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Uniform Gâteaux Smoothness of Higher Order on Separable Banach Spaces

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1. Introduction

The aim of this paper is to show, among other things, that, in separable Banach spaces, the presence of the smoothness with the highest derivative Lipschitzian implies the uniform Gâteaux smoothness of degree 1 up. More exactly:

Theorem 4.1. *Let X be a separable Banach space such that for some nonnegative integer n its norm is n times differentiable on $X\setminus\{0\}$ and its n-th derivative is Lipschitzian on the unit sphere. Then X admits an equivalent norm which is, on any fixed annulus around the origin, $n+1$ times uniformly Gâteaux differentiable, and whose derivatives of degree less than $n+1$ are Lipschitzian there.*

The proof of this theorem uses countably many integral convolutions, an implicit function theorem, and appropriate chain rules. Its predecessors are papers [3] and [4].

Let $p > 1$ be an odd integer. Then the space L_p, with a sigma finite measure, is separable and its canonical norm is $p-1$ times differentiable with the highest derivative Lipschitzian on the unit sphere [2, Theorem V.1.1]. Thus we get, from our theorem, an equivalent norm on L_p which is p times uniformly Gâteaux differentiable. This was proved, in a different way, by Troyanski in [9]. A more general case of Orlicz spaces considered by Maleev in [7] is also covered by our theorem.

Our theorem cannot be extended to nonseparable spaces: Let $p > 1$ be again an odd integer and let Γ be an uncountable set. Then the canonical norm $\ell_p(\Gamma)$ is still $p-1$ times differentiable with the highest derivative Lipschitzian on the unit sphere [2, Theorem V.1.1]. Yet this space does not admit any p times Gâteaux differentiable equivalent norm or bump function [9], [8], [5, Section 4].

The first part of the proof (dealing with the integral convolutions) of the above theorem yields:

Theorem 4.2. *Let X be a separable Banach space such that for some nonnegative integer n there exists an n times differentiable bump function on X, with its n-th derivative Lipschitzian. Then X admits an $n+1$ times uniformly Gâteaux differentiable bump whose derivatives of degree less than $n+1$ are Lipschitzian.*

An inspection of the process of subsequent integral convolutions used in the proofs reveals that the constructed equivalent norm or bump can be done, respectively, as close to the original norm or bump as we wish. Further, if the norm or bump in the assumptions of our theorems is moreover $\mathcal{C}^{(k)}$-smooth, with $k \in \{n+1, n+2, \dots\} \cup \{\infty\}$, then, respectively, the constructed norm or bump is moreover $\mathcal{C}^{(k)}$-smooth. Thus we cover [4], where $n = 0$ was considered. There are also local versions of our theorems when “Lipschitzian” and “uniformly” are replaced by “locally Lipschitzian” and “locally uniformly” respectively.

The above theorems can be completed by the following reverse results (for the notation see below):

Proposition 3.10. *Let $(X, \|\cdot\|)$ be a (not necessarily separable) Banach space, with dimension at least 2, and let $n \in \mathbb{N}$. Then the following three assertions are equivalent:*

(i) $\|\cdot\| \in \mathcal{G}^{(n)}(S_X)$ *and* $\|\cdot\|^{(n)}$ *is Lipschitzian on the unit sphere* S_X.

(ii) $\|\cdot\| \in \mathcal{LIP}^{(n)}(S_X)$.

(iii) $\|\cdot\| \in \mathcal{LIP}^{(n)}(A)$ *for any annulus A around* 0.

The above assertions are implied by

(iv) $\|\cdot\| \in \mathcal{UG}^{(n+1)}(S_X)$.

From [6] a variant of the above follows: *If a bounded continuous function f on a (not necessarily separable) Banach space is $n+1$ times uniformly Gâteaux differentiable, then all the derivatives $f^{(k)}$, $k = 1, \dots, n$, are Lipschitzian.*

2. Preliminaries

Throughout this paper, we will work with real Banach spaces. The topological dual of X is denoted by X^*. The closed unit ball in X is denoted by B_X. The unit sphere in X is denoted by S_X. Further we put $U(x,\delta) = \{y \in X : \|y-x\| < \delta\}$, $B(x,\delta) = \{y \in X : \|y-x\| \le \delta\}$, $U(\Omega,\delta) = \bigcup_{x\in\Omega} U(x,\delta)$, where $\delta > 0$, $x \in X$, and $\Omega \subset X$. The symbols $\mathbb{N}$ and $\mathbb{R}$ stand for the set of positive integers and the set of real numbers respectively. The symbol $\mathrm{dom} f$ denotes the domain of the mapping f. The symbol $\mathrm{Lip}_\Omega(f)$ denotes a Lipschitz constant of a mapping f with respect to the set Ω. Given $n \in \mathbb{N}$, the symbol $\mathcal{L}^{(n)}(X,Y)$ denotes the (Banach) space of all n-linear bounded operators from X to Y endowed with the norm

$$\|L\| = \sup\{\|L(h_1,\dots,h_n)\| : h_1,\dots,h_n \in B_X\}, \quad L \in \mathcal{L}^{(n)}(X,Y).$$

If $n = 1$, we write $\mathcal{L}(X,Y)$ instead of $\mathcal{L}^{(1)}(X,Y)$. Further we put $\mathcal{L}^{(n)}(X) = \mathcal{L}^{(n)}(X,\mathbb{R})$. Thus $X^* = \mathcal{L}^{(1)}(X)$. We note that there is a canonical isometry between $\mathcal{L}(X,\mathcal{L}^{(n-1)}(X,Y))$ and $\mathcal{L}^{(n)}(X,Y)$. So, in what follows, we will identify these two spaces.

Definition 2.1. Let X, Y be Banach spacesand Ω a subset of X. We say that a mapping f from X into Y is *Gâteaux differentiable at* $x \in X$ if

(i) there is $\delta > 0$ such that $U(x,\delta) \subset \mathrm{dom} f$, and

(ii) there is $l_x \in \mathcal{L}(X,Y)$ such that for every $h \in X$

$$\lim_{\tau\to 0} \left\| \frac{f(x+\tau h) - f(x)}{\tau} - l_x(h) \right\| = 0.$$

We then denote $f'(x) = l_x$. We say that f is *Gâteaux differentiable on* Ω if f is Gâteaux differentiable at every point $x \in \Omega$. Then we write $f \in \mathcal{G}^{(1)}(\Omega)$. The Gâteaux differentiability of the n-th order, the symbol $f^{(n)}(x)$, and the inclusion $f \in \mathcal{G}^{(n)}(\Omega)$ for $n = 2, 3, \dots$ are defined by induction.

We say that a mapping f is *uniformly Gâteaux differentiable on* Ω if

(i) $f \in \mathcal{G}^{(1)}(\Omega)$,

(ii) there is $\delta > 0$ such that $U(\Omega,\delta) \subset \mathrm{dom} f$, and

(iii)

$$\lim_{\tau\to 0} \sup_{x\in\Omega} \left\| \frac{f(x+\tau h) - f(x)}{\tau} - f'(x)(h) \right\| = 0,$$

whenever $h \in X$.

Then we write $f \in \mathcal{UG}^{(1)}(\Omega)$. The uniform Gâteaux differentiability of the order $n > 1$, as well as the inclusion $f \in \mathcal{UG}^{(n)}(\Omega)$ are defined by induction.

Finally, for $\Omega \subset X$ we write $f \in \mathcal{LIP}^{(0)}(\Omega)$, provided that f satisfies the Lipschitz condition on the set Ω. For $n \in \mathbb{N}$ we write $f \in \mathcal{LIP}^{(n)}(\Omega)$ if

(i) $f \in \mathcal{G}^{(n)}(\Omega)$, and
(ii) $f^{(k)} \in \mathcal{LIP}^{(0)}(\Omega)$ for $k = 0, \dots, n$.

If there is no doubt what the set Ω is, we write $\mathcal{UG}^{(n)}$, $\mathcal{LIP}^{(n)}$ instead of $\mathcal{UG}^{(n)}(\Omega)$, $\mathcal{LIP}^{(n)}(\Omega)$.

We can easily check that for a mapping $f \in \mathcal{UG}^{(1)}(\Omega),\ f \in \mathcal{LIP}^{(0)}(\Omega)$, the following equivalences hold :

$$\begin{aligned} f' \in \mathcal{UG}^{(n-1)}(\Omega) &\Longleftrightarrow f \in \mathcal{UG}^{(n)}(\Omega) \quad \text{for all} \quad n \geq 2 \\ f' \in \mathcal{LIP}^{(n-1)}(\Omega) &\Longleftrightarrow f \in \mathcal{LIP}^{(n)}(\Omega) \quad \text{for all} \quad n \geq 1. \end{aligned} \tag{2.1}$$

Also we use a standard definition of Fréchet smoothness (c.f. [1]). Let $\Omega \subset X$ be an open set and $n \in \mathbb{N}$. We say that the mapping f is $\mathcal{C}^{(n)}$*−smooth* on Ω if it is $n-$times Fréchet differentiable at every $x \in \Omega$ and the mapping $x \mapsto f^{(n)}(x)$ from Ω to $\mathcal{L}^n(X, Y)$ is continuous on Ω. It is easy to see that f is $\mathcal{C}^{(n)}-$smooth on Ω if and only if it is $n-$times Fréchet differentiable at every point of Ω and for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\begin{aligned} \Big\| \frac{1}{\tau}[f^{(n-1)}(z+\tau h_n)(h_1,\dots,h_{n-1}) - f^{(n-1)}(z)(h_1,\dots,h_{n-1})] \\ - L(h_1,\dots,h_n)\Big\| < \varepsilon \end{aligned} \tag{2.2}$$

whenever $0 \neq \tau \in (-\delta, \delta),\ z \in \Omega,\ \|z - x\| < \delta$, and $h_1, \dots, h_n \in B_X$. We say that f is $\mathcal{C}^{(\infty)}$-smooth on Ω if it is $\mathcal{C}^{(n)}$-smooth on Ω for every $n \in \mathbb{N}$. The $\mathcal{C}^{(0)}$-smoothness means the continuity and we put $f^{(0)} = f$.

Remark 2.2. Since norms are not differentiable at the origin, we say that a norm is differentiable (in Gâteaux or Fréchet sense), if it is differentiable at all nonzero points.

DEFINITION 2.3. A real valued function f on a Banach space X is called a *bump* if its support $\operatorname{supp} f := \{x \in X : f(x) \neq 0\}$ is nonempty and bounded.

