Diffusion approximation of controlled branching processes using limit theorems for random step processes*

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Abstract

A controlled branching process (CBP) is a modification of the standard Bienaymé-Galton-Watson process in which the number of progenitors in each generation is determined by a random mechanism. We consider a CBP starting from a random number of initial individuals. The main aim of this paper is to provide a Feller diffusion approximation for critical CBPs. A similar result by considering a fixed number of initial individuals by using operator semigroup convergence theorems has been previously proved in [16]. An alternative proof is now provided making use of limit theorems for random step processes.

Keywords: Controlled branching processes; Weak convergence theorem; Martingale differences; Diffusion processes; Stochastic differential equation; Random step processes.

1 Introduction

Let $\{X_{n,j} : n = 0, 1, \dots ; j = 1, 2, \dots\}$ be a sequence of independent and identically distributed (i.i.d.), non-negative and integer-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let also $\{\phi_n(k) : k = 0, 1, \dots\}$, for $n = 0, 1, \dots$, be a sequence of stochastic processes which consist of independent non-negative integer-valued random variables on $(\Omega, \mathcal{F}, P)$ with the same one-dimensional distributions. Therefore, these random variables are identically distributed for each fixed k. Furthermore, let us assume that $\{X_{n,j} : n = 0, 1, \dots ; j = 1, 2, \dots\}$ and $\{\phi_n(k) : n = 0, 1, \dots ; k = 0, 1, \dots\}$ are independent.

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A controlled branching process (CBP) is defined recursively as

$$Z_n = \sum_{j=1}^{\phi_{n-1}(Z_{n-1})} X_{n-1,j}, \qquad n = 1, 2, \ldots, \tag{1}$$

where $\sum_{j=1}^{0}$ is defined as 0 and Z_0 is a non-negative, integer-valued, square-integrable random variable which is independent of $\{X_{n,j} : n = 0, 1, \ldots; j = 1, 2, \ldots\}$ and $\{\phi_n(k) : n = 0, 1, \ldots; k = 0, 1, \ldots\}$.

Here, Z_n denotes the size of the n-th generation of a population and $X_{n-1,j}$ is the offspring size of the j-th individual in the $(n-1)$-th generation. We will assume that the mean $m = E[X_{0,1}]$ and variance $\sigma^2 = Var[X_{0,1}]$ are both finite.

The class of CBPs is a very general family of stochastic processes that collect as particular cases the simplest branching model, namely the standard Bienaymé–Galton–Watson (BGW) process, a branching processes with immigration or Galton-Watson processes with migration, among others (details will be given in Section 3). The monograph [7] provides an extensive description of its probabilistic theory.

The research of functional weak limit theorems for branching processes has attracted a lot of interest for many years ago. It was firstly formulated for a BGW process by [3] and proved by [11] and [13]. These results have been extended to another classes of branching processes. For instance, a wide literature exists around weak convergence results for branching processes with -nonhomogeneous-immigration (BPI) since the pioneer work by [17], see also [1] and references therein, and for the case of time homogeneous immigration, see [15] and references therein. In this paper we focus our attention on a weak convergence theorem for a critical CBP with a random initial number of individuals and assuming finite second order moment on this initial value. A similar result was already established for a single CBP in [16], and for an array of CBPs in [6], by assuming fixed initial numbers of progenitors using infinitesimal generators results for their proofs. Inspired in the paper [1] on BPI we will use limit theorems for random step processes towards a diffusion process provided in [9] to obtain an alternative proof. The scheme of it follows similar steps to the ones in [1]. An important feature of a CBP is that the value of Z_n conditioned on the knowledge of the previous generation, $Z_{n-1} = k$, is a random sum of random variables, namely $\sum_{j=1}^{\phi_n(k)} X_{n-1,j}$, instead of a non-random sum as in the case of a BPI. This leads to handle the proofs of each steps using conditioning arguments different from those used in [1].

Apart from this introduction, the paper is organized as follows. In Section 2 we provide the notation and some auxiliary results about the behaviour of the first and second moments of the process. Section 3 gathers the main theorem. The proof of the result is presented in Section 4. For the ease of reading the paper, additional results are presented in the Appendix.

2 Notation and auxiliary results

For all $n = 0, 1, \ldots$, we denote

$$\varepsilon(k) \quad = \quad E[\phi_n(k)],$$

$$\nu^2(k) \;=\; Var[\phi_n(k)],$$

for each $k = 0, 1, \ldots,$ and assume all finite. It is easy to obtain that for $n = 1, 2, \ldots,$

$$E[Z_n|\mathcal{F}_{n-1}] \;=\; m\varepsilon(Z_{n-1}), \quad (2)$$
$$Var[Z_n|\mathcal{F}_{n-1}] \;=\; \sigma^2\varepsilon(Z_{n-1}) + m^2\nu^2(Z_{n-1}), \quad (3)$$

where $\mathcal{F}_n$ is the $\sigma-$algebra generated by the random variables $Z_0, Z_1, \ldots, Z_n,\ n \geq 1$ (see Proposition 3.5 in [7]).

We introduce the quantities

$$\tau_m(k) = E[Z_{n+1}Z_n^{-1}|Z_n = k] = m\varepsilon(k)k^{-1},\ k \geq 1. \quad (4)$$

The quantity $\tau_m(k)$ represents a mean growth rate. Intuitively, it can be interpreted as an average offspring per individual for a generation of size k.

Assuming that $\lim_{k\to\infty} \tau_m(k) = \tau_m$ exists, the process can be classified as:

$$\tau_m < 1 \quad \text{subcritical;}\ \tau_m = 1 \quad \text{critical;}\ \tau_m > 1 \quad \text{supercritical.}$$

We are interested in critical CBPs that satisfy the following hypotheses:

A1) $\tau_m(k) = 1 + k^{-1}\alpha + o(k^{-1}),\ \ k > 0,\ \alpha > 0,$

A2) $\nu^2(k) = o(k)$, as $k \to \infty$.

It was studied in [5] the behavior of critical CBPs with $P(\phi_0(0) = 0) = 1$, i.e. 0 is an absorbing state, and satisfying that $P(X_{0,1} = 0) > 0$ or $P(\phi_0(k) = 0) > 0,\ k = 1, 2, \ldots$ In particular, it was established that under A1) and A2), if $\alpha > \sigma^2/(2m)$ and an assumption on conditional moments holds, then $P(Z_n \to \infty) > 0$. In the present paper we will consider critical CBPs, $\{Z_n\}_{n\geq 0}$, satisfying the above conditions, but with a reflecting barrier at zero, namely, $P(\phi_n(0) > 0) > 0$. Thus $\{Z_n\}_{n\geq 0}$ will have a finite number of returns to the state zero till the explosion to infinity, i.e. $P(Z_n \to \infty) = 1$.

