Defective Galton-Watson processes *

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Abstract

The Galton-Watson process is a Markov chain modelling the population size of independently reproducing particles giving birth to k offspring with probability p_k, $k \geq 0$. In this paper we consider *defective* Galton-Watson processes having defective reproduction laws, so that $\sum_{k\geq 0} p_k = 1 - \varepsilon$ for some $\varepsilon \in (0,1)$. In this setting, each particle may send the process to a graveyard state Δ with probability ε. Such a Markov chain, having an enhanced state space $\{0,1,\ldots\} \cup \{\Delta\}$, gets eventually absorbed either at 0 or at Δ. Assuming that the process has avoided absorption until the observation time t, we are interested in its trajectories as $t \to \infty$ and $\varepsilon \to 0$.

Keywords: branching process; defective distribution; Galton-Watson process with killing; conditional limit theorems.

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1 Introduction

The classical Galton-Watson process (GW-process) is a discrete time Markov chain $Z = \{Z(t)\}_{t\geq 0}$ with the state space $\{0, 1, \ldots\}$ defined recursively by

$$Z(0) = 1, \quad Z(t+1) = \sum_{j=1}^{Z(t)} \nu_{t,j}, \quad t = 0, 1, \ldots, \tag{1}$$

where $\nu_{t,j} \stackrel{d}{=} \nu$ are independent random variables with a common distribution

$$f(s) = Es^{\nu} = \sum_{k\geq 0} p_k s^k. \tag{2}$$

In terms of probability generating functions, the branching property (1) yields

$$Es^{Z(t)} = f(t, s), \quad f(0, s) = s, \quad f(t+1, s) = f(f(t, s)), \quad t \geq 0. \tag{3}$$

There are two types of trajectories for this simple demographic model, unless $p_1 = 1$. A GW-process either becomes extinct at time $T_0 = \inf\{t \geq 1 : Z(t) = 0\}$ or $Z(t) \to \infty$, as $t \to \infty$. It is well known that the corresponding probability of extinction $q = P(T_0 < \infty)$ is given by the smallest non-negative root of the equation $f(s) = s$, see [2, Ch I.5]. Much of the theory of branching processes is devoted to the limit behavior of $Z(t)$ conditioned on $T_0 > t$ as $t \to \infty$, see [6].

This paper deals with *defective GW-processes* having $f(1) \in (0, 1)$. We treat the defect $\varepsilon = 1 - f(1)$ of the reproduction law (2) as the probability that a given particle existing at time t sends the Markov chain at time $t+1$ to an additional graveyard state Δ. Thus, a defective GW-process becomes a Markov chain with a countable state space $\mathbb{N}_\Delta = \{0, 1, \ldots\} \cup \{\Delta\}$. Two of the states are absorbing: the process either becomes extinct at time T_0, or is stopped at time $T_\Delta = \inf\{t \geq 1 : Z(t) = \Delta\}$. If $T = T_0 \wedge T_\Delta$ denotes the ultimate absorption time, then for some $q \in [0,1)$,

$$P(T_0 < \infty) = q, \quad P(T_\Delta < \infty) = 1 - q, \quad P(T < \infty) = 1.$$

Applying the graveyard absorption properties

$$\Delta + x = \Delta, \quad x \in \mathbb{N}_\Delta, \qquad s^\Delta = 0, \quad s \geq 0, \qquad \sum_{j=1}^{\Delta} x_j = \Delta, \quad x_j \in \mathbb{N}_\Delta,$$

to the recursion (1), we obtain again (3) implying $f(q) = q$. Clearly, $P(Z(t) = \Delta) = 1 - f(t,1)$, and if $q = 0$, then $T = T_\Delta$. It is straightforward to see that

$$E(s^{Z(t)}; T_\Delta > t) = f(t,s), \quad E(s^{Z(t)}; T > t) = E(s^{Z(t)}; T_0 > t) = f(t,s) - f(t,0),$$

since

$$E(s^{Z(t)}; T \leq t) = E(s^{Z(t)}; T_0 \leq t) = P(Z(t) = 0) = f(t,0).$$

This implies,

$$\begin{aligned} P(t < T_\Delta < \infty) &= f(t,1) - q, \\ P(t < T_0 < \infty) &= q - f(t,0), \\ P(T > t) &= f(t,1) - f(t,0). \end{aligned}$$

The main aim of this paper is to provide, for the first time, results on the asymptotic distribution of $Z(t-k)$ conditioned on the survival event $\{T > t\}$ as $t \to \infty$, with $k \in [0,t]$ either being fixed or going to infinity. Note that since the process Z becomes absorbed at time T with probability one, it is natural to examine the nature of this convergence. In Section 2 we provide some asymptotic results for the sequence $f(t,\cdot)$ as $t \to \infty$, assuming that the reproduction law $f(\cdot)$ is fixed. We find that with fixed $f(\cdot)$, there are two different asymptotic regimes depending on whether $\gamma = f'(q)$ is positive or equals zero. Moreover, from these results we derive limit theorems for distribution of $Z(t-k)$ conditioned on the survival event $\{T > t\}$ as $t \to \infty$. The proofs of the results of Section 2 are collected in Section 5.

In realistic settings, the defect ε of the reproduction law is small and therefore it is interesting to find asymptotic results as $t \to \infty$ and $\varepsilon \to 0$. This is a difficult issue to be addressed without further assumptions on the reproduction law. For this reason, as a first approach, in Sections 3 and 4 we consider sequences of defective GW-processes $(Z_n)_{n\geq 1}$ governed by reproduction laws $f_n(\cdot)$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and $f_n(s) \to \hat{f}(s)$ uniformly over $s \in [0,1]$, provided $\hat{f}(1) = 1$. Under these assumptions, we prove that the key parameter determining the limit behaviour is not γ as in Section 2, but rather $\hat{m} = \hat{f}'(1)$. We assume $\hat{m} > 1$ and even study the case $\hat{m} = \infty$. The proofs of the results of Sections 3 and 4 are collected in Section 6.

The main difference between the results of Sections 3 and 4 is in the restrictions put upon the reproductions laws $\{f_n(\cdot)\}_{n\geq 1}$. In Section 3 we assume that $f_n(\cdot)$ can be written in terms of a common probability generating function $\hat{f}(\cdot)$ and a scale parameter r_n such that $r_n \to 1$ so that $\varepsilon_n \to 0$. While in Section 4, we examine a certain parametric family of GW-processes in order to gain some knowledge in the general case, when the main restriction of Section 3 is removed. The advantage of these, so-called theta-branching processes, is that their reproduction generating functions have explicit iterations. The results in Section 4 can be also seen as a continuation of the study of this family initiated in [9].

Earlier, a special subclass of the defective GW-processes, the so-called GW-processes with killing, was studied in [5, 7]. A GW-process with killing has a reproduction law of the form $f(s) = g(\alpha s)$, where $g(\cdot)$ is a non-defective generating function and $\alpha \in (0, 1)$. In this case $f(1) \in (0, 1)$ and $f(s_0) = 1$ for $s_0 = 1/\alpha > 1$. To see a counterexample violating the latter restriction, consider

$$f_0(s) = 1 - (p_1\sqrt{1-s} + 1 - p_1)^2, \quad s \in [0, 1], \tag{4}$$

having $f_0(1) = p_1(2 - p_1)$ and

$$f_0(t, s) = 1 - (p_1^t\sqrt{1-s} + 1 - p_1^t)^2.$$

Since $f_0'(1) = \infty$, the generating function $f_0(s)$ is not defined for $s > 1$. Example (4) belongs to the above mentioned family of theta-branching processes. A broad class of continuous time defective branching processes was investigated in [8].