3. Basic facts on differentiability

LEMMA 3.1. *Let X be a Banach space and $f : \Omega \to \mathbb{R}$ be a function defined on an open subset $\Omega \subset X$ satisfying the following three assumptions:*

(i) *f is n times Fréchet differentiable on the set Ω.*

(ii) *For all $x \in \Omega$ and $h_1, \ldots, h_{n+1} \in X$ the directional derivative*

$$D_{h_{n+1}} f^{(n)}(x)(h_1, \ldots, h_n) := \lim_{\tau \to 0} \frac{1}{\tau}[f^{(n)}(x + \tau h_{n+1})(h_1, \ldots, h_n) - f^{(n)}(x)(h_1, \ldots, h_n)]$$

exists.

(iii) *The mapping $x \mapsto D_{h_{n+1}} f^{(n)}(x)(h_1, \ldots, h_n)$ is continuous on Ω for any $h_1, \ldots, h_{n+1} \in X$.*

Then for every $x \in \Omega$ the mapping $\varphi_x : X^{n+1} \to \mathbb{R}$ defined by

$$\varphi_x(h_1, \ldots, h_{n+1}) = D_{h_{n+1}} f^{(n)}(x)(h_1, \ldots, h_n)$$

is an $(n+1)$-linear and symmetric form in variables $h_1, \ldots, h_{n+1}$.

Proof. It is a consequence of the well known Schwarz theorem from analysis. ∎

LEMMA 3.2. *Let X, Y be Banach spaces, $\Omega \subset X$ an open set, and $f : X \to Y$ a mapping.*

(i) *Let $f \in \mathcal{LIP}^{(0)}(U(\Omega, \delta))$ for some $\delta > 0$ and $f \in \mathcal{UG}^{(1)}(\Omega)$. Then $x \mapsto f'(x)(h)$ is uniformly continuous on Ω for every fixed $h \in X$.*

(ii) *Let $f \in \mathcal{G}^{(1)}(\Omega)$ and $x \mapsto f'(x)(h)$ be uniformly continuous on Ω for every fixed $h \in X$. Then $f \in \mathcal{UG}^{(1)}(\{x : B(x, \delta) \subset \Omega\})$ whenever $\delta > 0$.*

Proof. (i) Fix $h \in X$ and suppose that $f \in \mathcal{LIP}^{(0)}(U(\Omega, \delta))$ for some $\delta > 0$. Then for sufficiently small $\tau \neq 0$

$$\left\| \frac{1}{\tau}[f(x + \tau h) - f(x)] - \frac{1}{\tau}[f(z + \tau h) - f(z)] \right\| \to 0 \tag{3.1}$$

whenever $x, z \in \Omega$ and $\|x - z\| \to 0$. Then (3.1) and the inequality

$$\begin{aligned}\|f'(x)(h) - f'(z)(h)\| \le &\left\| f'(x)(h) - \frac{1}{\tau}[f(x+\tau h) - f(x)] \right\| \\ &+ \left\| \frac{1}{\tau}[f(z+\tau h) - f(z)] - f'(z)(h) \right\| \\ &+ \left\| \frac{1}{\tau}[f(x+\tau h) - f(x)] - \frac{1}{\tau}[f(z+\tau h) - f(z)] \right\|,\end{aligned}$$

complete the proof.

(ii) Fix $\varepsilon > 0$, $h \in X$, $\delta > 0$. Then we can find $r > 0$ such that $\{x : B(x,\delta) \subset \Omega\} + rh \subset \Omega$, and $\|f'(x)(h) - f'(z)(h)\| < \varepsilon$, whenever $x, z \in \Omega$ and $\|x - z\| < r$. Now, pick an arbitrary element y^* from the dual unit ball B_{Y^*}. Clearly $y^* \circ f \in \mathcal{G}^{(1)}(\Omega)$ and $(y^* \circ f)'(x)(h) = \langle y^*, f'(x)(h)\rangle$. Using the mean value theorem, we have

$$\begin{aligned}|\langle y^*, f(x+\tau h) - f(x) - &f'(x)(\tau h)\rangle| \\ &= |\langle y^*, f(x+\tau h)\rangle - \langle y^*, f(x)\rangle - \langle y^*, f'(x)(\tau h)\rangle| \\ &= |\tau||\langle y^*, f'(x+\theta\tau h)(h) - f'(x)(h)\rangle| \\ &\le |\tau|\|f'(x+\theta\tau h)(h) - f'(x)(h)\| \le |\tau|\varepsilon.\end{aligned}$$

whenever $B(x,\delta) \subset \Omega$ and $|\tau| < r$. Here $\theta \in (0,1)$ is a constant dependent on x, h, y^*, and τ. Hence

$$|\langle y^*, f(x+\tau h) - f(x) - f'(x)(\tau h)\rangle| \le \varepsilon|\tau|$$

whenever $B(x,\delta) \subset \Omega$, $|\tau| < r$, and $y^* \in B_{Y^*}$. Therefore

$$\|f(x+\tau h) - f(x) - f'(x)(\tau h)\| \le \varepsilon|\tau|$$

whenever $B(x,\delta) \subset \Omega$, $|\tau| < r$, which completes the proof. ∎

Remark 3.3. By Lemma 3.2 (ii), if $f : X \to Y$ satisfies $f \in \mathcal{LIP}^{(n)}(\Omega)$, then $f \in \mathcal{UG}^{(n)}(\{x : B(x,\delta) \subset \Omega\})$ whenever $\delta > 0$.

LEMMA 3.4. *Let X, Y and Z be Banach spaces and U, V be open subsets in X and Y respectively. Further let f be a mapping from U into Y and g be a Lipschitzian mapping from V into Z. Assume that f is Gâteaux differentiable at $x \in U$, that $y := f(x) \in V$, and that g is Gâteaux differentiable at y. Then the composition $g \circ f$ is Gâteaux differentiable at x and for all $h \in X$*

$$(g \circ f)'(x)(h) = g'(f(x))(f'(x)(h)).$$

Proof. For $h \in X$ and $\tau \neq 0$ we have

$$\frac{1}{\tau}[(g\circ f)(x+\tau h)-(g\circ f)(x)] = \frac{1}{\tau}\big[g\big(f(x)+f'(x)(\tau h)\big)-g\big(f(x)\big)\big] + \frac{1}{\tau}\big[g\big(f(x)+f'(x)(\tau h)+o(\tau)\big)-g\big(f(x)+f'(x)(\tau h)\big)\big].$$

Since g is Lipschitzian, the last term tends to zero when $\tau \to 0$. Thus

$$\frac{1}{\tau}[(g\circ f)(x+\tau h)-(g\circ f)(x)] \to g'(f(x))(f'(x)(h))$$

as $\tau \to 0$. The linearity of the derivative is clear. ∎

Lemma 3.5. *Let X, Y and Z be Banach spaces, let $\Omega \subset X$, $\Omega' \subset Y$ be open sets, and let $n \in \mathbb{N}$. Further let $f : X \to Y$, $g : Y \to Z$ be two mappings such that $f \in \mathcal{LIP}^{(n)}(\Omega)$, $g \in \mathcal{LIP}^{(n)}(\Omega')$ and $f(\Omega) \subset \Omega'$. Then $g \circ f \in \mathcal{LIP}^{(n)}(\Omega)$.*

Proof. The conclusion is obviously true for $n = 0$. Consider $n \geq 1$ and assume that the conclusion was verified for $n-1$. Further define a mapping $o : \mathcal{L}(Y,Z)\times\mathcal{L}(X,Y) \to \mathcal{L}(X,Z)$ by $o(u,v) := u\circ v$, $u \in \mathcal{L}(Y,Z)$, $v \in \mathcal{L}(X,Y)$. It is easy to check that the composition mapping o is $\mathcal{LIP}^{(\infty)}$. Indeed, for $h, a \in \mathcal{L}(Y,Z)$ and $k, b \in \mathcal{L}(X,Y)$ we have

$$\begin{aligned} o'(u,v)(h,k) &= h\circ v + u\circ k\\ o''(u,v)((h,k),(a,b)) &= h\circ b + a\circ k\\ o'''(u,v) &\equiv 0. \end{aligned}$$

Let γ denote the mapping $x \mapsto \big((g'\circ f)(x), f'(x)\big)$, $x \in \Omega$. By Lemma 3.4, the derivative $(g\circ f)'$ exists and

$$\begin{aligned} (g\circ f)'(x)(h) &= g'(f(x))[f'(x)h] = ([g'(f(x)]\circ[f'(x)])(h)\\ &= \{o([g'(f(x))], f'(x))\}(h). \end{aligned}$$

Hence

$$(g\circ f)'(x) = o([g'(f(x))], f'(x)) = (o\circ\gamma)(x),$$

and so $(g\circ f)' = o\circ\gamma$. By the induction assumption, $\gamma \in \mathcal{LIP}^{(n-1)}(\Omega)$. Hence, by the induction assumption used again, $o\circ\gamma \in \mathcal{LIP}^{(n-1)}(\Omega)$, i.e., $(g\circ f)' \in \mathcal{LIP}^{(n-1)}(\Omega)$, and, by (2.1), $g\circ f \in \mathcal{LIP}^{(n)}(\Omega)$. ∎

Lemma 3.6. *Let X, Y and Z be Banach spaces, $\Omega \subset X$, $\Omega' \subset Y$ open sets and $n \in \mathbb{N}$. Further let $f : X \to Y$, $g : Y \to Z$ be two mappings such that $f(\Omega) \subset \Omega'$, that*

(i) $f \in \mathcal{UG}^{(n)}(\Omega) \cap \mathcal{LIP}^{(n-1)}(U(\Omega, \varepsilon))$ *for some* $\varepsilon > 0$, *and that*

(ii) $g \in \mathcal{LIP}^{(n)}(\Omega')$.

Then for every $\delta > 0$, $g \circ f \in \mathcal{UG}^{(n)}(\{x : B(x, \delta) \subset \Omega\}) \cap \mathcal{LIP}^{(n-1)}(\Omega)$.

