Supplementary material of "Robust estimation in controlled branching processes: Bayesian estimators via disparities"

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Appendix

In this appendix we study the existence and continuity of the EDAP and MDAP functions introduced in Section 3. To this end, we consider as σ-field on $\Gamma \times \Omega$ the product of the σ-fields on Γ and Ω, where Γ is taken with the Borel σ-field induced by the topology generated by the l_1-metric. In the following, let denote the l_r-norm, $r \geq 1$, by $||\cdot||_r$, that is, if $h = \{h_k\}_{k\in\mathbb{N}_0}$ is a sequence of real numbers, then $||h||_r = (\sum_{k=0}^{\infty} |h_k|^r)^{1/r}$; when $||h||_r < \infty$, h is said to belong to l_r.

Our first proposition is concerned with the existence and continuity of the EDAP functional. We note here that the existence is immediate under the assumption that the prior distribution belongs to $\mathcal{L}^1$.

Proposition 1. *Under the assumption that $D(q,\cdot)$ is a continuous function on Θ for every $q \in \widetilde{\Gamma}$, then, for each $n \in \mathbb{N}$ fixed:*

(i) $\overline{T}_n(q)$ exists finitely with probability one and $\overline{T}_n(q)$ is a random variable.

(ii) If $D(\cdot,\theta)$ is continuous in $\widetilde{\Gamma}$ with respect to the l_1-metric for each $\theta \in \Theta$, then $\overline{T}_n(\cdot)$ is almost surely continuous on $\widetilde{\Gamma}$ with respect to the l_1-metric; that is, if $q_j \to q$ in l_1, then $\overline{T}_n(q_j) \to \overline{T}_n(q)$, as $j \to \infty$, with probability one. Moreover, $\overline{T}_n$ is a random variable.

The proof is given in Section 1 of this Supplementary Material.

For the existence and measurability of MDAP functional we will consider the following subclass of the family $\widetilde{\Gamma}$. For each $\omega \in \Omega$, let $\Gamma^+_{n,\omega}$ be a subclass $\Gamma^+_{n,\omega} \subseteq \widetilde{\Gamma}$ which satisfies the following condition: there exists a compact set $C^+_{n,\omega} \subseteq \Theta$ such that for every $q \in \Gamma^+_{n,\omega}$,

$$\inf_{\theta \in \Theta \setminus C^+_{n,\omega}} g_n(q,\omega,\theta) > g_n(q,\omega,\theta^+),$$

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for some $\theta^+ \in C^+_{n,\omega}$, and with $g_n(q,\omega,\theta) = \Delta_{n-1}(\omega)D(q,\theta) - \log(\pi(\theta))$, for each $q \in \Gamma$ and $\theta \in \Theta$.

Proposition 2. *Let $n \in \mathbb{N}$ be fixed and Θ be a complete and separable subset of $\mathbb{R}$. Assume that for every $q \in \Gamma^+_{n,\omega}$, $D(q,\cdot)$ is a continuous function on Θ, then:*

(i) $\widetilde{T}_n(q)$ exists finitely with probability one. In addition, if $\widetilde{T}_n(q)(\omega)$ is unique for each $\omega \in \Omega$, then $\widetilde{T}_n(q)$ is a random variable.

(ii) The function $\widetilde{T}_n(\cdot)$ is continuous in q; that is, $\widetilde{T}_n(q_j) \to \widetilde{T}_n(q)$ with probability one as $j \to \infty$, as $q_j \to q$ in the sense that $\sup_{\theta\in\Theta} |D(q_j,\theta) - D(q,\theta)| \to 0$. Moreover, $\widetilde{T}_n$ is a random variable.

The proof is available in Section 1 of this Supplementary Material.

We summarize several observations concerning the properties of disparities in the following remark.

Remark 1. (a) *First, we notice that if Θ is a compact set, we can choose $C^+_{n,\omega} = \Theta$, for each $n \in \mathbb{N}$ and $\omega \in \Omega$, and $\Gamma^+_{n,\omega} = \widetilde{\Gamma}$. Moreover, is such a case it is also complete and separable. Finally, the assumption of compactness can be also removed using conditions similar to those in Cheng and Vidyashankar (2006), p.1885.*

