof D is nothing but a small “twisted tube” inside B_X). Nevertheless, some important applications of smooth negligibility do not require such accurate instruments as a diffeomorphism deleting just a single point, and it is often enough to use diffeomorphisms which remove a small bounded set, as in the statement of Theorem 4.4. Indeed, this theorem will allow us to deduce two interesting corollaries. The first one is the failure of Brouwer's theorem in

infinite dimensions even for smooth self-mappings of balls or starlike bodies; this is a particular case (the non-Lipschitz one) of the main result of the preceding section. Second, we deduce from the above characterization that the support of the bump functions which violate Rolle's theorem can always be assumed to be a smooth starlike body. We will show this later on. Let us first say a few words about the proofs of Theorems 4.1 and 4.4.

SKETCH OF THE PROOFS OF THEOREMS 4.1 AND 4.4

The idea behind the proof of Theorem 4.1 is as follows. First we build a twisted tube T of infinite length in the interior of the unit ball B_X , with a beginning but with no end. This twisted tube can be thought of as directed by an ever-winding infinite path p that gets lost in the infinitely many dimensions of our space X . In technical words, one can construct a diffeomorphism π between a straight (unbounded) half-cylinder C and a twisted (bounded) tube T contained in B_X . The tube T is going to be the support of a smooth bump function f that does not satisfy Rolle's theorem. In order to define such a function f we only have to make it strictly increase in the direction which is tangent to the leading path p at each point of the tube T . The graph of f would thus represent an ever-ascending stairway built upon our twisted tube, with a beginning but no end.

The spirit of the proof that (1) implies (2) in Theorem 4.4 is not very different. We will make use of the diffeomorphism π between a straight (unbounded) half-cylinder C and a bounded twisted tube T contained in B_X . If we consider a straight closed half-cylinder C' contained in the interior of C and directed by the same line as C , it is elementary that there is a diffeomorphism $g : X \rightarrow X \setminus C'$ so that g restricts to the identity outside C' . In fact this is true even in the plane. Now, by composing this diffeomorphism g with the diffeomorphisms π and π^{-1} that give us an appropriate coordinate system in the twisted tube $T = \pi(C)$, we get a diffeomorphism $f : X \rightarrow X \setminus T'$, where $T' = \pi(C')$ is a smaller closed twisted tube inside T , and f restricts to the identity outside the unit ball. The precise definition of f would be $f(x) = \pi(g(\pi^{-1}(x)))$ if $x \in T$, and $f(x) = x$ if $x \in X \setminus T$. If we take $D = T'$ we are done.

The following lemma guarantees the existence of bounded infinite twisted tubes in all infinite-dimensional Banach spaces.

LEMMA 4.5. *There are universal constants $M > 0$ (large) and $\varepsilon > 0$ (small) such that, for every infinite-dimensional Banach space X , if we consider the decomposition $X = H \oplus [z]$ (where $H = \text{Ker } z^*$ for some $z^* \in X^*$)*

with $z^*(z) = \|z^*\| = \|z\| = 1$) and the open half-cylinder C of diameter 2ε , directed by z , and with base on H , $C = \{x + tz \in X : \|x\| < \varepsilon, t > 0\}$, then there exists an injection $\pi : C \rightarrow B_X$ which is a C^∞ diffeomorphism onto its image. The image $T = \pi(C)$ is thus a bounded open set which we will call a bounded open infinitely twisted tube in X . Moreover, the first derivatives of the mappings $\pi : C \rightarrow T$ and $\pi^{-1} : T \rightarrow C$ are both uniformly bounded by M .

Let us give a glimpse of the idea behind the proof of this key lemma. Let $(x_n)_{n=0}^\infty$ be a normalized basic sequence in X with biorthogonal functionals $(x_n^*)_{n=0}^\infty \subset X^*$ (that is, $x_n^*(x_k) = \delta_{n,k} = 1$ if $n = k$, and 0 otherwise) satisfying $\|x_n^*\| \leq 3$. Consider the following piecewise affine arc: $p = [0, x_1] \cup [x_1, x_2] \cup \dots \cup [x_{n-1}, x_n] \cup \dots$; p is an ever-twisting path that gets lost in the infinitely many dimensions of X . If we defined T as the set of points whose distance to p is less than or equal to some suitable small positive number then we would almost have the tube we want, only we would not know how to get the required diffeomorphism π . With some care, by smoothing out the broken line p and considering a neighbourhood of paths which are parallel and close enough to the smooth p , both the tube T and the diffeomorphism π can be constructed. We refer the reader to [13] for the details of the proof.

Let us now see what we can do with Lemma 4.5.

PROOF OF THEOREM 4.1

Consider the diffeomorphism $\pi : C \rightarrow T \subset B_X$ from Lemma 4.5. Take a C^p smooth (Lipschitz) non-negative bump function φ on H so that the support of φ is contained in the base of C , that is, $\varphi(x) = 0$ whenever $\|x\| \geq \varepsilon/2$, for instance. Pick a C^∞ smooth real function $\mu : \mathbb{R} \rightarrow [0, 1]$ such that $\mu(t) = 0$ for $t \leq 1$, $0 < \mu(t) < 1$ for $t > 1$ and $0 < \mu'(t) < 1$ for all $t > 1$. Then define $g : X = H \oplus [z] \rightarrow \mathbb{R}$ by

$$g(x, t) = \varphi(x)\mu(t).$$

It is plain that g is a C^p smooth (Lipschitz) function such that $g'(x, t) \neq 0$ for every $x \in \text{int}(\text{supp} f)$, that is, for every x such that $g(x, t) \neq 0$ (take into account that the interior of the support of g coincides in this case with the open support of g , that is the set of points at which g does not vanish). Indeed,

$$g'(x, t)(0, 1) = \frac{\partial g}{\partial t}(x, t) = \varphi(x)\mu'(t)$$

and therefore $g'(x, t)(0, 1) = 0$ if and only if $\varphi(x) = 0$ or $\mu'(t) = 0$, which happens if and only if $\varphi(x) = 0$ or $\mu(t) = 0$, that is to say, $g(x, t) = 0$. Now let us define $f : X \rightarrow \mathbb{R}$ by

$$f(y) = \begin{cases} g(\pi^{-1}(y)) & \text{if } y \in T; \\ 0 & \text{if } y \notin T. \end{cases}$$

It is clear that f is a well defined C^p smooth (Lipschitz) function, and $\text{supp}(f) = \pi(\text{supp}(g)) \subset T$, from which it follows that f has a bounded support. We claim that $f'(y) \neq 0$ whenever $y \in \text{int}(\text{supp} f)$, that is, f does not satisfy Rolle's theorem. Indeed, if $y \in \text{int}(\text{supp} f)$ then $\pi^{-1}(y) = (x, t) \in \text{int}(\text{supp} g)$ and therefore $g'(x, t)(0, 1) \neq 0$. But then

$$f'(y) = g'(x, t) \circ D\pi^{-1}(y) \neq 0,$$

because $D\pi^{-1}(y)$ is a linear isomorphism. ■

PROOF OF THEOREM 4.4

First of all let us choose a number $\varepsilon > 0$, a cylinder C , a bounded twisted tube T , and a diffeomorphism $\pi : C \rightarrow T$ from Lemma 4.5.

Let B be a C^∞ smooth convex body in the plane \mathbb{R}^2 whose boundary contains the set

$$\left\{ (s, t) : t = -1, |s| \leq \frac{\varepsilon}{4} \right\} \cup \left\{ (s, t) : |s| = \frac{\varepsilon}{2}, t \geq -1 + \frac{\varepsilon}{4} \right\},$$

and let q_B be the Minkowski functional of B . Define $B' = \frac{1}{2}B = \{(s, t) : q_B(s, t) \leq \frac{1}{2}\}$. Let $\theta : (\frac{1}{2}, \infty) \rightarrow [0, \infty)$ be a C^∞ smooth real function so that $\theta'(t) < 0$ for $\frac{1}{2} < t < 1$, $\theta(t) = 0$ for $t \geq 1$, and $\lim_{t \rightarrow 1/2^+} \theta(t) = +\infty$. Now define $\varphi : \mathbb{R}^2 \setminus B' \rightarrow \mathbb{R}^2$ by

$$\varphi(s, t) = (\varphi_1(s, t), \varphi_2(s, t)) = (s, t + \theta(q_B(s, t))).$$

It is elementary to check that φ is a C^∞ diffeomorphism from $\mathbb{R}^2 \setminus B'$ onto \mathbb{R}^2 so that φ restricts to the identity outside the band B .

Next, recall that since X has a C^p smooth bump then it has a C^p bounded starlike body A as well. If $X = H \oplus [z]$, take $W = A \cap H$, which is a C^p bounded starlike body in H , and denote by q_W its Minkowski functional. We can assume that $W \subseteq B(0, 1)$, that is, $\|x\| \leq q_W(x)$ for all $x \in H$. Let us define

$$\psi(x, t) = q_B(q_W(x), t)$$

for all $(x, t) \in X = H \oplus [z]$. It is clear that ψ is a continuous function which is positive-homogeneous and C^p smooth away from the half-line $L = \{(x, t) \in X : x = 0, t \geq 0\}$. Then the sets

$$U = \{(x, t) \in X : \psi(x, t) \leq 1\}, \quad U' = \{(x, t) \in X : \psi(x, t) \leq \frac{1}{2}\}$$

are cylindrical C^p starlike bodies whose characteristic cones are the half-line L . If we define

$$h(x, t) = (x, (\varphi^{-1})_2(q_W(x), t))$$

for $(x, t) \in X = H \oplus [z]$, it is not difficult to realize that h is a C^p diffeomorphism from X onto $X \setminus U'$ so that h restricts to the identity outside U . The inverse of h is given by

$$h^{-1}(x, t) = (x, t + \theta(\psi(x, t))).$$

Now consider the point $p_0 = (0, 2) \in X = H \oplus [z]$ and the cylindrical bodies $V := p_0 + U$ and $V' := p_0 + U'$, and put $g(x, t) = h(x, t - 2)$. Then $g : X \rightarrow X \setminus V'$ is a C^p diffeomorphism such that g is the identity outside V . Note that, since $W \subseteq B(0, 1)$, we have that $V' \subset V \subset C = \{(x, t) \in X : \|x\| < \varepsilon, t > 0\}$. Let us define

$$f(x) = \begin{cases} \pi(g(\pi^{-1}(x))) & \text{if } x \in T; \\ x & \text{otherwise.} \end{cases}$$

It is then clear that f is a C^p diffeomorphism from X onto $X \setminus T'$, where $T' = \pi(V')$ is a smaller closed twisted tube inside $\pi(V) \subseteq T$, and f restricts to the identity outside the larger tube $\pi(V) \subset T$, which is contained in B_X . This completes the proof that (1) implies (2).

Conversely, if there is such an f as in (2), we can assume that $f(0) \neq 0$ and take $x^* \in X^*$ so that $x^*(f(0)) \neq 0$; then the function $b : X \rightarrow \mathbb{R}$ defined by $b(x) = x^*(x - f(x))$ is a C^p smooth bump on X . ■

KILLING SINGULARITIES

Do not be afraid, this paragraph does not contain any totalitarian propaganda. Here we will present the two promised applications of Theorem 4.4, both of which have in common the following principle: if you have a mapping with a single singular point or an isolated set of singularities that bother you, you can just kill them by composing your map with some deleting diffeomorphisms. In this way you obtain a new map which is as close as you

want to the old one but does not have the adverse properties created by the singular points you eliminate.

For instance, if you want a smooth bump function g which does not satisfy Rolle's theorem and whose support is a smooth starlike body A , by composing the Minkowski functional of this body with a real bump function you get a function h whose support is A and whose derivative vanishes only at the origin and outside A ; then, by composing h with a diffeomorphism f which extracts a small set containing the origin and which restricts to the identity outside A , you get a map g with the required properties.