Proof. Suppose first that $n = 1$. Then, by Lemma 3.5, $g \circ f \in \mathcal{LIP}^{(0)}(\Omega)$, and, by Lemma 3.6 (i), for fixed $h \in X$ we have $\|f'(x)(h) - f'(z)(h)\| \to 0$ whenever $x, z \in \Omega$ and $\|x - z\| \to 0$. By Lemma 3.4, the composition $g \circ f$ is Gâteaux differentiable and $(g \circ f)'(x)(h) = g'(f(x))(f'(x)(h))$ if $x \in \Omega$, $h \in X$. Fix $h \in X$ and calculate

$$\begin{aligned}
\|(g \circ f)'(x)(h) - (g \circ f)'(z)(h)\| &= \|g'(f(x))(f'(x)(h)) - g'(f(z))(f'(z)(h))\| \\
&\le \|g'(f(x)) - g'(f(z))\| \, \|f'(x)(h)\| + \|g'(f(z))\| \, \|f'(x)(h) - f'(z)(h)\| \\
&\le \mathrm{Lip}(g')\big[\mathrm{Lip}(f)\big]^2 \|h\| \, \|x - z\| + \mathrm{Lip}(g) \|f'(x)(h) - f'(z)(h)\| \to 0,
\end{aligned}$$

whenever $x, z \in \Omega$, $\|x - z\| \to 0$. Then, by Lemma 3.2 (ii), $g \circ f \in \mathcal{UG}^{(1)}(\{x : B(x, \delta) \subset \Omega\})$ for every $\delta > 0$.

Now, assume that $n \ge 2$ and that we have verified the claim for $n - 1$. Fix $\delta > 0$ and consider $f \in \mathcal{UG}^{(n)}(\Omega) \cap \mathcal{LIP}^{(n-1)}(U(\Omega, \varepsilon))$, $g \in \mathcal{LIP}^{(n)}(\Omega')$. By Lemma 3.5, $g \circ f \in \mathcal{LIP}^{(n-1)}(\Omega)$. By the induction assumption, $g' \circ f \in \mathcal{UG}^{(n-1)}(\{x : B(x, \delta) \subset \Omega\})$. Note that also $f' \in \mathcal{UG}^{(n-1)}(\{x : B(x, \delta) \subset \Omega\})$. Then the mapping $x \mapsto ((g' \circ f)(x), f'(x))$ is in $\mathcal{UG}^{(n-1)}(\{x : B(x, \delta) \subset \Omega\})$. Finally, imitating the proof of Lemma 3.5, we get that $g \circ f \in \mathcal{UG}^{(n)}(\{x : B(x, \delta) \subset \Omega\})$. ∎

Lemma 3.7. *Let X be an at least two-dimensional Banach space, let $x \in X$, $x' \in X$, and assume that $\|x\| \le \|x'\|$. Then there exists $z \in X$, with $\|z\| = \|x\|$, and such that*

$$\|(1 - t)x + tz\| \ge \frac{1}{3}\|x\| \quad \text{for all} \quad t \in \mathbb{R}, \tag{3.2}$$

and

$$\|(1 - t)x' + tz\| \ge \frac{1}{3}\|x\| \quad \text{for all} \quad t \in [0, 1]. \tag{3.3}$$

Proof. First of all, we prove the following

CLAIM. Let $u, v \in X$ be such that $\|v\| \leq \|u\|$ and $\|v\| \leq \|v+su\|$ for every $s \in \mathbb{R}$. Then for every $t \in \mathbb{R}$ we have

$$\|(1-t)u + tv\| \geq \frac{1}{3}\|v\|.$$

Indeed, consider $t \in \mathbb{R}$. If $\|t(u-v)\| \leq \frac{2}{3}\|u\|$, then

$$\begin{aligned}\|(1-t)u + tv\| &= \|u - t(u-v)\| \geq \|u\| - \|t(u-v)\| \\ &\geq \|u\| - \frac{2}{3}\|u\| = \frac{1}{3}\|u\| \geq \frac{1}{3}\|v\|.\end{aligned}$$

If $\|t(u-v)\| > \frac{2}{3}\|u\|$, then

$$|t| > \frac{2}{3}\frac{\|u\|}{\|u-v\|} \geq \frac{2}{3}\cdot\frac{1}{2} = \frac{1}{3},$$

and so

$$\|(1-t)u + tv\| = |t|\left\|v - \frac{t-1}{t}u\right\| \geq \frac{1}{3}\|v\|.$$

This proves the claim. □

Now, consider a two-dimensional subspace X_2 of X such that $x, x' \in X_2$. We find $f \in X_2^*$ for which $f \neq 0$, $f(x) = 0$, and $f(x') \geq 0$. Let $B = \{u \in X_2 : \|u\| \leq \|x\|\}$, a compact subset of X_2. Find $z \in B$ such that $f(z) = \max f(B)$. Then surely

$$u \in X_2 \quad \text{and} \quad f(u) \geq f(z) \Longrightarrow \|u\| \geq \|x\|. \tag{3.4}$$

Since $z \in B$ is not contained in the interior of the set B, we have

$$\|z\| = \|x\|. \tag{3.5}$$

Now, since $f(z+sx) = f(z) + sf(x) = f(z)$ for every $s \in \mathbb{R}$, the implication (3.4) yields that

$$\|z\| \leq \|z + sx\| \quad \text{for all} \quad s \in \mathbb{R}. \tag{3.6}$$

By (3.5), (3.6) and by our Claim the property (3.2) is satisfied.

In order to prove the property (3.3), consider first the case when

$$f(x') \geq f(z). \tag{3.7}$$

Then for every $t \in [0,1]$ we have

$$f((1-t)x' + tz) = (1-t)f(x') + tf(z) \geq f(z).$$

Using this and the implication (3.4), we have $\|(1-t)x' + tz\| \geq \|x\|$, which is (3.3). Second, suppose that

$$0 \leq f(x') < f(z). \tag{3.8}$$

Find an element y lying in the line determined by the points z and x', and satisfying $f(y) = 0$. Then

$$x' = (1-\tau)y + \tau z \tag{3.9}$$

for some $\tau \in [0,1)$. If $\|y\| < \|z\|$, then y is contained in the interior of the set B and, because $z \in B$, by (3.9) the point x' lies in the interior of B as well. This gives us an inequality $\|x'\| < \|x\|$, which is in a contradiction with the assumption of our lemma. Therefore

$$\|z\| \leq \|y\|. \tag{3.10}$$

Further for every $s \in \mathbb{R}$ we have $f(z+sy) = f(z) + sf(y) = f(z)$. Hence, by (3.4),

$$\|z\| \leq \|z+sy\| \quad \text{for all} \quad s \in \mathbb{R}. \tag{3.11}$$

Now, by (3.10), (3.11) and by our Claim, for every $t \in \mathbb{R}$ we have

$$\|(1-t)y + tz\| \geq \frac{1}{3}\|z\| = \frac{1}{3}\|x\|.$$

The property (3.3) now follows from the fact that the line determined by the points x' and z is identical with the line determined by the points y and z. ∎

Lemma 3.8. *Let X, Y be Banach spaces, X be at least two-dimensional, $n \in \mathbb{N}$, and $0 < \alpha < 1$. Further assume that $f : X\setminus\{0\} \to Y$ is n times Gâteaux differentiable mapping with $f^{(n)}$ Lipschitzian on the annulus $B_X \setminus \frac{\alpha}{3^n} B_X$. Then $f, f', \ldots, f^{(n-1)}$ are all Lipschitzian on the annulus $B_X \setminus \alpha B_X$.*

Proof. Assume first $n = 1$. We have to show that f is Lipschitzian on $B_X \setminus \alpha B_X$. Since f' is Lipschitzian on $B_X \setminus (\alpha/3) B_X$, we have that the number

$$C = \sup\{\|f'(y)\| : \ y \in B_X \setminus \frac{\alpha}{3} B_X\}$$

is finite. Take any $x, x' \in B_X \setminus \alpha B_X$. If $[x, x'] \subset B_X \setminus (\alpha/3) B_X$, then the mean value theorem guarantees that

$$\|f(x) - f(x')\| \leq \sup\{\|f'(\xi)\| : \ \xi \in [x, x']\}\|x - x'\| \leq C\|x - x'\|.$$

Second, assume that $[x, x'] \cap (\alpha/3)B_X \neq \emptyset$, say $\|tx+(1-t)x'\| \leq (\alpha/3)$ for some $t \in (0, 1)$. By Lemma 3.7, there is $z \in B_X \backslash \alpha B_X$ so that $[x, z] \cap (\alpha/3)B_X = \emptyset$ and $[z, x'] \cap (\alpha/3)B_X = \emptyset$. Thus

$$\|f(x) - f(x')\| \leq \|f(x) - f(z)\| + \|f(z) - f(x')\| \leq C(\|x - z\| + \|z - x'\|) \leq 4C.$$

But

$$\frac{\alpha}{3} \geq \|tx + (1-t)x'\| \geq \|x'\| - t\|x - x'\| > \alpha - \|x - x'\|,$$

and so $\|x - x'\| > (2\alpha/3)$. Therefore

$$\|f(x) - f(x')\| \leq 4C = 4C \cdot \frac{3}{2\alpha} \cdot \frac{2\alpha}{3} < \frac{6C}{\alpha}\|x - x'\|.$$

For $n > 1$ we proceed by induction, using what we have just proved. ∎

LEMMA 3.9. *Let X, Y be Banach spaces, $n \in \{0\} \cup \mathbb{N}$, let $f : X \to Y$ be positively homogeneous mapping bounded on S_X. If $f \in \mathcal{UG}^{(n+1)}(S_X)$, then $f^{(n)}$ is Lipschitzian on every annulus around* 0.

Proof. Note that the homogeneity of the mapping implies that it is bounded on any annulus around 0 and that for every $\alpha > 0,\ x \in S_X,\ k \in \{1, \ldots, n+1\}$, $f^{(k)}(\alpha x)$ exists and

$$f^{(k)}(\alpha x) = \frac{1}{\alpha^{k-1}} f^{(k)}(x).$$

We will proceed by induction on n. Let first $n = 0$ and $f \in \mathcal{UG}^{(1)}(S_X)$. By the Baire category argument from [8, Remark 2.1] f' is bounded on S_X. Again by the homogeneity of f the derivative f' is bounded on every annulus around 0. Now from the proof of the Lemma 3.8 it is clear that f is Lipschitzian on every annulus around 0. Further, assume that the assertion is true for $n = k,\ k \geq 0$. Then according our assumption $f^{(k-1)}$ is Lipschitzian on every annulus around 0. Consequently $f^{(k)}$ is bounded on every annulus around 0. To finish the proof, we will apply the first part of the proof to $g := f^{(k)}$.

