# Branching processes in varying environment with generation dependent immigration*

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## Abstract

A branching process in varying environment with generation-dependent immigration is a modification of the standard branching process in which immigration is allowed and the reproduction and immigration laws may vary over the generations. This flexibility makes the process more appropriate to model real populations due to the fact that the stability in the reproductive capacity and in the immigration laws are not usually fulfilled. In this setting, we study the extinction problem, providing a necessary and sufficient condition for the certain extinction of the population. The asymptotic behaviour of the model is analysed for those processes with critical offspring distributions, according with the classification established in [16], and when the immigration means stabilize to a positive value. More specifically, we establish that the asymptotic distribution of the process -under a suitable normalization- belongs to the gamma distribution family.

**Keywords**: branching process; varying environment; inhomogeneous immigration; extinction; asymptotic distribution.

## 1 Introduction

Since the appearance of the pioneer branching model, the Bienaymé-Galton-Watson process (BGWP), and motivated by the complexity of the problems arisen in diverse fields,

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1

new branching processes have been proposed with the aim of giving an answer to them. In particular, special attention has been paid to the problem of modelling migratory movements. Within the class of discrete time models, it is worthwhile to mention the branching process with immigration, the branching process with time-dependent immigration, the branching process with immigration only after empty generation, branching processes with migration or the branching process with immigration in varying or random environment (for their description, see the pioneer works [11, 12], [4], [22], [21] and [18], respectively, or [6] for a review of them) and the age-dependent branching process with immigration or its generalization, the Sevast'yanov branching process with immigration (see [15] and [28], respectively, for their description), in continuous time among others.

There are several works where some of those models were considered for modelling in real situations. For instance, in [9] a multitype continuous-time branching process with immigration is used to describe a retrial queue, where the birth and death of the individuals represent the arrival and departure of a customer, respectively, and the situations when a customer is immediately served after the arrival into the system are modelled through the immigration. In the context of cell kinetics, a two-type age-dependent branching process with emigration is used to model the emigration of oligodendrocyte cells cultured in vitro out of the field of observation in [30]; more recently, in [7], a controlled two-type branching process with binomial control is used to describe its discrete branching structure as a result of an embedding of the aforesaid model. Another example is the application of an age-dependent branching process with time-inhomogeneous immigration for modelling the progression of Leukemia in mice in [13]; indeed, this process models the number of Leukemia cells in the blood and the immigration process represents the influx of cells from other tissues of the body.

This work is concerned with the branching process in a varying environment with generation-dependent immigration (BPVEI). This model represents a twofold generalization of the BGWP: on the one hand, we allow the reproductive capacity of the individuals to change over the generations and on the other hand, an inhomogeneous immigration is included into the probability model. These features make the process more appealing from a practical standpoint. Firstly, it is more natural to assume that the probability distribution governing the reproduction, referred as offspring distribution, may change over the time because the reproduction capacity of the individuals might be affected by factors such as, for instance, the resource supply or the weather conditions, which may vary over seasons. An example of such situations is the genus *Bicyclus* of butterflies of the sub-Saharan Africa, whose reproduction capacity changes from the wet season to the dry-season due to the adaptation to the environment (see [10], p.49, for further details). Secondly, the interest of considering an inhomogeneous immigration is due to the fact that in practical situations it is unknown whether the distribution of the influx of individuals in the population is constant or changes over the generations, and hence, the assumption of an homogeneous immigration might lead to a misspecification of the model. An example of such populations is a renewal process for cell populations of the central nervous system which comprise stem cells, progenitor cells and terminally differentiated cells. In such populations, only the stem and progenitor cells have reproductive capacity, whereas

the terminally differentiated cells arise as a result of the maturation of progenitor cells. Since any of these types of cells could die at any time, an immigration of stem cells -and their development into progenitor cells- supplies the amount of cells needed to maintain the tissues. Moreover, the reproduction and immigration rates of stem cells is believed to depend on the renewing tissue and on the time according to the experimental evidences, and hence the need of an inhomogeneous immigration in the model (see [29] for further details).

While a great number of papers on branching processes in varying environment (BPVE), or on branching processes with inhomogenous immigration is available (see, for instance, [3], [2], [16] or [17] for the first model and [4] and [24, 25, 26, 27] for the second one), in relation to the research on the BPVEI one only can find the works [5] and [14]. The former concerns a central limit theorem and a law of the iterated logarithm for the BPVEI in the supercritical case according to the classification in [16], whereas in the second paper the author obtained a Feller diffusion approximation when assuming certain convergence of the parameters of both the offspring and immigration laws in the critical case. The paper [20] also deals with a BPVE, but with homogeneous immigration and considering a very specific offspring distributions for such a study.

The main goals of this work are to analyse the extinction problem and the asymptotic behaviour for the general family of BPVEIs. Previously to the study of those questions, we establish some basic properties on the moments and the probability generating functions involved in the model. In relation to the extinction, it is important to highlight that although 0 is not an absorbing state of the process, it is possible to state conditions for the population size to reach the state 0 and to remain there forever. As we illustrate, this result also holds true when either the offspring distribution or the immigration laws are homogeneous. In order to examine the asymptotic behaviour of the process, we focus our attention on the family of BPVEIs with critical offspring distributions according to the classification in [16]. For those processes, we prove their convergence in distribution -once suitably normalized- to a gamma distribution when the immigration means and the normalized second factorial moments of the offspring distributions converge to positive values.

Apart from this introduction, this paper consists of three sections and an appendix. In Section 2, we describe the probability model and some basic properties regarding the moments and probability generating functions of the process. Section 3 is devoted to the extinction problem and Section 4 is dedicated to the study of the asymptotic behaviour of the process. In order to ease the reading, we dedicate an appendix to the proofs of the results.

# 2 Probability model and basic properties

In order to define a BPVEI, let us consider two independent families of $\mathbb{N}_0$-valued random variables, $\{X_{nj} : n \in \mathbb{N}_0; j \in \mathbb{N}\}$ and $\{I_n : n \in \mathbb{N}_0\}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Assume that both of them are made up by independent random variables such that for each $n \in \mathbb{N}_0$

fixed, $X_{nj}$, $j \in \mathbb{N}$, are identically distributed according to the probability generating function (p.g.f.) $f_n(s) = \sum_{k=0}^{\infty} f_n[k]s^k$, with $f_n(0) < 1$ and $f_n[k] = P[X_{n1} = k]$, and $I_n$ is distributed according to the p.g.f. $h_n(s) = \sum_{k=0}^{\infty} h_n[k]s^k$, with $h_n(0) < 1$ and $h_n[k] = P[Y_n = k]$, for $n, k \in \mathbb{N}_0$. The process $\{Z_n\}_{n \in \mathbb{N}_0}$ defined recursively as

$$Z_0 = 0, \quad Z_{n+1} = \sum_{j=1}^{Z_n + I_n} X_{nj}, \quad n \in \mathbb{N}_0,$$

is called *branching process in a varying environment* $v = \{f_0, f_1, f_2, \ldots\}$ *with generation-dependent immigration* and initial value $Z_0 = 0$. The distributions $f_n = \{f_n[k]\}_{k \in \mathbb{N}_0}$ and $h_n = \{h_n[k]\}_{k \in \mathbb{N}_0}$ are called offspring distribution or reproduction law and immigration law, respectively, of the $n$-th generation. Notice that we use the same notation for the probability distributions and the corresponding p.g.f.s.

