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Moufang $H^\star$-algebras

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We recall that a nonassociative algebra V over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) is said to be an $H^\star$-algebra when V is also a Hilbert space over $\mathbb{K}$ (the inner product of which is denoted by $(\,|\,)$) and is endowed with an algebra involution $\star$ which is linear in the real case and conjugate-linear in the complex case, and satisfies

$$(xy|z) = (x|zy^\star) = (y|x^\star z)$$

for all x, y, $z \in V$. $H^\star$-algebras were introduced by W. Ambrose [2] in a complex associative setting in a slightly different way. The real associative case was studied by I. Kaplansky [39] (see also [5, 10, 18]). Today the structure theory of several particular classes of nonassociative $H^\star$-algebras is well-known [47, 48, 3, 4, 35, 49, 43, 36, 37, 25, 32, 26, 29, 7, 11, 12, 34, 8, 22, 30, 42, 33]. There is also a complete determination of a wide class of ternary $H^\star$-structures [40, 41, 18, 12, 20, 17, 16, 23, 24, 58], as well as a germinal theory for both binary and ternary arbitrary $H^\star$-structures (see [32, 34, 31, 38, 6, 9, 14, 15, 56, 50, 13, 54, 53, 45] and [19, 21, 57, 51, 52, 58], respectively). The reader is referred to Section E of [44] for a complete survey on $H^\star$-theory.

Let V be an $H^\star$-algebra. Then the left annihilator

$$\operatorname{Lann} V = \{x \in V : \ xV = 0\}$$

coincides with the right annihilator

$$\operatorname{Rann} V = \{x \in V : \ Vx = 0\}$$

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[32, Proposition 2]. We denote it by $\operatorname{Ann} V$. There is a decomposition of V as an orthogonal direct sum

$$V = \overline{\operatorname{Lin} V^2} \oplus \operatorname{Ann} V\,, \tag{1}$$

where $\overline{\operatorname{Lin} V^2}$ is the closed linear span of the set $\{xy : x, y \in V\}$ (see [32, Proposition 2] and [34, Theorem 1]). Moreover $\overline{\operatorname{Lin} V^2}$, under a suitable involution and the restriction of the inner product and the algebraic operations of V, is an $H^\star$-algebra with zero annihilator in itself [25, 32, 34]. On the other hand, if V has zero annihilator, then V splits as an orthogonal direct sum

$$V = \overline{\bigoplus_{\alpha \in \mathcal{A}} I_\alpha}\,, \tag{2}$$

where $\{I_\alpha\}$ is the family of all the minimal closed ideals of V, being furthermore every I_α a topologically simple $H^\star$-algebra with the structure inherited from V [32, 34, 38]. We recall that an $H^\star$-algebra W is called *topologically simple* if it has nonzero product and its unique closed two-sided ideals are 0 and W.

The decompositions given by (1) and (2) reduce the study of every class of nonassociative $H^\star$-algebras defined by identities to the determination of those $H^\star$-algebras in the class that are topologically simple. This note is devoted to the description of $H^\star$-algebras satisfying some of the Moufang identities. We recall that an *alternative algebra* is a nonassociative algebra satisfying the alternative identities given by

$$x^2 y = x(xy), \qquad (yx)x = yx^2.$$

It is well known that every alternative algebra satisfies the following identities

$$y((xz)x) = ((yx)z)x \tag{3}$$

$$(xy)(zx) = (x(yz))x \tag{4}$$

$$(x(yx))z = x(y(xz)) \tag{5}$$

(see [55, 46]). These identities are called *right*, *middle*, and *left Moufang* identity, respectively.

In this note we prove that $H^\star$-algebras satisfying some of the Moufang identities are in fact alternative. A similar result on alternativeness of middle Moufang algebras has been recently obtained in [27] and [28] for division and composition middle Moufang algebras. Applying among other results a deep theorem of M. Slater [55, Theorem 9, p. 194], alternative $H^\star$-algebras can be completely described [44, p. 148] (see also [43]). It follows that the result in this note concludes the structure theory of Moufang $H^\star$-algebras.

1. The results

Let A be a nonassociative algebra over a field F and $g : A \times A \longrightarrow F$ a symmetric bilinear form. We recall that g is said to be *associative* if

$$g(xy, z) = g(x, yz)$$

for all x, y, $z \in A$.

Lemma 1.1. *Let A be a nonassociative algebra over a field F such that there exists a nondegenerate associative symmetric bilinear form on A. Then the following assertions are equivalent:*

(i) *A satisfies the right Moufang identity.*
(ii) *A satisfies the middle Moufang identity.*
(iii) *For all x, z, $u \in A$ the equality $((xz)x)u = x(z(xu))$ holds.*

Proof. Let $g : A \times A \longrightarrow F$ be a nondegenerate associative symmetric bilinear form, and let x, y, z, u be in A. Then we have

$$\begin{aligned} g\Big(y\big((xz)x\big) - \big((yx)z\big)x, u\Big) &= g\big((xz)x, uy\big) - g\big(z, (xu)(yx)\big) \\ &= g\Big(z, \big(x(uy)\big)x - (xu)(yx)\Big) . \end{aligned} \tag{6}$$