PROPOSITION 3.10. *Let $(X, \|\cdot\|)$ be a (not necessarily separable) Banach space, with dimension, at least* 2, *and let $n \in \mathbb{N}$. Then the following three assertions are equivalent.*

(i) $\|\cdot\| \in \mathcal{G}^{(n)}(S_X)$ *and* $\|\cdot\|^{(n)}$ *is Lipschitzian on the unit sphere* S_X.

(ii) $\|\cdot\| \in \mathcal{LIP}^{(n)}(S_X)$.

(iii) $\|\cdot\| \in \mathcal{LIP}^{(n)}(A)$ *for any annulus* A *around* 0.

The above assertions are implied by

(iv) $\|\cdot\| \in \mathcal{UG}^{(n+1)}(S_X)$.

Proof. To prove the equivalence of the first three assertions, it is enough to show the implication (i)⇒(iii). Fix any $0<\alpha<1$. We will show that the n-th derivative $\|\cdot\|^{(n)}$ is Lipschitzian on $B_X \setminus \alpha B_X$. So fix such an α. Let C be a Lipschitz constant of $\|\cdot\|^{(n)}$ on S_X, and fix $x_0 \in S_X$. Take any $x, y \in S_X$ and any $\alpha < t \leq 1$. Then, using the homogeneity of the norm, we have :

$$\begin{aligned}\big\|\|\cdot\|^{(n)}(x) - \|\cdot\|^{(n)}(ty)\big\| \leq& \big\|\|\cdot\|^{(n)}(x) - \|\cdot\|^{(n)}(y)\big\| \\ &+ \big\|\|\cdot\|^{(n)}(y) - \|\cdot\|^{(n)}(ty)\big\| \\ \leq& C\|x-y\| + \big\|\|\cdot\|^{(n)}(y)\big\|\big(t^{1-n}-1\big).\end{aligned}$$

Now,

$$\|x-y\| \leq \|x-ty\| + \|ty-y\| = \|x-ty\| + \|x\| - \|ty\| \leq 2\|x-ty\|,$$

$$\big\|\|\cdot\|^{(n)}(y)\big\| \leq C\|x_0-y\| + \big\|\|\cdot\|^{(n)}(x_0)\big\| \leq 2C + \big\|\|\cdot\|^{(n)}(x_0)\big\| =: C_1\,,$$

and

$$t^{1-n} - 1 = (1-t)\frac{\big(1+t+\cdots+t^{n-2}\big)}{t^{n-1}} \leq \|x-ty\| \cdot \frac{n-1}{\alpha^{n-1}}.$$

Thus

$$\big\|\|\cdot\|^{(n)}(x) - \|\cdot\|^{(n)}(ty)\big\| \leq C_2\|x-ty\|,$$

where we have put $C_2 = 2C + C_1(n-1)\alpha^{1-n}$.

Now, take any $x, y \in S_X$ and any $\alpha < t \leq s \leq 1$. Then $\frac{t}{s} > \alpha$ and so, by the first paragraph,

$$\Big\|\|\cdot\|^{(n)}(x) - \|\cdot\|^{(n)}\big(\frac{t}{s}y\big)\Big\| \leq C_2\Big\|x - \frac{t}{s}y\Big\|.$$

Hence,

$$\begin{aligned}\big\|\|\cdot\|^{(n)}(sx) - \|\cdot\|^{(n)}(ty)\big\| &= s^{1-n}\Big\|\|\cdot\|^{(n)}(x) - \|\cdot\|^{(n)}\big(\frac{t}{s}y\big)\Big\| \\ &\leq s^{1-n}C_2\Big\|x-\frac{t}{s}y\Big\| = s^{-n}C_2\|sx-ty\| \\ &\leq \alpha^{-n}C_2\|sx-ty\|.\end{aligned}$$

Therefore $\|\cdot\|^{(n)}$ is Lipschitzian on the annulus $B_X\backslash\alpha B_X$. Now, applying Lemma 3.8, we get (iii).

(iv)$\Rightarrow$(i). This implication is a direct consequence of Lemma 3.9. ∎

Remark 3.11. It is easy to check that a norm $\|\cdot\|$ on a Banach space$X$ belongs to $\mathcal{UG}^{(n)}$ with respect to any annulus around 0, provided that $\|\cdot\| \in \mathcal{UG}^{(n)}(S_X)$.

4. Existence of uniform Gâteaux differentiable renorming

Theorem 4.1. *Let X be a separable Banach space such that for some nonnegative integer n its norm is n times differentiable on $X\setminus\{0\}$ and its n-th derivative is Lipschitzian on the unit sphere. Then X admits an equivalent norm which is, on any fixed annulus around the origin, $n+1$ times uniformly Gâteaux differentiable and whose derivatives of degree less than $n+1$ are Lipschitzian there.*

Proof. In our proof we will freely follow the constructions from [4] and [3]. Also note that, by Proposition 3.10, we can suppose that the derivatives of our norm $\|\cdot\|$ are Lipschitzian on every annulus around 0. So we may not care about a concrete annulus.

STEP 1. Basic construction. Consider a $\mathcal{C}^{(\infty)}$-smooth function $\phi_0:\mathbb{R}\to\mathbb{R}$ such that $\phi_0\geq 0$, $\operatorname{supp}\phi_0\subset[-1/2,1/2]$ and $\int\phi_0=1$. Then define the functions $\phi_j(t)=2^j\phi_0(2^jt), j\in\mathbb{N}, t\in\mathbb{R}$. Since X is separable, there is a countable set $S=\{x_j\in B_X : j\in\mathbb{N}\}$, such that $\overline{S}=B_X$. Now, we will define a sequence of functions $\{f_m : X\to\mathbb{R}\}_{m=1}^{\infty}$ by

$$f_m(x)=\int_{\mathbb{R}^m}\Big\|x-\sum_{j=1}^m t_jx_j\Big\|\prod_{j=1}^m\phi_j(t_j)dt_1\dots dt_m,\quad x\in X. \tag{4.1}$$

Note that the functions f_m are well defined since the integrand is continuous on $\mathbb{R}^m$ and zero outside the compact space

$$\left[-\frac{1}{4},\frac{1}{4}\right]\times\left[-\frac{1}{8},\frac{1}{8}\right]\times\dots\times\left[-\frac{1}{2^{m+1}},\frac{1}{2^{m+1}}\right].$$

It is easy to see that

$$|f_m(x)-f_m(y)|\leq\|x-y\| \tag{4.2}$$

for all $m \in \mathbb{N}, x, y \in X$. Hence f_m are 1-Lipschitzian functions for each $m \in \mathbb{N}$. Now, assume $n < m$ and $x \in X$. Then, using the fact that $\int \phi_j = 1$, we get

$$
\begin{aligned}
|f_n(x) - f_m(x)| &\leq \int_{\mathbb{R}^m} \Big\| \sum_{j=n+1}^{m} t_j x_j \Big\| \prod_{j=1}^{m} \phi_j(t_j) dt_1 \dots dt_m \\
&\leq \sum_{j=n+1}^{m} 2^{-j-1} \to 0,
\end{aligned}
$$

as $n \to \infty$ and $m \to \infty$, uniformly for $x \in X$. Thus, we can put $f(x) := \lim_{m\to\infty} f_m(x)$, $x \in X$. It is easy to see that f is convex, 1-Lipschitzian, and

$$
\|x\| - \frac{1}{2} \leq f(x) \leq \|x\| + \frac{1}{2} \tag{4.3}
$$

for all $x \in X$.

STEP 2. The functions f_m are $\mathcal{C}^{(n)}$-smooth on $X \setminus \frac{1}{2}B_X$, and for $k = 1, \dots, n$ we have

$$
\begin{aligned}
&f_m^{(k)}(x)(h_1, \dots, h_k) \\
&\quad = \int_{\mathbb{R}^m} \| \cdot \|^{(k)}(x - \sum_{j=1}^{m} t_j x_j)(h_1, \dots, h_k) \prod_{j=1}^{m} \phi_j(t_j) dt_1 \dots dt_m
\end{aligned} \tag{4.4}
$$

where $x \in X \setminus \frac{1}{2}B_X$ and $h_1, \dots, h_k \in X$.

For $k = 0$ the claim is trivial. Assume that the claim was verified for $k-1$ where $k \in \{1, \dots, n\}$. Fix any $x \in X \setminus \frac{1}{2}B_X$ and any $\epsilon > 0$. Since the k-th derivative $\| \cdot \|^{(k)}$ is Lipschitzian, there is $\delta \in (0, \frac{1}{2}\|x\| - \frac{1}{4})$ such that

$$
\Big\| \| \cdot \|^{(k)}(z - \sum_{j=1}^{m} t_j x_j) - \| \cdot \|^{(k)}(x - \sum_{j=1}^{m} t_j x_j) \Big\| < \varepsilon \tag{4.5}
$$

whenever $z \in X$, $\|z - x\| < 2\delta$, $m \in \mathbb{N}$ and $t_j \in \left[-\frac{1}{2^{j+1}}, \frac{1}{2^{j+1}}\right]$, $j = 1, \dots, m$. Then, if $0 \neq \tau \in (-\delta, \delta)$, $y \in X$, $\|y - x\| < \delta$, $h_1, \dots, h_k \in B_X$, $m \in \mathbb{N}$, using

(2.2), (4.4), (4.5), and the Newton-Leibniz formula, we have

$$\begin{aligned}&\Big|\frac{1}{\tau}[f_m^{(k-1)}(y+\tau h_k)(h_1,\dots,h_{k-1}) - f_m^{(k-1)}(y)(h_1,\dots,h_{k-1})]\\&\qquad - \int_{\mathbb{R}^m} \|\cdot\|^{(k)}(x-\sum_{j=1}^m t_jx_j)(h_1,\dots,h_k)\prod_{j=1}^m \phi_j(t_j)dt_1\dots dt_m\Big|\\&\quad = \Big|\int_{\mathbb{R}^m}\Big\{\frac{1}{\tau}[\|\cdot\|^{(k-1)}(y+\tau h_k-\sum_{j=1}^m t_jx_j)(h_1,\dots,h_{k-1})\\&\qquad - \|\cdot\|^{(k-1)}(y-\sum_{j=1}^m t_jx_j)(h_1,\dots,h_{k-1})]\\&\qquad - \|\cdot\|^{(k)}(x-\sum_{j=1}^m t_jx_j)(h_1,\dots,h_k)\Big\}\prod_{j=1}^m \phi_j(t_j)dt_1\dots dt_m\Big|\\&\quad = \Big|\int_{\mathbb{R}^m}\int_0^1 [\|\cdot\|^{(k)}(y+\theta\tau h_k-\sum_{j=1}^m t_jx_j)(h_1,\dots,h_k)\\&\qquad - \|\cdot\|^{(k)}(x-\sum_{j=1}^m t_jx_j)(h_1,\dots,h_k)]d\theta\prod_{j=1}^m \phi_j(t_j)dt_1\dots dt_m\Big| < \varepsilon.\end{aligned}$$

Thus, by (2.2) and by the induction assumption, f_m is $\mathcal{C}^{(n)}$-smooth on $X\setminus\frac{1}{2}B_X$ for every $n \in \mathbb{N}$.