Intuitively, $Z_n$ represents the number of individuals in the $n$-th generation, $I_n$ is the number of immigrants in the $n$-th generation and $X_{nj}$ the number of offspring of the $j$-th progenitor in the $n$-th generation. Observe that the assumption $Z_0 = 0$ indicates that the population starts with the offspring of the immigrants in the initial generation. Moreover, in the development of this model one distinguishes two phases in each generation: the immigration phase, when new individuals migrate into the population, and the reproduction phase, when the progenitors have their offspring. Notice that we define the process as a controlled branching process in varying environment (see [8]) with an inhomogeneous control function $\phi_n(k) = k + I_n$, with $k, n \in \mathbb{N}_0$.

It is easy to verify that a BPVEI is an inhomogeneous Markov chain. We shall fix the notation for the remainder of the paper. Let us denote by $F_n$ the p.g.f. of the variable $Z_n$, $n \in \mathbb{N}_0$, and $f_{k,n} = f_{k+1} \circ \ldots \circ f_n$, for $k = -1, 0, \ldots, n-1$, with the convention that $f_{n,n}(s) = s$, $s \in [0, 1]$. Let us write $m_n = E[X_{n1}]$ and $\sigma_n^2 = Var[X_{n1}]$ for the offspring mean and variance in generation $n$, respectively, and assume that both of them are positive and finite; analogously, denote by $\alpha_n = E[I_n]$ and $\beta_n^2 = Var[I_n]$ the expectation and the variance of the number of immigrants in the $n$-th generation, which are assumed to satisfy $0 < \alpha_n < \infty$ and $0 < \beta_n^2 < \infty$. For $n \in \mathbb{N}_0$, we also write $\mu_n = \prod_{i=0}^{n} m_i$, and $\nu_n = f_n''(1)/f_n'(1)^2$, with the convention $\mu_{-1} = 1$.

Having set the notation, in the following result we establish the relations between the p.g.f.s and the moments of the BPVEI and the p.g.f.s and moments of the reproduction and immigration laws, some of which will be useful for the analysis that we develop in the following sections.

**Proposition 1.** *The following expressions hold true:*

(a) $F_{n+1}(s) = \prod_{i=0}^{n} h_i(f_{i-1,n}(s))$, *for* $n \in \mathbb{N}_0$ *and* $s \in [0, 1]$.

(b) *For* $n \in \mathbb{N}_0$,

(i) $E[Z_{n+1}] = \sum_{i=0}^{n} \alpha_{n-i} \prod_{j=0}^{i} m_{n-j}$.

(ii) $Var[Z_{n+1}] = \sum_{i=0}^{n} \beta_i^2 \prod_{j=i}^{n} m_j^2 + \sum_{i=0}^{n} \prod_{j=i+1}^{n} m_j^2 \sigma_i^2 \left( \alpha_i + \sum_{k=0}^{i} \alpha_{i-k} \prod_{l=0}^{k} m_{i-l} \right)$.

A BPVEI can be also seen in the following way: each immigrant, when incorporating into the population and independently of the other immigrants, produces a BPVE with initial value the unity. The next proposition formally describes this idea.

**Proposition 2.** *Let us consider the independent processes $\{\widetilde{Z}_k^{(j)}\}_{k\in\mathbb{N}_0}$, $j \in \mathbb{N}_0$, defined as:*

$$\widetilde{Z}_0^{(j)} = I_j, \quad \widetilde{Z}_{k+1}^{(j)} = \sum_{i=1}^{\widetilde{Z}_k^{(j)}} X_{ki}^{(j)}, \quad k \in \mathbb{N}_0, \tag{1}$$

*where $\{X_{ki}^{(j)} : k \in \mathbb{N}_0; i \in \mathbb{N}; j \in \mathbb{N}_0\}$ is a family of independent random variables such that for each $k \in \mathbb{N}_0$ and $j \in \mathbb{N}_0$ fixed, $X_{ki}^{(j)}$, $i \in \mathbb{N}$ are distributed according to the p.g.f. $f_{k+j}$. Then*

$$Z_n \overset{d}{=} \sum_{j=0}^{n-1} \widetilde{Z}_{n-j}^{(j)}, \quad n \in \mathbb{N},$$

*where $\overset{d}{=}$ denotes "equal in distribution".*

Intuitively, the previous proposition can be interpreted as follows: in each generation $j$, the immigrants $I_j$ generate a BPVE $\{\widetilde{Z}_k^{(j)}\}_{k\in\mathbb{N}_0}$, with environment $v_j = \{f_j, f_{j+1}, \ldots\}$ and which starts with those immigrants $I_j$. Then, the distribution of the number of individuals of the BPVEI at generation $n$ is equal to the distribution of the sum of the total number of descendants in generation $n-j$ of the BPVEs produced by the immigrants in generation $j$, where $j$ goes from the initial generation to the $(n-1)$-st generation. This representation will be useful in Proposition 3, where we establish a necessary and sufficient condition for the certain extinction of the process.

# 3   Extinction problem

In this section we address the extinction problem of a BPVEI. It is well known that in the BGWP the incorporation of immigration independent of the generation into the model prevents from the extinction of the population. Similarly, for the BPVEI it is easy to see that 0 is not an absorbing state. Indeed, for each $n \in \mathbb{N}_0$, since $h_n(f_n(0)) < 1$,

$$P[Z_{n+1} > 0 | Z_n = 0] = 1 - P\left[\sum_{i=0}^{I_n} X_{ni} = 0\right] = 1 - \sum_{k=0}^{\infty} f_n(0)^k P[I_n = k] = 1 - h_n(f_n(0)) > 0.$$

However, in this case we can establish some conditions under which the process reaches the state 0 and stays there forever, which can be interpreted as the extinction of the population. To that end, let us denote the probability of this event by

$$q = P\left[\bigcup_{n=0}^{\infty} \bigcap_{j=n}^{\infty} \{Z_j = 0\}\right].$$

**Proposition 3.** *The probability $q = 1$ if and only if*

$$\lim_{n \to \infty} f_{-1,n}(0) = 1 \quad and \quad \sum_{j=0}^{\infty}(1 - h_j(f_j(0))) < \infty. \tag{2}$$

Note that $f_{-1,n}$ would be the p.g.f. of $Z_n$ if $I_0 = 1$ a.s. and $I_n = 0$ a.s. for all $n \in \mathbb{N}$, i.e., if the population starts with the offspring of a single immigrant in the initial generation and there is no immigration after that time. Thus, intuitively, this result indicates that the a.s. extinction of the population happens when the process without immigration becomes extinct with probability one and from some generation, the probability that there are new immigrants in the population and they produce any offspring in the following generations is negligible. This interpretation is illustrated in the examples provided below.