Notice that under A1), $\varepsilon(k) = (k+\alpha)m^{-1} + o(1),\ k \geq 1$, and, for simplicity in the posterior calculations, we will also assume throughout the paper that $\varepsilon(0) = \alpha m^{-1}$.

Remark 2.1. *The controlled branching process we are considering is such that migration may take place in the next generation no matter the size of the current generation (when there are no individuals in the populations only immigration is possible). BGW processes with immigration at 0 were considered firstly in [4] and [14].*

In next result we calculate the first and second moments of a CBP which verifies A1) and A2).

Proposition 2.1. *Let $\{Z_n\}_{n\geq 0}$ be a CBP with $E[Z_0^2] < \infty$ and satisfying hypotheses A1) and A2). It is verified as $k \to \infty$ that*

$$E[Z_k] = O(k) \ \textit{and}\ E[Z_k^2] = O(k^2).$$

Proof. From A1) it follows that there exists $C_1 > 0$ such that $|m\varepsilon(k) - k - \alpha| \leq C_1$ for all $k > 0$, so that $|H_{k-1}| \leq C_1$ a.s. where $H_{k-1} = m\varepsilon(Z_{k-1}) - Z_{k-1} - \alpha$. Then, applying (2), we deduce

$$E[Z_k] = E[m\varepsilon(Z_{k-1})] \leq E[Z_{k-1}] + E[|H_{k-1}|] + \alpha \leq E[Z_0] + k(C_1 + \alpha), \quad k \geq 1. \qquad (5)$$

Using (3) we have

$$E[Var[Z_k \mid \mathcal{F}_{k-1}]] \leq m^{-1}\sigma^2(E[Z_0] + k(C_1 + \alpha)) + m^2 E[\nu^2(Z_{k-1})].$$

Now, from A2), we have that there exists $C_2 > 0$ such that $\nu^2(k) \leq C_2 k$ for all $k > 0$, so that $E[\nu^2(Z_{k-1})] \leq C_2 E[Z_{k-1}] + \nu^2(0)$. Consequently, letting $C_3 = 3\max\{m^{-1}\sigma^2 E[Z_0], m^2 C_2 E[Z_0],\ m^2\nu^2(0)\}$ and $C_4 = 2\max\{m^{-1}\sigma^2(C_1+\alpha),\ m^2 C_2(C_1+\alpha)\}$, we have

$$E[Var[Z_k \mid \mathcal{F}_{k-1}]] \leq C_3 + C_4 k. \qquad (6)$$

Let $C_5 = 2\max\{C_1^2,\ 4C_1 E[Z_0]\}$ and $C_6 = 4C_1(C_1+\alpha)$, it follows that

$$\begin{aligned} Var[E[Z_k \mid \mathcal{F}_{k-1}]] &= Var[Z_{k-1}] + Var[H_{k-1}] + 2Cov[Z_{k-1}, H_{k-1}] \\ &\leq Var[Z_{k-1}] + E[|H_{k-1}|^2] + 2E[|H_{k-1}|Z_{k-1}] + 2E[|H_{k-1}|]E[Z_{k-1}] \\ &\leq Var[Z_{k-1}] + C_5 + C_6 k. \end{aligned}$$

Hence,

$$\begin{aligned} Var[Z_k] &= E[Var[Z_k \mid \mathcal{F}_{k-1}]] + Var[E[Z_k \mid \mathcal{F}_{k-1}]] \\ &\leq (C_3 + C_5)k + 2^{-1}(C_4 + C_6)k(k+1) + Var[Z_0]. \end{aligned}$$

The latter inequality proves that $E[Z_k^2] = O(k^2)$.

The following lemma, which can be easily verified, will be useful to establish certain relationships among the random variables $\{X_{n,j} :\ n = 0, 1, \ldots;\ j = 1, 2, \ldots\}$.

Lemma 2.1. *Let $\{Y_n : n = 1, 2, \ldots\}$ be a sequence of i.i.d. random variables with zero mean and finite variance, denoted by σ^2. Let denote $S_l = \sum_{j=1}^{l} Y_j$, $l = 1, 2, \ldots$ and for $j = 1, \ldots, l$ $\tilde{S}^j(l) = \sum_{j' \neq j}^{l} Y_{j'}$, and $M > 0,\ M \in \mathbb{R}$. It is verified that*

$$E\left[\sum_{j=1}^{l} Y_j^2 \mathbb{I}_{\{|\tilde{S}^j(l)| > M\}}\right] \leq \frac{l^2\sigma^4}{M^2}$$

and

$$E\left[\left(\sum_{j,j',j\neq j'}^{l} Y_j Y_{j'}\right)^2\right] = 2l(l-1)\sigma^4.$$

3 Main result

We introduce for each $n \in \mathbb{N}$, a stochastic process $W_n(t) = n^{-1} Z_{\lfloor nt \rfloor}$, for $t \geq 0$, $t \in \mathbb{R}$, $\lfloor \cdot \rfloor$ denoting the integer part. It is easy to see that $\{W_n\}_{n\geq 1}$ is a sequence of random functions that take values in $D_{[0,\infty)}[0,\infty)$, which is the space of non-negative functions on $[0,\infty)$ that are right continuous and have left limits. We also denote by $C_c^\infty[0,\infty)$ the space of infinitely differentiable functions on $[0,\infty)$ which have a compact support. Throughout the paper "$\xrightarrow{\mathcal{D}}$" denotes the convergence of random functions in the Skorokhod topology.

Theorem 3.1. *Let $\{Z_n\}_{n\geq 0}$ be a CBP with $E[Z_0^2] < \infty$, satisfying hypotheses A1) and A2). Then, $W_n \xrightarrow{\mathcal{D}} W$, as $n \to \infty$, being W a non-negative diffusion process, with generator $Tf(x) = \alpha f'(x) + \frac{1}{2} x\sigma^2 m^{-1} f''(x)$, for $f \in C_c^\infty[0,\infty)$. The process W is the pathwise unique solution of the stochastic differential equation*

$$\mathrm{d}W(t) = \alpha \mathrm{d}t + \sqrt{\sigma^2 m^{-1} (W(t))^+} \mathrm{d}\mathcal{W}(t), \qquad t \geq 0, \tag{7}$$

with initial value $W(0) = 0$, denoting $x^+ = \max\{x, 0\}$, $x \in \mathbb{R}$, and where $\mathcal{W}$ is a standard Wiener process.