STEP 3. The function f defined in STEP 1 is $\mathcal{C}^{(n)}$-smooth on $X \setminus \frac{1}{2}B_X$ and

$$\|f_m^{(k)}(x) - f^{(k)}(x)\| \to 0 \quad \text{as} \quad m\to\infty,$$

for $k = 0,\dots,n$ and $x \in X \setminus \frac{1}{2}B_X$.

The claim for $k = 0$ was proved in STEP 1. Let $k \in \{1,\dots,n\}$ and assume that the claim was verified for $k-1$. Fix $x \in X\setminus\frac{1}{2}B_X$. Since $\|\cdot\|^{(k)}$ is Lipschitz,

$$\Big\|\|\cdot\|^{(k)}(x-\sum_{j=1}^m t_jx_j) - \|\cdot\|^{(k)}(x-\sum_{j=1}^n t_jx_j)\Big\| \to 0$$

as $m,n \to \infty$ uniformly for $\{t_j\}_{j=1}^\infty$ where $t_j \in \big[-\frac{1}{2^{j+1}}, \frac{1}{2^{j+1}}\big]$. Then, using STEP 2, we get that

$$\|f_m^{(k)}(x) - f_n^{(k)}(x)\| \to 0 \quad \text{as} \quad m,n\to\infty.$$

Thus, when we put $L_k = \lim_{m\to\infty} f_m^{(k)}(x)$, then L_k belongs to $\mathcal{L}^{(k)}(X)$. Fix $\varepsilon > 0$ and let δ be that chosen in STEP 2. Take $0 \neq \tau \in (-\delta, \delta)$, $h_1, \dots, h_k \in B_X$, and $y \in X$ such that $\|y - x\| < \delta$. Then

$$\left| \frac{1}{\tau}[f_m^{(k-1)}(y + \tau h_k)(h_1, \dots, h_{k-1}) - f_m^{(k-1)}(y)(h_1, \dots, h_{k-1})] \right. \\ \left. - f_m^{(k)}(x)(h_1, \dots, h_k) \right| < \varepsilon$$

for all $m \in \mathbb{N}$. Now, let $m \to \infty$. By our induction assumption, we get

$$\left| \frac{1}{\tau}[f^{(k-1)}(y + \tau h_k)(h_1, \dots, h_{k-1}) - f^{(k-1)}(y)(h_1, \dots, h_{k-1})] \right. \\ \left. - L_k(h_1, \dots, h_k) \right| \leq \varepsilon.$$

By (2.2), f is $\mathcal{C}^{(k)}$-smooth, $f^{(k)}(x) = L_k$ and $\|f_m^{(k)}(x) - f^{(k)}(x)\| \to 0$ as $m \to \infty$ whenever $x \in X \setminus \frac{1}{2}B_X$.

STEP 4. Let $x \in X \setminus \frac{1}{2}B_X$, $h_1, \dots, h_n \in X$, $i \in \mathbb{N}$, $m \geq i$. Then the directional derivative $D_{x_i} f_m^{(n)}(x)(h_1, \dots, h_n)$ exists.

Indeed, by STEP 2, we can calculate

$$\begin{aligned} &D_{x_i} f_m^{(n)}(x)(h_1, \dots, h_n) \\ &\quad = \lim_{\tau\to 0} \frac{1}{\tau}\left[f_m^{(n)}(x + \tau x_i)(h_1, \dots, h_n) - f_m^{(n)}(x)(h_1, \dots, h_n)\right] \\ &\quad = \lim_{\tau\to 0} \frac{1}{\tau}\left[\int_{\mathbb{R}^m} \|\cdot\|^{(n)}(x + \tau x_i - \sum_{j=1}^m t_j x_j)(h_1, \dots, h_n) \prod_{j=1}^m \phi_j(t_j) dt_1 \dots dt_m \right. \\ &\qquad \left. - \int_{\mathbb{R}^m} \|\cdot\|^{(n)}(x - \sum_{j=1}^m t_j x_j)(h_1, \dots, h_n) \prod_{j=1}^m \phi_j(t_j) dt_1 \dots dt_m \right] \\ &\quad = \lim_{\tau\to 0} \int_{\mathbb{R}^m} \|\cdot\|^{(n)}(x - \sum_{j=1}^m t_j x_j)(h_1, \dots, h_n) \\ &\qquad\qquad \prod_{\substack{j=1 \\ j\neq i}}^m \phi_j(t_j) \frac{\phi_i(t_i + \tau) - \phi_i(t_i)}{\tau} dt_1 \dots dt_m \\ &\quad = \int_{\mathbb{R}^m} \|\cdot\|^{(n)}(x - \sum_{j=1}^m t_j x_j)(h_1, \dots, h_n) \prod_{\substack{j=1 \\ j\neq i}}^m \phi_j(t_j) \phi_i'(t_i) dt_1 \dots dt_m. \end{aligned}$$

Here we used the substitution $t_i - \tau \to t_i$, the fact that ϕ_0 is $\mathcal{C}^{(1)}$-smooth, and the Lebesgue's dominated convergence theorem. Thus, we have

$$
\begin{aligned}
&D_{x_i} f_m^{(n)}(x)(h_1, \dots, h_n) \\
&= \int_{\mathbb{R}^m} \|\cdot\|^{(n)}(x - \sum_{j=1}^{m} t_j x_j)(h_1, \dots, h_n) \prod_{\substack{j=1 \\ j \neq i}}^{m} \phi_j(t_j)\phi_i'(t_i)dt_1 \dots dt_m.
\end{aligned} \tag{4.6}
$$

STEP 5. The directional derivative $D_{x_i} f^{(n)}(x)(h_1, \dots, h_n)$ exists for all $x \in X \setminus \frac{1}{2}B_X$, $i \in \mathbb{N}$, and $h_1, \dots, h_n \in X$.

Fix $x \in X \setminus \frac{1}{2}B_X$, $i \in \mathbb{N}$ and $h_1, \dots, h_n \in X$. Let Ω denote a suitable annulus around 0 and containing the point x. Since $\|\cdot\|^{(n)}$ is Lipschitzian on Ω (see Proposition 3.10), there is $\delta > 0$ such that

$$
\left\| \|\cdot\|^{(n)}(x + \tau x_i - \sum_{j=1}^{m_1} t_j x_j) - \|\cdot\|^{(n)}(x + \tau x_i - \sum_{j=1}^{m_2} t_j x_j) \right\| \to 0 \tag{4.7}
$$

as $m_1, m_2 \to \infty$ uniformly for $\tau \in (-\delta, \delta)$, and

$$
(t_1, \dots, t_m, \dots) \in \left[-\frac{1}{4}, \frac{1}{4}\right] \times \dots \times \left[-\frac{1}{2^{m+1}}, \frac{1}{2^{m+1}}\right] \times \cdots
$$

Define the functions

$$
\varphi(\tau) = f^{(n)}(x + \tau x_i)(h_1, \dots, h_n)
$$

and

$$
\varphi_m(\tau) = f_m^{(n)}(x + \tau x_i)(h_1, \dots, h_n)
$$

where $\tau \in (-\delta, \delta)$. We calculate

$$
\varphi_m'(\tau) = D_{x_i} f_m^{(n)}(x + \tau x_i)(h_1, \dots, h_n), \quad \tau \in (-\delta, \delta).
$$

By (4.6) and (4.7), we have that

$$
|\varphi_{m_1}'(\tau) - \varphi_{m_2}'(\tau)| \longrightarrow 0 \quad \text{as} \quad m_1, m_2 \to \infty
$$

uniformly for $\tau \in (-\delta, \delta)$. Moreover, $\varphi_m(0) \to \varphi(0)$ as $m \to \infty$ by STEP 3. Thus, according to a well known theorem from mathematical analysis, we know that for all $\tau \in (-\delta, \delta)$ the derivative $\varphi'(\tau)$ exists and

$$
\varphi'(\tau) = \lim_{m \to \infty} \varphi_m'(\tau), \quad \tau \in (-\delta, \delta).
$$

In particular

$$\begin{aligned} D_{x_i} f^{(n)}(y)(h_1, \dots, h_n) &= \varphi'(0) = \lim_{m\to\infty} \varphi'_m(0) \\ &= \lim_{m\to\infty} D_{x_i} f^{(n)}_m(y)(h_1, \dots, h_n). \end{aligned} \tag{4.8}$$

STEP 6. The directional derivative $D_{h_{n+1}} f^{(n)}(x)(h_1, \dots, h_n)$ exists for all $x \in X \setminus \frac{1}{2}B_X$ and all $h_1, \dots, h_n, h_{n+1} \in X$.