**Remark 1.** *(a) It is worthwhile to mention that Proposition 3 can be applied even when either the immigration or the offspring distribution are independent of the generation. Indeed, the second condition can also hold true in those cases as we show below, but not for the BGWP with homogeneous immigration.*

*(b) In [16], Theorem 1 provides necessary and sufficient criteria for the first condition in (2) to hold under a mild assumption on the family $\{X_{nj} : n \in \mathbb{N}_0; j \in \mathbb{N}\}$. In the particular case that the offspring distribution is independent of the generation, it is well known that the aforementioned condition holds if and only if $m \leq 1$, where $m$ is the offspring mean (see [1], p.4).*

*(c) The second requirement in (2) is not difficult to verify. For example, using the mean value theorem, one can prove that a sufficient condition is that the sequence of immigration means is bounded and the series $\sum_{j=0}^{\infty}(1 - f_j(0))$ converges.*

**Examples 1.** *(a) Let us consider a BPVEI with $h_n(s) = 2^{-1}(1 + s)$ for $n = 0, 1$, and $h_n(s) = 1 - n^{-2} + n^{-2}s$, $n \geq 2$, whose immigration mean satisfies $\alpha_n = n^{-2} \to 0$. Observe that for any environment $v = \{f_0, f_1, \ldots\}$, this process fulfils the second condition in Proposition 3. Indeed,*

$$\sum_{n=0}^{\infty}(1 - h_n(f_n(0))) = 1 - \frac{f_0(0) + f_1(0)}{2} + \sum_{n=2}^{\infty}\frac{1 - f_n(0)}{n^2} \leq 1 + \sum_{n=2}^{\infty}\frac{1}{n^2} < \infty.$$

*Thus, if $v$ is such that $f_{-1,n}(0) \to 1$, as $n \to \infty$, then $q = 1$ and otherwise, $0 \leq q < 1$. Note that one can even choose an offspring distribution independent of the generation.*

*(b) Let us consider a BPVEI with $h_n(s) = 2^{-1}(1 + s)$, whose immigration mean is $\alpha_n = 2^{-1}$, $n \in \mathbb{N}_0$ and $f_n(s) = 2^{-1}(s + 1)$, for $n = 0, 1$, and $f_n(s) = 1 - n^{-2} + n^{-2}s$, for $n \geq 2$. Note that in this case, the immigration is independent of the generation and the iteration of the offspring p.g.f.s leads to $f_{-1,n}(s) = 1 - 4^{-1}(n!)^{-2} + 4^{-1}(n!)^{-2}s$, $n \geq 1$, from which one obtains $f_{-1,n}(0) \to 1$, as $n \to \infty$. Moreover, the process*

6

*also satisfies the second condition in Proposition 3, as we show below, and as a consequence, $q = 1$.*

$$\sum_{n=0}^{\infty}(1 - h_n(f_n(0))) = \sum_{n=0}^{\infty}\frac{1 - f_n(0)}{2} = \frac{1}{2} + \sum_{n=2}^{\infty}\frac{1}{2n^2} < \infty.$$

*(c) Let us consider a BPVEI with the same immigration laws as in the previous example but offspring distributions $f_n(s) = 2^{-1}(s + 1)$, for $n = 0, 1$, and $f_n(s) = 1 - n^{-1} + n^{-1}s$, for $n \geq 2$. With the same arguments as before, one can show that $0 \leq q < 1$ since $f_{-1,n}(s) = 1 - 4^{-1}(n!)^{-1} + 4^{-1}(n!)^{-1}s$, $n \geq 1$, implying $f_{-1,n}(0) \to 1$, as $n \to \infty$, and the second condition in Proposition 3 does not hold true due to the fact that*

$$\sum_{n=0}^{\infty}(1 - h_n(f_n(0))) = \sum_{n=0}^{\infty}\frac{1 - f_n(0)}{2} = \frac{1}{2} + \sum_{n=2}^{\infty}\frac{1}{2n} = \infty.$$
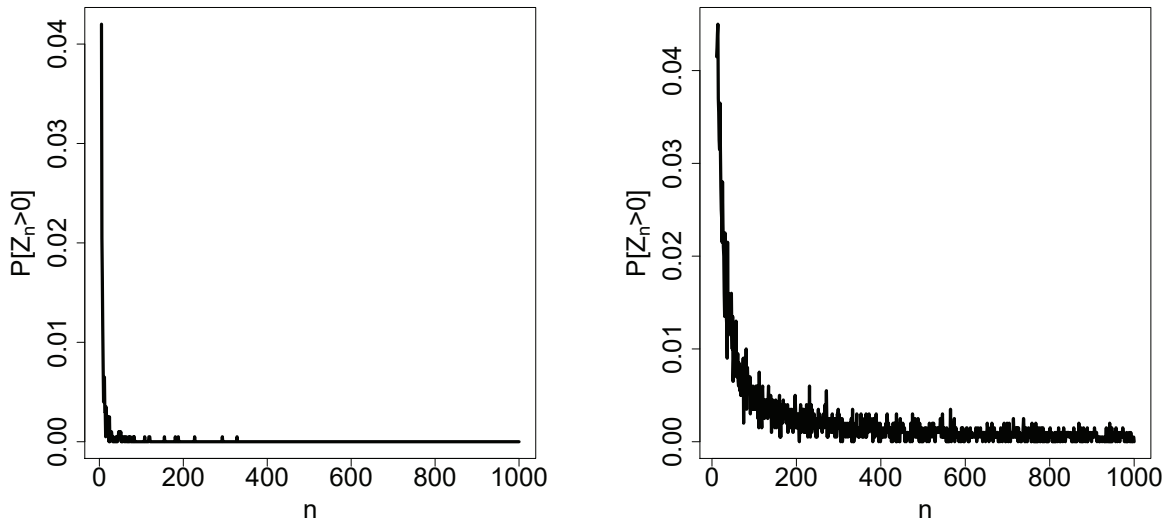


Figure 1: Comparison of the evolution of the probability $P[Z_n > 0]$ for the models described in Examples 1 *(b)* (left) and 1 *(c)* (right). The probabilities $P[Z_n > 0]$, $n \in \mathbb{N}$, were estimated by Monte Carlo methods at each generation $n \in \mathbb{N}$. To that end, the first 1000 generations of 2000 processes following the models in the aforementioned examples were simulated using the statistical software and programming environment R (see [23]).