Remark 3.1. *Taking into account Theorem A2 in Appendix, the stochastic differential equation (SDE) (7) has a pathwise unique solution $\{X(t)^{(x)}\}_{t\geq 0}$ for all initial values $X(0)^{(x)} = x \in \mathbb{R}$. Moreover if $x \geq 0$, then $X(t)^{(x)} \geq 0$ almost surely for all $t \geq 0$.*

Remark 3.2. *Notice that the result in Theorem 3.1 is also valid as $\alpha = 0$.*

As was mentioned in the Introduction, particular subclasses of branching models can be recovered from a CBP by introducing specific control variables. Consequently, Theorem 3.1 leads us to already known and not yet established diffusion approximation results for different kind of branching models.

1) *BGW process.* A BGW process is a CBP by considering $\phi_n(k) = k$ a.s. for each k. Taking into account Remark 3.2, in the case $\alpha = 0$ and $m = E[X_{0,1}] = 1$ the result provides an alternative proof of the weak convergence result for the BGW process (see [2], p. 388) for a non-array version.

2) *BPI.* A BPI can be written as a special case of a CBP, by setting $\phi_n(k) = k + I_n$, where $\{I_n\}_{n\geq 0}$ are non–negative integer–valued i.i.d. random variables and independent of the the offspring variables (writing in this way the immigrants give rise to offspring at the same generation as their arrival and with the same probability law as $X_{0,1}$). For this case, by considering $m = E[X_{0,1}] = 1$ ($\varepsilon(k) = k + E[I_0]$ and $\nu^2(k) = Var[I_0]$), we obtain an analogous result to that in [1].

3) *The Galton–Watson process with migration (GWMP).* Let the random variables $\{X_{n,i} : n = 0, 1, \ldots, i = 1, 2, \ldots\}$ be the offspring variables as defined previously and $\{I_n\}_{n\geq 0}$ be non–negative integer–valued i.i.d. (immigration) random variables, independent from the offspring variables. The discrete time homogeneous Markov chain $\{Z_n\}_{n\geq 0}$ is called a Galton–Watson

process with migration (GWMP) if for $n = 0, 1, \ldots$

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} + M_{n+1}, \tag{8}$$

with Z_0 being a non-negative integer-valued random variable independent of the offspring and immigration variables, and where for $p + q + r = 1$

$$M_{n+1} = \begin{cases} -X_{n,1}\mathbb{I}_{\{Z_n>0\}} & \text{with probability } p, \quad \text{(emigration)} \\ 0 & \text{with probability } q, \quad \text{(no migration)} \\ I_{n+1} & \text{with probability } r, \quad \text{(immigration)} \end{cases} \tag{9}$$

is the migration component. In each generation there are three possible scenarios: (i) the offspring of one individual is removed (emigration) with probability p; (ii) there is no migration with probability q; or (iii) I_n individuals join the population (immigration) with probability r. This model was introduced in [18]. It can be seen as a CBP with $\phi_n(k) = (k + M_n)^+$ where $\{M_n\}_{n\geq 0}$ is a sequence of i.i.d. random variables, such that

$$P(M_n = -1) = p, \; P(M_n = 0) = q + rP(I_n = 0), \;\; \text{and } P(M_n = \eta) = rP(I_n = \eta), \; \eta = 1, 2, \ldots,$$

where $p + q + r = 1$ for $p, q, r \in (0, 1)$. Let denote $a = E[I_0]$ and $b = E[I_n^2]$, both assumed finite. Consequently, the next theorem establishes for the first time a diffusion limit process for a GWMP.

Theorem 3.2. *Let $\{Z_n\}_{n\geq 0}$ be a GWMP (written as a CBP) with $E[Z_0^2] < \infty$. Then, if $ra - p > 0$, $W_n \xrightarrow{\mathcal{D}} W$, as $n \to \infty$, being $W_n = n^{-1}Z_{\lfloor nt \rfloor}$ and W a non-negative diffusion process with generator $Tf(x) = (ra - p)f'(x) + \frac{1}{2}x\sigma^2 m^{-1} f''(x)$, for $f \in C_c^\infty[0, \infty)$.*

The proof follows from Theorem 3.1 due to the fact that it can be checked that $\varepsilon(k) = k + (ra - p)$ and $\nu^2(k) = p + rb - (ra - p)^2$, and therefore A1) and A2) in such a theorem hold.

4 Proof of the main result

In order to prove Theorem 3.1, we will establish previously the weak convergence of random step processes defined from a martingale difference created from the CBP. For simplicity in the presentation of the calculations we will assume that $\tau_m(k) = 1 + k^{-1}\alpha$, as $k > 0$.

We introduce the following sequence of martingale differences $\{M_k\}_{k\geq 1}$ with respect the filtration $\{\mathcal{F}_k\}_{k\geq 0}$ as:

$$M_k = Z_k - E[Z_k \mid \mathcal{F}_{k-1}] = Z_k - Z_{k-1} - \alpha, \; k \geq 1.$$

Consider the random step processes:

$$\mathcal{M}_n(t) = \frac{1}{n}\left(Z_0 + \sum_{k=1}^{\lfloor nt \rfloor} M_k\right) = \frac{1}{n}Z_{\lfloor nt \rfloor} - \frac{\lfloor nt \rfloor}{n}\alpha, \quad t \geq 0, \quad n \in \mathbb{N}. \tag{10}$$

Theorem 4.1. *Let $\{Z_n\}_{n\geq 0}$ be a CBP with $E[Z_0^2]<\infty$, satisfying hypotheses A1) and A2). It is verified that*

$$\mathcal{M}_n \xrightarrow{\mathcal{D}} \mathcal{M}, \qquad \text{as } n\to\infty,$$

where the limit process $\mathcal{M}$ is the pathwise unique solution of

$$\mathrm{d}\mathcal{M}(t)=\sqrt{m^{-1}\sigma^2(\mathcal{M}(t)+\alpha t)^+}\mathrm{d}\mathcal{W}(t), \quad t\geq 0, \text{ with initial value } \mathcal{M}(0)=0. \tag{11}$$

Proof As was done in [1], we prove the result by applying Theorem A1 in Appendix with $\mathcal{U}=\mathcal{M},\ U_n(k)=n^{-1}M_k,\ k\in\mathbb{N}$, $U_n(0)=n^{-1}Z_0$, $\mathcal{F}_n(k)=\mathcal{F}_k, k\geq 0$, where $n\in\mathbb{N}$ (yielding $\mathcal{U}_n=\mathcal{M}_n, n\in\mathbb{N}$, as well), and with coefficient functions $\beta:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ and $\gamma:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ given by

$$\beta(t,x)=0, \quad \gamma(t,x)=\sqrt{m^{-1}\sigma^2\left(x+\alpha t\right)^+}, \quad t\geq 0, \quad x\in\mathbb{R}.$$