Defective GW-processes arise naturally in the framework of some special non-defective GW-processes with countably many types. For example, the authors of [3] construct an embedded defective GW-process in which absorption in the graveyard state corresponds to local survival of the GW-process with countably many types, and absorption in state 0 corresponds to its global extinction. In another multi-type setting [10], the defect ε is treated as the probability of a favorable mutation allowing a population of viruses to escape extinction. Some other biological examples, where these processes apply as models, can be found in [5].

Notice that the defective GW-processes can be put into the framework of ϕ-branching processes using a random control function

$$\phi(k) = \begin{cases} k & \text{with probability} \quad (1-\varepsilon)^k, \\ \Delta & \text{with probability} \quad 1-(1-\varepsilon)^k, \end{cases} \qquad k \geq 0,$$

cf. [11]. Indeed, in the defective case, the branching property (1) can be rewritten as

$$Z(t+1) = \sum_{j=1}^{\phi_t(Z(t))} \tilde{\nu}_{t,j}, \quad t = 0, 1, \ldots,$$

where $\phi_t(\cdot) \stackrel{d}{=} \phi(\cdot)$. Here the common distribution of the random variables $\tilde{\nu}_{t,j}$ has a proper probability generating function $f(\cdot)/f(1)$. For a given small value of ε, the control function gets a chance to stop the growth of a non-defective GW-process, when the population size k becomes inverse-proportional to ε, that is when the stopping probability $1-(1-\varepsilon)^k$ is approximated by $1-e^{-\varepsilon k}$.

2 Limit theorems with fixed reproduction law

In this section we assume that the defective reproduction law $f(\cdot)$ is fixed while the observation time t tends to infinity. Recall that $q \in [0,1)$ is defined by $q = f(q)$ and $\gamma = f'(q)$. Observe that $\gamma \in [0,1)$, and denote

$$l = \min\{k \geq 0 : p_k > 0\}.$$

Clearly, $q = 0$ if and only if $l \geq 1$, and $\gamma = 0$ if and only if $l \geq 2$. Define $\pi_t = \gamma^t$ for $l = 0, 1$, and

$$\pi_t = \prod_{k=0}^{t-1} p_l{}^{l^k} = p_l^{a_t}, \quad a_t = \frac{l^t - 1}{l - 1}, \quad t \geq 1,$$

for $l \geq 2$. Observe that given $l \geq 1$, the minimal t-th generation size is l^t and

$$P(Z(t) = l^t) = \pi_t.$$

Proposition 1. *Consider iterations $f(t, \cdot)$ of a defective probability generating function $f(\cdot)$.*

(a) If $\gamma > 0$, then for each $s \in [0, 1]$,

$$f(t, s) - q \sim (s - q)H(s)\pi_t, \quad t \to \infty,$$

where $H(\cdot)$ is a generating function defined as

$$H(s) = \prod_{j=0}^{\infty} h(f(j, s)), \quad h(s) = \frac{f(s) - q}{(s - q)\gamma},$$

and having $H(q) = 1$, $H(1) < \infty$.

(b) If $\gamma = 0$, then for each $s \in [0, 1]$,

$$f(t, s) \sim (sR(s))^{l^t}\pi_t, \quad t \to \infty,$$

where $R(\cdot)$ is a generating function defined as

$$R(s) = \prod_{j=0}^{\infty} (b(f(j, s)))^{l^{-j-1}}, \quad b(s) = \frac{f(s)}{p_l s^l},$$

and having

$$1 = R(0) < R(1) < p_l^{-1/(l-1)}.$$

Proposition 1 indicates that there are two different asymptotic regimes depending on whether $\gamma > 0$ or $\gamma = 0$. It is worthwhile to note that Proposition 1-a and 1-b are analogous results to Theorem 2 in [2, Ch I.11] and Proposition 3 in [1], respectively, for non-defective GW-processes.

An immediate consequence of Proposition 1-a is

$$\gamma^{-t} P(T > t) \to qH(0) + (1-q)H(1), \quad t \to \infty, \tag{5}$$

which implies

$$P(T = t + k | T \geq t) \to (1-\gamma)\gamma^k, \quad k \geq 1.$$

As it is shown next by Theorem 2, devoted to the case $\gamma > 0$, relation

$$\frac{(s-q)H(s) + qH(0)}{(1-q)H(1) + qH(0)} = \sum_{j \geq 1} v_j s^j \tag{6}$$

defines an important proper distribution $(v_j)_{j\geq 1}$. Indeed, Theorem 2-a is the counterpart result of Theorem 1 in [2, Ch I.14] for non-defective GW-processes and Theorem 2-b is a multivariate analogue of Theorem 2-a.

Theorem 2. *Consider a defective GW-process with* $\gamma > 0$*.*

(a) The asymptotic relation (5) *holds, and for* $0 \leq k \leq t, j \geq 1$*,*

$$P(Z(t-k) = j | T > t) \to v_{k,j}, \quad t \to \infty,$$

where $(v_{k,j})_{j\geq 1}$ *is a proper probability distribution defined by*

$$v_{k,j} = v_j \gamma^{-k} (f(k,1)^j - f(k,0)^j), \tag{7}$$

so that $v_{0,j} \equiv v_j$ *are given by* (6).

(b) For $j_0 \geq 1, \ldots, j_k \geq 1$, $k \geq 0$,

$$P(Z(t) = j_0, \ldots, Z(t-k) = j_k | T > t) \to v_{k,j_k} Q^{(k)}_{j_k, j_{k-1}} \cdots Q^{(1)}_{j_1, j_0}, \qquad t \to \infty,$$

where

$$Q^{(k)}_{ij} = \frac{f(k-1,1)^j - f(k-1,0)^j}{f(k,1)^i - f(k,0)^i} P_{ij}, \quad \sum_{j \geq 1} Q^{(k)}_{ij} = 1, \quad i \geq 1,$$

is a transformation of the time-homogeneous transition probabilities

$$P_{ij} = P(Z(t+1) = j | Z(t) = i).$$

We see that in the case $\gamma > 0$, the conditional branching process asymptotically behaves as a time-inhomogeneous Markov chain. Observe that given $q \in (0,1)$, the limit towards the past

$$Q^{(k)}_{ij} \to \frac{P_{ij} j q^{j-i}}{\gamma i}, \qquad k \to \infty,$$

recovers the well known formula for the so-called Q-process, see [2, Ch I.14] and [9].

On the other hand, for $\gamma = 0$, Proposition 1-b gives a much faster decay of the tail distribution

$$P(T > t) \sim \pi_t R(1)^{l^t} = p_l^{-\frac{1}{l-1}} \rho^{l^t}, \quad t \to \infty, \tag{8}$$

where $\rho = p_l^{\frac{1}{l-1}} R(1) \in (0,1)$. This yields $P(T = t | T \geq t) \to 1$. The next Theorem 3 establishes a conditional weak law of large numbers for $l^{t-k} Z(t-k)$ as $t \to \infty$.