# 4 Asymptotic behaviour

In this section, we analyse the limiting behaviour of BPVEIs focusing our attention on those with critical offspring distributions according to the classification established in [16]

for BPVEs, that is, with reproduction laws satisfying

$$\sum_{k=0}^{n} \frac{\nu_k}{\mu_{k-1}} \to \infty \quad \text{and} \quad \frac{1}{\mu_n} = o\left(\sum_{k=0}^{n} \frac{\nu_k}{\mu_{k-1}}\right), \quad \text{as } n \to \infty. \tag{3}$$

**Remark 2.** *Note that in* (3) *the convergence of the sequence* $\{\mu_n\}_{n\in\mathbb{N}_0}$ *is not required; the limit may not exist, or it may be 0, infinity, or any positive value. Condition* (3) *is rather technical and has been established taking into account the asymptotic behaviour of the BPVE in comparison with a BGWP. A deeper discussion on it can be found in [16]. A sufficient condition for* (3) *is that* $\{\mu_n\}_{n\in\mathbb{N}_0}$ *and* $\{f_n''(1)\}_{n\in\mathbb{N}_0}$ *are bounded sequences from below and above with lower bounds which are greater than 0. For instance, under the condition given in [14] for the strong criticality, that is,* $\sum_{n=0}^{\infty} |1 - m_n| < \infty$, *one obtains that* $\{\mu_n\}_{n\in\mathbb{N}_0}$ *converges to a finite and positive limit. As a consequence, if the variances of the offspring distributions are bounded and its infimum is greater than 0, the previous sufficient condition for* (3) *is fulfilled.*

Moreover, it is important to remark that in contrast to the BGWP, for BPVEs one might encounter different growth rates (see [19]). In order to avoid that situation, we consider the following regularity condition introduced in [16] for the BPVE and which is satisfied by a wide class of probability distributions: for every $\epsilon > 0$ there is a constant $c_\epsilon < \infty$ such that for all $n \in \mathbb{N}_0$,

$$E[X_{n1}^2; X_{n1} > c_\epsilon(1 + E[X_{n1}])] \leq \epsilon E[X_{n1}^2; X_{n1} \geq 2]. \tag{4}$$

The following result is a counterpart of Theorem VI.7.4 in [1] for BGWPs with immigration independent of the generation and establishes the asymptotic distribution of a BPVEI suitably normalized when the immigration means stabilize.

**Theorem 4.** *Let* $\{Z_n\}_{n\in\mathbb{N}_0}$ *be a BPVEI satisfying* (3) *and* (4) *and denote*

$$a_{n+1} = \frac{\mu_n}{2} \sum_{k=0}^{n} \frac{\nu_k}{\mu_{k-1}}, \quad n \in \mathbb{N}_0.$$

*Assume that* $\nu_n \to \nu > 0$ *and* $\alpha_n \to \alpha > 0$, *as* $n \to \infty$, $0 < \tau = \inf_{n\in\mathbb{N}_0} h_n(0)$ *and* $\sup_{n\in\mathbb{N}_0} \beta_n^2 < \infty$. *Then, the asymptotic distribution of* $Z_n/a_n$ *is a gamma distribution with parameters* $2\alpha/\nu$ *and* 1.

**Corollary 5.** *Let* $\{Z_n\}_{n\in\mathbb{N}_0}$ *be a BPVEI satisfying the conditions of Theorem 4, then*

$$P[Z_n > 0] \to 1, \quad \text{as } n \to \infty.$$

Now, we shall provide a description of the relation between the asymptotic behaviour of the BPVEI given by Theorem 4 and the asymptotic behaviour of the BPVEs $\{\widetilde{Z}_k^{(j)}\}_{n\in\mathbb{N}_0}$

introduced in Proposition 2. First, note that if $\{Z_n\}_{n \in \mathbb{N}_0}$ is a BPVEI with critical offspring distribution, then the processes $\{\widetilde{Z}_k^{(j)}\}_{n \in \mathbb{N}_0}$ are critical BPVEs. On the other hand, at each generation $j$, $\alpha_j$ individuals are expected to immigrate to the population. Since from Theorem 4 in [16] one has that the probability that the BPVE produced by one immigrant at generation $j$, $\{\bar{Z}_n^{(j)}\}_{n \in \mathbb{N}_0}$, is not extinct after $n$ generations is $P[\overline{Z}_n^{(j)} > 0] \sim 2\left(\sum_{k=j}^{j+n} \frac{\nu_k}{\mu_{k-1}}\right)^{-1}$, as $n \to \infty$, then one obtains that $P[\widetilde{Z}_n^{(j)} > 0] \sim 2\alpha_j \left(\sum_{k=j}^{j+n} \frac{\nu_k}{\mu_{k-1}}\right)^{-1}$. Moreover, Theorem 4 in [16] provides the asymptotic distribution of those BPVEs; indeed, for each $j \in \mathbb{N}_0$, each one of these BPVEs satisfies that the distribution of the process $\bar{Z}_n^{(j)}/a_n^{(j)}$ conditioned on the event $\{\bar{Z}_n^{(j)} > 0\}$ converges to an exponential distribution with mean equal to one, where

$$a_n^{(j)} = \frac{\mu_{j+n}}{2\mu_{j-1}} \sum_{k=j}^{j+n} \frac{\nu_k}{\mu_{k-1}}, \quad n \geq j.$$

We also remark here that Theorem 4 can be applied even in the cases when one of the distributions of the model -either the offspring distribution or the immigration law- is inhomogeneous but the other one is homogeneous. However, Theorem 4 cannot be applied for the model considered in [20] due to the fact that in that situation $\nu_n \to \infty$, as $n \to \infty$.

# Appendix

## Proof of Proposition 1

*(a)* Using the independence of the offspring variables and the independence between the offspring and immigration phases, one has

$$F_{n+1}(s) = E\left[E\left[s^{\sum_{i=1}^{Z_n+I_n} X_{ni}} | Z_n + I_n\right]\right] = E[f_n(s)^{Z_n+I_n}] = F_n(f_n(s))h_n(f_n(s)), \quad (5)$$

and by iterating the previous formula one obtains the expression in *(a)*.