Firstly, we check that the SDE (11) has a pathwise unique strong solution $\left\{\mathcal{M}(t)^{(x)}\right\}_{t\geq 0}$ for all initial values $\mathcal{M}(0)^{(x)}=x\in\mathbb{R}$. In fact, notice that if $\left\{\mathcal{M}(t)^{(x)}\right\}_{t\geq 0}$ is a strong solution of the SDE (11) with initial value $\mathcal{M}(0)^{(x)}=x\in\mathbb{R}$, then, by Itô's formula, the process $\mathcal{P}(t)=\mathcal{M}(t)^{(x)}+\alpha t$, $t\geq 0$, is a solution of the SDE

$$\mathrm{d}\mathcal{P}(t)=\alpha\mathrm{d}t+\sqrt{m^{-1}\sigma^2\mathcal{P}(t)^+}\mathrm{d}\mathcal{W}(t), \quad t\geq 0, \text{ with initial value } \mathcal{P}(0)=x. \tag{12}$$

Conversely, if $\{\mathcal{P}(t)^{(x)}\}_{t\geq 0}$ is a strong solution of the SDE (12) with initial value $\mathcal{P}^{(x)}(0)=x\in\mathbb{R}$, then, by Itô's formula, the process $\mathcal{M}(t)=\mathcal{P}(t)^{(x)}-\alpha t$, $t\geq 0$, is a strong solution of the SDE (11) with initial value $\mathcal{M}(0)=x$. Notice that SDE (12) is the same as SDE (7), consequently the SDE (12) and therefore the SDE (11) as well admit a pathwise unique strong solution with arbitrary initial value, and

$$\{\mathcal{M}(t)+\alpha t\}_{t\geq 0}\overset{\mathcal{D}}{=}\{W(t)\}_{t\geq 0}. \tag{13}$$

Let us see that $E\left[(U_n(k))^2\right]<\infty$ for all $n=1,2\ldots$ and $k=0,1,2,\ldots$. Indeed, taking into account (6) in Proposition 2.1,

$$E\left[(U_n(k))^2\right]=n^{-2}E\left[M_k^2\right]=E[Var[Z_k\mid\mathcal{F}_{k-1}]]\leq\frac{M_1+M_2k}{n^2}<\infty \tag{14}$$

and, by the assumption in the statement of the theorem, $E\left[(U_n(0))^2\right]=n^{-2}E\left[Z_0^2\right]<\infty$, for $n=1,2,\ldots$ Moreover, $U_n(0)=n^{-1}Z_0\xrightarrow{\text{a.s.}}0$ as $n\to\infty$, especially $U_n(0)\xrightarrow{\mathcal{D}}0$ as $n\to\infty$.

For conditions (i), (ii) and (iii) of Theorem A1 in Appendix, we have to check that for each $T>0$, $T\in\mathbb{R}$, as $n\to\infty$:

a) $\sup_{t\in[0,T]}\left|\frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor}E\left[M_k\mid\mathcal{F}_{k-1}\right]-0\right|\xrightarrow{\text{P}}0.$

b) $\sup_{t\in[0,T]}\left|\frac{1}{n^2}\sum_{k=1}^{\lfloor nt\rfloor}E\left[M_k^2\mid\mathcal{F}_{k-1}\right]-\int_0^t\frac{\sigma^2}{m}\left(\mathcal{M}_n(s)+\alpha s\right)^+\mathrm{d}s\right|\xrightarrow{\text{P}}0.$

c) For all $\theta>0$, $\frac{1}{n^2}\sum_{k=1}^{\lfloor nT\rfloor}E\left[M_k^2\mathbb{I}_{\{|M_k|>n\theta\}}\mid\mathcal{F}_{k-1}\right]\xrightarrow{\text{P}}0.$

Since $E\left[M_k \mid \mathcal{F}_{k-1}\right]=0$, $n\in\mathbb{N}$, $k\in\mathbb{N}$, a) holds.

Let us check b).

For each $s>0$, $s\in\mathbb{R}$ and $n\in\mathbb{N}$, we have:

$$\mathcal{M}_n(s)+\alpha s=\frac{1}{n}Z_{\lfloor ns\rfloor}+\frac{ns-\lfloor ns\rfloor}{n}\alpha,$$

thus $(\mathcal{M}_n(s)+\alpha s)^+=\mathcal{M}_n(s)+\alpha s$. Now, we have, for all $t>0$ and $n\in\mathbb{N}$,

$$\begin{aligned}\int_0^t(\mathcal{M}_n(s)+\alpha s)^+\,\mathrm{d}s &= \sum_{k=0}^{\lfloor nt\rfloor-1}\int_{k/n}^{(k+1)/n}\left(\frac{1}{n}Z_k+\frac{ns-k}{n}\alpha\right)\mathrm{d}s\\ &\quad+\int_{\lfloor nt\rfloor/n}^{t}\left(\frac{1}{n}Z_{\lfloor nt\rfloor}+\frac{ns-\lfloor nt\rfloor}{n}\alpha\right)\mathrm{d}s\\ &= \frac{1}{n^2}\sum_{k=0}^{\lfloor nt\rfloor-1}Z_k+\frac{nt-\lfloor nt\rfloor}{n^2}Z_{\lfloor nt\rfloor}+\frac{\alpha}{2n^2}\lfloor nt\rfloor\\ &\quad+\frac{\alpha}{n}\left(\frac{n}{2}\left(t^2-\frac{\lfloor nt\rfloor^2}{n^2}\right)-\lfloor nt\rfloor\left(t-\frac{\lfloor nt\rfloor}{n}\right)\right)\\ &= \frac{1}{n^2}\sum_{k=0}^{\lfloor nt\rfloor-1}Z_k+\frac{nt-\lfloor nt\rfloor}{n^2}Z_{\lfloor nt\rfloor}+\frac{\lfloor nt\rfloor+(nt-\lfloor nt\rfloor)^2}{2n^2}\alpha.\end{aligned}$$