On the other hand,

$$\begin{aligned} g\Big(z, \big(x(uy)\big)x - (xu)(yx)\Big) &= g\big(xz, x(uy)\big) - g\big(z(xu), yx\big) \\ &= g\Big(\big((xz)x\big)u - x\big(z(xu)\big), y\Big) . \end{aligned} \tag{7}$$

Since g is nondegenerate, (6) and (7) yield the equivalence of Assertions (i), (ii), and (iii). ∎

Theorem 1.2. *Let V be an $H^\star$-algebra. Then the following assertions are equivalent:*

(i) *V satisfies the right Moufang identity.*
(ii) *V satisfies the middle Moufang identity.*
(iii) *For all x, z, $u \in V$ the equality $((xz)x)u = x(z(xu))$ holds.*
(iv) *V satisfies the left Moufang identity.*
(v) *V is an alternative algebra.*

Proof. Let $g : V \times V \longrightarrow \mathbb{K}$ be the symmetric bilinear form defined by $g(x,y) = (x|y^\star) + (y|x^\star)$ for all x, $y \in V$. A routine calculation shows that g is associative and nondegenerate. Now Lemma 1.1 yields the equivalence of (i), (ii), and (iii).

Assume that the equivalent Conditions (i) and (iii) are fulfilled. By (i), for x, y, z, $t \in V$ we have

$$\begin{aligned}((xy)x - x(yx) \,|\, zt) &= \Big(z^\star\big((xy)x\big) \,|\, t\Big) - \Big(z^\star\big(x(yx)\big) \,|\, t\Big)\\ &= \Big(\big((z^\star x)y\big)x \,|\, t\Big) - \Big(z^\star \,|\, t\big(x(yx)\big)^\star\Big)\\ &= \Big(z^\star \,|\, \big((tx^\star)y^\star\big)x^\star\Big) - \Big(z^\star \,|\, \big((tx^\star)y^\star\big)x^\star\Big) = 0\,.\end{aligned}$$

Without loss of generality we can assume that $\operatorname{Ann} V = \{0\}$. Then, since the linear span of $\{zt : z, t \in V\}$ is dense in V (see (1)), and $(\cdot|\cdot)$ is nondegenerate, we obtain that $(xy)x = x(yx)$ for all x, $y \in V$. In this way we have proved that V is a flexible algebra. By (iii), V is left Moufang.

Now assume that Condition (iv) is fulfilled. Since the opposite algebras of left Moufang $H^\star$-algebras are right Moufang $H^\star$-algebras, and these last algebras are flexible, we obtain that V is also flexible. Then, applying again that V is left Moufang, we realize that Assertion (iii) holds for V.

Finally, we assume that V satisfies the equivalent Assertions (i)-(iv), and show that V is alternative. In view of (1), (2), and [9, Theorem 1], we can assume that V is topologically simple and complex. We already know that V is flexible, and that, consequently, for x in V the equality $x^2x = xx^2$ holds. On the other hand, by Assertion (ii), we have $x^2x^2 = (xx^2)x$ for every $x \in V$. Since there exists a nondegenerate associative symmetric bilinear form on V, it follows from [1] (see also [25, p. 39]) that V is a noncommutative Jordan $H^\star$-algebra. Then, by [32, Theorems 2 and 4] and [2], either V is anticommutative or there exists an approximate unit for V. Assume that V is anticommutative. Then, by Assertion (i), we have $R_x^3 = 0$ for every x in V, where R_x stands for the operator of right multiplication by x on V. Therefore, if x is in V and satisfies $x^\star = \varepsilon x$ for $\varepsilon \in \{+,-\}$, then for every y in V we have $0 = (R_x^4(y)|y) = \|R_x^2(y)\|^2$, and hence $0 = (R_x^2(y)|y) = \varepsilon\|R_x(y)\|^2$, so that $x \in \operatorname{Ann} V = \{0\}$. Since $V = \operatorname{Sym}(V,\star) \oplus \operatorname{Sk}(V,\star)$, where $\operatorname{Sym}(V,\star)$ (respectively, $\operatorname{Sk}(V,\star)$) means the real vector subspace of all self-adjoint (respectively, skew-adjoint) elements of V, we deduce $V = 0$, which contradicts that V is topologically simple. Now assume that V has an approximate unit (say $\{e_\lambda\}_{\lambda\in\Lambda}$). Then, making $z = e_\lambda$ in (3) and $y = e_\lambda$ in (5), and taking limits in λ, the alternativeness of V follows. ∎

As we said in the introduction, alternative $H^\star$-algebras are well-understood. Indeed, *the topologically simple alternative nonassociative $H^\star$-algebras are: the algebra of complex octonions, in the complex case; and the same algebra regarded as a real algebra, together with the two real octonions algebras, in the real case* (see [44, p. 148]). On the other hand *every topologically simple associative $H^\star$-algebra is isomorphic to the $H^\star$-algebra $\mathcal{HS}(H)$ of all Hilbert-Schmidt operators on a suitable Hilbert space H, which is complex in the complex case, and either real, complex, or quaternionic in the real case* (see [2] and [39]).

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