On the other hand, suppose you want a smooth retraction r from a bounded starlike body A of a Banach space X onto its boundary ∂A . This is impossible if X is finite-dimensional, but otherwise you can use the following trick: it is trivial that there is a smooth retraction h from $A \setminus \{0\}$ onto ∂A ; then take a diffeomorphism f which removes from X a small subset containing the origin and restricts to the identity outside A . The composition $r = h \circ f$ gives the required retraction.

Let us formalize these ideas.

THE FAILURE OF BROUWER'S THEOREM IN INFINITE DIMENSIONS, REVISITED

Next we give a proof of the following particular case (the non-Lipschitz one) of Theorem 3.1. The Lipschitz case is much harder to handle because the known diffeomorphisms which remove points, small balls, or (as in our case) small twisted tubes from infinite-dimensional Banach spaces are not Lipschitz, so that the deleting diffeomorphisms approach does not work in this case.

COROLLARY 4.6. *Let X be an infinite-dimensional Banach space and let A be a C^p smooth bounded starlike body. Then:*

- (1) *The boundary ∂A is C^p contractible.*
- (2) *There is a C^p smooth retraction from A onto ∂A .*
- (3) *There exists a C^p smooth mapping $\varphi : A \rightarrow A$ without approximate fixed points.*

Proof. Let $f : X \rightarrow X \setminus D$ be the diffeomorphism from Theorem 4.4. We may assume that the origin belongs to the deleted set D and that $B_X \subseteq A$, so that f restricts to the identity outside A . Then the formula

$$R(x) = \frac{f(x)}{\mu_A(f(x))},$$

where μ_A is the Minkowski functional of A , defines a C^p smooth retraction from A onto the boundary ∂A . This proves (2).

Once we have such a retraction it is easy to prove parts (1) and (3): the formula $\varphi(x) = -R(x)$ defines a C^p smooth self-mapping of A without approximate fixed points. On the other hand, if we pick a non-decreasing C^∞ function $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ so that $\zeta(t) = 0$ for $t \leq \frac{1}{4}$ and $\zeta(t) = 1$ for $t \geq \frac{3}{4}$, then the formula

$$H(t, x) = R((1 - \zeta(t))x),$$

for $t \in [0, 1]$, $x \in \partial A$, defines a C^p homotopy joining the identity to a constant on ∂A , that is, H contracts the pseudosphere ∂A to a point. ■

THE SUPPORT OF THE BUMPS THAT VIOLATE ROLLE'S THEOREM

The bump function constructed in the proof of Theorem 4.1 has a weird support, namely a twisted tube. Some readers (including the authors) might judge this fact rather unpleasant and wonder whether it is possible to construct a bump function which does not satisfy Rolle's theorem and whose support is a nicer set, such as a ball or a starlike body. To comfort ourselves let us first recall that in infinite dimensions there is no topological difference between a tube (whether it is twisted or not) and a ball or a starlike body (see Theorems 2.1 and 2.7). Furthermore, Theorem 4.4 allows us to show that for a given C^p smooth bounded starlike body A in an infinite-dimensional Banach space X , it is always possible to construct a C^p smooth bump function on X which does not satisfy Rolle's theorem and whose support is precisely the body A .

COROLLARY 4.7. *Let X be an infinite-dimensional Banach space with a C^p smooth bounded starlike body A . Then there exists a C^p smooth bump function g on X whose support is precisely the body A , and with the property that $g'(x) \neq 0$ for all x in the interior of A .*

Proof. Let μ_A be the Minkowski functional of A . We may assume that $B_X \subseteq A$. By Theorem 4.4 there is a closed subset D of A and a C^p diffeomorphism $f : X \rightarrow X \setminus D$ which is the identity outside A . It can be assumed that the origin belongs to D . Then the function $h : X \rightarrow \mathbb{R}$ defined by

$$h(x) = \mu_A(f(x))$$

is C^p smooth on X , restricts to the gauge μ_A outside A , and has the remarkable property that $h'(x) \neq 0$ for all $x \in X$ (indeed, $h'(x) = \mu'_A(f(x)) \cdot f'(x)$ is non-zero everywhere because $\mu'_A(y) \neq 0$ whenever $y \neq 0$, $0 \notin f(X)$, and $f'(x)$ is a linear isomorphism at each point x).

Now, take a C^∞ real function $\theta : \mathbb{R} \rightarrow [0, 1]$ such that $\theta(t) > 0$ for $t \in (-1, 1)$, $\theta = 0$ outside $[-1, 1]$, $\theta(t) = \theta(-t)$, $\theta(0) = 1$, and $\theta'(t) < 0$ for all $t \in (0, 1)$. Then, if we define $g : X \rightarrow \mathbb{R}$ by

$$g(x) = \theta(h(x)),$$

it is immediately checked that g is a C^p smooth bump on X which does not satisfy Rolle's theorem and whose support is precisely the body A . ■

5. HOW SMALL CAN THE SET OF GRADIENTS OF A BUMP BE?

In this section and the following one we will be involved in trying to answer the following natural question. If $b : X \rightarrow \mathbb{R}$ is a smooth bump on a Banach space X , how many tangent hyperplanes does its graph have? In other words, if we denote the cone generated by its set of gradients by

$$\mathcal{C}(b) = \{\lambda b'(x) : x \in X, \lambda \geq 0\},$$

what is the (topological) size of $\mathcal{C}(b)$?

As we will see, this problem is strongly related to a similar question about the size of the cones of tangent hyperplanes to starlike bodies in X . Namely, if A is a smooth bounded starlike body in X , how many tangent hyperplanes does A have? More precisely, if we denote the cone of hyperplanes which are tangent to A at some point of its boundary ∂A by

$$\mathcal{C}(A) = \{x^* \in X : x + \text{Ker } x^* \text{ is tangent to } \partial A \text{ at some point } x \in \partial A\},$$

what is the size of $\mathcal{C}(A)$?

It may be helpful to make some previous general considerations about these questions.

To begin with, as a consequence of Ekeland's variational principle [27], it is easily seen that if $b : X \rightarrow \mathbb{R}$ is a Gâteaux smooth and continuous bump function on a Banach space X then the norm-closure of $b'(X)$ is a neighbourhood of 0 in X^* . If, in addition, X is finite-dimensional, and b is C^1 smooth, then $b'(X)$ is a compact neighbourhood of 0 in X^* , and in particular 0 is an interior point of $b'(X)$.

However, as we already know, the classical Rolle's theorem is false in a Banach space X whenever there are smooth bumps in X , and this fact has some interesting consequences on the question about the minimal size of the cones of gradients $\mathcal{C}(b)$. Indeed, by using the main result of the preceding

section, one can construct smooth bump functions whose sets of gradients lack not only the point zero, but also any prescribed finite-dimensional linear subspace of the dual space, so that they violate Rolle's theorem in a quite strong way, as we will see later on.

If we restrict the scope of our search to classic Banach spaces, much stronger results are available. On the one hand, if $X = c_0$ the size of $\mathcal{C}(b)$ can be really small. Indeed, as a consequence of P. Hájek's work [40] on smooth functions on c_0 we know that if b is C^1 smooth with a locally uniformly continuous derivative (note that there are bump functions with this property in c_0), then $b'(X)$ is contained in a countable union of compact sets in X^* (and in particular $\mathcal{C}(b)$ has empty interior). On the other hand, if X is non-reflexive and has a Fréchet norm, there are Fréchet smooth bumps b on X so that $\mathcal{C}(b)$ has empty interior [6], [42].

In the reflexive case, however, the problem is far from being settled. To begin with, the cone $\mathcal{C}(b)$ cannot be very small, since it is going to be a residual subset of the dual X^* . Indeed, as a consequence of Stegall's variational principle, for every Banach space X having the Radon-Nikodym Property (RNP) it is not difficult to see that $\mathcal{C}(b)$ is a residual set in X^* . Thus, for infinite-dimensional Banach spaces X enjoying RNP (such is the case of reflexive ones and, of course, ℓ_2) one can hardly expect a better answer to the question about the minimal size of the cones of gradients of smooth bumps than the following one: there are smooth bumps b on X such that the cones $\mathcal{C}(b)$ have empty interior in X^* .

In this section we will settle the question as to how small the sets of gradients $\mathcal{C}(b)$ can be for a smooth bump b on the Hilbert space ℓ_2 . Namely, we will construct C^1 smooth bumps b on ℓ_2 so that the cones of gradients $\mathcal{C}(b)$ have empty interior. Furthermore, these strange bumps can be made to uniformly approximate the norm of ℓ_2 .

As we will see in Section 8, this result will allow us to answer the corresponding question about the minimal size of the cone of tangent hyperplanes, $\mathcal{C}(A)$, to a smooth starlike body A in the Hilbert space.

We begin by showing how one can use one of the main results of the preceding section to construct smooth bump functions whose sets of gradients lack not only the point zero, but any pre-set finite-dimensional linear subspace of the dual space, thus violating Rolle's theorem in a quite strong way.

THEOREM 5.1. *Let X be an infinite-dimensional Banach space and W a finite-dimensional subspace of X^* . The following statements are equivalent.*

- (1) *X has a C^p smooth (Lipschitz) bump function.*
- (2) *X has a C^p smooth (Lipschitz) bump function f so that $\mathcal{C}(f) \cap W = \{0\}$ and, moreover, $\{f'(x) : x \in \text{int}(\text{supp}(f))\} \cap W = \emptyset$.*

Proof. We only need to prove that (1) implies (2). We can write $X = Y \oplus Z$, where $Y = \bigcap_{w^* \in W} \ker w^*$ and $\dim Z = \dim W$ is finite. Let us pick a C^p smooth (Lipschitz) bump function $\varphi : Y \rightarrow \mathbb{R}$ such that $\varphi'(y) = 0$ if and only if $y \notin \text{int}(\text{supp}(\varphi))$ (the existence of such a bump φ is guaranteed by the main theorem of the preceding section). Let θ be a C^∞ smooth Lipschitz bump function on Z so that $\theta'(z) = 0$ whenever $\theta(z) = 0$. Then the function $f : X = Y \oplus Z \rightarrow \mathbb{R}$ defined by $f(y, z) = \varphi(y)\theta(z)$ is a C^p smooth (Lipschitz) bump which satisfies $\{f'(x) : x \in \text{int}(\text{supp}(f))\} \cap W = \emptyset$. Indeed, if $(y, z) \in Y \oplus Z$ we have

$$f'(y, z) = (\theta(z)\varphi'(y), \varphi(y)\theta'(z)) \in X^* = Y^* \oplus Z^* = Y^* \oplus W.$$

If $(y, z) \in \text{int}(\text{supp}(f))$, then $\theta(z)\varphi'(y) \neq 0$, and hence $f'(y, z) \notin W$ and $\mathcal{C}(f) \cap W = \{0\}$. ■

The following theorem and its corollary are the main results of this section. This theorem is also the keystone for the construction of a smooth bounded starlike body whose cone of tangent hyperplanes has empty interior (see Section 8).

THEOREM 5.2. *Let $\|\cdot\|$ denote the usual hilbertian norm of ℓ_2 . There are C^1 functions $f_\varepsilon : \ell_2 \rightarrow (0, \infty)$, $0 < \varepsilon < 1$, which are Lipschitz on bounded sets and have Lipschitz derivatives, so that:*

- (1) $\lim_{\varepsilon \rightarrow 0} f_\varepsilon(x) = \|x\|^2$ uniformly on ℓ_2 ;
- (2) $\lim_{\varepsilon \rightarrow 0} f'_\varepsilon(x) = 2x$ uniformly on ℓ_2 (that is, the derivatives of the f_ε uniformly approximate the derivative of the squared norm of ℓ_2); and
- (3) the cones $\mathcal{C}(f_\varepsilon)$ generated by the sets of gradients of the f_ε have empty interior, and $f'_\varepsilon(x) \neq 0$ for all $x \in \ell_2$, $0 < \varepsilon < 1$.