(b) *Taking into account Proposition 2, the existence and uniqueness of $\theta^{+D}_n = \widetilde{T}_n(\hat{p}_n)$ for $n \geq n_0$ is guaranteed if we assume additionally that there exists $n_0 \in \mathbb{N}$ such that $\hat{p}_n(\omega) \in \Gamma^+_{n,\omega}$, and $\hat{\theta}^{+D}_n(\omega)$ is unique, for each $n \geq n_0$, and for each $\omega \in \{Z_n \to \infty\}$. Analogous assumption will be considered when it is required that $\widetilde{T}_n(q)$, $q \in \Gamma$, exists and is unique, that is, there exists some $n_0 \in \mathbb{N}$ such that $q \in \Gamma^+_{n,\omega}$, and $\widetilde{T}_n(q)(\omega)$ is unique, for each $n \geq n_0$, and $\omega \in \Omega$.*

(c) *In Proposition 2 (i), the assumption that $\widetilde{T}_n(q)(\omega)$ is unique for each $q \in \Gamma$ and $\omega \in \Omega$ can be weakened by assuming that $\{\theta \in \Theta : \theta = \widetilde{T}_n(q)(\omega)\}$ is a closed subset in Θ. In such a case, by Theorem 4.1 in Wagner (1977) one has that there exists a version of $\widetilde{T}_n$ which is measurable.*

(d) *In González et al. (2017), conditions that guarantee the continuity of $D(q,\cdot)$ on Θ, for each $q \in \Gamma$, are established. In particular, it is sufficient to assume that such a disparity measure is defined by a bounded function $G(\cdot)$.*

(e) *It is known that a sufficient condition for the disparity measure to be bounded is the boundedness of the function $G(\cdot)$. In some cases, when this condition does not hold, it is possible to re-define the disparity measure as a disparity corresponding to a bounded function $G(\cdot)$ with no change in the values of the function $D(q,\cdot)$, for any $q \in \Gamma$. For instance, for the negative exponential disparity, one can also consider the function $\overline{G}(\delta) = e^{-\delta} - 1$, which is bounded and satisfies that $NED(q,\theta) = \sum_{k=0}^{\infty} \overline{G}(\delta(q,\theta,k))p_k(\theta)$, for each $q \in \Gamma$ and $\theta \in \Theta$.*

(f) *In González et al. (2017), it is also proved that the boundedness of the derivative of the function $G(\cdot)$, $G'(\cdot)$, is a sufficient condition for* $\sup_{\theta\in\Theta}|D(q_j,\theta)-D(q,\theta)|\to 0$, *as* $j\to\infty$, *if* $q_j\to q$ *in* l_1.

(g) *For the Hellinger distance, the continuity of $D(q,\cdot)$ is deduced without the boundedness of $G(\cdot)$. Moreover, it is also a bounded disparity in $\Gamma\times\Theta$ despite not being defined by a bounded function $G(\cdot)$ and satisfies the uniform convergence in (f) (see González et al. (2017)).*

Our next result exhibits the strong relation between the behaviour of the EDAP and MDAP functions with their frequentist counterpart. Indeed, it establishes that the EDAP and MDAP functions can be approximated a.s. by the corresponding disparity function up to an error of certain order, on the set $\{Z_n\to\infty\}$.

Theorem 3. *Suppose that Assumption 1, and (c) in Assumption 2 hold. Let $q\in\Gamma^*$ satisfy (a) in Assumption 2, and (a) in Assumption 4. Then, the following convergences hold:*

(i) The EDAP function satisfies

$$|\overline{T}_n(q)-T(q)|=o\left(\Delta_{n-1}^{-1/2}\right)\qquad a.s.\ on\ \{Z_n\to\infty\}.$$

(ii) If Θ is complete and separable, (b) in Assumption 2 holds, and q satisfies that there exists some $n_0\in\mathbb{N}$ such that $q\in\Gamma^+_{n,\omega}$, and $\widetilde{T}_n(q)(\omega)$ is unique, for each $n\geq n_0$, and $\omega\in\Omega$, then the MDAP function satisfies

$$|\widetilde{T}_n(q)-T(q)|=o\left(\Delta_{n-1}^{-1/2}\right)\qquad a.s.\ on\ \{Z_n\to\infty\}.$$

We refer the reader to Section 1 in this Supplementary Material for the proof of this theorem.