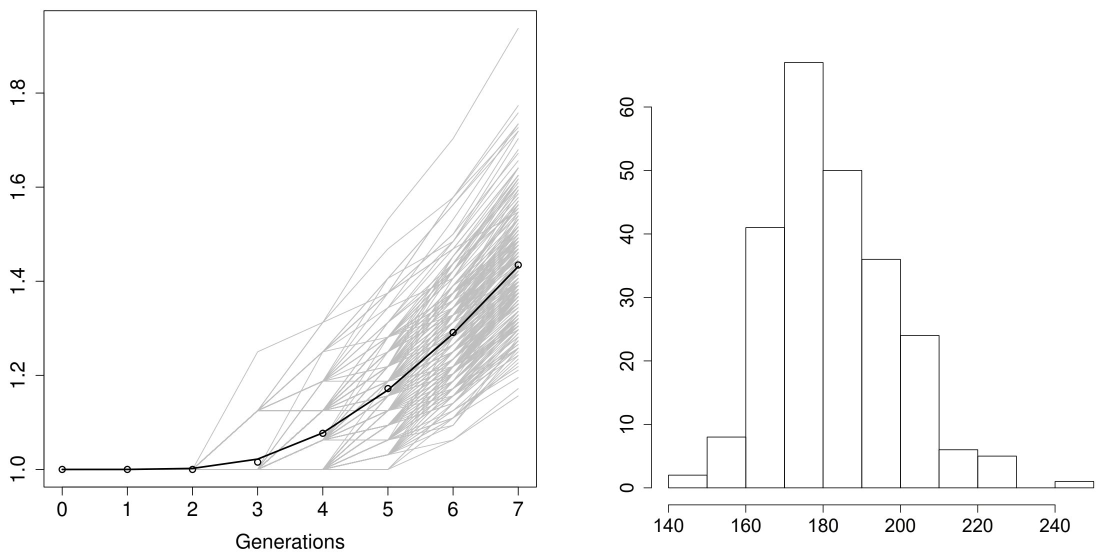

Figure 1: Simulation results for $f(s) = 0.7s^2 + 0.2s^3$ and $t = 7$. *Left panel.* Grey lines represent the vectors $(Z(0), 2^{-1}Z(1), \ldots, 2^{-t}Z(t))$ for 240 successful simulations having $T > t$. The thick black line shows the limit vector $(c(t), c(t-1), \ldots, c(0))$ suggested by Theorem 3, which provides with a good approximation for the average trajectory (shown by circles) even for the small observation time $t = 7$. *Right panel.* The histogram presents the observed values $Z(t)$ in the successful simulations.

Theorem 3. *Consider a defective GW-process with* $\gamma = 0$. *Then the asymptotic relation* (8) *holds and for the normalized process* $Y(t) = l^{-t}Z(t)$, *we have the following results concerning its expectation and variance.*

(a) If $f'(1) < \infty$, *then uniformly over* $0 \le k \le t$,

$$E(Y(k)|T > t) - c(t-k) \to 0, \quad t \to \infty,$$

where in terms of $\bar{R}(s) = R'(s)/R(s)$,

$$c(k) = 1 + f(k,1)\bar{R}(f(k,1)), \quad k = 0, 1, \ldots, \tag{9}$$

is a strictly decreasing sequence with

$$1 < \ldots < c(k+1) < c(k) < c(k-1) < \ldots < c(1) < c(0) < \infty.$$

(b) If $f''(1) < \infty$, *then uniformly over* $0 \le k \le t$,

$$Var(Y(k)|T > t) \to 0, \quad t \to \infty.$$

According to Theorem 3-b, if $f''(1) < \infty$, then conditionally on $T > t$, we have convergence in probability $Y(t-k) \to c(k)$ as $k \ge 0$ is fixed and $t \to \infty$, and convergence in probability $Y(k) \to 1$ as $t - k \to \infty$. This indicates that being conditioned on survival, the reproduction regime prefers the minimal offspring number l, especially at early times (see Figure 1).

3 Extendable defective GW-processes

Suppose $f(r) = r$ for some $r > 1$, so that necessarily $f(1) < 1$ (see Figure 2). In this case the corresponding defective GW-process Z could be called an extendable GW-process because the usual range $0 \le s \le 1$ for the reproduction generating function $f(s)$ can be extended to $0 \le s \le r$. The transformed function

$$\hat{f}(s) = r^{-1} f(rs), \quad s \in [0,1], \quad \hat{f}(1) = 1,$$

generates a proper reproduction distribution $\hat{p}_k = r^{k-1} p_k$ with mean $\hat{m} = \hat{f}'(1) = f'(r)$. Denote by $\hat{Z} = \{\hat{Z}(t)\}_{t \ge 0}$ the GW-process with the reproduction law $\hat{f}(\cdot)$. If $\hat{m} \in (1, \infty)$, then by Theorem 3 in [2, Ch I.10], there exists a sequence $C(t) \to \infty$, $t \to \infty$ such that $\hat{Z}(t)/C(t) \to W$ a.s., where $P(W > 0) = 1 - \hat{q}$ and $\hat{q} = q/r$. In this case, for any given $\lambda \ge 0$, we have a positive finite limit

$$E(e^{-\lambda \hat{Z}_n(t)/C(t)} | \hat{T}_0 > t) \to \Psi(\lambda), \quad t \to \infty, \tag{10}$$

where $\Psi(\lambda) = E(e^{-\lambda W} | W > 0)$. On the other hand, if $\hat{m} = \infty$, then by [4],

$$P(b^{-t} \ln \hat{Z}(t) \le u | \hat{T}_0 > t) \to \psi(u), \quad u \in (0, \infty), \tag{11}$$

provided the following condition holds

$$g'(x) = a x^{b-1} (1 + O(x^{\delta})), \quad x \to 0, \quad a > 0, \quad b > 1, \quad \delta > 0.$$

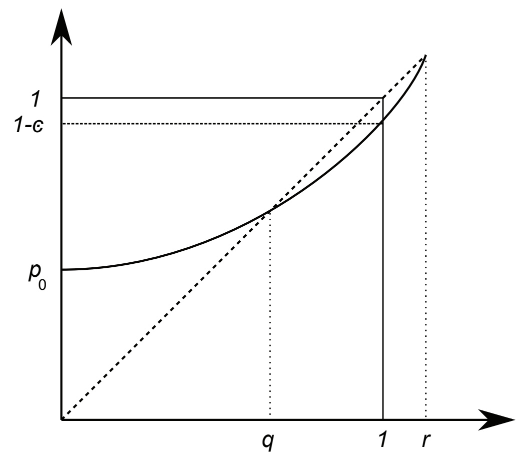

Figure 2: Extendable generating function $f(\cdot)$.

Here $g(\cdot) = G_{-1}(\cdot)$ is the inverse function of $G(x) = 1 - \hat{f}(1-x)$, and the limit $\psi(\cdot)$ in (11) is continuous and strictly monotonic increasing function such that

$$\psi(u) \to 0, \quad u \to 0+, \qquad \psi(u) \to 1, \quad u \to \infty.$$

Theorem 4. *Let $\hat{f}(\cdot)$ be a probability generating function for a proper reproduction law. Consider a sequence of defective GW-processes $\{Z_n\}_{n\geq 1}$ corresponding to the sequence of reproduction laws*

$$f_n(s) = r_n \hat{f}(s/r_n), \quad r_n > 1, \quad n \geq 1, \tag{12}$$

and with absorption time T_n.

(a) Suppose $\hat{m} \in (1, \infty)$ so that (10) holds. If for some sequence $t_n \to \infty$,

$$(r_n - 1)C(t_n) \to x \in (0, \infty),$$

then

$$P\left(T_n > t_n\right) \to (1-\hat{q})\Psi\left(x\right),$$

and for each $\lambda \geq 0$,

$$E(e^{-\lambda Z_n(t_n)/C(t_n)}|T_n > t_n) \to \Psi(\lambda + x)/\Psi(x), \quad n \to \infty. \tag{13}$$

(b) Suppose $\hat{m} = \infty$ *and* (11) *holds. If for some sequence* $t_n \to \infty$,

$$b^{-t_n} \ln(r_n - 1)^{-1} \to y, \quad y \in (0, \infty), \quad n \to \infty,$$

then

$$P\left(T_n > t_n\right) \to (1-\hat{q})\psi(y),$$

and for $u \in [0, y]$,

$$P(b^{-t_n} \ln Z_n(t_n) \leq u|T_n > t_n) \to \psi(u)/\psi(y), \quad n \to \infty.$$

Theorem 4-a should be compared to [7, Theorem 3.4] concerning a sequence of GW-processes with killing: if Z_n has a reproduction law of the form $f_n(s) = \hat{f}(\alpha_n s)$, where $\hat{f}(1) = 1$, $\hat{f}'(1) \in (1, \infty)$, and

$$(1 - \alpha_n)C(t_n) \to (\hat{m} - 1)x/\hat{m}, \quad n \to \infty,$$

then the same weak convergence result (13) holds. The proof of Theorem 4 given in Section 6 is more straightforward than the proof of [7, Theorem 3.4], which demonstrates the advantage of dealing with the extendable GW-processes.