*(b) (i)* It can be easily obtained using a recursion on $E[Z_{n+1}] = m_n(E[Z_n]+\alpha_n)$, which follows from

$$E[Z_{n+1}|Z_n] = E\left[\sum_{i=1}^{Z_n+I_n} X_{ni}|Z_n\right] = m_n(\alpha_n + Z_n) \quad a.s.$$

*(b) (ii)* To that end, similarly to (5) we have

$$Var[Z_{n+1}] = E[Var[Z_{n+1}|Z_n + I_n]] + Var[E[Z_{n+1}|Z_n + I_n]]$$
$$= E[(Z_n + I_n)\sigma_n^2] + Var[(Z_n + I_n)m_n]$$
$$= (E[Z_n] + \alpha_n)\sigma_n^2 + (Var[Z_n] + \beta_n^2)m_n^2,$$

and to obtain the desired formula we use a recursion in the following expression

$$\frac{Var[Z_{n+1}]}{\prod_{j=0}^n m_j^2} = \frac{Var[Z_n]}{\prod_{j=0}^{n-1} m_j^2} + \frac{\beta_n^2 m_n^2 + \sigma_n^2(E[Z_n] + \alpha_n)}{\prod_{j=0}^n m_j^2}.$$

## Proof of Proposition 2

Let $\widetilde{F}_n^{(j)}$ be the p.g.f. of $\widetilde{Z}_n^{(j)}$, $n, j \in \mathbb{N}_0$. Then, for $s \in [0, 1]$ and $j \in \mathbb{N}_0$, $\widetilde{F}_0^{(j)}(s) = h_j(s)$ and for each $k \in \mathbb{N}_0$,

$$\widetilde{F}_{k+1}^{(j)}(s) = E\left[E\left[s^{\widetilde{Z}_{k+1}^{(j)}}|\widetilde{Z}_k^{(j)}\right]\right] = E\left[(f_{k+j}(s))^{\widetilde{Z}_k^{(j)}}\right] = \widetilde{F}_k^{(j)}(f_{k+j}(s)) = h_j(f_{j-1,j+k}(s)).$$

As a consequence, since the processes $\{\widetilde{Z}_k^{(j)}\}_{k \in \mathbb{N}_0}$, $j \in \mathbb{N}_0$, are independent, one has

$$E\left[s^{\sum_{j=0}^{n-1}\widetilde{Z}_{n-j}^{(j)}}\right] = \prod_{j=0}^{n-1} E\left[s^{\widetilde{Z}_{n-j}^{(j)}}\right] = \prod_{j=0}^{n-1} h_j(f_{j-1,n-1}(s)) = F_n(s).$$

$\square$

## Proof of Proposition 3

First, one has $q = \lim_{n \to \infty} P\left[\cap_{j=n}^{\infty}\{Z_j = 0\}\right]$ and

$$P\left[\cap_{j=n}^{\infty}\{Z_j = 0\}\right] = P[Z_n = 0]\prod_{j=n}^{\infty} P\left[Z_{j+1} = 0|Z_j = 0\right] = F_n(0)\prod_{j=n}^{\infty} h_j(f_j(0)).$$

Thus, a necessary and sufficient condition for $q = 1$ is that both limits $F_n(0) \to 1$ and $\prod_{j=n}^{\infty} h_j(f_j(0)) \to 1$, as $n \to \infty$. Now, we prove that this condition is equivalent to that established in Proposition 3.

Note that $\prod_{j=n}^{\infty} h_j(f_j(0)) \to 1$, as $n \to \infty$ holds if and only if $\sum_{j=n}^{\infty} \log(h_j(f_j(0))) \to 0$, as $n \to \infty$, which is equivalent to the condition $\sum_{j=0}^{\infty}(1 - h_j(f_j(0))) < \infty$.

Regarding the first limit, from Proposition 1 (a) one has that if $F_n(0) \to 1$, then $h_i(f_{i-1,n}(0)) \to 1$, for each $i = 0, 1, \ldots, n$, which implies $f_{-1,n}(0) \to 1$, as $n \to \infty$.

Conversely, assume that $f_{-1,n}(0) \to 1$, as $n \to \infty$, and $\sum_{j=0}^{\infty}(1 - h_j(f_j(0))) < \infty$. To obtain that $F_n(0) \to 1$, as $n \to \infty$, taking into account the representation of the BPVEI described in Proposition 2, it is enough to prove that $P\left[\sum_{i=0}^{n-1} \widetilde{Z}_{n-i}^{(i)} > 0 \text{ i.o.}\right] = 0$, writing "i.o." to mean infinitely often. To that end, let us define the random variable $L = \max\{j \in \mathbb{N} : \widetilde{Z}_1^{(j)} > 0\}$. Since $P[\widetilde{Z}_1^{(j)} > 0] = 1 - h_j(f_j(0))$, $j \in \mathbb{N}_0$, from Borel-Cantelli's lemma,

$$P\left[\widetilde{Z}_1^{(j)} > 0 \text{ f.o.}\right] = 1,$$

where "f.o." stands for finitely often, and $L < \infty$ a.s. Consequently,

$$P\left[\sum_{i=0}^{n-1}\widetilde{Z}_{n-i}^{(i)} > 0 \text{ i.o.}\right] = P\left[\sum_{i=0}^{n-1}\widetilde{Z}_{n-i}^{(i)} > 0 \text{ i.o.}, L < \infty\right]$$

$$= \sum_{k=0}^{\infty} P\left[\sum_{i=0}^{n-1}\widetilde{Z}_{n-i}^{(i)} > 0 \text{ i.o.}, L = k\right]$$

$$\leq \sum_{k=0}^{\infty}\sum_{j=0}^{k} P\left[\widetilde{Z}_n^{(j)} > 0 \text{ i.o.}\right].$$

11

and for each $j \in \mathbb{N}_0$ fixed,

$$
\begin{aligned}
P\left[\widetilde{Z}_n^{(j)} > 0 \text{ i.o.}\right] &= 1 - P\left[\widetilde{Z}_n^{(j)} = 0 \text{ f.o.}\right] \\
&= 1 - \lim_{n \to \infty} \widetilde{F}_n^{(j)}(0) \\
&= 1 - \lim_{n \to \infty} h_j(f_{j-1,j+n-1}(0)),
\end{aligned}
$$

where recall that $\widetilde{F}_n^{(j)}(s) = h_j(f_{j-1,j+n-1}(s))$ is the p.g.f. of $\widetilde{Z}_n^{(j)}$, $n \in \mathbb{N}$. Now, since $f_{-1,n}(0) \to 1$, as $n \to \infty$, $f_{j,n}(0) \to 1$ and $h_j(f_{j-1,n}(0)) \to 1$, for each $j \in \mathbb{N}_0$, which implies $P\left[\sum_{i=0}^{n-1} \widetilde{Z}_{n-i}^{(i)} > 0 \text{ i.o.}\right] = 0$.