Let us fix $\varepsilon > 0$, $x \in X \setminus \frac{1}{2}B_X$, $h_1, \dots, h_n, h_{n+1} \in B_X$. Since the set S is dense in the unit ball, we can find a sequence $\{z_j\}_{j=1}^{\infty} \subset S$ such that $z_j \to h_{n+1}$ as $j \to \infty$. Let $C > 0$ be a Lipschitz constant of $f^{(n)}$ with respect to some suitable annulus Ω around 0, containing x. Note that, by Proposition 3.10, $f \in \mathcal{LIP}^{(n)}(\Omega)$. Find $j \in \mathbb{N}$ such that

$$\|h_{n+1} - z_j\| \le \frac{\varepsilon}{2C}.$$

Using this and the Lipschitzness of $f^{(n)}$, we have

$$\begin{aligned} \limsup_{\tau,\tau'\to 0} \Big| &\frac{1}{\tau}\big[f^{(n)}(x+\tau h_{n+1})(h_1,\dots,h_n) - f^{(n)}(x)(h_1,\dots,h_n)\big] \\ &- \frac{1}{\tau'}\big[f^{(n)}(x+\tau' h_{n+1})(h_1,\dots,h_n) - f^{(n)}(x)(h_1,\dots,h_n)\big]\Big| \\ \le \lim_{\tau,\tau'\to 0} \Big| &\frac{1}{\tau}\big[f^{(n)}(x+\tau z_j)(h_1,\dots,h_n) - f^{(n)}(x)(h_1,\dots,h_n)\big] \\ &- \frac{1}{\tau'}\big[f^{(n)}(x+\tau' z_j)(h_1,\dots,h_n) - f^{(n)}(x)(h_1,\dots,h_n)\big]\Big| \\ &+ 2C\|h_{n+1} - z_j\| = 2C\|h_{n+1} - z_j\| \le \varepsilon. \end{aligned}$$

Hence $D_{h_{n+1}} f^{(n)}(x)(h_1, \dots, h_n)$ exists. Moreover, we have

$$D_{h_{n+1}} f^{(n)}(x)(h_1, \dots, h_n) = \lim_{j\to\infty} D_{z_j} f^{(n)}(x)(h_1, \dots, h_n) \tag{4.9}$$

for all $x \in X \setminus \frac{1}{2}B_X$, $h_1, \dots, h_n, h_{n+1} \in X$. Also, the limit (4.9) exists uniformly for $h_1, \dots, h_n$ from any bounded subset of X and locally uniformly in the sense that for each $x \in X \setminus \frac{1}{2}B_X$ and any annulus Ω around 0 such that $\overline{\Omega} \subset X \setminus \frac{1}{2}B_X$, $x \in \Omega$, the limit (4.9) exists uniformly with respect to Ω. Indeed, if $y \in \Omega,\ r > 0$, and $h_1, \dots, h_n \in rB_X$, then

$$\begin{aligned}
&|D_{h_{n+1}} f^{(n)}(y)(h_1, \dots, h_n) - D_{z_j} f^{(n)}(y)(h_1, \dots, h_n)| \\
&\qquad = \lim_{\tau \to 0} \Big| \frac{1}{\tau}[f^{(n)}(y + \tau h_{n+1})(h_1, \dots, h_n) - f^{(n)}(y)(h_1, \dots, h_n)] \\
&\qquad\quad - \frac{1}{\tau}[f^{(n)}(y + \tau z_j)(h_1, \dots, h_n) - f^{(n)}(y)(h_1, \dots, h_n)] \Big| \\
&\qquad \le r^n \mathrm{Lip}_\Omega(f^{(n)}) \|h_{n+1} - z_j\|.
\end{aligned}$$

STEP 7. For any fixed $i \in \mathbb{N}$ and any $h_1, \dots, h_n \in X$ the mapping $x \mapsto D_{x_i} f^{(n)}(x)(h_1, \dots, h_n)$ is Lipschitzian on any annulus Ω around 0 such that $\overline{\Omega} \subset X \setminus \frac{1}{2}B_X$.

Fix $\alpha, \beta \in \mathbb{R}$, $\alpha > \beta > \frac{1}{2}$, $h_1, \dots, h_n \in X$, and $= i \in \mathbb{N}$. Let $C > 0$ be a Lipschitz constant of the n-th derivative $\|\cdot\|^{(n)}$ with respect to the set $(\alpha + \frac{1}{2})B_X \setminus (\beta - \frac{1}{2})B_X$, see Proposition 3.10. If $x, y \in \alpha B_X \setminus \beta B_X$, then we have

$$\begin{aligned}
&\big|D_{x_i} f^{(n)}(x)(h_1, \dots, h_n) - D_{x_i} f^{(n)}(z)(h_1, \dots, h_n)\big| \\
&\qquad = \lim_{m \to \infty} \Big| \int_{\mathbb{R}^m} \Big[\|\cdot\|^{(n)}(x - \sum_{j=1}^m t_j x_j)(h_1, \dots, h_n) \\
&\qquad\quad - \|\cdot\|^{(n)}(y - \sum_{j=1}^m t_j x_j)(h_1, \dots, h_n)\Big] \prod_{\substack{j=1 \\ j \ne i}}^m \phi_j(t_j) \phi_i'(t_i) dt_1 \dots dt_m \Big| \\
&\qquad \le C 2^{2i} \int_{\mathbb{R}} |\phi_0'(s)| ds\ \|x - y\|\ \|h_1\| \dots \|h_n\|.
\end{aligned}$$

Thus

$$\begin{aligned}
&\big|D_{x_i} f^{(n)}(x)(h_1, \dots, h_n) - D_{x_i} f^{(n)}(y)(h_1, \dots, h_n)\big| \\
&\qquad \le C 2^{2i} \int_{\mathbb{R}} |\phi_0'(s)| ds\ \|x - y\|\ \|h_1\| \dots \|h_n\|.
\end{aligned} \tag{4.10}$$

STEP 8. For any $h_1, \dots, h_n, h_{n+1} \in X$ the function

$$x \mapsto D_{h_{n+1}} f^{(n)}(x)(h_1, \dots, h_n)$$

is uniformly continuous on an arbitrary annulus Ω around 0 such that $\overline{\Omega} \subset X \setminus \frac{1}{2}B_X$, (even uniformly with respect to $h_1, \dots, h_n \in B_X$).

Fix such an annulus Ω, $\varepsilon > 0$, and $h_{n+1} \in B_X$. As in STEP 6, we can find a sequence $\{z_j\}_{j=1}^{\infty} \subset S$ such that $z_j \to h_{n+1}$. By (4.9), there is $k \in \mathbb{N}$ such that

$$\left|D_{h_{n+1}} f^{(n)}(x)(h_1, \dots, h_n) - D_{z_k} f^{(n)}(x)(h_1, \dots, h_n)\right| < \varepsilon/3$$

whenever $x \in \Omega$ and $h_1, \dots, h_n \in B_X$. Further, by STEP 7, there is $\delta > 0$ such that

$$\left|D_{z_k} f^{(n)}(x)(h_1, \dots, h_n) - D_{z_k} f^{(n)}(y)(h_1, \dots, h_n)\right| < \varepsilon/3$$

whenever $x, y \in \Omega$, $\|x - y\| < \delta$, and $h_1, \dots, h_n \in B_X$. Thus, if $x, y \in \Omega$, $\|x - y\| < \delta$, $h_1, \dots, h_n \in B_X$, then

$$\begin{aligned}
&\left|D_{h_{n+1}} f^{(n)}(x)(h_1, \dots, h_n) - D_{h_{n+1}} f^{(n)}(y)(h_1, \dots, h_n)\right| \\
&\qquad \le \left|D_{h_{n+1}} f^{(n)}(x)(h_1, \dots, h_n) - D_{z_k} f^{(n)}(x)(h_1, \dots, h_n)\right| \\
&\qquad\quad + \left|D_{z_k} f^{(n)}(x)(h_1, \dots, h_n) - D_{z_k} f^{(n)}(y)(h_1, \dots, h_n)\right| \\
&\qquad\quad + \left|D_{z_k} f^{(n)}(y)(h_1, \dots, h_n) - D_{h_{n+1}} f^{(n)}(y)(h_1, \dots, h_n)\right| \\
&\qquad < 3 \cdot \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

STEP 9. f is $n+1$ times Gâteaux differentiable on $X \setminus \frac{1}{2}B_X$. Moreover $f \in \mathcal{UG}^{(n+1)}(\Omega)$ for any annulus Ω around 0 such that $\overline{\Omega} \subset X \setminus \frac{1}{2}B_X$.

From the assumptions of our Theorem, using Proposition 3.10, and the construction of f and its derivatives, we get that $f \in \mathcal{LIP}^{(n)}(\Omega)$ on any annulus Ω around 0 and such that $\overline{\Omega} \subset X \setminus \frac{1}{2}B_X$. Then, by Lemma 3.2 (ii), $f \in \mathcal{UG}^{(n)}(\Omega)$ for any annulus Ω, with $\overline{\Omega} \subset X \setminus \frac{1}{2}B_X$. If $x \in X \setminus \frac{1}{2}B_X$, then by STEP 6 and 8, and by Lemma 3.1, the mapping $(h_1, \dots, h_n, h_{n+1}) \mapsto D_{h_{n+1}} f^{(n)}(x)(h_1, \dots, h_n)$ is bounded, $(n+1)$-linear, and symmetric. Hence f is $n+1$ times Gâteaux differentiable on $X \setminus \frac{1}{2}B_X$. Now, it remains to show that $f \in \mathcal{UG}^{(n+1)}(\Omega)$, i.e., that$f^{(n)} \in \mathcal{UG}^{(1)}(\Omega)$ for any annulus Ω around 0 and such that $\overline{\Omega} \subset X \setminus \frac{1}{2}B_X$. Fix such an annulus Ω, $\varepsilon > 0$, and $k \in B_X$. By STEP 8, for sufficiently small $\delta > 0$, we have

$$\sup_{h_1, \dots, h_n \in B_X} \left|D_k f^{(n)}(x + \tau k)(h_1, \dots, h_n) - D_k f^{(n)}(x)(h_1, \dots, h_n)\right| < \varepsilon$$

whenever $x \in \Omega$ and $|\tau| < \delta$. Then, using the mean value theorem, we obtain

$$\begin{aligned}
&\big\|f^{(n)}(x+\tau k) - f^{(n)}(x) - D_{\tau k} f^{(n)}(x)(\cdot,\dots,\cdot)\big\| \\
&\quad = \sup_{h_1,\dots,h_n \in B_X} \big|f^{(n)}(x+\tau k)(h_1,\dots,h_n) - f^{(n)}(x)(h_1,\dots,h_n) \\
&\qquad\qquad - D_{\tau k} f^{(n)}(x)(h_1,\dots,h_n)\big| \\
&\quad \le |\tau| \sup_{\substack{h_1,\dots,h_n \in B_X \\ \theta\in[0,1]}} \big|D_k f^{(n)}(x+\theta\tau k)(h_1,\dots,h_n) - D_k f^{(n)}(x)(h_1,\dots,h_n)\big| \\
&\quad < |\tau|\varepsilon,
\end{aligned}$$

whenever $x \in \Omega$ and $|\tau| < \delta$. Therefore $f \in \mathcal{UG}^{(n+1)}(\Omega)$.