4 Explicit limits for defective theta-branching processes

As was pointed out in the Introduction, the main assumption of Section 3 is quite restrictive on the mode of convergence $f_n(\cdot) \to \hat{f}(\cdot)$, namely, condition (12) requires that the sequence $f_n(\cdot)$ has a common shape of the reproduction laws and only a scale parameter $r_n \to 1$ is changing as $n \to \infty$. In this section we take a step towards a more general setting for the convergence $f_n(\cdot) \to \hat{f}(\cdot)$. We focus on the parametric family of the theta-branching processes introduced in [9]. Our Propositions 5, 6 and 7 give explicit expressions for the corresponding limit distributions.

Proposition 5 is a counterpart of Theorem 4-a in terms of a sequence of extendable GW-processes whose generating functions are explicitly characterized by four parameters

$$(\theta_n, q_n, \gamma_n, r_n) \in (0, 1] \times [0, 1) \times (0, 1) \times (1, \infty)$$

as follows

$$f_n(t, s) = r_n - \left[\gamma_n^t (r_n - s)^{-\theta_n} + (1 - \gamma_n^t)(r_n - q_n)^{-\theta_n}\right]^{-1/\theta_n}, \quad s \in [0, r_n],$$

In agreement with our previous notation, q_n is the extinction probability and $\gamma_n = f_n'(q_n)$. These defective GW-processes have the defect value

$$\varepsilon_n = \left[\gamma_n (r_n - 1)^{-\theta_n} + (1 - \gamma_n)(r_n - q_n)^{-\theta_n}\right]^{-1/\theta_n} - (r_n - 1).$$

Proposition 5. *Fix a triplet* $(\theta, q, \gamma) \in (0, 1] \times [0, 1) \times (0, 1)$ *and consider the above* *described sequence of defective theta-branching processes* $\{Z_n\}_{n\geq 1}$ *with*

$$(\theta_n, \gamma_n, q_n, r_n) \to (\theta, \gamma, q, 1), \quad n \to \infty.$$

Denote $m_n = f_n'(1) = \gamma_n^{-1/\theta_n}$, *and assume that for some* $t_n \to \infty$,

$$(r_n - 1)m_n^{t_n} \to x \in (0, \infty), \quad n \to \infty. \tag{14}$$

(a) As $n \to \infty$,

$$P\left(T_n > t_n\right) \to (1-q)\Psi\left(x\right),$$

where

$$\Psi(\lambda) = 1 - \left[1 + (1-q)^{\theta}\lambda^{-\theta}\right]^{-1/\theta}, \quad \lambda \geq 0. \tag{15}$$

(b) If $k \geq 0$ *and* $t_n - k \to \infty$, *then for each* $\lambda \geq 0$,

$$E\left(\exp\{-\lambda m_n^{k-t_n} Z_n(t_n - k)\}|T_n > t_n\right) \to \frac{\Psi\left(x+\lambda\right)}{\Psi\left(x\right)}, \quad n \to \infty.$$

Under the conditions of Proposition 5 we have $f_n(s) \to \hat{f}(s)$, where

$$\hat{f}(s) = 1 - \left[\gamma(1-s)^{-\theta} + (1-\gamma)(1-q)^{-\theta}\right]^{-1/\theta}. \tag{16}$$

For the corresponding supercritical GW-process having the offspring mean $\hat{m} = \gamma^{-1/\theta}$, it is straightforward to check that the limit Laplace transform

$$E(e^{-\lambda \hat{Z}(t)\hat{m}^{-t}}|\hat{T}_0 > t) = 1 - \frac{1 - \hat{f}(t, e^{-\lambda\gamma^{t/\theta}})}{1 - \hat{f}(t, 0)} \to \Psi(\lambda), \quad t \to \infty,$$

is given by (15). Since

$$\varepsilon_n \sim (\gamma^{-1/\theta} - 1)(r_n - 1), \quad n \to \infty,$$

the first part of Proposition 5 essentially says that for a given small ε, the absorption time T of a defective theta-branching process with $\theta \in (0, 1]$ is of order $\theta \log_\gamma \varepsilon$. Observe that the new normalization $m_n^{t_n}$ may not be asymptotically equivalent to the normalization $\hat{m}^{t_n}$ suggested by Theorem 4-a under an additional "xlogx" condition.

The next two propositions deal with two different sequences $f_n(\cdot)$ converging to the same limit reproduction law given by

$$\hat{f}(s) = 1 - (1-q)^{1-\gamma}(1-s)^{\gamma}, \quad s \in [0, 1], \tag{17}$$

with $q \in [0, 1)$, $\gamma \in (0, 1)$, $\hat{f}(1) = 1$, and $\hat{m} = \hat{f}'(1) = \infty$. Plugging $s = \exp\{-\lambda e^{-u\gamma^{-t}}\}$ into

$$\hat{f}(t, s) = 1 - (1-q)^{1-\gamma^t}(1-s)^{\gamma^t},$$

it is straightforward to find a convergence

$$P\left(\gamma^t \ln \hat{Z}(t) \le u | \hat{T}_0 > t\right) \to 1 - e^{-u}, \quad u \ge 0$$

to a standard exponential distribution. Observe that both propositions are counterparts of Theorem 4-b. Proposition 6 deals with the family of reproduction laws depending on three parameters, while Proposition 7 handles a more complicated four-parameter case.

Proposition 6. *Consider a sequence of defective GW-processes* $\{Z_n\}_{n\ge1}$ *having the* *following reproduction laws*

$$f_n(s) = r_n - (r_n - q_n)^{1-\gamma_n}(r_n - s)^{\gamma_n}, \quad s \in [0, r_n),$$

with $(q_n, \gamma_n, r_n) \in [0,1) \times (0,1) \times (1,\infty)$. *Suppose that for some* $(q, \gamma) \in [0,1) \times (0,1)$,

$$(q_n, \gamma_n, r_n) \to (q, \gamma, 1) \quad n \to \infty,$$

and that for some $t_n \to \infty$,

$$\gamma_n^{t_n} \ln(r_n - 1)^{-1} \to y \in (0,\infty), \quad n \to \infty. \tag{18}$$

(a) As $n \to \infty$,

$$P(T_n > t_n) \to (1-q)(1-e^{-y}). \tag{19}$$

(b) If $k \geq 0$ *and* $t_n - k \to \infty$, *then*

$$P\left(\gamma_n^{t_n - k} \ln Z_n(t_n - k) \leq u | T_n > t_n\right) \to \frac{1 - e^{-u}}{1 - e^{-y}}, \quad 0 \leq u \leq y. \tag{20}$$

Since in this parametric case the defect size has the asymptotic value

$$\varepsilon_n \sim (1-q)^{1-\gamma}(r_n - 1)^{\gamma}, \quad n \to \infty,$$

the first part of Proposition 6 essentially says that for a given small defect value ε, the absorption time of a defective theta-branching process with $\theta \in (0,1]$ is of order $\ln\ln\varepsilon^{-1}$.