It is verified that, for $t>0$ and $n\in\mathbb{N}$,

$$\begin{aligned}\frac{1}{n^2}\sum_{k=1}^{\lfloor nt\rfloor}E\left[M_k^2\mid\mathcal{F}_{k-1}\right] &= \frac{1}{n^2}\sum_{k=1}^{\lfloor nt\rfloor}Var[Z_k\mid\mathcal{F}_{k-1}]\\ &= \frac{1}{n^2}\sum_{k=1}^{\lfloor nt\rfloor}\left(m^2\nu^2(Z_{k-1})+\frac{\sigma^2}{m}(Z_{k-1}+\alpha)\right)\\ &= \frac{m^2}{n^2}\sum_{k=1}^{\lfloor nt\rfloor}\nu^2(Z_{k-1})+\frac{\lfloor nt\rfloor\alpha\sigma^2}{n^2m}+\frac{\sigma^2}{n^2m}\sum_{k=1}^{\lfloor nt\rfloor}Z_{k-1}.\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{1}{n^2}\sum_{k=1}^{\lfloor nt\rfloor}E\left[M_k^2\mid\mathcal{F}_{k-1}\right] - \int_0^t\frac{\sigma^2}{m}(\mathcal{M}_n(s)+\alpha s)^+\,\mathrm{d}s &= \frac{m^2}{n^2}\sum_{k=1}^{\lfloor nt\rfloor}\nu^2(Z_{k-1})+\frac{\lfloor nt\rfloor\alpha\sigma^2}{n^2m}\\ &\quad-\frac{\sigma^2(nt-\lfloor nt\rfloor)}{mn^2}Z_{\lfloor nt\rfloor}-\frac{\sigma^2}{m}\frac{\lfloor nt\rfloor+(nt-\lfloor nt\rfloor)^2}{2n^2}\alpha.\end{aligned}$$

Since for each $T>0, T\in\mathbb{R}$,

$$\sup_{t\in[0,T]}\frac{\lfloor nt\rfloor}{n^2}\leqslant\frac{T}{n}\to 0,\qquad \text{as } n\to\infty$$

$$\sup_{t\in[0,T]}\frac{\lfloor nt\rfloor+(nt-\lfloor nt\rfloor)^2}{2n^2}\leqslant\frac{T}{2n}+\frac{1}{2n^2}\to 0,\qquad \text{as } n\to\infty,$$

in order to show b), it suffices to prove that for each $T > 0, T \in \mathbb{R}$,

$$\frac{1}{n^2} \sup_{t\in[0,T]} \big((nt - \lfloor nt \rfloor) Z_{\lfloor nt \rfloor}\big) \le \frac{1}{n^2} \sup_{t\in[0,T]} Z_{\lfloor nt \rfloor} \xrightarrow{P} 0 \quad \text{as } n \to \infty. \tag{15}$$

and

$$\frac{m^2}{n^2} \sup_{t\in[0,T]} \sum_{k=1}^{\lfloor nt \rfloor} \nu^2(Z_{k-1}) \xrightarrow{P} 0 \quad \text{as } n \to \infty. \tag{16}$$

First we check (15). For each $k \in \mathbb{N}$, we have $Z_k = Z_{k-1} + M_k + \alpha$, thus

$$Z_k = Z_0 + \sum_{j=1}^{k} M_j + k\alpha,$$

and hence, for each $t > 0,\ t \in \mathbb{R}$ and $n \in \mathbb{N}$, we get

$$Z_{\lfloor nt \rfloor} = \left| Z_{\lfloor nt \rfloor} \right| \leqslant Z_0 + \sum_{j=1}^{\lfloor nt \rfloor} |M_j| + \lfloor nt \rfloor \alpha.$$

Consequently, in order to prove (15), it suffices to show

$$\frac{1}{n^2} \sup_{t\in[0,T]} \sum_{j=1}^{\lfloor nt \rfloor} |M_j| \leqslant \frac{1}{n^2} \sum_{j=1}^{\lfloor nT \rfloor} |M_j| \xrightarrow{P} 0, \quad \text{as } n \to \infty.$$

By (14), $E[M_k^2] = O(k)$, as $k \to \infty$, and therefore by Jensen's inequality, $E[|M_k|] = O(k^{1/2})$, as $k \to \infty$, and hence

$$E\left[\frac{1}{n^2} \sum_{j=1}^{\lfloor nT \rfloor} |M_j|\right] = \frac{1}{n^2} \sum_{j=1}^{\lfloor nT \rfloor} O\left(j^{1/2}\right) = O\left(n^{-1/2}\right) \to 0, \quad \text{as } n \to \infty.$$

Thus we obtain $n^{-2} \sum_{j=1}^{\lfloor nT \rfloor} |M_j| \xrightarrow{P} 0$ as $n \to \infty$ implying (15).

Now, taking into account hypothesis A2), for each fixed $\epsilon > 0$ there exists $K(\epsilon) \in \mathbb{N}$ such that $\nu^2(k) \le \epsilon k$, for all $k \ge K(\epsilon)$. Consequently, as $E[Z_k] = O(k)$ by Proposition 2,

$$\begin{aligned} E\left[\frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} \nu^2(Z_{k-1})\right] &= E\left[\frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} \nu^2(Z_{k-1})(\mathbb{I}_{\{Z_{k-1} < K(\epsilon)\}} + \mathbb{I}_{\{Z_{k-1} \ge K(\epsilon)\}})\right] \\ &\le \frac{\lfloor nT \rfloor}{n^2} \max\{E[\nu^2(j)], j = 0, 1, \ldots, K(\epsilon) - 1\} \qquad (17) \\ &\quad + \epsilon E\left[\frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} Z_{k-1}\right] \qquad (18) \end{aligned}$$

$$= \quad o(1) + \epsilon O(1), \tag{19}$$

and taking $\epsilon \to 0$, we have (16).

Let us check c).

We follow the separation argument of the proof of Theorem 2.2 in [10]. We write

$$M_k = \sum_{j=1}^{\phi_{k-1}(Z_{k-1})} (X_{k-1,j} - m) + m(\phi_{k-1}(Z_{k-1}) - \varepsilon(Z_{k-1})).$$

Let denote $N_k = \sum_{j=1}^{\phi_{k-1}(Z_{k-1})} (X_{k-1,j} - m)$. It is verified for each $n, k \in \mathbb{N}$, and $\theta > 0, \theta \in \mathbb{R}$ that

$$M_k^2 \leq 2\left(N_k^2 + m^2(\phi_{k-1}(Z_{k-1}) - \varepsilon(Z_{k-1}))^2\right),$$

and

$$\mathbb{I}_{\{|M_k|>n\theta\}} \leq \mathbb{I}_{\{|N_k|>n\theta/2\}} + \mathbb{I}_{\{|\phi_{k-1}(Z_{k-1})-\varepsilon(Z_{k-1})|>n\theta/2m\}}.$$

Hence

$$M_k^2\mathbb{I}_{\{|M_k|>n\theta\}} \leq 2N_k^2\mathbb{I}_{\{|N_k|>n\theta/2\}} + 2N_k^2\mathbb{I}_{\{|\phi_{k-1}(Z_{k-1})-\varepsilon(Z_{k-1})|>n\theta/2m\}} + 2m^2(\phi_{k-1}(Z_{k-1}) - \varepsilon(Z_{k-1}))^2.$$