Moreover, the functions $\psi_\varepsilon = (f_\varepsilon)^{1/2}$ are C^1 smooth and Lipschitz, with Lipschitz derivatives. Note, in particular, that $\lim_{\varepsilon \rightarrow 0} \psi_\varepsilon = \|\cdot\|$ uniformly on ℓ_2 , the cones of gradients $\mathcal{C}(\psi_\varepsilon)$ have empty interior, and $\psi'_\varepsilon(x) \neq 0$ for all $x \in \ell_2$. Besides, for every $r > 0$, the derivatives ψ'_ε approximate the derivative of the norm uniformly on the set $\{x \in \ell_2 : \|x\| \geq r\}$ as ε goes to 0.

COROLLARY 5.3. *There is a C^1 Lipschitz bump function b on ℓ_2 (with Lipschitz derivative) satisfying that the cone $\mathcal{C}(b)$ generated by its set of gradients has empty interior, and $b'(x) \neq 0$ for every x in the interior of its support.*

SKETCH OF THE PROOFS OF THEOREM 5.2 AND COROLLARY 5.3

We will make use of the following restatement of a strong result due to S. A. Shkarin [50].

THEOREM 5.4. (Shkarin) *There is a C^∞ diffeomorphism φ from ℓ_2 onto $\ell_2 \setminus \{0\}$ such that all the derivatives $\varphi^{(n)}$ are uniformly continuous on ℓ_2 , and $\varphi(x) = x$ for $\|x\| \geq 1$.*

Let us consider, for $0 < \varepsilon < 1$, the diffeomorphism $\varphi_\varepsilon : \ell_2 \rightarrow \ell_2 \setminus \{0\}$, $\varphi_\varepsilon(x) = \varepsilon\varphi(x/\varepsilon)$, and the function $U \equiv U_\varepsilon : \ell_2 \rightarrow \mathbb{R}$ defined by $U(x) = \varepsilon^2 + \|\varphi_\varepsilon(x)\|^2$. Now, we define the functions $U_n : \ell_2 \rightarrow \mathbb{R}$ by $U_n(x) = \frac{1}{2^{2n}}U(2^n x)$, whenever $x \in \ell_2$. We identify ℓ_2 with the infinite sum $\sum_2 \ell_2 \equiv \ell_2 \oplus_2 \ell_2 \oplus_2 \ell_2 \cdots$, where an element $x = (x_n)$ belongs to $\sum_2 \ell_2$ if and only if every x_n is in ℓ_2 and $\sum_n \|x_n\|^2 < \infty$, being $\|x\|^2 = \sum_n \|x_n\|^2$. Then, we define the function $f \equiv f_\varepsilon : \sum_2 \ell_2 \rightarrow \mathbb{R}$ by

$$f(x) = \sum_n U_n(x_n), \text{ where } x = (x_n)_n.$$

It can be checked that f has the properties of the statement of Theorem 5.2. We refer the reader to [14] for the details. We only mention that in order to see that the cones of gradients $\mathcal{C}(f_\varepsilon)$ have empty interior, it suffices to note that the set $\{\lambda f'(x) = \lambda(U'_n(x_n)) : x = (x_n) \in \sum_2 \ell_2, \lambda > 0\}$ is contained in $\{z = (z_n) \in \sum_2 \ell_2 : z_n \neq 0 \text{ for every } n \in \mathbb{N}\}$, which has empty interior in $(\sum_2 \ell_2)^* = \sum_2 \ell_2$.

In order to prove Corollary 5.3, we consider a C^∞ function $\theta : \mathbb{R}^+ \rightarrow \mathbb{R}$, $\theta'(t) < 0$ for $t \in (0, 1)$, and $\text{supp } \theta = (0, 1]$. Then, we can define a required bump function as the composition $b(x) = \theta(f(x))$. ■

6. HOW LARGE CAN THE RANGE OF A DERIVATIVE BE?

In this section we continue our study of the topological size of the cone of gradients of a bump, focusing on the opposite question; namely, for a smooth bump function b on an infinite-dimensional Banach space X , what is the maximal size of $\mathcal{C}(b)$? And what is that of $b'(X)$?

More generally, if X and Y are Banach spaces, let $\mathcal{L}(X, Y)$ stand for the Banach space of all bounded linear operators from X to Y . Is it possible to have a Fréchet (resp. Gâteaux) smooth surjection $f : X \rightarrow Y$ such that f vanishes outside a bounded set and $f'(X) = \mathcal{L}(X, Y)$?

We will see that, when X and Y are separable and X is infinite-dimensional, there exists a uniformly Gâteaux smooth function f from X to Y , with bounded support, so that $f'(X)$ contains the unit ball of the Banach space $\mathcal{L}(X, Y)$. We obtain as a corollary that every separable Banach space X has a uniformly Gâteaux smooth bump b so that $b'(X)$ contains the dual unit ball of X^* and, as a consequence, there is a continuous Gâteaux smooth bump g so that $g'(X) = X^*$. In the Fréchet smooth case, we obtain that if a Banach space X has a Fréchet smooth bump and $\text{dens } X = \text{dens } \mathcal{L}(X, Y)$, then there is a Fréchet smooth function $f : X \rightarrow Y$ with bounded support so that $f'(X) = \mathcal{L}(X, Y)$. One corollary to this result is that if a Banach space X has a Fréchet smooth bump, then X has a Fréchet smooth bump b so that $b'(X) = X^*$. Another corollary states that for every separable infinite-dimensional Banach space Y and every $n \in \mathbb{N}$, there is a Fréchet smooth function $f : \mathbb{R}^n \rightarrow Y$, with bounded support, so that $f'(\mathbb{R}^n) = \mathcal{L}(\mathbb{R}^n, Y)$.

We also provide conditions on a pair of Banach spaces X and Y which ensure the existence of a C^p smooth surjection $f : X \rightarrow Y$ such that f vanishes outside a bounded set and the derivatives of f are all surjections. We prove that if X has a C^p smooth bump with bounded derivatives and $\text{dens } X = \text{dens } \mathcal{L}_s^m(X; Y)$ then there exists another C^p smooth function $f : X \rightarrow Y$, with bounded derivatives, so that f vanishes outside the unit ball of X and $f^{(k)}(X)$ contains the unit ball of $\mathcal{L}_s^k(X; Y)$ for all $k = 0, 1, \dots, m$ (notice that this conclusion is in fact equivalent to the assumption on X); in particular, this implies that there is also a C^p smooth surjection $b : X \rightarrow Y$ so that $b^{(k)}(X) = \mathcal{L}_s^k(X; Y)$ for all $k = 0, 1, \dots, m$. Here, $1 \leq p \leq \infty$, $m \in \mathbb{N}$, $\mathcal{L}_s^k(X; Y)$ stands for the space of k -linear symmetric mappings from X into Y , and $\text{dens } X$ denotes the character of density of a Banach space X . Note in particular that for $m = 0$ we identify $Y = \mathcal{L}_s^0(X, Y)$ and we obtain a C^p smooth surjection b from X onto Y , thus recovering a result of Bates's [15]. For some classical spaces X and Y , such as the ℓ_p , c_0 and L_p , we also say when the above conditions for the existence of smooth functions with surjective derivatives are fulfilled.

THEOREM 6.1. *Let X and Y be separable Banach spaces, where X is infinite-dimensional. Then, there is a uniformly Gâteaux smooth Lipschitz function $b : X \rightarrow Y$ with bounded support so that $b(X)$ contains the unit ball*

of Y and $b'(X)$ contains the unit ball of $\mathcal{L}(X, Y)$. Consequently, there is also a continuous Gâteaux smooth function $g : X \rightarrow Y$ with bounded support so that g and g' are surjections, that is, $g(X) = Y$ and $g'(X) = \mathcal{L}(X, Y)$.

A consequence of this theorem is that there is no upper bound for the range of the set of gradients of a continuous Gâteaux smooth bump on a separable Banach space.

COROLLARY 6.2. *Every separable Banach space X has a uniformly Gâteaux smooth Lipschitz bump b such that $b'(X)$ contains the dual unit ball B_{X^*} . Consequently, X has a continuous Gâteaux smooth bump g so that $g'(X) = X^*$.*

The following result concerns Fréchet smooth functions. It was proved in [6] that if a Banach space has a C^1 smooth and Lipschitz bump, then the space has a C^1 smooth and Lipschitz bump satisfying that the set of gradients covers the dual unit ball. The proof of this result, as well as the proof given below for the C^p smooth case, strongly rely on the existence of a smooth bump function with *bounded derivatives*. This requirement allows us to obtain smooth functions with *continuous* surjective derivatives. If one is not interested in the continuity of the first derivative, one can dispense with that assumption, obtaining similar results on the existence of Fréchet smooth bumps whose sets of gradients cover the dual unit ball. Notice that it is still an open problem whether every Banach space with a Fréchet smooth bump has a Fréchet smooth bump with bounded derivative as well.

THEOREM 6.3. *Let X be a Banach space with a Fréchet smooth bump and Y a Banach space so that $\text{dens } X = \text{dens } \mathcal{L}(X, Y)$. Then, there exists a Fréchet smooth function $g : X \rightarrow Y$ so that g has bounded support, $g'(X) = \mathcal{L}(X, Y)$ and, when X is infinite dimensional, also $g(X) = Y$.*

COROLLARY 6.4. *Let X be a Banach space with a Fréchet smooth bump. Then, X has a Fréchet smooth bump b so that $b'(X) = X^*$.*

COROLLARY 6.5. *Let Y be an infinite dimensional and separable Banach space and $n \in \mathbb{N}$. Then, there is a Fréchet smooth and Lipschitz function $b : \mathbb{R}^n \rightarrow Y$ with bounded support such that $b'(\mathbb{R}^n)$ contains the unit ball of the space $\mathcal{L}(\mathbb{R}^n, Y)$.*

Consequently, there is a Fréchet smooth function $g : \mathbb{R}^n \rightarrow Y$ with bounded support so that $g'(\mathbb{R}^n) = \mathcal{L}(\mathbb{R}^n, Y)$.

Next we deal with the following question. When can one construct a C^p smooth mapping f between two Banach spaces X and Y such that f has a bounded support and the derivatives $f^{(k)}$, $k = 0, 1, \dots, p$, are all surjections (that is, $f(X) = Y$ and $f^{(k)}(X) = \mathcal{L}_s^k(X; Y)$ for all $k = 1, \dots, p$, where $\mathcal{L}_s^k(X; Y)$ is the space of k -linear symmetric and continuous mappings from X into Y)?

To begin with, it should be noted that, even in the simplest case when $Y = \mathbb{R}$, there are very smooth separable Banach spaces X for which this is not possible at all, since the spaces $\mathcal{L}_s^k(X)$ need not be separable for $k \geq 2$ (here we denote $\mathcal{L}_s^k(X; \mathbb{R}) = \mathcal{L}_s^k(X)$, the space of k -linear symmetric and continuous forms on X , which is isomorphic to $\mathcal{P}^{(k)}(X)$, the space of k -homogeneous and continuous polynomials on X). For instance, if $X = \ell_2$ then no C^2 smooth bump b on X has the property that $b^{(2)}(X) = \mathcal{L}_s^2(X)$; indeed, since $b^{(2)}$ is continuous and X is separable, $b^{(2)}(X)$ is separable as well and hence cannot fill all of $\mathcal{L}_s^2(X)$, which is nonseparable (to see this, notice that the mapping $a = (a_n) \mapsto A(x, y) = \sum_{n=1}^{\infty} a_n x_n y_n$ defines an isometric embedding of ℓ_{∞} into the space of bilinear forms on $X = \ell_2$). More generally, it is known that if $X = \ell_p$ then the spaces $\mathcal{L}_s^k(X)$ are separable if and only if $k < [p]$, where $[p]$ is the integer part of p .