Proposition 7. *Consider a sequence of defective GW-processes* $\{Z_n\}_{n\geq 1}$ *having the following reproduction laws*

$$f_n(s) = A_n - \left[\gamma_n(A_n - s)^{|\theta_n|} + (1-\gamma_n)(A_n - q_n)^{|\theta_n|}\right]^{1/|\theta_n|}, \quad s \in [0, A_n],$$

where $(\theta_n, q_n, \gamma_n, A_n) \in (-1, 0) \times [0, 1) \times (0, 1) \times [1, \infty)$. *Suppose that for some* $(\gamma, q) \in (0, 1) \times [0, 1)$,

$$(\theta_n, \gamma_n, q_n, A_n) \to (0, \gamma, q, 1), \quad n \to \infty,$$

in such a way that for some $t_n \to \infty$,

$$|\theta_n| \ln(A_n - 1)^{-1} \to a \in (0, \infty], \tag{21}$$

$$\gamma_n^{t_n} |\theta_n|^{-1} \to y \in (0, \infty), \quad n \to \infty. \tag{22}$$

(a) As $n \to \infty$,

$$P\left(T_n > t_n\right) \to (1-q)(1 - e^{-y(1-e^{-a})}).$$

(b$_1$*) If* $k \geq 0$ *is fixed , then putting* $\hat{u}(x) = -x \ln(1 - u/x)$,

$$P\left(\gamma_n^{t_n - k} \ln Z_n(t_n - k) \leq \hat{u}(y\gamma^{-k}) | T_n > t_n\right) \to \frac{1 - e^{-u}}{1 - e^{-y(1-e^{-a})}}, \quad 0 \leq u < y(1-e^{-a}).$$

(b$_2$*) If* $k \to \infty$, $t_n - k \to \infty$, *then*

$$P\left(\gamma_n^{t_n - k} \ln Z_n(t_n - k) \leq u | T_n > t_n\right) \to \frac{1 - e^{-u}}{1 - e^{-y(1-e^{-a})}}, \quad 0 \leq u < y(1 - e^{-a}).$$

Here, $\varepsilon_n \sim (1-q)(1-\gamma)^{1/|\theta_n|}$ and by Proposition 7-a, given a small defect value ε, the absorption time is again of order $\ln \ln \varepsilon^{-1}$. If $A_n \equiv 1$, then $a = \infty$, and convergence in

Proposition 7-a is given by (20). To see a connection of the convergence in Proposition 7-b$_1$ to that of Proposition 7-b$_2$, notice that $\hat{u}(x) \to u$, as $x \to \infty$.

Observe that in Propositions 6 and 7, the absorption time is of the same order. Moreover, the asymptotic distribution of the processes conditioned upon survival and equally normalized is a truncated exponential distribution in Proposition 6-b, with $k \geq 0$ fixed, as well as in Proposition 7-b_2, as $k \to \infty$. However, the exponential distribution resulting in Proposition 6-b has mean equal to one, whereas the mean of its counterpart in 7-b_2 is equal to $(1 - e^{-a})^{-1}$, where a is defined in (21). In both cases, the support of the corresponding truncated distribution depends on the rate of convergence of $\varepsilon_n \to 0$.

5 Proofs of Proposition 1 and Theorems 2 and 3

5.1 Proof of Proposition 1

Assume $\gamma > 0$. Putting

$$H_t(s) = \frac{f(t,s) - q}{(s-q)\gamma^t}, \quad 0 \leq s \leq 1, \quad t \geq 1,$$

observe that

$$H_t(s) = \prod_{j=0}^{t-1} h(f(j,s)), \quad h(s) = \frac{f(s) - q}{(s-q)\gamma}.$$

It is easy to check that $h(\cdot)$ is a generating function with $h(q) = 1$. (In fact, $\frac{f(s)-f(q)}{s-q}$ is a tail generating function naturally linked to the reproduction law $f(\cdot)$, see [8].) It follows that $H_t(\cdot)$ is also a generating function such that $H_t(q) = 1$.

Since $h(f(t,s)) < 1$ for $s < q$, and $h(f(t,s)) > 1$ for $s > q$, we conclude that

$H_{t+1}(s) < H_t(s)$ for $s < q$, and $H_{t+1}(s) > H_t(s)$ for $s > q$. Due to this monotonicity property, we have $H_t(s) \to H(s)$, as $t \to \infty$, where the limit function $H(s)$ has the stated form.

To finish the proof of Proposition 1-a it remains to show that $H(1) < \infty$ or equivalently,

$$\sum_{j=1}^{\infty}(h(f(j,1)) - 1) < \infty.$$

The last is indeed true because

$$h(f(t,1)) - 1 \leq \left(1 - \frac{\varepsilon}{1-q}\right)^t c, \quad t > t_0,$$

for some finite c and t_0. This upper bound is justified using two observations: on one hand, we have

$$\frac{h(s) - h(q)}{s-q} \to \frac{f''(q)}{\gamma} \in (0, \infty), \quad s \to q,$$

and on the other hand,

$$f(t,1) - q \leq (1-q)\left(1 - \frac{\varepsilon}{1-q}\right)^t,$$

which is due to the following convexity property of $f(\cdot)$

$$f(s) \leq q + (s-q)\frac{1-q-\varepsilon}{1-q}, \quad s \in [q,1].$$

Assume now $\gamma = 0$, or equivalently $l \geq 2$. By iterating the function $f(s) = p_l s^l b(s)$, we get the following representation

$$f(t,s)=\pi_t(sR_t(s))^{l^t},\quad R_t(s)=\prod_{j=1}^{t}\big(b(f(j-1,s))\big)^{l^{-j}},\quad t\geq 0. \tag{23}$$

A straightforward adjustment to the defective case $f(1)<1$ of the argument used in [1, Prop. 3] shows that the sequence of monotonely increasing functions $R_t(\cdot)$ has a well defined limit

$$R(s)=\lim_{t\to\infty}R_t(s)=\prod_{j=1}^{\infty}b(f(j-1,s))^{l^{-j}},\quad s\in[0,1],$$

and moreover, that

$$\lim_{t\to\infty}\left(R_t(s)/R(s)\right)^{l^t}=1.$$

This proves the main assertion of Proposition 1-b. It remains to verify the stated upper bound for $R(1)$ which in terms of $\rho=p_l^{\frac{1}{l-1}}R(1)$, is equivalent to the inequality $\rho<1$. Since $f(t,1)\to q=0$, the relation

$$f(t,1)\sim\pi_tR(1)^{l^t}=p_l^{-\frac{1}{l-1}}\rho^{l^t},\quad t\to\infty$$

indeed implies that $\rho<1$. This also gives (8).

5.2 Proof of Theorem 2

We will need the following relations

$$P(T > t|Z(k) = i) = f^i(t-k, 1) - f^i(t-k, 0), \tag{24}$$

$$E(s^{Z(k)}|T > t) = \frac{f(k, sf(t-k, 1)) - f(k, sf(t-k, 0))}{f(t, 1) - f(t, 0)}, \tag{25}$$

holding for $0 \le k \le t < \infty$, $s \in [0, 1]$. Relation (24) follows from

$$\{T > t\} = \{T_\Delta > t\} \setminus \{T_0 \le t\}$$

and

$$P(T_\Delta > t|Z(k) = i) = P(Z(t-k) \neq \Delta)^i = f^i(t-k, 1),$$

$$P(T_0 \le t|Z(k) = i) = P(Z(t-k) = 0)^i = f^i(t-k, 0).$$

Relation (25) is obtained using (24) as follows

$$\begin{aligned} E(s^{Z(t)}|T > t+k) &= \frac{E(s^{Z(t)}P(T > t+k|Z(t)))}{P(T > t+k)} \\ &= \frac{E((sf(k,1))^{Z(t)}) - E((sf(k,0))^{Z(t)})}{f(t+k,1) - f(t+k,0)} \\ &= \frac{f(t, sf(k,1)) - f(t, sf(k,0))}{f(t+k,1) - f(t+k,0)}. \end{aligned}$$

Applying (25) and Proposition 1-a, we get

$$\begin{aligned}E(s^{Z(t-k)}|T>t) &= \frac{f(t-k, sf(k,1)) - f(t-k, sf(k,0))}{f(t-k, f(k,1)) - f(t-k, f(k,0))} \\ &\to \frac{(sf(k,1)-q)H(sf(k,1)) - (sf(k,0)-q)H(sf(k,0))}{(f(k,1)-q)H(f(k,1)) - (f(k,0)-q)H(f(k,0))}.\end{aligned}$$