In consequence, to check c) we will prove, as $n \to \infty$,

c.1) $\frac{1}{n^2}\sum_{k=1}^{\lfloor nT \rfloor} E\left[N_k^2\mathbb{I}_{\{|N_k|>n\theta\}} \mid \mathcal{F}_{k-1}\right] \xrightarrow{P} 0$ for all $\theta > 0, \theta \in \mathbb{R}$.

c.2) $\frac{1}{n^2}\sum_{k=1}^{\lfloor nT \rfloor} E\left[N_k^2\mathbb{I}_{\{|\phi_{k-1}(Z_{k-1})-\varepsilon(Z_{k-1})|>n\theta\}} \mid \mathcal{F}_{k-1}\right] \xrightarrow{P} 0$ for all $\theta > 0, \theta \in \mathbb{R}$.

c.3) $\frac{1}{n^2}\sum_{k=1}^{\lfloor nT \rfloor} E\left[(\phi_{k-1}(Z_{k-1}) - \varepsilon(Z_{k-1}))^2 \mid \mathcal{F}_{k-1}\right] \xrightarrow{P} 0.$

In what follows let $\theta > 0, \theta \in \mathbb{R}$ be fixed.

Let us see c.3). It is verified that

$$\begin{aligned}\frac{1}{n^2}\sum_{k=1}^{\lfloor nT \rfloor} E\left[(\phi_{k-1}(Z_{k-1}) - \varepsilon(Z_{k-1}))^2 \mid \mathcal{F}_{k-1}\right] &= \frac{1}{n^2}\sum_{k=1}^{\lfloor nT \rfloor} Var\left[\phi_{k-1}(Z_{k-1}) \mid \mathcal{F}_{k-1}\right] \\ &= \frac{1}{n^2}\sum_{k=1}^{\lfloor nT \rfloor} \nu^2(Z_{k-1}) \xrightarrow{P} 0, \text{ as } n \to \infty.\end{aligned}$$

This latter was proved by considering (19).

Now, we check c.1). By the properties of conditional expectation with respect to a σ - algebra, we get for all $n, k \in \mathbb{N}$,

$$E\left[N_k^2\mathbb{I}_{\{|N_k|>n\theta\}} \mid \mathcal{F}_{k-1}\right] = F_{n,k}\left(Z_{k-1}\right),$$

where on $\{Z_{k-1} = z\}$, with $z = 0, 1, \ldots$

$$F_{n,k}(z) = E\left[S_k(z)^2\mathbb{I}_{\{|S_k(z)|>n\theta\}}\right], \text{ where } S_k(z) = \sum_{j=1}^{\phi_{k-1}(z)} (X_{k-1,j} - m).$$

Consider the decomposition $F_{n,k}(z) = A_{n,k}(z) + B_{n,k}(z)$ with

$$\begin{aligned} A_{n,k}(z) &= E\left[\sum_{j=1}^{\phi_{k-1}(z)} (X_{k-1,j} - m)^2 \mathbb{I}_{\{|S_k(z)|>n\theta\}}\right], \\ B_{n,k}(z) &= E\left[\sum_{j,j',j\neq j'}^{\phi_{k-1}(z)} (X_{k-1,j} - m)(X_{k-1,j'} - m)\mathbb{I}_{\{|S_k(z)|>n\theta\}}\right]. \end{aligned}$$

Again, this technique is original from [10], see proof of Theorem 2.2. Now, let denote $S_{k,l} = \sum_{j=1}^{l}(X_{k-1,j} - m)$, $k = 1, 2, \ldots$, $l = 0, 1, \ldots$. It is verified the inequality, for $j \in \{1, \ldots, l\}$

$$|S_{k,l}| \leq |X_{k-1,j} - m| + |\tilde{S}_k^j(l)|, \text{ with } \tilde{S}_k^j(l) = \sum_{j'\neq j}^{l}(X_{k-1,j'} - m).$$

We have, using Lemma 2.1,

$$\begin{aligned} A_{n,k}(z) &= E\left[E\left[\sum_{j=1}^{\phi_{k-1}(z)} (X_{k-1,j} - m)^2 \mathbb{I}_{\{|S_k(z)|>n\theta\}}\middle| \phi_{k-1}(z)\right]\right] \\ &= \sum_{l=0}^{\infty} E\left[\sum_{j=1}^{l}(X_{k-1,j} - m)^2 \mathbb{I}_{\{|S_{k,l}|>n\theta\}}\right] P(\phi_{k-1}(z) = l) \\ &\leq \sum_{l=0}^{\infty}\sum_{j=1}^{l} \Big(E\left[(X_{k-1,j} - m)^2 \mathbb{I}_{\{|X_{k-1,j}-m|>n\theta/2\}}\right] \\ &\quad + E[(X_{k-1,j} - m)^2 \mathbb{I}_{\{|\tilde{S}_k^j(l)|>n\theta/2\}}]\Big) P(\phi_{k-1}(z) = l) \\ &\leq \sum_{l=0}^{\infty}\left(lE\left[(X_{0,1} - m)^2 \mathbb{I}_{\{|X_{0,1}-m|>n\theta/2\}}\right] + \frac{4l^2\sigma^4}{n^2\theta^2}\right) P(\phi_{k-1}(z) = l) \\ &= \varepsilon(z)E\left[(X_{0,1} - m)^2 \mathbb{I}_{\{|X_{0,1}-m|>n\theta/2\}}\right] + \frac{4\sigma^4}{n^2\theta^2}E[(\phi_0(z))^2] \end{aligned}$$

Therefore

$$A_{n,k}(z) \leq A_{n,k}^{(1)}(z) + A_{n,k}^{(2)}(z),$$

with

$$\begin{aligned} A_{n,k}^{(1)}(z) &= \varepsilon(z)E\left[(X_{0,1} - m)^2 \mathbb{I}_{\{|X_{0,1}-m|>n\theta/2\}}\right], \\ A_{n,k}^{(2)}(z) &= (\nu^2(z) + (\varepsilon(z))^2)\frac{4\sigma^4}{n^2\theta^2}. \end{aligned}$$

Using (5) in Proposition 2.1, it is verified that for $n \in \mathbb{N}$

$$E\left[\frac{1}{n^2}\sum_{k=1}^{\lfloor nT\rfloor} A_{n,k}^{(1)}(Z_{k-1})\right] = \frac{1}{n^2}\sum_{k=1}^{\lfloor nT\rfloor} E[\varepsilon(Z_k)]E\left[(X_{0,1} - m)^2 \mathbb{I}_{\{|X_{0,1}-m|>n\theta/2\}}\right]$$

$$
\begin{aligned}
&= \frac{1}{n^2}\left(\sum_{k=1}^{\lfloor nT\rfloor} O(k)\right) E\left[(X_{0,1}-m)^2 \mathbb{I}_{\{|X_{0,1}-m|>n\theta/2\}}\right] \\
&= O(1) E\left[(X_{0,1}-m)^2 \mathbb{I}_{\{|X_{0,1}-m|>n\theta/2\}}\right].
\end{aligned}
$$