The above argument clearly shows that $\text{dens } X = \text{dens } \mathcal{L}_s^m(X; Y)$ is a necessary condition for a pair of Banach spaces X and Y to have a C^p smooth function f from X onto Y so that $f^{(k)}(X) = \mathcal{L}_s^k(X)$ for all $k = 0, 1, \dots, m$. The next result (which can be regarded as a generalization of both one of the main theorems in [6] and another in [15]) tells us that if the Banach space X has a C^p smooth bump with bounded derivatives then this condition is sufficient as well. In the following statement we use the convention $\mathcal{L}_s^0(X; Y) = Y$ and $f^{(0)} = f$.

THEOREM 6.6. *Let $m, p \in \{0, 1, 2, \dots, \infty\}$, and let X, Y be Banach spaces with $\dim X = \infty$. The following are equivalent:*

- (1) *X has a C^p smooth bump function with bounded derivatives, and $\text{dens } X = \text{dens } \mathcal{L}_s^m(X; Y)$.*
- (2) *There is a C^p smooth function $f : X \rightarrow Y$, with bounded derivatives and bounded support, such that $f^{(k)}(X)$ contains the unit ball of $\mathcal{L}_s^k(X; Y)$ for every $k = 0, 1, \dots, m$.*

In particular, if X satisfies condition (1) then there is another C^p smooth function b from X onto Y with bounded support so that its derivatives are

all surjections up to the degree m , that is, $b^{(k)}(X) = \mathcal{L}_s^k(X; Y)$, for $k = 0, 1, \dots, m$.

Notice that when $m = 0$ and $\text{dens } X = \text{dens } Y$ we recover a particular case of a result of S. M. Bates's [15].

COROLLARY 6.7. *Let X and Y be Banach spaces with $\text{dens } X \geq \text{dens } Y$, $\dim X = \infty$, and assume that X has a C^p smooth bump function with bounded derivatives ($p = 1, 2, \dots, \infty$). Then there is a C^p smooth surjection $f : X \rightarrow Y$ whose support is in the unit ball of X ; moreover, if we additionally assume that $\text{dens } X = \text{dens } \mathcal{L}(X, Y)$, the derivative f' is a continuous surjection as well, that is, $f'(X) = \mathcal{L}(X, Y)$.*

When $m = 1$ and $Y = \mathbb{R}$ Theorem 6.6 yields the following improvement of one of the main results in [6].

COROLLARY 6.8. *Let X be an infinite-dimensional Banach space and $p \in \mathbb{N} \cup \{\infty\}$. The following are equivalent:*

- (1) X has a C^p smooth bump function with bounded derivatives;
- (2) X has a C^p smooth bump function f , with bounded derivatives, so that $f'(X)$ contains the unit ball of X^* .

In either case, there exists another C^p smooth bump b on X so that $b'(X) = X^$.*

It should be noted that if a Banach space X satisfies condition (1) of Theorem 6.6 for $p \geq 2$ then it is superreflexive (see [27]). Let us mention that condition $\text{dens } X = \text{dens } \mathcal{L}_s^k(X, Y)$ is strongly related to Gonzalo and Jaramillo indexes $\ell(X)$, $\ell(Y)$ and $u(X)$, $u(Y)$ concerning upper and lower estimates of the Banach spaces X and Y (see [37] and [28]). For instance, it is proved in [28] that if a Banach space has an unconditional and shrinking basis then $\mathcal{L}_s^k(X)$ is separable if and only if every $T \in \mathcal{L}_s^k(X)$ is weakly sequentially continuous which is equivalent to the fact that the Banach space $\mathcal{L}_s^k(X)$ has a monomial basis. Also, it is proved that (a) if X has a shrinking basis and $(k - 1)u(X^*) < \ell(X)$, then $\mathcal{L}_s^k(X)$ has a monomial basis (and thus it is separable); (b) if X has an unconditional and shrinking basis and $u(X) < k$, then $\mathcal{L}_s^k(X)$ contains ℓ_∞ .

When X is one of the classic Banach spaces c_0 , or ℓ_r , $1 < r < \infty$, and we apply Theorem 6.6 we get the following result.

- COROLLARY 6.9. (1) c_0 has a C^n smooth bump b with $b'(c_0) = \ell_1$ if and only if $n = 1$.
- (2) For r an even integer, the space ℓ_r has a C^∞ smooth bump b with $b^{(k)}(\ell_r) = \mathcal{L}_s^k(\ell_r)$ for $k = 1, 2, \dots, m$ if and only if $m < r$.
- (3) If r is not an even integer, ℓ_r has a C^m smooth bump b with $b^{(k)}(\ell_r) = \mathcal{L}_s^k(\ell_r)$ for $k = 1, 2, \dots, m$ if and only if $m < r$.

Notice that according to a result of Hájek [40], no C^2 -smooth bump b on c_0 has the property that $b'(c_0) = \ell_1$ and assertion (1) in the above corollary follows.

The classical Banach space $L_r[0, 1]$, $r \geq 1$, contain a complemented copy of ℓ_2 . Thus $\mathcal{L}_s^2(L_r[0, 1])$ and contain ℓ_∞ and the best we can expect for these spaces is the following result.

- COROLLARY 6.10. (1) For r an even integer the space $L_r[0, 1]$ has a C^∞ smooth bump so that $b'(L_r[0, 1]) = L_{r'}[0, 1]$, $1/r + 1/r' = 1$.
- (2) If r is not an even integer, the space $L_r[0, 1]$ has a C^m smooth bump b so that $b'(L_r[0, 1]) = L_{r'}[0, 1]$ if and only if $m < r$.

In the vector valued case let us mention that $\mathcal{L}(c_0, \ell_1)$ is separable, and $\mathcal{L}_s^k(\ell_r, \ell_q)$ is separable if and only if $kq < r$ (see [28]). Thus we obtain for these spaces the following result.

- COROLLARY 6.11. (1) There is a C^1 smooth function $f : c_0 \rightarrow \ell_1$ with bounded support so that $f(c_0) = \ell_1$ and $f'(c_0) = \mathcal{L}(c_0; \ell_1)$.
- (2) When $mq < r$, there is a C^m smooth function $f : \ell_r \rightarrow \ell_q$ with bounded support so that $f^{(k)}(\ell_r) = \mathcal{L}_s^k(\ell_r; \ell_q)$ for $k \in \{0, 1, \dots, m\}$.

What about Theorem 6.6 when X and Y are finite-dimensional? In this case, an analogous result is available which provides us with Peano functions from \mathbb{R}^k to \mathbb{R}^m which in fact are derivatives of smooth functions.

PROPOSITION 6.12. For every $k, m \in \mathbb{N}$, there exists a C^1 smooth Lipschitz function $f : \mathbb{R}^k \rightarrow \mathbb{R}^m$ so that f vanishes outside a bounded set and the unit ball of $\mathcal{L}(\mathbb{R}^k, \mathbb{R}^m)$ is contained in $f'(\mathbb{R}^k)$. In particular, for every $m \in \mathbb{N}$ there is a continuous path $g : [0, 1] \rightarrow \mathbb{R}^m$ whose image contains the unit ball of \mathbb{R}^m and so that g is the derivative of a C^1 smooth Lipschitz path $f : [0, 1] \rightarrow \mathbb{R}^m$.

As for the proofs of these results, all we can say here is that they are all alike, in fact the main ideas are always the same, but the various technical details involved in each of them make it impossible to give a general proof that applies in all the cases. As a sample, let us prove Theorem 6.6.

PROOF OF THEOREM 6.6

It is clear that (2) implies (1). Let us see that (1) implies (2) too. Assume that $\text{dens } X = \text{dens } \mathcal{L}_s^m(X; Y) = \kappa$. Then it is easily seen that $\text{dens } X = \text{dens } \mathcal{L}_s^k(X; Y) = \kappa$ for all k with $0 \leq k \leq m$. It is enough to see that for any k with $0 \leq k \leq m$ there exists a C^p smooth function $g : X \rightarrow Y$ with support on B_X so that $g^{(k)}(X)$ contains the unit ball of $\mathcal{L}_s^k(X; Y)$. Indeed, once this is shown, we can take a disjoint sequence of balls of the same diameter, $2r$, contained in the unit ball B_X , say $B(z_n, r)$, $n = 0, 1, 2, 3, \dots$, and C^p smooth functions $b_0, b_1, b_2, \dots, b_m$, with support in B_X and taking values in Y , so that $b_k^{(k)}(X)$ contains the unit ball of $\mathcal{L}_s^k(X; Y)$ for every k ; then the function $f : X \rightarrow Y$ defined by

$$f(x) = \sum_{k=0}^m r^k b_k\left(\frac{x - z_k}{r}\right)$$

is clearly a C^p smooth bump with the property that $f^{(k)}(X) = b_k^{(k)}(X)$ contains the unit ball of $\mathcal{L}_s^k(X; Y)$ for every k .

So let us prove that for a fixed k with $0 \leq k \leq p$ there exists a C^p smooth function $g : X \rightarrow Y$ with support on B_X so that $g^{(k)}(X)$ contains the unit ball of $\mathcal{L}_s^k(X; Y)$.

Since X has a C^p bump function with bounded derivatives, by composing it with a suitable real function, we can obtain a C^p function $h : X \rightarrow [0, 1]$ such that for some $M \geq 3$ and $0 \leq M_0 \leq M_1 \leq \dots \leq M_j \leq M_{j+1} \leq \dots$, we have $h(x) = 1$ whenever $\|x\| \leq 2$, $h(x) = 0$ if $\|x\| \geq M$, and $\|h^{(j)}\|_\infty = \sup_{x \in X} \|h^{(j)}(x)\| \leq M_j$. Let us fix ε , where $0 < 2M\varepsilon < \frac{1}{2}$, and select a $2M\varepsilon$ -separated collection of points $(z_\alpha)_{\alpha \in \Gamma}$ in $\frac{1}{2}B_X$ with $\text{card}(\Gamma) = \kappa = \text{dens } X$. The balls $B(z_\alpha, M\varepsilon)$, $\alpha \in \Gamma$, are all disjoint and contained in $B = B_X$. We define chains of balls

$$U_j^s := B_{(\alpha_1, \alpha_2, \dots, \alpha_j)} = z_{\alpha_1} + \varepsilon z_{\alpha_2} + \dots + \varepsilon^{j-1} z_{\alpha_j} + \varepsilon^j B$$

for $s = (\alpha_1, \alpha_2, \dots, \alpha_j, \dots) \in \Gamma^{\mathbb{N}}$. There is a bijection between the chains of balls (U_j^s) and the set of sequences $\Gamma^{\mathbb{N}}$; besides, the intersection of any chain of these balls consists exactly of the point $\bigcap_{j=1}^\infty B_{(\alpha_1, \alpha_2, \dots, \alpha_j)} = \sum_{j=1}^\infty \varepsilon^{j-1} z_{\alpha_j}$.

Now, since $\text{dens}(\mathcal{L}_s^k(X; Y)) = \text{card}(\Gamma)$, we can take a family $(Q_\alpha)_{\alpha \in \Gamma}$ which is dense in the unit ball of $\mathcal{L}_s^k(X; Y)$, and a corresponding family $(P_\alpha)_{\alpha \in \Gamma}$ of k -homogeneous polynomials from X into Y so that Q_α is the k th derivative of P_α for each α . Notice that in the case $k = 0$ we are dealing with a dense subset $(y_\alpha)_{\alpha \in \Gamma}$ of Y .