STEP 10. Using the construction from [4, Theorem 1, Step 6] we "improve" the function f by defining a new function $g : X \to \mathbb{R}$ as follows

$$g(x) = \int_{\mathbb{R}} f(sx)\eta(s)ds, \quad x \in X, \tag{4.11}$$

where $\eta : \mathbb{R} \to [0,\infty)$ is a fixed $\mathcal{C}^{(\infty)}$-smooth function, with support in $[1,2]$, and such that $\int_{\mathbb{R}} \eta = 1$. It is clear that g is well defined on all of X, that it is convex, 2-Lipschitzian, and that it satisfies (see (4.3))

$$\|x\| - \frac{1}{2} \le g(x) \le 2\|x\| + \frac{1}{2}, \quad \text{for all} \quad x \in X. \tag{4.12}$$

STEP 11. Similarly, as in STEP 2, it can be shown that g is a $\mathcal{C}^{(n)}$-smooth function on $X \setminus \frac{1}{2}B_X$. Moreover

$$g^{(n)}(x)(h_1,\dots,h_n) = \int_{\mathbb{R}} f^{(n)}(sx)(h_1,\dots,h_n)s^n\eta(s)ds, \tag{4.13}$$

for all $x \in X \setminus \frac{1}{2}B_X$, $h_1,\dots,h_n \in X$. From (4.13) it even follows that $g \in \mathcal{LIP}^{(n)}(\Omega)$ for any annulus Ω around 0 such that $\overline{\Omega} \subset X \setminus \frac{1}{2}B_X$. See also [4, Theorem 1, Step 12].

STEP 12. We claim that:

(i) For $x \in X \setminus \frac{1}{2}B_X$, $h_1,\dots,h_n,h_{n+1} \in X$ the directional derivative $D_{h_{n+1}} g^{(n)}(x)(h_1,\dots,h_n)$ exists and

$$\begin{aligned}
&D_{h_{n+1}} g^{(n)}(x)(h_1,\dots,h_n) \\
&\qquad = \int_{\mathbb{R}} D_{h_{n+1}} f^{(n)}(sx)(h_1,\dots,h_n)s^{n+1}\eta(s)ds.
\end{aligned} \tag{4.14}$$

(ii) The mapping $X^{n+1} \ni (h_1, \dots, h_n, h_{n+1}) \mapsto D_{h_{n+1}} g^{(n)}(x)(h_1, \dots, h_n)$ is bounded, $(n+1)$-linear, and symmetric for every $x \in X \setminus \frac{1}{2}B_X$.

(iii) The function g is $n+1$ times Gâteaux differentiable on $X \setminus \frac{1}{2}B_X$ and moreover $g \in \mathcal{UG}^{(n+1)}(\Omega)$ for any annulus Ω around 0 such that $\overline{\Omega} \subset X \setminus \frac{1}{2}B_X$.

(i) The integral in (4.14) exists by STEP 8. Fix $x \in X \setminus \frac{1}{2}B_X$, $h_1, \dots, h_n$, $h_{n+1} \in X$ and $\varepsilon > 0$. Then, using STEP 8, we can find $\delta > 0$ and an annulus Ω around 0 such that $\overline{\Omega} \subset X \setminus \frac{1}{2}B_X$, $[1,2]x + 2\delta B_X \subset \Omega$, and

$$\sup_{\theta \in [0,1]} \big| D_{h_{n+1}} f^{(n)}(sx + \theta\tau s h_{n+1})(h_1, \dots, h_n) - D_{h_{n+1}} f^{(n)}(sx)(h_1, \dots, h_n) \big| < \varepsilon / 2^{n+1}$$

whenever $0 \neq |\tau| < \delta$ and $s \in [1,2]$. Suppose that $0 \neq |\tau| < \delta$, then, using the Newton-Leibniz integral formula, we have

$$\begin{aligned}
&\left| \frac{1}{\tau} [g^{(n)}(x + \tau h_{n+1})(h_1, \dots, h_n) - g^{(n)}(x)(h_1, \dots, h_n)] \right. \\
&\qquad \left. - \int_{\mathbb{R}} D_{h_{n+1}} f^{(n)}(sx)(h_1, \dots, h_n) s^{n+1} \eta(s) ds \right| \\
&= \left| \int_{\mathbb{R}} \frac{1}{\tau} [f^{(n)}(sx + \tau s h_{n+1})(h_1, \dots, h_n) - f^{(n)}(sx)(h_1, \dots, h_n)] s^n \eta(s) ds \right. \\
&\qquad \left. - \int_{\mathbb{R}} D_{h_{n+1}} f^{(n)}(sx)(h_1, \dots, h_n) s^{n+1} \eta(s) ds \right| \\
&\leq \int_{\mathbb{R}} \int_0^1 \big| D_{h_{n+1}} f^{(n)}(sx + \theta\tau s h_{n+1})(h_1, \dots, h_n) \\
&\qquad - D_{h_{n+1}} f^{(n)}(sx)(h_1, \dots, h_n) \big| d\theta s^{n+1} \eta(s) ds \\
&< \int_{\mathbb{R}} (\varepsilon / 2^{n+1}) s^{n+1} \eta(s) ds \leq \varepsilon.
\end{aligned}$$

This proves (i). (ii) follows directly from (i) and because we already know that the mapping $(h_1, \dots, h_n, h_{n+1}) \mapsto D_{h_{n+1}} f^{(n)}(x)(h_1, \dots, h_n)$ is bounded, $(n+1)$-linear, and symmetric for all $x \in X \setminus \frac{1}{2}B_X$. (iii) can be shown similarly as in STEP 9, using this step and (4.14).

STEP 13. Consider the set

$$U = \{x \in X : g(x) \leq 7\}$$

From the properties of g it directly follows that U is convex, closed, and bounded. Since $g(0) \leq 1/2$ (see (4.12)), the interior of U contains 0. Let p denote Minkowski functional of U. It is easy to check that p satisfies all the properties of an equivalent norm on X but the symmetry. Thus, we can find $a, b > 0$ such that

$$a\|x\| \leq p(x) \leq b\|x\|, \quad x \in X. \tag{4.15}$$

STEP 14. The functional p is $\mathcal{C}^{(n)}$-smooth on $X \setminus \{0\}$ and

$$p'(x) = \left[g'\left(\frac{x}{p(x)}\right)\left(\frac{x}{p(x)}\right)\right]^{-1} g'\left(\frac{x}{p(x)}\right), \quad x \in X \setminus \{0\}, \tag{4.16}$$

Fix any $x \in X$ such that $g(x) = 7$. First, let us verify that $g'(x)(x) \neq 0$. Define the function $\sigma : \mathbb{R} \to \mathbb{R}$ by $\sigma(\tau) = g(\tau x)$, $\tau \in \mathbb{R}$. It is easy to see that σ is a convex function, and $\sigma'(1) = g'(x)(x)$. Since $\sigma(0) = g(0) < \sigma(1)$, from the convexity of σ it follows that $0 \neq \sigma'(1) = g'(x)(x)$. Now, from the implicit function theorem we can derive (4.16). This formula together with STEP 11 and [1, Theorem 4.7.1. or Corollary 5.4.5.], guarantees that p is $\mathcal{C}^{(n)}$-smooth on $X \setminus \{0\}$.

STEP 15. If $\|\cdot\| \in \mathcal{LIP}^{(n)}(S_X)$, then $p \in \mathcal{UG}^{(n+1)}(\Omega) \cap \mathcal{LIP}^{(n)}(\Omega)$ whenever $n \in \{0\} \cup \mathbb{N}$, $\Omega = \{x : \alpha < p(x) < \beta\}$, $0 < \alpha < \beta$.

The case $n = 0$ was proved in [4]. Let $n \geq 1$. Assume that our claim is true for $n - 1$, and assume that $\|\cdot\| \in \mathcal{LIP}^{(n)}(S_X)$. Define the auxillary mappings

$$\alpha(t) = \frac{1}{t}, \quad 0 \neq t \in \mathbb{R}; \qquad \beta(t, x) = t \cdot x, \quad t \in \mathbb{R}, \quad x \in X;$$

$$\gamma(x) = ((\alpha \circ p)(x), x), \quad 0 \neq x \in X; \qquad \delta(t, \xi) = t \cdot \xi, \quad t \in \mathbb{R}, \quad \xi \in x^*;$$

$$r(x) = \frac{x}{p(x)}, \quad 0 \neq x \in X; \qquad \psi(x) = g'(x)(x), \quad x \in X.$$

Integration by parts gives

$$\psi(x) = g'(x)(x) = -\int_{\mathbb{R}} f(sx)[s\eta'(s)]'ds, \quad x \in X.$$

Thus $\psi \in \mathcal{LIP}^{(n)}(\Omega)$ for any annulus Ω around 0 such that $\overline{\Omega} \subset X \setminus \frac{1}{2}B_X$. Further it is easy to see that $r(X \setminus \{0\}) \subset 8B_X \setminus 3B_X$ and that $\psi \geq \frac{13}{2}$ on the annulus $8B_X \setminus 3B_X$.

Since $\|\cdot\| \in \mathcal{LIP}^{(n-1)}(S_X)$, then by our induction assumption we have that $p \in \mathcal{UG}^{(n)}(\Omega) \cap \mathcal{LIP}^{(n-1)}(\Omega)$ whenever $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta$. As $\alpha \in \mathcal{LIP}^{(\infty)}(\Omega)$ on any closed bounded set $\Omega \subset (0, +\infty)$, by Lemma 3.6 we have $\alpha \circ p \in \mathcal{UG}^{(n)}(\Omega) \cap \mathcal{LIP}^{(n-1)}(\Omega)$ whenever $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta$. Thus $\gamma \in \mathcal{UG}^{(n)}(\Omega) \cap \mathcal{LIP}^{(n-1)}(\Omega)$ whenever $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta$. Now, as $r = \beta \circ \gamma$ and $\beta \in \mathcal{LIP}^{(\infty)}(\Omega')$ whenever $\Omega' \subset \mathbb{R} \times X$ is bounded, by Lemma 3.6 we have $r \in \mathcal{UG}^{(n)}(\Omega) \cap \mathcal{LIP}^{(n-1)}(\Omega)$ whenever $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta$. Again, by Lemma 3.10, $\alpha \circ \psi \circ r \in \mathcal{UG}^{(n)}(\Omega) \cap \mathcal{LIP}^{(n-1)}(\Omega)$ whenever $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta$. Finally, we define $\omega(x) = \big((\alpha \circ \psi \circ r)(x), (g' \circ r)(x)\big)$, $0 \neq x \in X$. Then we can write $p' = \delta \circ \omega$.