$\square$

**Remark 3.** *An alternative way to prove that $f_{-1,n}(0) \to 1$, as $n \to \infty$, and $\sum_{j=0}^{\infty}(1 - h_j(f_j(0))) < \infty$ imply $F_n(0) \to 1$, as $n \to \infty$, and based on the use of probability generating functions is provided below.*

*By applying Proposition 1 (a) and taking into account the monotonicity of probability generating functions and the fact that $0 \le f(s) \le 1$, for each $s \in [0,1]$, we have that for any $k \in \{0, \dots, n-1\}$,*

$$
F_{n+1}(0) = \prod_{i=0}^{n} h_i(f_{i-1,n}(0)) \ge \prod_{i=0}^{k} h_i(f_{i-1,n}(0)) \prod_{i=k+1}^{\infty} h_i(f_i(0)).
$$

*Since $\prod_{i=k+1}^{\infty} h_i(f_i(0)) \to 1$, as $k \to \infty$, for each $\epsilon > 0$ there exists $k_0 = k_0(\epsilon) \in \mathbb{N}$ such that $\prod_{i=k_0+1}^{\infty} h_i(f_i(0)) \ge 1 - \epsilon$, and consequently, for any $n > k_0$ we obtain*

$$
F_{n+1}(0) \ge (1 - \epsilon) \prod_{i=0}^{k_0} h_i(f_{i-1,k_0-1}(f_{k_0-1,n}(0)))
$$

*It is not difficult to see that $f_{-1,n}(0) \to 1$ implies that $f_{k,n}(0) \to 1$, as $n \to \infty$, for each $k \in \mathbb{N}$ fixed. Indeed, if there exists a sequence $\{n_j\}_{j \in \mathbb{N}}$ such that $f_{k,n_j}(0) \to p \in [0,1]$, as $j \to \infty$, then by the continuity of generating functions $f_{-1,k-1}(p) = \lim_{j \to \infty} f_{-1,k-1}(f_{k-1,n_j}(0)) = 1$, which implies $p = 1$. Thus, we have $\lim_{n \to \infty} F_{n+1}(0) \ge (1 - \epsilon)$, for all $\epsilon > 0$, which proves the assertion.*

In order to prove Theorem 4, we make use of the shape functions corresponding to the p.g.f.s of the reproduction laws. The *shape function* of the p.g.f. $f_k$, $k \in \mathbb{N}_0$, is the function $\varphi_k : [0,1) \to \mathbb{R}$ satisfying

$$
\frac{1}{1 - f_k(s)} = \frac{1}{m_k(1-s)} + \varphi_k(s), \quad s \in [0,1). \tag{6}
$$

This function satisfies $\varphi_k(s) \ge 0$, for $0 \le s < 1$, and can be extended by setting $\varphi_k(1) = \nu_k/2$. The reader is referred to Section 3 in [16] for further details about the properties

of these functions. Then, using the fact that $f_{k,n} = f_{k+1} \circ f_{k+1,n}$, for $k = 0, \ldots, n-1$, by iterating (6) one obtains:

$$\frac{1}{1 - f_{k,n}(s)} = \frac{\mu_k}{\mu_n(1-s)} + \mu_k \sum_{l=k+1}^{n} \frac{\varphi_l(f_{l,n}(s))}{\mu_{l-1}}, \quad s \in [0,1). \tag{7}$$

To prove Theorem 4, we also need the next lemma; its proof is similar to the one of Lemma 7 in [16] and is omitted.

**Lemma 6.** *Let $i = 0, \ldots, n$ be fixed. Let $\{Z_n\}_{n \in \mathbb{N}_0}$ be a BPVEI satisfying (3) and (4), then*

$$\sup_{s \in [0,1]} \left| \sum_{k=i}^{n} \frac{\varphi_k(f_{k,n}(s))}{\mu_{k-1}} - \sum_{k=i}^{n} \frac{\varphi_k(1)}{\mu_{k-1}} \right| = o \left( \sum_{k=i}^{n} \frac{\varphi_k(1)}{\mu_{k-1}} \right), \quad as \ n \to \infty.$$

## Proof of Theorem 4

We shall prove that the Laplace transform of $Z_n/a_n$ converges to the Laplace transform of a gamma distribution with parameters $2\alpha/\nu$ and 1. Let us fix $\lambda > 0$ and for simplicity, let us denote $s_n = e^{-\lambda/a_n}$, $n \in \mathbb{N}$. From Proposition 1 *(a)* and using a Taylor expansion for the functions $\log h_i$ at 1, for $i = 0, \ldots, n$, one has

$$F_{n+1}(s_{n+1}) = \exp \left\{ \sum_{i=0}^{n} \log(h_i(f_{i-1,n}(s_{n+1}))) \right\}$$

$$= \exp \left\{ - \sum_{i=0}^{n} \alpha_i(1 - f_{i-1,n}(s_{n+1})) \right.$$

$$\left. + \frac{1}{2} \sum_{i=0}^{n} \left( \frac{h_i''(\xi_{in})}{h_i(\xi_{in})} - \frac{h_i'(\xi_{in})^2}{h_i(\xi_{in})^2} \right) (1 - f_{i-1,n}(s_{n+1}))^2 \right\},$$

with $f_{i-1,n}(s_{n+1}) < \xi_{in} < 1$, $i = 0, \ldots, n$, $n \in \mathbb{N}_0$. Thus, the result yields by proving the following convergences, as $n \to \infty$,

$$\sum_{i=0}^{n} \alpha_i(1 - f_{i-1,n}(s_{n+1})) \to \log(1+\lambda)^{\frac{2\alpha}{\nu}}, \tag{8}$$

$$\sum_{i=0}^{n} \left( \frac{h_i''(\xi_{in})}{h_i(\xi_{in})} - \frac{h_i'(\xi_{in})^2}{h_i(\xi_{in})^2} \right) (1 - f_{i-1,n}(s_{n+1}))^2 \to 0. \tag{9}$$

We shall start with (8). To prove this result, for each $N, n \in \mathbb{N}_0$, $n \geq N$, let us denote

$$S_{1n}^{(N)} = \sum_{i=0}^{N-1} \alpha_i(1 - f_{i-1,n}(s_{n+1})), \quad \text{and} \quad S_{2n}^{(N)} = \sum_{i=N}^{n} \alpha_i(1 - f_{i-1,n}(s_{n+1})).$$

13

By applying (7),

$$S_{2n}^{(N)} = \sum_{i=N}^{n} \frac{\alpha_i}{\frac{\mu_{i-1}}{\mu_n(1-s_{n+1})} + \mu_{i-1}\sum_{l=i}^{n}\frac{\varphi_l(f_{l,n}(s_{n+1}))}{\mu_{l-1}}}$$

$$= \sum_{i=N}^{n} \frac{\alpha_i}{\varphi_i(f_{i,n}(s_{n+1}))} \cdot \frac{\varphi_i(f_{i,n}(s_{n+1}))}{\mu_{i-1}} \cdot \frac{1}{\frac{1}{\mu_n(1-s_{n+1})} + \sum_{l=i}^{n}\frac{\varphi_l(f_{l,n}(s_{n+1}))}{\mu_{l-1}}},$$

where the functions $\varphi_l(\cdot)$, $l = 0, \ldots, n$, were introduced in (6). First, we deal with this latter term.