Next, for every $n \geq 1$ we can define $\delta_n = \varepsilon^{n^2-1}$, and

$$g_n(x) = \sum_{(\alpha_1, \alpha_2, \dots, \alpha_n) \in \Gamma^n} \delta_n h\left(\frac{x - \sum_{i=1}^n \varepsilon^{i-1} z_{\alpha_i}}{\varepsilon^n}\right) P_{\alpha_n}(x)$$

for all $x \in X$. It is clear that g_n is C^p smooth with bounded derivatives, and its support is in B . Notice also that every $x \in X$ has a neighbourhood V_x so that all but one of the terms in the sum defining $g_n(y)$ are zero for $y \in V_x$. Besides, we have that $g_n^{(k)}(B_{(\alpha_1, \alpha_2, \dots, \alpha_n)}) = \delta_n Q_{\alpha_n}$ for all $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \Gamma^n$. Bearing in mind that the i th derivative of h is uniformly bounded by M_i , the construction of g_n , and the fact that if a k -homogeneous polynomial P has its k th derivative Q bounded by 1 then all the derivatives of P are bounded by 1 as well (this is an immediate inductive application of the mean value theorem), we can estimate the norm of the j th derivative of g_n as follows

$$\|g_n^{(j)}(x)\| \leq \delta_n \sum_{i=0}^j \binom{j}{i} \frac{M_i}{\varepsilon^{ni}} \leq \delta_n \varepsilon^{-nj} M_j \sum_{i=0}^j \binom{j}{i} \leq 2^j M_j \varepsilon^{n(n-j)-1}$$

for all $x \in X$ and $n \in \mathbb{N}$, and for every j with $0 \leq j \leq p$.

Since the series $\sum_{n=1}^{\infty} 2^j M_j \varepsilon^{n(n-j)-1}$ are convergent for all $j = 0, 1, 2, \dots$, this implies that the series of derivatives $\sum_{n=1}^{\infty} g_n^{(j)}(x)$ converge uniformly on X (for all $0 \leq j \leq p$), and therefore the function $g : X \rightarrow Y$ defined by

$$g(x) = \sum_{n=1}^{\infty} g_n(x)$$

is C^p smooth with bounded derivatives, and $g^{(j)}(x) = \sum_{n=1}^{\infty} g_n^{(j)}(x)$.

Let us now see that $g^{(k)}(X)$ contains the unit ball of $\mathcal{L}_s^k(X; Y)$. By the construction of the g_n and g it is clear that

$$g^{(k)}(\partial B_{(\alpha_1, \dots, \alpha_k)}) = Q_{\alpha_1} + \delta_2 Q_{\alpha_2} + \dots + \delta_n Q_{\alpha_n}$$

for every chain of balls $(B_{(\alpha_1, \dots, \alpha_n)})_{n \in \mathbb{N}}$; then, for $x := \bigcap_{n=1}^{\infty} B_{(\alpha_1, \alpha_2, \dots, \alpha_n)}$, by the continuity of $g^{(k)}$ we get that $g^{(k)}(x) = \sum_{n=1}^{\infty} \delta_n Q_{\alpha_n}$. Since $(Q_{\alpha})_{\alpha \in \Gamma}$ is dense in the unit ball of $\mathcal{L}_s^k(X; Y)$ it is clear that every Q in this ball can be written as a series $Q = \sum_{n=1}^{\infty} \delta_n Q_{\alpha_n}$ for some sequence $(\alpha_n) \in \Gamma^{\mathbb{N}}$, so we can conclude that $g^{(k)}(X)$ contains the unit ball of $\mathcal{L}_s^k(X; Y)$.

Finally, in order to obtain a C^p smooth surjection $b : X \rightarrow Y$ such that $b^{(k)}(X) = \mathcal{L}_s^k(X; Y)$ for every $k = 0, 1, \dots, m$, we only have to take a sequence $(\alpha_1, \alpha_2, \dots, \alpha_n, \dots) \in \Gamma^{\mathbb{N}}$ with $\alpha_i \neq \alpha_j$ if $i \neq j$, a C^p smooth function $f : X \rightarrow Y$ with support in B_X and such that $f^{(k)}(X)$ contains the unit ball of $\mathcal{L}_s^k(X; Y)$ for every $k = 0, 1, 2, \dots, m$, and put

$$b(x) = \sum_{n=1}^{\infty} n f\left(\frac{x - z_{\alpha_n}}{\varepsilon}\right)$$

for all $x \in X$. ■

7. WHAT DOES THE RANGE OF A DERIVATIVE LOOK LIKE?

While in the preceding sections we have been concerned about the topological size of the ranges of the derivatives of a bump function, now we will look at the *shape* of those ranges. Several questions arise naturally. For instance, the range of the derivative of a C^1 smooth bump on a Banach space X is obviously a connected set containing the origin, but: Are there any restrictions on its shape? May it fail to be simply connected?

By this time the reader may have formed an opinion of his own as to what bumps are capable of, and he or she has probably guessed the following meta-theorem: everything can happen with a bump, at least in infinite dimensions. We will not disappoint his expectations, he or she is right.

The first answer to such questions was provided by the work of Borwein, Fabian, Kortezev and Loewen [23]. They constructed a C^1 smooth bump b on the plane \mathbb{R}^2 so that the range of its derivative, $b'(\mathbb{R}^2)$ is not simply connected (for instance, a circular corona). In fact they showed that $b'(\mathbb{R}^n)$ can happen to fill in any reasonably looking closed figure containing the origin as an interior point (however, giving full details of what reasonably looking means would not be very reasonable at this moment). In a subsequent work [24] Borwein, Fabian and Loewen extended this result to the infinite-dimensional setting, establishing the following theorem.

THEOREM 7.1. (Borwein-Fabian-Loewen) *Let X be an infinite-dimensional Banach space with a Lipschitz C^1 -smooth bump. Let $\Omega \subset X^*$ be an open connected set containing the origin and satisfying this property:*

There exists a summable sequence a_0, a_1, a_2, \dots of positive numbers such that every $\eta \in \bar{\Omega}$ can be expressed as $\lim_{i \rightarrow \infty} \xi_i$ for some sequence $0 = \xi_0, \xi_1, \xi_2, \dots$ in Ω such that $\|\xi_{i+1} - \xi_i\| < a_i$, and that the linear segment $\text{co}\{\xi_i, \xi_{i+1}\}$ lies in Ω for every $i = 0, 1, 2, \dots$

Then there exists a Lipschitz C^1 -smooth bump $b: X \rightarrow [0, 1]$ so that $b'(X) = \bar{\Omega}$.

On the other hand we have been informed that Thierry Gaspari has independently obtained several sufficient conditions for a subset of a dual space (in any dimension, finite or infinite) to be filled in by the range of a derivative of a C^1 smooth bump [34].

Next we study the same problem in the case of higher order derivatives and, by using Theorem 6.6 above, we establish some results that generalize Theorem 7.1. The proofs we sketch here are different (even in the case of a first derivative) from the original ones in [23], [24].

In what follows we will be using the same notation as in the preceding section. We begin with a lemma which tells us that, for a polygonal arc P in the space of symmetric n -linear forms $\mathcal{L}_s^n(X)$ one can always find a bump whose n -th derivative's range contains a suitable neighborhood of P and is contained in a (larger, but not much larger) neighborhood of P . This lemma (which holds true in any dimension) is our main tool to construct bumps with a prescribed range of derivatives.

LEMMA 7.2. *Let $p \in \{0, 1, \dots, \infty\}$ and X be a Banach space with a C^p smooth bump with bounded derivatives. Assume that $\text{dens } X = \text{dens } \mathcal{L}_s^n(X)$, for some $n \leq p$. Consider a polygonal arc P in $\mathcal{L}_s^n(X)$ from 0 to any point Q . Then, there is a constant $M > 0$ (which only depends on the space and not on the polygonal) so that for any $\varepsilon > 0$ there exists a C^p smooth bump g with bounded derivatives and support in the unit ball of X satisfying that*

$$\begin{aligned} \|g^{(k)}\|_\infty &\leq 4\varepsilon, \quad \text{for } k = 0, 1, \dots, n-1, \\ P + \frac{\varepsilon}{M} B_{\mathcal{L}_s^n(X)} &\subset g^{(n)}(X) \subset P + 2\varepsilon B_{\mathcal{L}_s^n(X)}, \\ \text{and } g^{(n)}(\delta B_X) &= Q, \quad \text{for some } \delta > 0. \end{aligned}$$

Moreover, if $n < i \leq p$, the i -th derivative $g^{(i)}$ is bounded by a constant which only depends on i, ε, M and the length of the polygonal.

Proof. If X has a C^p smooth bump with bounded derivatives, by composing this bump with a suitable C^∞ bump on \mathbb{R} , we obtain a C^p smooth bump b_1 with bounded image, bounded derivatives and $b_1(r B_X) = 1$, for some $r > 0$. For a given element $R \in B_{\mathcal{L}_s^n(X)}$, we consider the associated polynomial S whose n -th derivative is R and the product $b_2 = b_1 S$. Notice that $b_2^{(n)}(r B_X) = R$. On the other hand, by the main results of the preceding section we know that X has a C^p smooth bump b_3 with bounded derivatives so that $b_3^{(n)}(X)$ contains the unit ball of $\mathcal{L}_s^n(X)$, denoted by $B_{\mathcal{L}_s^n(X)}$. By summing b_2 and a suitable translation of b_3 (with disjoint support from b_1) we obtain a C^p smooth bump h so that $h^{(n)}(r B_X) = R$ for some $r \in (0, 1)$, and $B_{\mathcal{L}_s^n(X)} \subset h^{(n)}(X)$. Up to elementary operations of dilation and constant multiplying we may additionally assume that the support of b is included in the unit ball B_X . Moreover, $h, h', \dots, h^{(n)}$ are bounded by a constant $M > 1$ while, for $n < i \leq p$, $h^{(i)}$ is bounded by a constant M_i . The constants M and M_i do not depend on the given $R \in B_{\mathcal{L}_s^n(X)}$; they only depend on X .

Now, let P be the given polygonal arc and $\varepsilon > 0$. We consider in the polygonal P the extreme points of the straight lines which form P . We shall denote this set by $\{Q_0 = 0, Q_1, \dots, Q_k = Q\}$. By adding or removing to this set more points of the polygonal, if necessary, we may assume that $\|Q_j - Q_{j-1}\| \leq \frac{2\varepsilon}{M}$, the polygonal P is included in $\cup_j (Q_j + \frac{2\varepsilon}{M} B_{\mathcal{L}_s^n(X)})$ and $k \frac{\varepsilon}{M} \leq l + 1$, where l denotes the length of the polygonal P . According to our previous considerations, there are C^p smooth bumps $h_j, j = 1, 2, \dots, k$, with bounded derivatives, support in the unit ball of X , and $\|h_j^{(i)}\|_\infty \leq 2\varepsilon$ for $i = 0, 1, \dots, n$, $\|h_j^{(i)}\|_\infty \leq \frac{2\varepsilon M_i}{M}$ for $i = n + 1, \dots, p$, and

$$\frac{2\varepsilon}{M} B_{\mathcal{L}_s^n(X)} \subset h_j^{(n)}(X) \subset 2\varepsilon B_{\mathcal{L}_s^n(X)},$$

$$h_j^{(n)}(r B_X) = Q_j - Q_{j-1}, \quad j = 1, \dots, k.$$

We then define our bump $g : X \rightarrow \mathbb{R}$ as

$$g(x) = \sum_{j=1}^k \left(\frac{r}{2}\right)^{(j-1)n} h_j\left(\left(\frac{r}{2}\right)^{j-1} x\right).$$

Notice that the support of g is included in the unit ball of X , and if we take $\delta = \frac{r^k}{2^k}$ then $g^{(n)}(\delta B_X) = Q$. Also,

$$\|g^{(i)}\|_\infty \leq \|h_1^{(i)}\|_\infty + \frac{r}{2} \|h_2^{(i)}\|_\infty + \dots + \left(\frac{r}{2}\right)^{k-1} \|h_k^{(i)}\|_\infty \leq 4\varepsilon, \quad i = 0, 1, \dots, n-1,$$

and $\bigcup_{j=1}^k (Q_j + \frac{2\varepsilon}{M} B_X) \subset g^{(n)}(X) \subset \bigcup_{j=1}^k (Q_j + 2\varepsilon B_X)$. This implies that $P + \frac{\varepsilon}{M} B_X \subset g^{(n)}(X) \subset P + 2\varepsilon B_X$. Finally, if $n < i \leq p$, then

$$\begin{aligned} \|h^{(i)}\|_\infty &\leq \sum_{j=1}^k \frac{2\varepsilon M_i}{M} \left(\frac{r}{2}\right)^{(j-1)(i-n)} = \frac{2\varepsilon M_i}{M} \frac{\left(\frac{2}{r}\right)^{(i-n)k} - 1}{\left(\frac{2}{r}\right)^{(i-n)} - 1} \\ &\leq \frac{4\varepsilon M_i}{M} \left(\frac{r}{2}\right)^{(i-n)((l+1)M\varepsilon^{-1})}. \end{aligned}$$

Now we can easily deduce that every open connected subset of an infinite-dimensional dual X^* that contains the origin (and so that X has a suitable smooth bump) can be regarded as the range of a higher order derivative of some bump. In particular we see that there are no restrictions on the *shape* of the ranges of derivatives of smooth bumps.