Now, it remains to show that $g' \circ r \in \mathcal{UG}^{(n)}(\Omega) \cap \mathcal{LIP}^{(n-1)}(\Omega)$, whenever $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta$. Once having this shown, then Lemma 3.6 guarantees that $p' \in \mathcal{UG}^{(n)}(\Omega) \cap \mathcal{LIP}^{(n-1)}(\Omega)$ whenever $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta$ as $\delta \in \mathcal{LIP}^{(\infty)}(\Omega')$ for every bounded set $\Omega' \subset \mathbb{R} \times X^*$. Thus, by (2.1), $p \in \mathcal{UG}^{(n+1)}(\Omega) \cap \mathcal{LIP}^{(n)}(\Omega)$.

To prove this, let us define

$$\mathcal{J}_k(x) := g^{(k)}\left(\frac{x}{p(x)}\right), \quad 0 \neq x \in X, \quad k = 1, 2, \ldots, n.$$

We will now proceed by an induction in descending order. First put $k = n$ and let us show that $\mathcal{J}_n \in \mathcal{UG}^{(1)}(\Omega) \cap \mathcal{LIP}^{(0)}(\Omega)$ whenever $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta$. Obviously $\mathcal{J}_n = g^{(n)} \circ r \in \mathcal{G}^{(1)}(\Omega)$ whenever $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta$. Let us show that for any fixed $h \in X$ $x \mapsto \mathcal{J}_n'(x)(h)$ is uniformly continuous on Ω whenever $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta$. Using Lemma 3.4, we calculate

$$\begin{aligned}
\mathcal{J}_n &= g^{(n+1)}(r(x))(r'(x)(h)) \\
&= g^{(n+1)}\left(\frac{x}{p(x)}\right)\left(\frac{p(x)h - (p'(x)(h))x}{p^2(x)}\right) \\
&= \frac{1}{p(x)} g^{(n+1)}\left(\frac{x}{p(x)}\right)(h) - \frac{p'(x)(h)}{p(x)} g^{(n+1)}\left(\frac{x}{p(x)}\right)\left(\frac{x}{p(x)}\right).
\end{aligned}$$

Since $g^{(n)} \in \mathcal{UG}^{(1)}(\Omega') \cap \mathcal{LIP}^{(0)}(\Omega')$ whenever Ω' is any annulus around 0 such that $\overline{\Omega'} \subset X \setminus \frac{1}{2}B_X$, by Lemma 3.6 for each fixed $h \in X$ we have $\|g^{(n+1)}(x)(h) - g^{(n+1)}(z)(h)\| \to 0$ as $x, z \in 8B_X \setminus 3B_X$ and $\|x - z\| \to 0$. If

$x, z \in \Omega$, then we have

$$\|r(x) - r(z)\| = \left\| \frac{x}{p(x)} - \frac{z}{p(z)} \right\| \leq \frac{\|x - z\|}{p(x)} + \left\| \frac{z}{p(z)} \right\| \cdot \frac{|p(z) - p(x)|}{p(x)}$$
$$\leq \frac{1}{\alpha}\|x - z\| + \frac{b}{a\,\alpha}\|x - z\|.$$

Using this and the fact that $\sup\{\|g^{(n+1)}(x)\| : x \in 8B_X \setminus 3B_X\} < +\infty$, resulting from the Lipschitzness of $g^{(n)}$, we get

$$\left\| \frac{1}{p(x)} g^{(n+1)}\left(\frac{x}{p(x)}\right)(h) - \frac{1}{p(z)} g^{(n+1)}\left(\frac{z}{p(z)}\right)(h) \right\| \to 0$$

whenever $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta,\ x, z \in \Omega,\ \|x - z\| \to 0$.

By STEP 12, using the integration by parts we get for $x \in X \setminus \frac{1}{2}B_X$, $h_1, \dots, h_n \in B_X$

$$g^{(n+1)}(x)(x)(h_1, \dots, h_n) = \int_{\mathbb{R}} f^{(n+1)}(sx)(x)(h_1, \dots, h_n) s^{n+1} \eta(s) ds$$
$$= -\int_{\mathbb{R}} f^{(n)}(sx)(h_1, \dots, h_n)[s^n \eta(s)]' ds.$$

Let $C > 0$ be the Lipschitz constant of $f^{(n)}$ with respect to $16B_X \setminus 3B_X$. Then

$$\|g^{(n+1)}(x)(x) - g^{(n+1)}(z)(z)\| \leq 2C\|x - z\| \int_{\mathbb{R}} |s^n \eta(s)]'| ds, \quad x, z \in 8B_X \setminus 3B_X.$$

Using this, and the fact that $g^{(n)} \in \mathcal{LIP}^{(0)}(8B_X \setminus 3B_X)$, we get

$$\left\| \frac{-p'(x)(h)}{p(x)} g^{(n+1)}\left(\frac{x}{p(x)}\right)\left(\frac{x}{p(x)}\right) - \frac{-p'(z)(h)}{p(z)} g^{(n+1)}\left(\frac{z}{p(z)}\right)\left(\frac{z}{p(z)}\right) \right\| \to 0$$

whenever $x, z \in \Omega$, $\|x - z\| \to 0$, $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta$. Consequently for every fixed $h \in X$ $\|\mathcal{J}_n'(x)(h) - \mathcal{J}_n'(z)(h)\| \to 0$ whenever $x, z \in \Omega$, $\|x - z\| \to 0$, $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta$. By Lemma 3.2, $\mathcal{J}_n \in \mathcal{UG}^{(1)}(\Omega) \cap \mathcal{LIP}^{(0)}(\Omega)$ whenever $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta$. Now, assume that for some $k \in \{n, n-1, \dots, 3, 2\}$ we have already showed that $\mathcal{J}_k \in \mathcal{UG}^{(n-k+1)} \cap \mathcal{LIP}^{(n-k)}(\Omega)$ whenever $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta$. We will show that $\mathcal{J}_{k-1} \in \mathcal{UG}^{(n-k+2)}(\Omega) \cap \mathcal{LIP}^{(n-k+1)}(\Omega)$, whenever $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta$. Since $\mathcal{J}_{k-1} = g^{(k-1)} \circ r$, then by

Lemma 3.4 $\mathcal{J}_{k-1} \in \mathcal{G}^{(1)}(\Omega)$ whenever $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta$. Moreover

$$\mathcal{J}'_{k-1}(x) = [g^{(k)}(r(x))] \circ [r'(x)] = [\mathcal{J}_k(x)] \circ [r'(x)], \quad x \in \Omega.$$

Here $r' \in \mathcal{UG}^{(n-1)}(\Omega) \cap \mathcal{LIP}^{(n-2)}(\Omega) \subset \mathcal{UG}^{(n-k+1)}(\Omega) \cap = \mathcal{LIP}^{(n-k)}(\Omega)$ whenever $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta$. Clearly also $x \mapsto (\mathcal{J}_k(x), r'(x))$ is in $\mathcal{UG}^{(n-k+1)}(\Omega) \cap \mathcal{LIP}^{(n-k)}(\Omega)$ whenever $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta$. Then, using the properties of the composition mapping "o" (used in the proof of Lemma 3.5), we can conclude by Lemma 3.6 that $\mathcal{J}'_{k-1} \in \mathcal{UG}^{(n-k+1)}(\Omega) \cap \mathcal{LIP}^{(n-k)}(\Omega)$ and consequently, by (2.1), $\mathcal{J}_{k-1} \in \mathcal{UG}^{(n-k+2)}(\Omega) \cap \mathcal{LIP}^{(n-k+1)}(\Omega)$ whenever $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta$. Thus, we proved that $g' \circ r$ $(= \mathcal{J}_1) \in \mathcal{UG}^{(n)}(\Omega) \cap \mathcal{LIP}^{(n-1)}(\Omega)$ whenever $\Omega = \{x : \alpha < p(x) < \beta\}$ and $0 < \alpha < \beta$. Finally putting $|||x||| := p(x) + p(-x)$, $x \in X$ we get an equivalent norm satisfying the conclusion of our Theorem. ∎

An inspection of the first nine steps of the proof of Theorem 4.1 yields the following

THEOREM 4.2. *Let X be a separable Banach space such that for some nonnegative integer n there exists an n times differentiable bump on X, with its n-th derivative Lipschitzian. Then X admits an $n+1$ times uniformly Gâteaux differentiable bump whose derivatives of degree less than $n+1$ are Lipschitzian.*

From our Theorem 4.1 we can also derive the following two propositions.

PROPOSITION 4.3. *Let X be a separable Banach space which admits a bounded $\mathcal{UG}^{(2)}$-smooth bump. Then X admits an equivalent norm, $\mathcal{UG}^{(2)}$-smooth on any annulus around 0 and whose first derivative is Lipschitzian on the unit sphere.*

Proof. Let b denote a bounded $\mathcal{UG}^{(2)}$-smooth bump on X. By [6] we know that b is Fréchet smooth with Lipschitzian derivative on X. Then, by [2, Theorem V.3.2], X admits a norm whose first derivative is Lipschitzian on the unit sphere. Finally, our Theorem 4.1 gives an equivalent $\mathcal{UG}^{(2)}$-smooth norm with first derivative Lipschitzian on the unit sphere. ∎

PROBLEM. Let X be a nonseparable Banach space which admits a bounded $\mathcal{UG}^{(2)}$-smooth bump. Does then X admit an equivalent $\mathcal{UG}^{(2)}$-smooth norm?

Proposition 4.4. *Let X be a separable Banach spacecontaining no isomorphic copy of c_0, and assume that X admits a $\mathcal{C}^{(2)}$-smooth bump. Then X admits an equivalent norm which is $\mathcal{UG}^{(2)}$ smooth on each annulus around 0 and whose first derivative is Lipschitzian on the unit sphere.*

Proof. First note that if X admits a $\mathcal{C}^{(2)}$-smooth bump, then its first derivative must be locally Lipschitzian on X. By [2, Theorem V.3.1], X then admits a smooth bump whose first derivative is Lipschitzian on X. Then, by [2, Theorem V.3.2], X admits a norm whose first derivative is Lipschitzian on the unit sphere S_X. Consequently, Theorem 4.1 completes the proof. ∎

Let us mention that it is unknown whether a separable Banach spaceadmits a $\mathcal{C}^{(2)}$-smooth norm if it admits a $\mathcal{C}^{(2)}$-smooth bump.

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