In particular,

$$E(s^{Z(t)}|T>t) \to \frac{(s-q)H(s)+qH(0)}{(1-q)H(1)+qH(0)} = \sum_{j=1}^{\infty} v_j s^j.$$

Thus, $P(Z(t-k)=j|T>t) \to v_{k,j}$ with

$$\sum_{j=1}^{\infty} v_{k,j}s^j = \frac{(sf(k,1)-q)H(sf(k,1)) - (sf(k,0)-q)H(sf(k,0))}{(f(k,1)-q)H(f(k,1)) - (f(k,0)-q)H(f(k,0))}.$$

Modifying the denominator by a repeated use of the relation

$$(f(s)-q)H(f(s)) = \gamma(s-q)H(s),$$

we find

$$\begin{aligned}\sum_{j=1}^{\infty} v_{k,j}s^j &= \gamma^{-k}\frac{(sf(k,1)-q)H(sf(k,1)) - (sf(k,0)-q)H(sf(k,0))}{(1-q)H(1)+qH(0)} \\ &= \gamma^{-k}\left(\sum_{j=1}^{\infty} v_j(sf(k,1))^j - \sum_{j=1}^{\infty} v_j(sf(k,0))^j\right),\end{aligned}$$

which implies (7) thereby finishing the proof of Theorem 2-a.

Turning to the proof of Theorem 2-b, observe that

$$P(T > t | Z(t) = j_0, \ldots, Z(t-k) = j_k) = 1,$$

implying

$$P(Z(t) = j_0, \ldots, Z(t-k) = j_k; T > t) = P(Z(t) = j_0, \ldots, Z(t-k) = j_k).$$

Similarly, by (24),

$$P(Z(t-k) = j_k; T > t) = P(Z(t-k) = j_k)(f(k,1)^{j_k} - f(k,0)^{j_k}),$$

which gives

$$P(Z(t-k) = j_k) \sim v_{k,j_k}(f(k,1)^{j_k} - f(k,0)^{j_k})^{-1} P(T > t).$$

Therefore, by the Markov property,

$$\begin{aligned} P(Z(t) = j_0, \ldots, Z(t-k) = j_k | T > t) &\sim v_{k,j_k} \frac{P_{j_k,j_{k-1}} \cdots P_{j_1,j_0}}{f(k,1)^{j_k} - f(k,0)^{j_k}} \\ &= v_{k,j_k} Q^{(k)}_{j_k,j_{k-1}} \cdots Q^{(1)}_{j_1,j_0}. \end{aligned}$$

Finally, observe that $(Q^{(k)}_{ij})_{j \geq 1}$ is a proper distribution with the probability generating function

$$\sum_{j=1}^{\infty} Q^{(k)}_{ij} s^j = \frac{f(sf(k-1,1))^i - f(sf(k-1,0))^i}{f(k,1)^i - f(k,0)^i}.$$

5.3 Proof of Theorem 3

Recall notation $\bar{R}(s) = R'(s)/R(s)$ and observe that

$$\overline{R}(s) = \frac{d}{ds}\ln R(s) = \sum_{j=0}^{\infty} \frac{1}{l^{j+1}} \frac{b'(f(j,s))f'(j,s)}{b(f(j,s))},$$

where $f'(j,s) = \frac{d}{ds}f(j,s)$. Put furthermore, $\bar{R}_t(s) = \frac{R_t'(s)}{R_t(s)}$ for $s \in [0,1]$ and $t \geq 0$. Using (23), we obtain

$$\bar{R}_t(s) = \frac{d}{ds}\ln R_t(s) = \sum_{j=0}^{t-1} \frac{1}{l^{j+1}} \frac{b'(f(j,s))f'(j,s)}{b(f(j,s))}.$$

Lemma 8. *Assume $\gamma = 0$, $f'(1) < \infty$, and put*

$$\delta_t = \sum_{j=t}^{\infty} \gamma_0 \cdots \gamma_{j-1}, \quad \gamma_i = f'(f(i,1)).$$

Then $\delta_t \to 0$ as $t \to \infty$ and

$$\bar{R}(s) - \bar{R}_t(s) < \frac{f'(1)\delta_t}{p_l}, \quad s \in [0,1].$$

Proof. Using the expressions for $\bar{R}(s)$ and $\bar{R}_t(s)$, as well as the inequality $b(s) \geq 1$, we see that indeed

$$\bar{R}(s) - \bar{R}_t(s) = \sum_{j=t}^{\infty} \frac{b'(f(j,s))f'(j,s)}{b(f(j,s))l^{j+1}} \leq b'(1)\sum_{j=t}^{\infty} f'(j,1) < \frac{f'(1)\delta_t}{p_l}.$$

The fact that $\delta_t < \infty$ follows from $\gamma_i \to 0$ as $i \to \infty$, which, in turn, is a consequence of $\gamma = 0$. □

Lemma 9. *Assume $f'(1) < \infty$, $\gamma = 0$. The sequence* (9) *is strictly decreasing.*

Proof. It suffices to show that

$$1 + f(s)\bar{R}(f(s)) < 1 + s\bar{R}(s), \quad s \in [0, 1].$$

Using the definition of $R(\cdot)$ given in Proposition 1 it is easy to verify the equality

$$f(s)R(f(s)) = p_l(sR(s))^l,$$

which entails

$$\ln f(s) + \ln R(f(s)) = \ln p_l + l \ln s + l \ln R(s).$$

After differetiating

$$\frac{f'(s)}{f(s)} + \bar{R}(f(s))f'(s) = \frac{l}{s} + l\bar{R}(s),$$

we find

$$1 + f(s)\bar{R}(f(s)) = \frac{(\ln p_l s^l)'}{(\ln f(s))'}(1 + s\bar{R}(s)),$$

where $\frac{(\ln p_l s^l)'}{(\ln f(s))'} < 1$, since

$$(\ln p_l s^l)' < (\ln p_l s^l)' + (\ln b(s))' = (\ln f(s))'.$$

□

Lemma 10. *If $\gamma = 0$, then*

$$\frac{f'(t,s)s}{f(t,s)} = l^t(1 + s\bar{R}_t(s)),$$
$$\frac{f''(t,s)s^2}{f(t,s)} = l^{2t}(1 + s\bar{R}_t(s))^2 + l^t(s^2\bar{R}'_t(s) - 1).$$

Proof. Both relations are straightforward corollaries of formula (23). □

Assuming $\gamma = 0$, we first prove Theorem 3-a using Lemmas 8, 9 and 10, and then turn to the proof of Theorem 3-b.

Let $f'(1) < \infty$. From (25), we compute the conditional expectation

$$E\left(Z(k)|T > t\right) = \frac{f'(k, f(t-k,1))f(t-k,1)}{f(k, f(t-k,1))},$$

and applying the first relation in Lemma 10, we find

$$E\left(Y(k)|T > t\right) = 1 + f(t-k,1)\bar{R}_k(f(t-k,1)).$$

Thus the difference

$$c(t-k) - E\left(Y(k)|T > t\right) = f(t-k,1)(\bar{R}(f(t-k,1)) - \bar{R}_k(f(t-k,1))$$

is non-negative and bounded from above by a constant times $f(t-k,1)\delta_k$, see Lemma 8. By the monotonocity of the sequences $\{f(j,1)\}_{j\geq 0}$ and $\{\delta_j\}_{j\geq 1}$, we have for all $1 \leq k, k' \leq t$,

$$f(t-k,1)\delta_k \leq \max_{0\leq k\leq k'} f(t-k,1)\delta_k + \max_{k'\leq k\leq t} f(t-k,1)\delta_k \leq f(t-k',1)\delta_0 + \delta_{k'}.$$

The obtained upper bound goes to 0 as first $t \to \infty$ and then $k' \to \infty$. This proves the uniform convergence stated in Theorem 3-a.