Let $0 < \varepsilon < \frac{2\alpha}{\nu}$. We shall prove that there exist $J_0 = J_{0,\varepsilon}$, $N_0 = N_{0,\varepsilon} \in \mathbb{N}$ such that $N_0 > J_0$ and for $n \geq N_0$ and $J_0 \leq i \leq n$,

$$\left| \frac{\alpha_i}{\varphi_i(f_{i,n}(s_{n+1}))} - \frac{2\alpha}{\nu} \right| < \varepsilon. \tag{10}$$

From the proof of Lemma 7 in [16], one has that given $\varrho > 0$, there exists $0 < \eta = \eta_\varrho < 1$, such that

$$\sup_{t_i \leq t \leq 1} |\varphi_i(1) - \varphi_i(t)| < \varrho\varphi_i(1), \tag{11}$$

for $t_i = 1 - \frac{\eta}{1+m_i}$, $i \in \mathbb{N}_0$. To obtain (10), we prove that given $\varrho > 0$, there exists $N_1 = N_{1,\varrho}$, such that $f_{i,n}(s_{n+1}) \geq t_i$, for $0 \leq i \leq n$, $n \geq N_1$, or equivalently,

$$(1 - f_{i,n}(s_{n+1}))(1 + m_i) \leq \eta, \quad \text{for } 0 \leq i \leq n, n \geq N_1. \tag{12}$$

We write $s_{n+1} = 1 - \frac{\lambda}{a_{n+1}} + \frac{y_n}{a_{n+1}}$, with $y_n \to 0$; using $f'_{i,n}(1) = \prod_{j=i+1}^{n} f'_j(1) = \frac{\mu_n}{\mu_i}$ and a Taylor expansion for $f_{i,n}(\cdot)$ at 1, we have

$$(1 - f_{i,n}(s_{n+1}))(1 + m_i) \leq f'_{i,n}(1)\,(1 - s_{n+1})\,(1 + m_i)$$

$$\leq \frac{\mu_n}{\mu_i}\left(\frac{\lambda}{a_{n+1}} - \frac{y_n}{a_{n+1}}\right)(1 + m_i)$$

$$\leq \frac{\mu_n}{a_{n+1}}(\lambda - y_n)\left(\frac{1}{\mu_i} + \frac{1}{\mu_{i-1}}\right)$$

$$\leq (\lambda - y_n)\left(\frac{1}{a_{i+1}} + \frac{1}{a_i}\right).$$

Now, since $a_i \to \infty$, as $i \to \infty$, given $\epsilon > 0$, there exists $J_1 = J_{1,\epsilon}$ such that $a_i^{-1} \leq \epsilon(2\lambda)^{-1}$, for $i \geq J_1$, and consequently

$$(1 - f_{i,n}(s_{n+1}))(1 + m_i) \leq \frac{\epsilon}{\lambda}(\lambda - y_n), \quad n \geq J_1, \; J_1 \leq i \leq n,$$

and therefore,

$$\lim_{n \to \infty} \sup_{J_1 \leq i \leq n} (1 - f_{i,n}(s_{n+1}))(1 + m_i) \leq \epsilon. \tag{13}$$

14

On the other hand,

$$\sup_{0 \le i \le J_1 - 1} (1 - f_{i,n}(s_{n+1}))(1 + m_i) \le \frac{\mu_n}{a_{n+1}} (\lambda - y_n) \max_{0 \le i \le J_1 - 1} \left( \frac{1}{\mu_i} + \frac{1}{\mu_{i-1}} \right),$$

and hence,

$$\lim_{n \to \infty} \sup_{0 \le i \le J_1 - 1} (1 - f_{i,n}(s_{n+1}))(1 + m_i) = 0,$$

which with (13) and taking into account that $\epsilon$ is arbitrary gives us

$$\lim_{n \to \infty} \sup_{0 \le i \le n} (1 - f_{i,n}(s_{n+1}))(1 + m_i) = 0. \tag{14}$$

From this, (12) yields. Moreover, since $\varphi_i(1) \to \nu/2 > 0$, as $i \to \infty$, there exists $J_2 \in \mathbb{N}$ such that $\varphi_i(1) > 0$, for each $i \ge J_2$, and we can safely assume that $N_1 \ge J_2$. Taking these facts into account together with (11) and (12), one obtains

$$\sup_{n \ge N_1} \sup_{J_2 \le i \le n} \left| \frac{\varphi_i(f_{i,n}(s_{n+1}))}{\varphi_i(1)} - 1 \right| \le \sup_{n \ge N_1} \sup_{J_2 \le i \le n} \sup_{t_i \le t \le 1} \left| \frac{\varphi_i(t)}{\varphi_i(1)} - 1 \right| < \varrho. \tag{15}$$

Now, the claim in (10) yields from (15), the convergences $\alpha_i \to \alpha$ and $\varphi_i(1) = \nu_i/2 \to \nu/2$, as $i \to \infty$, and the inequality

$$\left| \frac{\alpha_i}{\varphi_i(f_{i,n}(s_{n+1}))} - \frac{2\alpha}{\nu} \right| \le \frac{\alpha_i}{\varphi_i(1)} \left| \frac{\varphi_i(1)}{\varphi_i(f_{i,n}(s_{n+1}))} - 1 \right| + \left| \frac{2\alpha_i}{\nu_i} - \frac{2\alpha}{\nu} \right|.$$