THEOREM 7.3. *Let $p \in \{0, 1, \dots, \infty\}$ and X be an infinite dimensional Banach space with a C^p smooth bump having bounded derivatives. Assume that $\text{dens } X = \text{dens } \mathcal{L}_s^n(X)$ for some $n \leq p$. If $U \subset \mathcal{L}_s^n(X)$ is a given open, bounded, connected set with $0 \in U$, then there is a C^p smooth bump h with bounded derivatives up to the order n so that the range of the n -th derivative $h^{(n)}$ is U .*

Proof. Let us consider a dense set D in U so that the cardinality of D is the density of X . Let us define

$$\mathcal{P} = \{P = \{Q_0 = 0, Q_1, \dots, Q_m\} : \text{ where } P \text{ is a polygonal within } U, \\ m \in \mathbb{N} \text{ and the vertices } Q_0, Q_1, \dots, Q_m \in D\}$$

Clearly $\text{card } \mathcal{P} = \text{dens } X$. For every rational number $0 < \varepsilon < 1$ and for every $P \in \mathcal{P}$ so that $P + 2\varepsilon B_{\mathcal{L}_s^n(X)} \subset U$, let us pick a bump $g_{P,\varepsilon}$ on X satisfying the conditions of the lemma. Let us relabel the family of these bumps as $\{g_\alpha\}_{\alpha \in \Gamma}$, where $\text{card } \Gamma = \text{dens } X$.

Now consider a family of $\frac{2}{3}$ -separated points $\{x_\alpha\}_{\alpha \in \Gamma}$ in B_X , and define

$$h(x) = \sum_{\alpha \in \Gamma} 4^{-n} g_\alpha(4(x - x_\alpha)).$$

Let us check that the bump h fulfills the required conditions. Note that for every $x \in X$ there is at most one non-null summand in the above equation, which is the same in a neighborhood of x . Thus, $h \in C^p$, $h, h', \dots, h^{(n-1)}$ are bounded by 1 and $h^{(n)}(X) \subset U$. Let us check that $U \subset h^{(n)}(X)$. Since U is connected and open, then it is connected by polygonals. For any $Q \in U$, there is a polygonal $P = \{Q_0 = 0, Q_1, \dots, Q_m = Q\}$ connecting 0 to Q and there is an $0 < \varepsilon < 1/2$ so that $P + 4\varepsilon B_X \subset U$. We may assume, by the density of D , that $Q_0, Q_1, \dots, Q_{m-1} \in D$. Also, take $Q'_m \in D$ so that $\|Q'_m - Q\| < \frac{\varepsilon}{M}$. The polygonal $P' = \{Q_0 = 0, Q_1, \dots, Q_{m-1}, Q'_m\} \in \mathcal{P}$ and, since $P' + 2\varepsilon B_{\mathcal{L}_s^n(X)} \subset P + 4\varepsilon B_X \subset U$, the associated bump $g_{P', \varepsilon}$ belongs to the family $\{g_\alpha\}$. By the lemma we have that $Q \in P' + \frac{\varepsilon}{M} B_X \subset g_{P', \varepsilon}^{(n)}(X) \subset h^{(n)}(X)$, so the proof is finished.

A careful modification of the above argument allows to prove the following generalization of Theorem 7.1.

THEOREM 7.4. *Let X be an infinite dimensional Banach space with a C^n smooth bump having bounded derivatives. Assume that $\text{dens } X = \text{dens } \mathcal{L}_s^n(X)$. Let $U \subset \mathcal{L}_s^n(X)$ be a pre-fixed open, bounded, connected set with $0 \in U$, so that for every $Q \in \partial U$ there exists a path from 0 to Q through points of U . Then, there is a C^n smooth bump h with bounded derivatives so that the image of the n -th derivative $h^{(n)}$ is the closure of U .*

We will not give the proof of this result here. We only mention that the proof bears some resemblance to that of Theorem 7.6 below, which deals with the finite-dimensional case.

Theorem 7.4 does not hold true when X is finite dimensional. Next we give an example of an open bounded subset $U \subset \mathbb{R}^2$ containing the origin and satisfying the condition given in Theorem 7.4, so that the closure of U cannot be the range of the first derivative of any C^1 smooth bump on \mathbb{R}^2 .

EXAMPLE 7.5. Consider the open sets of the plane

$$U_n = \left\{ (x, y) : 1 - \frac{1}{2n} < |x| < 1 - \frac{1}{2n+1}, |y| < 2 \right\}, \quad n \in \mathbb{N}$$

and

$$U = \bigcup_n U_n \cup \left\{ (x, y) : 1 < \max\{|x|, |y|\} < 2 \right\} \cup \left\{ (x, y) : |x| < \frac{1}{4}, |y| < 2 \right\}.$$

Obviously the closure of U satisfies the conditions required in Proposition 7.4. Assume the closure of U were the image of a continuously Fréchet smooth

bump $b : \mathbb{R}^2 \rightarrow \mathbb{R}$. Let us take points $(a_n, 0) \in U_n$ converging to $(1, 0) \in \partial U$. By the assumption, there is a bounded sequence of points (x_n, y_n) so that $b'(x_n, y_n) = (a_n, 0)$. By compactness, we may take for granted that the sequence (x_n, y_n) converges to some point (x, y) . By continuity, $b'(x, y) = (1, 0)$, and there is some $\delta > 0$ so that $A = b'((x, y) + \delta B) \subset (1, 0) + \frac{1}{2}B$, where B is the unit ball of the euclidean norm in \mathbb{R}^2 . Since b' is continuous, the set A should be connected. But this is a contradiction, since $\{(x_n, y_n)\}_{n \geq N} \subset A \subset \overline{U} \cap ((1, 0) + \frac{1}{2}B)$ for some $N \in \mathbb{N}$. ■

Nevertheless, we next show that, if for every $\varepsilon > 0$, there is a finite collection of open and connected subsets of U with diameter less than ε and covering U , then \overline{U} is the image of a C^1 smooth bump. The above example clearly shows that if we drop this condition the conclusion does not necessarily hold.

THEOREM 7.6. *Let us consider $n, m \in \mathbb{N}$ and an open bounded and connected subset $U \subset \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ containing the origin. Assume that, for every $\varepsilon > 0$, there is a finite family \mathcal{F}_ε of open (non-empty) subsets of U which covers U so that every $V \in \mathcal{F}_\varepsilon$ is connected and has diameter less than ε . Then, there is a C^1 smooth and Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with bounded support so that $b'(\mathbb{R}^n) = \overline{U}$.*

Proof. We denote by \mathcal{F}_k the finite open covering of U given in the hypothesis for $\varepsilon = \frac{1}{2^{k+4}}$. Now, for every open subset $V \in \mathcal{F}_k$, we select a point $T \in V$, and denote the set consisting of all the points obtained in this way by F_k . In order to avoid problems of notation we may consider that the selected points are all different and even that $F_k \cap F_j = \emptyset$, whenever $k \neq j$. Notice that, for every k , the finite set F_k is a $\frac{1}{2^{k+4}}$ -net of \overline{U} .

Let us denote by \mathcal{P}_k the family of all finite sequences $R = \{T_0 = 0, T_1, T_2, \dots, T_k\}$ where

- (1) $T_j \in F_j$, for $j = 1, \dots, k$,
- (2) the associated open sets $V_j \in \mathcal{F}_j$, so that $T_j \in V_j$, $j = 1, \dots, k$, satisfy that $V_j \cap V_{j+1} \neq \emptyset$ for $j = 1, \dots, k-1$.

Let us observe that if a sequence $R = \{0, T_1, \dots, T_{k+1}\} \in \mathcal{P}_{k+1}$, then $R' = \{0, T_1, \dots, T_k\} \in \mathcal{P}_k$. Now, we can adapt the proof of Theorem 7.4 as follows. We construct inductively the next family of functions from \mathbb{R}^n to \mathbb{R}^m :

1. For every $R = \{0, T_1\} \in \mathcal{P}_1$ there is a polygonal P_R and $0 < \varepsilon_R < 1/2^3$ so that $P_R + 2\varepsilon_R B_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)} \subset U$, where $B_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)}$ stands for the closed unit ball of $\mathcal{L}(\mathcal{R}^n, \mathcal{R}^m)$. By Lemma 7.2, there is a C^1 smooth function $g_R : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with support in the unit ball of X so that $\|g_R\|_\infty \leq 4\varepsilon_R$ and there is $0 < \delta_R < 1$ so that

$$g'_R(\delta_R B_X) = T_1 - T_0 = T_1,$$

and

$$P_R + \frac{\varepsilon_R}{M} B_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)} \subset g'_R(X) \subset P_R + 2\varepsilon_R B_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)}.$$

Note again that, in what follows, we do not use the first inclusion of the above display which is the difficult one and it is deduced in Lemma 7.2 from the result [7, Proposition 3.7].