Let $f''(1) < \infty$. To prove Theorem 3-b it suffices to show the inequality

$$Var(Y(k)|T > t) < c\, l^{-k} f(t-k, 1), \quad 0 \le k \le t,$$

for some constant c. From formula (25) one can obtain the following expression, where $s_0 = f(t-k, 1)$,

$$Var\,(Z(k)|T > t) = \frac{f''(k, s_0)s_0^2}{f(k, s_0)} + \frac{f'(k, s_0)s_0}{f(k, s_0)} - \left(\frac{f'(k, s_0)s_0}{f(k, s_0)}\right)^2,$$

so that by Lemma 10, we get

$$Var\,(Z(k)|T > t) = l^k f(t-k, 1)\left(\bar{R}_k(f(t-k, 1)) + f(t-k, 1)\bar{R}_k'(f(t-k, 1))\right).$$

Since we already know that $\bar{R}_t(s)$ is uniformly bounded by a constant, it remains to establish a similar property for the derivative $\bar{R}_t'(s)$, which satisfies

$$\bar{R}_t'(s) < \sum_{j=0}^{\infty} \frac{b''(f(j, s))f'(j, s)^2 + b'(f(j, s))f''(j, s)}{l^{j+1} b(f(j, s))},$$

and since $b''(s) \le f''(1)/p_l$, we obtain

$$\bar{R}_t'(s) < \frac{f''(1)}{lp_l} \sum_{j=0}^{\infty} \frac{f'(j, 1)^2}{l^j} + \frac{f'(1)}{lp_l} \sum_{j=0}^{\infty} \frac{f''(j, 1)}{l^j}.$$

We finish the proof by verifying that $\sum_{j=0}^{\infty} f''(j,1) < \infty$. Indeed, by the chain rule,

$$\begin{aligned} f''(j+1,1) &= \sum_{i=0}^{j} f'(i,1)^2 f''(f(i,1)) f'(f(i+1,1)) \cdots f'(f(j,1)) \\ &\leq f''(1) \sum_{i=0}^{j} \gamma_0^2 \cdots \gamma_{i-1}^2 \gamma_{i+1} \cdots \gamma_j, \end{aligned}$$

and because $\gamma_j \to 0$ as $j \to \infty$, we have

$$\sum_{j=0}^{\infty} \sum_{i=0}^{j} \gamma_0^2 \cdots \gamma_{i-1}^2 \gamma_{i+1} \cdots \gamma_j < \infty.$$

6 Proofs of Theorem 4 and Propositions 5, 6 and 7

For a sequence of defective GW-processes with reproduction laws $f_n(\cdot)$, we have

$$P(T_n > t) = f_n(t,1) - f_n(t,0),$$

and by (25),

$$E(e^{-\lambda Z_n(t-k)} | T_n > t) = \frac{f_n(t-k, e^{-\lambda} f_n(k,1)) - f_n(t-k, e^{-\lambda} f_n(k,0))}{f_n(t,1) - f_n(t,0)}, \qquad (26)$$

so that in particular,

$$E(e^{-\lambda Z_n(t)} | T_n > t) = \frac{f_n(t, e^{-\lambda}) - f_n(t,0)}{f_n(t,1) - f_n(t,0)}.$$

6.1 Proof of Theorem 4

Relation (12) is easily extended to the iterations of the generating functions

$$f_n(t,s) = r_n \hat{f}(t, s/r_n).$$

Therefore, if $\ln r_n \sim x/C(t_n)$, then

$$f_n(t_n, e^{-\lambda/C(t_n)}) = (1+o(1))\hat{f}(t_n, e^{-(\lambda+x+o(1))/C(t_n)}), \quad n\to\infty.$$

On the other hand, by (10) and

$$E(e^{-\lambda \hat{Z}(t)/C(t)}|\hat{T}_0 > t) = \frac{\hat{f}(t, e^{-\lambda/C(t)}) - \hat{f}(t,0)}{1-\hat{f}(t,0)},$$

we get

$$\hat{f}(t, e^{-\lambda/C(t)}) \to \hat{q} + (1-\hat{q})\Psi(\lambda), \quad t\to\infty.$$

This and the previous relation lead to the assertion of Theorem 4-a.

Turning to the proof of Theorem 4-b, observe that by (11),

$$P(e^{-ub^t}\hat{Z}(t) < z|\hat{T}_0 > t) \to \psi(u), \quad u\in(0,\infty), \quad z\in(0,\infty),$$

and therefore, for $\lambda \geq 0$,

$$\hat{f}(t, e^{-\lambda e^{-ub^t}}) \to \hat{q} + (1-\hat{q})\psi(u), \quad t\to\infty,$$

implying

$$\hat{f}(t, e^{-e^{-(u+o(1))b^t}}) \to \hat{q} + (1-\hat{q})\psi(u), \quad t\to\infty. \tag{27}$$

If for some sequence $t_n \to \infty$,

$$\ln(1/r_n) = -e^{-(y+o(1))b^{t_n}}, \quad y \in (0,\infty), \quad n \to \infty,$$

then for fixed positive λ and u, we can write

$$f_n(t_n, e^{-\lambda e^{-ub^{t_n}}}) = (1+o(1))\hat{f}(t_n, \exp\{-e^{-(u+o(1))b^{t_n}} - e^{-(y+o(1))b^{t_n}}\}), \quad n \to \infty.$$

Applying (27) we conclude that

$$f_n(t_n, e^{-\lambda e^{-ub^{t_n}}}) \to \hat{q} + (1-\hat{q})\psi(u \wedge y), \quad n \to \infty,$$

yielding

$$P(e^{-ub^{t_n}} Z(t_n) < z | T_n > t_n) \to \frac{\psi(u \wedge y)}{\psi(y)}, \quad u \in (0,\infty), \quad z \in (0,\infty),$$

and eventually for $u \in (0, y)$,

$$P(b^{-t_n} \ln Z_n(t_n) \leq u | T_n > t_n) \to \psi(u)/\psi(y), \quad n \to \infty.$$

6.2 Proof of Proposition 5

Here we deal with the sequence

$$f_n(t_n - k, s) = r_n - \left[\gamma_n^{t_n-k}(r_n - s)^{-\theta_n} + (1-\gamma_n^{t_n-k})(r_n - q_n)^{-\theta_n}\right]^{-1/\theta_n}, \tag{28}$$

assuming $\gamma_n \to \gamma \in (0,1)$, $\theta_n \to \theta \in (0,1]$, $q_n \to q \in [0,1)$, and $r_n \to 1$ so that (14) holds. Note that the convergence $\gamma_n \to \gamma \in (0,1)$ implies $\gamma_n^{t_n} \to 0$. Proposition 5-a directly follows from two relations

$$\begin{aligned} f_n(t_n,1) &= r_n - \left[\gamma_n^{t_n}(r_n-1)^{-\theta_n} + (1-\gamma_n^{t_n})(r_n-q_n)^{-\theta_n}\right]^{-1/\theta_n} \\ &\to 1-(1-q)\left[1+(1-q)^\theta x^{-\theta}\right]^{-1/\theta}, \\ f_n(t_n,0) &= r_n - \left[\gamma_n^{t_n} r_n^{-\theta_n} + (1-\gamma_n^{t_n})(r_n-q_n)^{-\theta_n}\right]^{-1/\theta_n} \to q. \end{aligned}$$

Turning to Proposition 5-b, let $k \geq 0$ and $t_n - k \to \infty$. In view of (26), we have to show that putting $\hat{\gamma}_n = \gamma_n^{\frac{t_n-k}{\theta_n}}$,

$$\begin{aligned} &f_n(t_n-k, e^{-\lambda\hat{\gamma}_n} f_n(k,1)) \to 1-(1-q)\left(1+(1-q)^\theta(\lambda+x)^{-\theta}\right)^{1/\theta}, \\ &f_n(t_n-k, e^{-\lambda\hat{\gamma}_n} f_n(k,0)) \to q. \end{aligned}$$