By applying the dominated convergence theorem we have

$$
E\left[\frac{1}{n^2}\sum_{k=1}^{\lfloor nT\rfloor} A^{(1)}_{n,k}(Z_{k-1})\right] \to 0, \text{ as } n\to\infty. \tag{20}
$$

It is also verified by using again (5) in Proposition 2.1 and A2) that

$$
\begin{aligned}
E\left[\frac{1}{n^2}\sum_{k=1}^{\lfloor nT\rfloor} A^{(2)}_{n,k}(Z_{k-1})\right] &= \frac{1}{n^2}\sum_{k=1}^{\lfloor nT\rfloor} E\left[\left(\nu^2(Z_{k-1}) + (\varepsilon(Z_{k-1}))^2\right)\frac{4\sigma^4}{n^2\theta^2}\right] \\
&= \frac{4\sigma^4}{n^4\theta^2}\sum_{k=1}^{\lfloor nT\rfloor} O(k^2) = O(n^{-1}) \to 0, \text{ as } n\to\infty. \qquad (21)
\end{aligned}
$$

Taking into account (20) and (21) we have that, as $n\to\infty$,

$$
\frac{1}{n^2}\sum_{k=1}^{\lfloor nT\rfloor} A_{n,k}(Z_{k-1}) \xrightarrow{P} 0.
$$

Let us now deal with $B_{n,k}(z)$. It is verified that, using Cauchy-Schwarz's inequality:

$$
\begin{aligned}
B_{n,k}(z) &= \sum_{l=0}^{\infty} E\left[\sum_{j,j',j\neq j'}^{l} (X_{k-1,j}-m)(X_{k-1,j'}-m)\mathbb{I}_{\{|S_{k,l}|>n\theta\}}\right] P(\phi_{k-1}(z)=l) \\
&\leq \sum_{l=0}^{\infty} \sqrt{E\left[\left(\sum_{j,j',j\neq j'}^{l} (X_{k-1,j}-m)(X_{k-1,j'}-m)\right)^2\right] E[\mathbb{I}_{\{|S_{k,l}|>n\theta\}}]} P(\phi_{k-1}(z)=l).
\end{aligned}
$$

Now, using Markov's inequality

$$
E[\mathbb{I}_{\{|S_{k,l}|>n\theta\}}] \leq \frac{Var[S_{k,l}]}{n^2\theta^2} = \frac{l\sigma^2}{n^2\theta^2},
$$

and using Lemma 2.1, we have

$$
B_{n,k}(z) \leq \sum_{l=0}^{\infty} \sqrt{2l^2\sigma^4 n^{-2}\theta^{-2} l\sigma^2}\, P(\phi_{k-1}(z)=l) = \frac{\sqrt{2}\sigma^3}{\theta n} E[(\phi_0(z))^{3/2}].
$$

By Lyapunov's inequality, $E[(\phi_{k-1}(z))^{3/2}]^{2/3} \leq E[(\phi_{k-1}(z))^2]^{1/2} = (\nu^2(z)+(\varepsilon(z))^2)^{1/2}$. Consequently, in order to prove, as $n \to \infty$,

$$\frac{1}{n^2}\sum_{k=1}^{\lfloor nT \rfloor} B_{n,k}(Z_{k-1}) \xrightarrow{P} 0,$$

is enough to check that

$$\frac{1}{n^3}\sum_{k=1}^{\lfloor nT \rfloor}(\nu^2(Z_{k-1}) + (\varepsilon(Z_{k-1}))^2)^{3/4} \xrightarrow{P} 0.$$

In fact, using hypotheses A1) and A2) and Proposition 2.1, we have

$$E\left[\frac{1}{n^3}\sum_{k=1}^{\lfloor nT \rfloor}(\nu^2(Z_{k-1}) + (\varepsilon(Z_{k-1}))^2)^{3/4}\right] = n^{-3}\sum_{k=1}^{\lfloor nT \rfloor} O(k^{3/2}) = O(n^{-1/2}).$$

Finally, we check c.2). We have that

$$E\left[N_k^2 \mathbb{I}_{\{|\phi_{k-1}(Z_{k-1})-\varepsilon(Z_{k-1})|>n\theta\}} \mid \mathcal{F}_{k-1}\right] = G_{n,k}(Z_{k-1}),$$

where on $\{Z_{k-1} = z\}$, with $z = 0, 1, \ldots$,

$$G_{n,k}(z) = E[S_k(z)^2 \mathbb{I}_{\{|\phi_{k-1}(z)-\varepsilon(z)|>n\theta\}}].$$

Now, again by Cauchy-Schwarz's inequality and Markov's inequality

$$\begin{aligned} G_{n,k}(z) &= \sum_{l=0}^{\infty} \mathbb{I}_{\{|l-\varepsilon(z)|>n\theta\}} E[S_{k,l}^2] P(\phi_{k-1}(z) = l) = \sigma^2 E[\phi_{k-1}(z)\mathbb{I}_{\{|\phi_{k-1}(z)-\varepsilon(z)|>n\theta\}}] \\ &\leq \sigma^2\sqrt{E[\phi_{k-1}^2(z)]P(|\phi_{k-1}(z)-\varepsilon(z)| > n\theta)} \leq \sigma^2 E[(\phi_0(z))^2]^{1/2}\left(\frac{\nu^2(z)}{n^2\theta^2}\right)^{1/2}. \end{aligned}$$

In consequence from

$$E\left[\frac{\sigma^2}{\theta n^3}\sum_{k=1}^{\lfloor nT \rfloor}\left(\nu^2(Z_{k-1}) + (\varepsilon(Z_{k-1}))^2\right)^{1/2}(\nu^2(Z_{k-1}))^{1/2}\right] = O(n^{-1/2}),$$

c.2) follows.

Finally, using the weak convergence of $\{\mathcal{M}_n\}_{n\geq 1}$, we will obtain weak convergence of $\{W_n\}_{n\geq 1}$.