If the cardinal of the set \mathcal{P}_1 is p_1 , we fix $M > 0$ so that there are p_1 points $\{x_R : R \in \mathcal{P}_1\}$ in $MB_{\mathbb{R}^n}$ with the property that

$$\|x_R - x_S\| > 2, \quad \text{if } R, S \in \mathcal{P}_1, R \neq S,$$

and

$$x_R + B_{\mathbb{R}^n} \subset MB_{\mathbb{R}^n}.$$

2. We select for every pair (T_1, T_2) , where $R = \{0, T_1, T_2\} \in \mathcal{P}_2$, a polygonal P_{T_1, T_2} from T_1 to T_2 so that P_{T_1, T_2} is included in a ball of radius $\frac{1}{2^4}$. Indeed, consider the associated open sets $V_1 \in \mathcal{F}_1$ and $V_2 \in \mathcal{F}_2$, so that $T_1 \in V_1$ and $T_2 \in V_2$. Then, by assumption V_1 and V_2 have non empty intersection. Thus, the union $V_1 \cup V_2$ is connected and has a diameter not bigger than $\frac{1}{2^4}$. Take $0 < \varepsilon_R < \frac{1}{2^4}$ so that $P_{T_1, T_2} + 2\varepsilon_R B_{\mathcal{L}^s(\mathbb{R}^n, \mathbb{R}^m)} \subset U$. By Lemma 7.2, there is a C^1 smooth function $f_R : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with support in the unit ball of X so that $\|f_R\|_\infty \leq 4\varepsilon_R$ and there is $0 < \gamma_R < 1$ with

$$f'_R(\gamma_R B_X) = T_2 - T_1$$

and

$$T_1 + f'_R(X) \subset P_{T_1, T_2} + 2\varepsilon_R B_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)}.$$

Denote by $\delta_1 = \min\{\delta_R : R \in \mathcal{P}_1\} > 0$ and p_2 the number of elements of \mathcal{P}_2 . Then select $\delta'_1 > 0$ small enough so that we can include p_2 disjoint balls of radius δ'_1 within a ball of radius δ_1 . Then, we define

$$g_R(x) = \delta'_1 f_R\left(\frac{x}{\delta'_1}\right).$$

The function $g_R : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is C^1 smooth, with support in $\delta'_1 B_{\mathbb{R}^n}$ and $\|g_R\|_\infty \leq \delta'_1 4\varepsilon_R \leq 4\varepsilon_R$. Also, there is $0 < \delta_R < \delta'_1$ with

$$g'_R(\delta_R B_X) = T_2 - T_1$$

and

$$T_1 + g'_R(\mathbb{R}^n) \subset P_{T_1, T_2} + 2\varepsilon_R B_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)}.$$

3. In general, for $k \geq 2$ and for every pair (T_{k-1}, T_k) , where $R = \{0, T_1, \dots, T_{k-1}, T_k\} \in \mathcal{P}_k$, we can select a polygonal P_{T_{k-1}, T_k} from T_{k-1} to T_k so that P_{T_{k-1}, T_k} is included in a ball of radius $\frac{1}{2^{k+2}}$. Indeed, consider the associated open sets $V_{k-1} \in \mathcal{F}_{k-1}$ and $V_k \in \mathcal{F}_k$, with $T_{k-1} \in V_{k-1}$ and $T_k \in V_k$. Then, the open sets V_{k-1} and V_k have non empty intersection and, by assumption, they are connected. Thus, the union $V_{k-1} \cup V_k$ is non empty, connected and has diameter not bigger than $\frac{1}{2^{k+2}}$. Take $0 < \varepsilon_R < \frac{1}{2^{k+2}}$ so that $P_{T_{k-1}, T_k} + 2\varepsilon_R B_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)} \subset U$. Define $\delta_{k-1} = \min\{\delta_R : R \in \mathcal{F}_{k-1}\} > 0$ and p_k the number of elements of F_k . Then select $\delta'_{k-1} > 0$ small enough so that we can include p_k disjoint balls of radius δ'_{k-1} within a ball of radius δ_{k-1} . Then, by Lemma 7.2, for every $R = \{0, T_1, \dots, T_{k-1}, T_k\} \in \mathcal{P}_k$, there is a C^1 smooth function $g_R : \mathbb{R}^n \rightarrow \mathbb{R}^m$ whose support is included in $\delta'_{k-1} B_X$ satisfying that

$$\|g_R\|_\infty \leq 4\varepsilon_R,$$

$$g'_R(\delta_R B_X) = T_k - T_{k-1}, \quad \text{for some } 0 < \delta_R < \delta'_{k-1}$$

$$\text{and } T_{k-1} + g'_R(\mathbb{R}^n) \subset P_{T_{k-1}, T_k} + 2\varepsilon_R B_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)}.$$

Due to the way we have selected the constants δ'_k , we can choose within MB_X a family of points $\{x_R : R \in \mathcal{P}_k, k \in \mathbb{N}\}$ with the properties (for $k = 1$, they have been already selected):

$$\|x_R - x_S\| > 2, \quad \text{if } R, S \in \mathcal{P}_1 \quad \text{and } R \neq S,$$

$$x_R + B_X \subset MB_X, \quad \text{if } R \in \mathcal{P}_1$$

and for $k \geq 2$,

$$\|x_R - x_S\| \geq 2\delta'_{k-1}, \quad \text{if } R, S \in \mathcal{P}_k,$$

$$\|x_R - x_{R'}\| \leq \delta_{k-1}, \quad \text{if } R' \leq R, \quad R' \in \mathcal{P}_{k-1},$$

$$(x_R + \delta'_{k-1} B_X) \cap (x_S + \delta'_{k-1} B_X) = \emptyset, \quad R, S \in \mathcal{P}_k$$

$$(x_R + \delta'_{k-1} B_X) \subset (x_{R'} + \delta_{k-1} B_X), \quad R \in \mathcal{P}_k, \quad R' \in \mathcal{P}_{k-1}.$$

Finally, as in Proposition 7.4, it can be checked that the function $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined as the sum $\sum_{k=1}^{\infty} h_k$, with

$$h_k(x) = \sum_{R \in \mathcal{P}_k} g_R(x - x_R), \quad x \in X,$$

and support within $MB_{\mathbb{R}^n}$, fulfills the required conditions. Notice, in particular, that $\cup_k F_k \subset h'(\mathbb{R}^n) \subset \overline{U}$, and then $\overline{U} = \overline{\cup_k F_k} = h'(\mathbb{R}^n)$. ■

8. SOME GEOMETRICAL PROPERTIES OF STARLIKE BODIES. THE FAILURE OF JAMES' THEOREM FOR STARLIKE BODIES

One of the deepest classical results of functional analysis is James' Theorem [41] on the characterization of reflexivity. Let us recall what James' theorem reads. A Banach space X is reflexive if and only if, for a given bounded convex body B in X , every continuous linear functional $T \in X^*$ attains its supremum on B . In this section we investigate to what extent this fundamental result can be generalized for starlike bodies.

There are two problems to be considered, one for each direction in the equivalence given by James' theorem. The difficult and more interesting part of James' theorem tells us that for every bounded convex body B in a non-reflexive Banach space X there exists $T \in X^*$ so that T does not attain its supremum on B . Since B is convex this amounts to saying that T does not attain any local extrema on B . Moreover, if B is smooth then this means that there is some one-codimensional subspace H of X so that the hyperplanes $y + H$ are not tangent to B at any point $y \in \partial B$. At this point we face two possible generalizations of this result for starlike bodies, one for each of those formulations (which, as we just said, are equivalent in the case of convex bodies, but not for starlike bodies). The first one yields a statement which is true but not very interesting; we call it a "weak form of James' theorem" for starlike bodies:

PROPOSITION 8.1. *Let A be a bounded starlike body in a nonreflexive Banach space X . Then there exists a continuous linear functional $T \in X^*$ such that T does not attain its supremum on A .*

However, when one considers the second formulation of the difficult part of James' theorem, things turn out very different in the case of starlike bodies. In this new setting it is natural to ask whether a "strong form of James' theorem" is true for starlike bodies (at least when they are smooth). By a

strong James' theorem we mean the following: if A is a bounded starlike body in a nonreflexive Banach space X , does there exist $T \in X^*$ so that T does not attain any local extrema on A ? For a smooth starlike body A the question should even be made stronger: is there some one-codimensional subspace H in X such that the hyperplanes $z + H$ are not tangent to A at any point $z \in \partial A$? Recall that in Section 5 we denoted the cone of hyperplanes which are tangent to a smooth starlike body A at some point of its boundary ∂A by

$$\mathcal{C}(A) = \{x^* \in X^* : x + \text{Ker } x^* \text{ is tangent to } \partial A \text{ at some point } x \in \partial A\}.$$

If μ_A is the Minkowski functional of A , then it is clear that

$$\mathcal{C}(A) = \{\mu'_A(x) : x \neq 0\}.$$

Hence, the above question is equivalent to the following one: if A is a smooth bounded starlike body in a nonreflexive Banach space X , is it true that $\mathcal{C}(A) \neq X^*$? Of course, if A is a convex body then the answer is “yes”, it satisfies this strong form of James' theorem.

We will show that both questions have negative answers, that is, a strong James' theorem fails for bounded starlike bodies, even when they are smooth, in (nonreflexive) Banach spaces:

THEOREM 8.2. *Let X be an infinite-dimensional Banach space. Then there exists a bounded starlike body $A \subset X$ such that every $T \in X^*$ attains infinitely many local maxima and minima on A .*

Moreover, if X has a separable dual then there exists a bounded C^1 smooth starlike body $A \subset X$ with the property that $\mathcal{C}(A) = X^$.*

It is worth mentioning that the result provided by Theorem 6.6 is the keystone for our proof of the “smooth” part of Theorem 8.2.

In fact Theorem 8.2 can be improved by showing that those weird smooth starlike bodies that do not satisfy James' theorem are not so scarce: every other starlike body in an infinite-dimensional separable Banach space can be approximated in the Hausdorff distance by such bodies. The following theorem formalizes this assertion.

THEOREM 8.3. *Let X be an infinite-dimensional Banach space with a separable dual X^* . Then, for every bounded starlike body A and every $\varepsilon > 0$ there exists a C^1 smooth starlike body D so that $|\mu_D(x) - \mu_A(x)| \leq \varepsilon$ for all x with $\|x\| \leq 1$, and $\mathcal{C}(D) = X^*$.*

If X is separable but X^ is not, the same conclusion holds replacing C^1 smoothness with Gâteaux smoothness.*

Now let us consider the other direction of the equivalence given by James' theorem, the "easy" part of this result. Namely, if X is reflexive, every bounded convex body $B \subset X$ satisfies that, for every $T \in X^*$, T attains its supremum on B . Equivalently, every one-codimensional subspace H of X has the property that $z + H$ supports and touches B at some point $z \in \partial B$. When B is smooth this means that $\mathcal{C}(B) = X^*$. Does this part of James' theorem remain true when one replaces the term "convex body" with "starlike body"?

The next result tells us precisely that, whatever the formulation we choose for this part of James' theorem, the answer to the above question is negative.

THEOREM 8.4. *In the Hilbert space ℓ_2 there exist a C^∞ smooth bounded starlike body A and a one-codimensional subspace H with the property that for no $y \in \partial A$ is the hyperplane $y + H$ tangent to A at y . In other words, $\mathcal{C}(A) \neq X^*$.*

It comes as no surprise that this result is a consequence of the failure of Rolle's theorem and the existence of deleting diffeomorphisms in infinite-dimensional Banach spaces. Indeed, James' theorem trivially implies that the classical Rolle's theorem is true for the class of convex functions in a Banach space X if and only if X is reflexive. Namely, for every Banach space X and every bounded convex body $B \subset X$, the following statements are equivalent:

- (1) X is reflexive,
- (2) For every continuous convex function $f : B \rightarrow \mathbb{R}$ such that $f = 0$ on ∂B , there exists $x_0 \in \text{int} B$ so that $0 \in \partial f(x_0)$,

where $\partial f(x)$ stands for the classical subdifferential of f at x , $\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq x^*(y - x) \text{ for all } y\}$. Hence, it is only natural that the failure of the "easy" part of James' theorem for starlike bodies is closely related to the failure of Rolle's theorem for bump functions in infinite dimensions.

The next result improves the above counterexample and fully answers the question as to how small the cone $\mathcal{C}(A)$ of tangent hyperplanes to a starlike body A in the Hilbert space can be, by constructing smooth bounded starlike bodies A in ℓ_2 so that $\mathcal{C}(A)$ have empty interior. As a matter of fact, the family of such starlike bodies happens to be dense.

Notice that, as in the case of bump functions, Stegall's variational principle implies that, if A is a bounded starlike body in a RNP Banach space then the cone of tangent hyperplanes to A , $\mathcal{C}(A)$, contains a subset of second category in X , so the best result one can get about the smallest possible size of the

cone of tangent hyperplanes to a starlike body in ℓ_2 is that there are indeed smooth bounded starlike bodies A in ℓ_2 so that $\mathcal{C}(A)$ have empty interior.