Let us denote

$$x_{j,n} = \frac{1}{\mu_n(1 - s_{n+1})} + \sum_{l=n-j}^{n} \frac{\varphi_l(f_{l,n}(s_{n+1}))}{\mu_{l-1}}, \quad j = -1, 0, \dots, n; n \in \mathbb{N}_0,$$

where the empty sum is considered to be 0; then, $b_n = x_{-1,n}, x_{0,n}, \dots, x_{n-1,n}, x_{n,n} = d_n$ is a partition of the interval $[b_n, d_n]$ and

$$\sum_{i=N}^{n} \frac{\varphi_i(f_{i,n}(s_{n+1}))}{\mu_{i-1}} \cdot \frac{1}{\frac{1}{\mu_n(1-s_{n+1})} + \sum_{l=i}^{n} \frac{\varphi_l(f_{l,n}(s_{n+1}))}{\mu_{l-1}}} = \sum_{i=N}^{n} x_{n-i,n}^{-1} (x_{n-i,n} - x_{n-i-1,n}).$$

Combining this with (10), for $N \ge J_0$, and $n \ge \max\{N_0, N\}$ one has

$$\left( \frac{2\alpha}{\nu} - \varepsilon \right) \sum_{i=N}^{n} x_{n-i,n}^{-1} (x_{n-i,n} - x_{n-i-1,n}) \le S_{2n}^{(N)} \le \left( \frac{2\alpha}{\nu} + \varepsilon \right) \sum_{i=N}^{n} x_{n-i,n}^{-1} (x_{n-i,n} - x_{n-i-1,n}),$$

$$\tag{16}$$

consequently, if one proves

$$\sum_{i=N}^{n} x_{n-i,n}^{-1} (x_{n-i,n} - x_{n-i-1,n}) \sim \int_{x_{-1,n}}^{x_{n-N,n}} \frac{1}{x} dx, \quad \text{as } n \to \infty, \tag{17}$$

15

then, by using Lemma 6, one obtains

$$\sum_{i=N}^{n} x_{n-i,n}^{-1}(x_{n-i,n} - x_{n-i-1,n}) \sim \int_{x_{-1,n}}^{x_{n-N,n}} \frac{1}{x} dx$$

$$= \log\left(1 + \mu_n(1 - s_{n+1}) \sum_{l=N}^{n} \frac{\varphi_l(f_{l,n}(s_{n+1}))}{\mu_{l-1}}\right)$$

$$\sim \log\left(1 + \frac{\mu_n \lambda}{2a_{n+1}} \sum_{l=N}^{n} \frac{\nu_l}{\mu_{l-1}}\right)$$

$$\rightarrow \log(1 + \lambda), \tag{18}$$

as $n \to \infty$. Observe that (17) follows from the integral test for convergence and the fact that given $\delta > 0$, there exists $N_2 = N_{2,\delta} \in \mathbb{N}$ such that

$$\sup_{0 \le i \le n} \left| \frac{x_{n-i,n}}{x_{n-i-1,n}} - 1 \right| < \delta, \quad \text{for each } n \ge N_2. \tag{19}$$

Indeed, from (15) one has that the sequence $\{\varphi_i(f_{i,n}(s_{n+1}))\}_{0 \le i \le n; n \in \mathbb{N}_0}$ is bounded and

$$\left| \frac{x_{n-i,n}}{x_{n-i-1,n}} - 1 \right| = \frac{\varphi_i(f_{i,n}(s_{n+1}))}{\frac{\mu_{i-1}}{\mu_n(1-s_{n+1})} + \mu_{i-1} \sum_{l=i+1}^{n} \frac{\varphi_l(f_{l,n}(s_{n+1}))}{\mu_{l-1}}} \le \frac{K_1 \mu_n(1 - s_{n+1})}{\mu_{i-1}},$$

where $K_1 > 0$ is an upper bound of the sequence $\{\varphi_i(f_{i,n}(s_{n+1}))\}_{0 \le i \le n; n \in \mathbb{N}_0}$. With similar arguments to those used before to establish (12), (19) yields.

Thus, from (16) and (18),

$$\left(\frac{2\alpha}{\nu} - \varepsilon\right) \log(1 + \lambda) \le \liminf_{n \to \infty} S_{2n}^{(J_0)} \le \limsup_{n \to \infty} S_{2n}^{(J_0)} \le \left(\frac{2\alpha}{\nu} + \varepsilon\right) \log(1 + \lambda)$$

hence, $\lim_{n \to \infty} S_{2n}^{(J_0)} = \log(1 + \lambda)^{\frac{2\alpha}{\nu}}$. For $S_{1n}^{(J_0)}$, one has $S_{1n}^{(J_0)} = o(1)$, as $n \to \infty$; indeed,

$$S_{1n}^{(J_0)} = \sum_{i=0}^{J_0-1} \alpha_i(1 - f_{i-1,n}(s_{n+1})) \le J_0 \max_{0 \le i < J_0} \left\{\frac{\alpha_i}{\mu_{i-1}}\right\} \mu_n(1 - s_{n+1}) \to 0,$$

which together with the previous limit gives us (8).

Finally, to establish (9), we have that

$$\sum_{i=0}^{n} \left| \frac{h_i''(\xi_{in})}{h_i(\xi_{in})} - \frac{h_i'(\xi_{in})^2}{h_i(\xi_{in})^2} \right| (1 - f_{i-1,n}(s_{n+1}))^2 \le$$

$$\le \left(\sum_{i=0}^{n} \alpha_i(1 - f_{i-1,n}(s_{n+1}))\right) \sup_{0 \le i \le n} \left\{ \frac{1 - f_{i-1,n}(s_{n+1})}{\alpha_i} \left| \frac{h_i''(\xi_{in})}{h_i(\xi_{in})} - \frac{h_i'(\xi_{in})^2}{h_i(\xi_{in})^2} \right| \right\},$$

16

and

$$\sup_{0 \le i \le n} \left\{ \frac{1 - f_{i-1,n}(s_{n+1})}{\alpha_i} \left| \frac{h_i''(\xi_{in})}{h_i(\xi_{in})} - \frac{h_i'(\xi_{in})^2}{h_i(\xi_{in})^2} \right| \right\} \le$$
$$\le \left( \frac{K_2 \sup_{n \in \mathbb{N}_0} h_n''(1)}{\tau} + \frac{K_3}{\tau^2} \right) \sup_{0 \le i \le n} \left\{ 1 - f_{i,n}(s_{n+1}) \right\},$$

with $K_2 > 0$ and $K_3 > 0$ being upper bounds of the sequences $\{1/\alpha_n\}_{n \in \mathbb{N}_0}$ and $\{\alpha_n\}_{n \in \mathbb{N}_0}$, respectively. Notice that $\sup_{n \in \mathbb{N}_0} \beta_n^2 < \infty$ implies $\sup_{n \in \mathbb{N}_0} h_n''(1) < \infty$. Furthermore, from (14), the convergence $\sup_{0 \le i \le n}\{1 - f_{i-1,n}(s_{n+1})\} \to 0$, as $n \to \infty$, is deduced, and as a consequence, using (8), (9) is obtained and the proof is finished.

$\square$

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