Proof of Theorem 3.1. A version of the continuous mapping theorem is applied (see Lemma 1 in Appendix). Let $D_{\mathbb{R}}[0,\infty)$ be the space of the real functions on $[0,\infty)$ that are right continuous and have left limits. For each $n \in \mathbb{N}$, by (10), $\{W_n(t)\}_{t\geq 0} = \Psi^{(n)}(\mathcal{M}_n)$, where the mapping $\Psi^{(n)} : D_{\mathbb{R}}[0,\infty) \to D_{\mathbb{R}}[0,\infty)$ is given by

$$\left(\Psi^{(n)}(f)\right)(t) = f\left(\frac{\lfloor nt \rfloor}{n}\right) + \frac{\lfloor nt \rfloor}{n}\alpha,$$

for $f \in D_{\mathbb{R}}[0,\infty)$ and $t \in [0,\infty)$. Indeed, for each $n \in \mathbb{N}$ and $t \geq 0$:

$$\left(\Psi^{(n)}(\mathcal{M}_n)\right)(t) = \mathcal{M}_n\left(\frac{\lfloor nt \rfloor}{n}\right) + \frac{\lfloor nt \rfloor}{n}\alpha = \frac{1}{n}Z_{\lfloor nt \rfloor} = W_n(t).$$

Further, taking into account (13), $W \overset{\mathcal{D}}{=} \Psi(\mathcal{M})$, where the mapping $\Psi : D_{\mathbb{R}}[0,\infty) \to D_{\mathbb{R}}[0,\infty)$ is given by

$$(\Psi(f))(t) = f(t) + \alpha t, \quad f \in D_{\mathbb{R}}[0,\infty), \quad t \in [0,\infty).$$

The measurability of $\Psi^{(n)}$, $n \in \mathbb{N}$ and Ψ likewise the conditions for applying Lemma 1 are proved in [1].

Remark 4.1. *As was pointed out in the Introduction, the proof of the main result follows similar steps as those given in [1]. One can check that similar formulas often appear being the roles of the immigration mean and the offspring variance in the BPI case played in the CBP by α and $m^{-1}\sigma^2$, respectively. However, new approaches by considering conditioning arguments are needed to dealt with b)- c) in Theorem 3.1, as a consequence that random sums of i.i.d. random variables arise in the proofs, see for instance the definition of $A_{n,k}(z)$ and $B_{n,k}(z)$, in p.12. An extra work is required to calculate the mathematical expectation of these quantities.*

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Appendix

Theorem A1. *Let $\beta : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ and $\gamma : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be continuous functions. Assume that uniqueness in the sense of probability law holds for the SDE*

$$\mathrm{d}\mathcal{U}(t) = \beta\left(t, \mathcal{U}(t)\right)\mathrm{d}t + \gamma\left(t, \mathcal{U}(t)\right)\mathrm{d}\mathcal{W}(t), \quad t \geq 0, \tag{22}$$

with initial value $\mathcal{U}(0) = u(0)$ for all $u(0) \in \mathbb{R}$, where $\mathcal{W} = \{\mathcal{W}(t)\}_{t\geq 0}$ is an one-dimensional standard Wiener process. Let $\mathcal{U} = \{\mathcal{U}(t)\}_{t\geq 0}$ be a solution of (22) with initial value $\mathcal{U}(0) = 0$.

For each $n \in \mathbb{N}$, let $\{U_n(k) : k = 0, 1, 2, \ldots\}$ be a sequence of real-valued random variables adapted to a filtration $\{\mathcal{F}_n(k) : k = 0, 1, 2, \ldots\}$, that is, $U_n(k)$ is $\mathcal{F}_n(k)$- measurable. Let

$$\mathcal{U}_n(t) := \sum_{k=0}^{\lfloor nt \rfloor} U_n(k), \quad t \geq 0, \quad n \in \mathbb{N}.$$

Suppose that $E\left[(U_n(k))^2\right] < \infty$ for all $n, k \in \mathbb{N}$, and $\mathcal{U}_n(0) \xrightarrow{\mathcal{D}} 0$ as $n \to \infty$. Suppose that for each $T > 0$

(i) $\sup_{t \in [0,T]} \left| \sum_{k=1}^{\lfloor nt \rfloor} E\left[U_n(k) \mid \mathcal{F}_n(k-1)\right] - \int_0^t \beta\left(s, \mathcal{U}_n(s)\right) \mathrm{d}s \right| \xrightarrow{P} 0$ *as* $n \to \infty$,

(ii) $\sup_{t \in [0,T]} \left| \sum_{k=1}^{\lfloor nt \rfloor} \mathrm{Var}\left[U_n(k) \mid \mathcal{F}_n(k-1)\right] - \int_0^t \left(\gamma\left(s, \mathcal{U}_n(s)\right)\right)^2 \ \mathrm{d}s \right| \xrightarrow{P} 0$ *as* $n \to \infty$,

(iii) $\sum_{k=1}^{\lfloor nT \rfloor} E\left[(U_n(k))^2 \mathbb{I}_{\{|U_n(k)| > \theta\}} \mid \mathcal{F}_n(k-1)\right] \xrightarrow{P} 0$ *as* $n \to \infty$ *for all* $\theta > 0$.

Then $\mathcal{U}_n \xrightarrow{\mathcal{D}} \mathcal{U}$ as $n \to \infty$.

The proof can be seen in Corollary 2.2. in [9].

Theorem A2. *Let a, b, c real constants such that $a > 0$. Consider the stochastic differential equation*

$$\mathrm{d}X(t) = (bX(t) + c)\mathrm{d}t + \sqrt{2aX(t)^+}\mathrm{d}\mathcal{W}_t, \ t \geq 0. \tag{23}$$

There exists a pathwise unique strong solution $\{X(t)^{(x)}\}_{t \geq 0}$ for all initial values $X(0)^{(x)} = x \in \mathbb{R}$. Moreover if $x \geq 0$, then $X(t)^{(x)} \geq 0$ almost surely for all $t \geq 0$. In the case $c \geq 0$, the solution of (23) defines diffusion process with generator

$$Tf(x) = (bx + c)f'(x) + axf''(x), \ f \in C_c^\infty[0, \infty),$$

where $C_c^\infty[0, \infty)$ is the space of infinitely differentiable functions on $[0, \infty)$ which have a compact support.

The proof can be seen in [8] p. 235.

Lemma A1. *Let S and T be two metric spaces, and $X, X_1, X_2, \cdots$ be random functions with values in S with $X_n \xrightarrow{\mathcal{D}} X$. Consider some measurable mappings $h, h_1, h_2, \cdots : S \to T$ and a measurable set $C \subset S$ with $X \in C$ a.s. such that $h_n(s_n) \to h(s)$ as $s_n \to s \in C$. Then $h_n(X_n) \xrightarrow{d} h(X)$.*

The previous version of the continuous mapping theorem can be found in Theorem 3.27 in [12].

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