The second convergence is easily obtained from (28) using the following limit that holds for $n \to \infty$ and each $k \geq 0$,

$$f_n(k,0) = r_n - (\gamma_n^k r_n^{-\theta_n} + (1-\gamma_n^k)(r_n-q_n)^{-\theta_n})^{-1/\theta_n} \to 1-(\gamma^k + (1-\gamma^k)(1-q)^{-\theta})^{-1/\theta}.$$

The first convergence is also obtained from (28) using the following asymptotic formulas. Since for each $k \geq 0$, $\gamma_n^{-k/\theta_n}(r_n-1) \to 0$ as $n \to \infty$, we have

$$1-f_n(k,1) \sim 1-r_n + (\gamma_n^k(r_n-1)^{-\theta_n})^{-1/\theta_n} \sim (r_n-1)(\gamma_n^{-k/\theta_n}-1).$$

Thus, for each $k \geq 0$,

$$r_n - e^{-\lambda\hat{\gamma}_n} f_n(k,1) \sim \lambda \gamma_n^{\frac{t_n-k}{\theta_n}} + (r_n-1)\gamma_n^{-k/\theta_n}, \quad \text{as } n \to \infty,$$

implying

$$\gamma_n^{t_n-k}\big(r_n - e^{-\lambda\hat{\gamma}_n} f_n(k,1)\big)^{-\theta_n} \sim \big(\lambda + (r_n-1)\gamma_n^{-\frac{t_n}{\theta_n}}\big)^{-\theta_n} \to (\lambda+x)^{-\theta}, \quad \text{as } n \to \infty.$$

6.3 Proof of Proposition 6

Here we deal with the sequence

$$f_n(t,s) = r_n - (r_n - q_n)^{1-\gamma_n^t}(r_n - s)^{\gamma_n^t},$$

as $\gamma_n \to \gamma \in (0,1)$, $q_n \to q \in [0,1)$, and $r_n \to 1$. We assume that (18) holds for some $t_n \to \infty$.

Condition (18) gives

$$(r_n - 1)^{\gamma_n^{t_n}} \to e^{-y},$$

which implies

$$f_n(t_n,1) = r_n - (r_n - q_n)^{1-\gamma_n^{t_n}}(r_n-1)^{\gamma_n^{t_n}} \to 1 - (1-q)e^{-y},$$
$$f_n(t_n,0) = r_n - (r_n - q_n)^{1-\gamma_n^{t_n}} r_n^{\gamma_n^{t_n}} \to q.$$

yielding Proposition 6-a.

Let $k \geq 0$ and $t_n - k \to \infty$. To prove Proposition 6-b it suffices to show that putting $\hat{r}_n = (r_n - 1)^{uy^{-1}\gamma_n^k}$,

$$E\left(e^{-\lambda \hat{r}_n Z_n(t_n-k)} | T_n > t_n\right) \to \frac{1-e^{-u}}{1-e^{-y}}, \quad n \to \infty,$$

for $\lambda \geq 0$ and $u \in [0, y]$. This in turn, follows from

$$f_n\left(t_n - k, e^{-\lambda \hat{r}_n} f_n(k,1)\right) \to 1 - (1-q)e^{-u},$$
$$f_n\left(t_n - k, e^{-\lambda \hat{r}_n} f_n(k,0)\right) \to q,$$

which we prove next. The first of these two relations is obtained as follows: using

$$1 - f_n(k,1) \sim (r_n - 1)^{\gamma_n^k}(1-q)^{1-\gamma_n^k},$$

and taking into account that $u \leq y$, we get

$$\left(r_n - e^{-\lambda \hat{r}_n} f_n(k,1)\right)^{\gamma_n^{t_n-k}} \sim \left(r_n - 1 + \lambda \hat{r}_n + (r_n-1)^{\gamma_n^k}(1-q)^{1-\gamma_n^k}\right)^{\gamma_n^{t_n-k}}$$
$$\sim (\lambda \hat{r}_n)^{\gamma_n^{t_n-k}} \to e^{-u},$$

and, as a consequence,

$$f_n\left(t_n - k, e^{-\lambda \hat{r}_n} f_n(k,1)\right) = r_n - (r_n - q_n)^{1-\gamma_n^{t_n-k}} \left(r_n - e^{-\lambda \hat{r}_n} f_n(k,1)\right)^{\gamma_n^{t_n-k}}$$
$$\to 1 - (1-q)e^{-u}.$$

The second relation follows from

$$f_n(k,0) = r_n - r_n^{\gamma_n^k}(r_n - q_n)^{1-\gamma_n^k} \to 1 - (1-q)^{1-\gamma^k}.$$

6.4 Proof of Proposition 7

Here we deal with the sequence

$$f_n(t,s) = A_n - \left[\gamma_n^t (A_n - s)^{|\theta_n|} + (1-\gamma_n^t)(A_n - q_n)^{|\theta_n|}\right]^{1/|\theta_n|},$$

as $\gamma_n \to \gamma \in (0,1)$, $q_n \to q \in [0,1)$, $A_n \to 1$, and $\theta_n \to 0$. We assume that (22) holds for some $t_n \to \infty$.

Propositions 7-a and 7-b$_2$ are proven similarly to Proposition 6. To prove Proposition 7-b$_1$, fix $k \geq 0$ and let $t_n - k \to \infty$. We write $\hat{u}(x) = -x \ln(1 - u/x)$ and also

$$\hat{\theta}_n = (1 - uy^{-1}\gamma_n^k)^{y\gamma_n^{-t_n}}.$$

It suffices to show that

$$E\left(e^{-\lambda\hat{\theta}_n Z_n(t_n-k)} | T_n > t_n\right) \to \frac{1-e^{-u}}{1-e^{-y(1-e^{-a})}}, \quad n \to \infty,$$

for $\lambda \geq 0$ and $u \in [0, y(1-e^{-a}))$, or in terms of generating functions,

$$f_n\left(t_n - k, e^{-\lambda\hat{\theta}_n} f_n(k,1)\right) \to 1 - (1-q)e^{-u},$$
$$f_n\left(t_n - k, e^{-\lambda\hat{\theta}_n} f_n(k,0)\right) \to q.$$

We finish the proof by checking only the first of these two relations.

Since

$$A_n - f_n(k,1) = \left[(A_n - q_n)^{|\theta_n|} - \gamma_n^k\left(1 - (A_n - 1)^{|\theta_n|}\right)\right]^{1/|\theta_n|} = \left[1 - \gamma^k(1-e^{-a}) + o(1)\right]^{1/|\theta_n|},$$

we get

$$\left(A_n - e^{-\lambda\hat{\theta}_n} f_n(k,1)\right)^{|\theta_n|} = \left(\left[1 - \gamma^k(1-e^{-a}) + o(1)\right]^{1/|\theta_n|} + (\lambda + o(1))\hat{\theta}_n\right)^{|\theta_n|}.$$

Using

$$\hat{\theta}_n^{|\theta_n|} \to 1 - uy^{-1}\gamma^k,$$

and $u < y(1-e^{-a})$, we obtain

$$\begin{aligned} f_n\left(t_n - k, e^{-\lambda\hat{\theta}_n} f_n(k,1)\right) &= 1 - (1-q)\left(1 - (u/y + o(1))\gamma_n^{t_n}\right)^{1/|\theta_n|}(1 + o(1)) \\ &\to 1 - (1-q)e^{-u}, \end{aligned}$$

since $(1-\gamma_n^{t_n})^{1/|\theta_n|} \to e^{-y}$ due to condition (22).

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