THEOREM 8.5. *There are C^1 smooth Lipschitz and bounded starlike bodies A_ε in ℓ_2 , $0 < \varepsilon < 1$, so that:*

- (1) *their Minkowski functionals μ_{A_ε} uniformly approximate the usual norm on bounded sets, that is, $\lim_{\varepsilon \rightarrow 0} \mu_{A_\varepsilon} = \|\cdot\|$ uniformly on bounded sets of ℓ_2 ; and*
- (2) *the cones $\mathcal{C}(A_\varepsilon)$ generated by the set of gradients of μ_{A_ε} have empty interior in ℓ_2 .*

To finish this work, let us give a sketch of the proofs of the strongest results stated in this section.

SKETCH OF THE PROOF OF THEOREM 8.5

We use the same notation as in the proof of Theorem 5.2. Take $0 < \delta < 1$ and consider, for

$$0 < \varepsilon \leq \frac{\delta}{2 + M + 2\|\varphi(0)\|\|\varphi'(0)\|} < \frac{\delta}{2},$$

the associated mapping $f_\varepsilon(x) \equiv f(x) = \sum_n U_n(x_n)$, where $x = (x_n) \in \sum_2 \ell_2$. Now define A_ε as the 1-level set for f , that is to say,

$$A_\varepsilon \equiv A = \{x \in \ell_2 : f(x) \leq 1\}.$$

Clearly A is a closed set with boundary

$$\partial A = \{x \in \ell_2 : f(x) = 1\},$$

and we have the inclusion

$$(1 - \varepsilon)B \subset A \subset B,$$

where B denotes the unit ball of ℓ_2 . It can be proved that A satisfies all the properties of the statement. To see that the cone of tangent hyperplanes to A has empty interior one checks the inclusion $\mathcal{C}(\mu_A) \subseteq \mathcal{C}(f)$ and, since $\mathcal{C}(f)$ has empty interior, so does $\mathcal{C}(\mu_A)$. We refer to [14] for the details.

SKETCH OF THE PROOF OF THEOREM 8.3

The main idea of the proof of this result is as follows. First we approximate our starlike body A by a C^1 smooth starlike body V . Then we modify V by creating a number of suitably located small flat patches on its boundary, and upon each of those patches we put a small C^1 smooth bump whose set of gradients is large enough. The starlike body D thus constructed will have the required properties. We will split the most technical part of the proof into several lemmas stated without proofs (see [7] for the details); then we will proceed with the final and more interesting part of the proof.

LEMMA 8.6. *Let X be a Banach space with separable dual X^* . For every bounded starlike body A and for every $\varepsilon > 0$ there exists a C^1 smooth starlike body $V = V_\varepsilon$ so that $|\mu_A(x) - \mu_V(x)| \leq \varepsilon$ for every $x \in B_X$.*

LEMMA 8.7. *Let X be a Banach space, and let A be a C^1 smooth bounded starlike body in X . For every $z^* \in X^*$ and $z \in X$ so that $z^*(z) = \|z\| = \|z^*\| = 1$, and for every $\varepsilon > 0$, $\delta > 0$, there exist a C^1 smooth starlike body $V = V_{z,\varepsilon}$ and $r \in (0, \delta)$ so that:*

- (1) $\mu_V(x) = \mu_A(x)$ for all x with $\|x\| = 1$ and $\|x - z\| \geq 2r$; that is, A and V coincide outside the cone $\{\lambda x : \lambda \geq 0, \|x - z\| < 2r, \|x\| = 1\}$;
- (2) $|\mu_V(x) - \mu_A(x)| \leq \varepsilon$ for all $x \in B_X$;
- (3) $\mu_V(x) = \mu_A(z)z^*(x)$ for all x such that $\|x\| = 1$ and $\|x - z\| \leq r$; that is, ∂V has a flat patch (of radius $r/\mu_A(z)$) parallel to the hyperplane $\ker z^*$ around the point $z' = \frac{1}{\mu_A(z)}z \in \partial A$.

Now, if upon the flat patch of the starlike body provided by Lemma 8.7 we build a suitable Lipschitz C^1 smooth bump whose set of gradients contains two times the unit ball of the dual of the hyperplane directing this flat patch (notice that the existence of this bump is guaranteed by 6.6), then we obtain the following.

LEMMA 8.8. *Let X be an infinite-dimensional Banach space, and let A be a C^1 smooth bounded starlike body in X . For every $z^* \in X^*$ and $z \in X$ so that $z^*(z) = \|z\| = \|z^*\| = 1$, consider the decomposition $X = H \oplus [z] = H \times \mathbb{R}$, where $H = \text{Ker } z^*$. Then, for every $\varepsilon > 0$, $\delta > 0$, there exist a C^1 smooth starlike body $W = W_{z,\varepsilon}$ and $r \in (0, \delta)$ so that:*

- (1) A and W coincide outside the half-cylinder $\{x = (h, t) \in X : \|h\| \leq r, t > 0\}$;

- (2) $|\mu_W(x) - \mu_A(x)| \leq \varepsilon$ whenever $\|x\| \leq 1$;
- (3) For every hyperplane F not containing any vector of the cone $\{x = (h, t) \in X : |t| > 2\|h\|\}$ there exists $y \in \partial W \cap \{x = (h, t) \in X : t > 0, \|h\| \leq r\}$ such that $y + F$ is tangent to ∂W at y .

Let A be a bounded starlike body in X . For a given $\varepsilon > 0$, $\varepsilon \leq 1/8$, we have to find a C^1 smooth starlike body $D = D_\varepsilon$ so that the cone of its tangent hyperplanes, $\mathcal{C}(D)$, fills the dual space X^* , and

$$|\mu_D(x) - \mu_A(x)| \leq \varepsilon$$

for every $x \in B_X$. Thanks to Lemma 8.6, we can assume that A is C^1 smooth. Let $\{z_\alpha\}_{\alpha \in I}$ be a ε -net on the unit sphere S_X . For every $\alpha \in I$ (by the Hahn-Banach theorem) we can choose a $z_\alpha^* \in X^*$ so that $z_\alpha^*(z_\alpha) = 1 = \|z_\alpha^*\|$. Let us denote $H_\alpha = \text{Ker } z_\alpha^*$. Now, for every $\alpha \in I$, by Lemma 8.8, we can take $r_\alpha > 0$ and a C^1 smooth starlike body W_α so that

- (1) A and W_α coincide outside the half-cylinder $\{x = (h, t) \in X = H_\alpha \oplus [z_\alpha] : \|h\| \leq r_\alpha, t > 0\}$;
- (2) $|\mu_{W_\alpha}(x) - \mu_A(x)| \leq \varepsilon$ whenever $\|x\| \leq 1$;
- (3) For every hyperplane F not containing any vector of the cone $\{(h, t) \in H_\alpha \oplus [z_\alpha] : |t| > 2\|h\|\}$ there exists $y \in \partial W_\alpha \cap \{(h, t) \in H_\alpha \oplus [z_\alpha] : t > 0, \|h\| \leq r_\alpha\}$ such that $y + F$ is tangent to ∂W_α at y ;

moreover, the r_α can be chosen small enough so that the sets

$$\partial W_\alpha \cap \{(h, t) \in H_\alpha \oplus [z_\alpha] : \|h\| \leq r_\alpha, t > 0\}$$

are pairwise disjoint. For each $\alpha \in I$, let us denote the gauge of W_α by μ_α .

Now consider the union of all these bodies,

$$D = \bigcup_{\alpha \in I} W_\alpha.$$

Let us see that D is a bounded C^1 smooth starlike body. Define $\psi : X \rightarrow (0, \infty)$ by

$$\psi(x) = \inf_{\alpha \in I} \mu_\alpha(x).$$

It is obvious that ψ is positive homogeneous, and it is not difficult to check that for every $z \in S_X$ there exists some $\delta > 0$ and some $\alpha \in I$ such that

$\psi(x) = \mu_\alpha(x)$ for all $x \in S_X$ with $\|x - z\| < \delta$; since every functional μ_α is C^1 smooth away from the origin, this implies that ψ is C^1 smooth in $X \setminus \{0\}$. Therefore $\{x \in X : \psi(x) \leq 1\}$ is a C^1 smooth starlike body, and it is easily checked that $D = \{x \in X : \psi(x) \leq 1\}$, so that ψ is the Minkowski functional of D . The fact that ψ is locally some of the μ_α also implies that for every $x \in S_X$ there is some $\alpha \in I$ so that

$$|\psi(x) - \mu_A(x)| = |\mu_\alpha(x) - \mu_A(x)| \leq \varepsilon,$$

which shows that D approximates A as it is required.

It only remains to prove that for every hyperplane F of X there is some $y \in \partial D$ such that $y + F$ is tangent to ∂D at y . Since for each α the bodies W_α and D are the same inside the half-cylinder $C_\alpha = \{h + tz_\alpha \in H_\alpha \oplus [z_\alpha] : \|h\| \leq r_\alpha, t > 0\}$, all the hyperplanes of X not containing any vector of $\{h + tz_\alpha \in H_\alpha \oplus [z_\alpha] : |t| > 2\|h\|\}$ are tangent to ∂W_α , and therefore tangent to ∂A too, at some point of $\partial W_\alpha \cap C_\alpha = \partial D \cap C_\alpha$. This means that the set

$$\bigcup_{\alpha \in I} \{T \in X^* : T(h + tz_\alpha) \neq 0 \text{ for all } h + tz_\alpha \in H_\alpha \oplus [z_\alpha] \text{ with } |t| > 2\|h\|\}$$

is contained in

$$\{T \in X^* : y + \text{Ker } T \text{ is tangent to } \partial D \text{ at some point } y \in \partial D\}.$$

Therefore, in order to conclude the proof we only have to check that

$$X^* \setminus \{0\} = \bigcup_{\alpha \in I} \{T \in X^* : T(h + tz_\alpha) \neq 0 \text{ for all } h + tz_\alpha \in H_\alpha \oplus [z_\alpha] \text{ with } |t| > 2\|h\|\}.$$

Consider any $T \in X^*$, $T \neq 0$; we may assume $\|T\| = 1$. Choose $z \in X$, $\|z\| = 1$, such that $T(z) > 1 - \varepsilon$, and take z_α such that $\|z - z_\alpha\| \leq \varepsilon$. We have that $|T(z_\alpha) - T(z)| \leq \|z - z_\alpha\| \leq \varepsilon$ and hence $T(z_\alpha) \geq T(z) - \varepsilon > 1 - 2\varepsilon > 0$. Then, for every $h + tz_\alpha \in H_\alpha \oplus [z_\alpha]$ with $t > 2\|h\| > 0$ we get

$$\begin{aligned} T(h + tz_\alpha) &= T(h) + tT(z_\alpha) > T(h) + t(1 - 2\varepsilon) \geq -\|h\| + t(1 - 2\varepsilon) \\ &> -\|h\| + 2\|h\|(1 - 2\varepsilon) = (1 - 4\varepsilon)\|h\| > 0; \end{aligned}$$

and in a similar way one checks that $T(h + tz_\alpha) < 0$ for all $h + tz_\alpha \in H_\alpha \oplus [z_\alpha]$ with $t < -2\|h\| < 0$. Therefore $T(h + tz_\alpha) \neq 0$ for all $h + tz_\alpha \in H_\alpha \oplus [z_\alpha]$ with $|t| > 2\|h\|$. This concludes the proof of Theorem 8.3 in the C^1 smooth case. ■

Let us finish with the observation that, in Theorem 8.3, the assumption X^* separable can be replaced with the requirement that X has smooth partitions of unity, and in such a case we can also improve the order of smoothness of the approximating bodies.

THEOREM 8.9. *Let X be an infinite-dimensional Banach space with C^p smooth partitions of unity. Then, for every bounded starlike body A and every $\varepsilon > 0$ there exists a C^p smooth starlike body D so that $|\mu_D(x) - \mu_A(x)| \leq \varepsilon$ for all x with $\|x\| \leq 1$, and the cone of tangent hyperplanes to D , $\mathcal{C}(D)$, fills the dual space X^* .*

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