Remark 2. *(a) We observe that if the offspring distribution belongs to the parametric family, that is $p=\boldsymbol{p}_{\theta_0}$ and $T(p)$ is unique, then $T(p)=\theta_0$; furthermore, if p satisfies the conditions of Theorem 3, one has that $\overline{T}_n(p)\to\theta_0$, and $\widetilde{T}_n(p)\to\theta_0$ a.s. on $\{Z_n\to\infty\}$.*

(b) If $T(q)$ exists and is unique for each $q\in\widetilde{\Gamma}$, under the assumption of the compactness of Θ, one has $\Gamma^*=\widetilde{\Gamma}$. The compactness assumption can be replaced with the assumption that q belongs to the subclass $\overline{\Gamma}\subseteq\Gamma$ which satisfies the following condition: there exists a compact set $\overline{C}\subseteq\Theta$ such that for every $\bar{q}\in\overline{\Gamma}$,

$$\inf_{\theta\in\Theta\setminus\overline{C}}D(\bar{q},\theta)>D(\bar{q},\theta^*),\qquad for\ some\ \theta^*\in\overline{C}.$$

1 Proof of properties of the EDAP and MDAP functions

Proof of Proposition 1

For *(i)*, observe that since $G(\cdot)$ is strictly convex, the disparity D is non-negative, and as a consequence, for each $\omega \in \Omega$, $|\overline{T}_n(q)(\omega)| < \infty$.

(ii) For the continuity of EDAP function, due to the facts that $e^{-\Delta_{n-1}(\omega)D(q_j,\theta)} \leq 1$, for each $\omega \in \Omega$, $j \in \mathbb{N}$, $\theta \in \Theta$, and $D(q_j, \theta) \to D(q, \theta)$, as $j \to \infty$, one has that

$$\int_\Theta e^{-\Delta_{n-1}(\omega)D(q_j,\theta)}\pi(\theta)d\theta \to \int_\Theta e^{-\Delta_{n-1}(\omega)D(q,\theta)}\pi(\theta)d\theta,$$

as $j \to \infty$, for each $n \in \mathbb{N}$, and $\omega \in \Omega$, and consequently, making use of a generalized version of the dominated convergence theorem (see Royden (1988), p.92), one has that, as $j \to \infty$,

$$\begin{aligned}\overline{T}_n(q_j)(\omega) &= \frac{\int_\Theta \theta e^{-\Delta_{n-1}(\omega)D(q_j,\theta)}\pi(\theta)d\theta}{\int_\Theta e^{-\Delta_{n-1}(\omega)D(q_j,\theta)}\pi(\theta)d\theta} \\ &\to \frac{\int_\Theta \theta e^{-\Delta_{n-1}(\omega)D(q,\theta)}\pi(\theta)d\theta}{\int_\Theta e^{-\Delta_{n-1}(\omega)D(q,\theta)}\pi(\theta)d\theta} \\ &= \overline{T}_n(q)(\omega).\end{aligned}$$

Now, by the continuity of $\overline{T}_n(\cdot)(\omega)$ for each $\omega \in \Omega$ and the measurability of $\overline{T}_n(q)$ for each $q \in \Gamma$, the measurability of $\overline{T}_n$ follows from Theorem 2 in Gowrisankaran (1972).

□

Proof of Proposition 2

(i) For each $\omega \in \Omega$, the finiteness and existence of $\widetilde{T}_n(q)(\omega)$ is immediate from the definition of the family $\Gamma^+_{n,\omega}$ and the continuity of the function $g_n(q,\omega,\cdot)$. For the measurability of $\widetilde{T}_n(q)$ we apply Theorem 4.5 in Debreu (1967) bearing in mind that $g_n(q,\cdot,\cdot)$ is measurable, $g_n(q,\omega,\cdot)$ is continuous for each $\omega \in \Omega$, and since $\widetilde{T}_n(q)(\omega)$ is unique, the function $S : \omega \in \Omega \mapsto S(\omega) = \arg\min_{\theta\in\Theta} g_n(q,\omega,\theta)$ is well defined.

For *(ii)*, let us denote $t_\omega = \widetilde{T}_n(q)(\omega)$ in order to ease the notation (it exists by *(i)* and is unique) and consider that all the limits are taken as $j \to \infty$ unless specified otherwise. First of all, from the continuity of $D(q,\cdot)$ and the fact that

$$\sup_{\theta\in\Theta} |D(q_j,\theta) - D(q,\theta)| \to 0, \tag{1}$$

it follows that if $\{\theta_n\}_{n\in\mathbb{N}}$ is a sequence in Θ converging to $\theta^* \in \Theta$, given $\epsilon > 0$, one can find $J = J(\epsilon) \in \mathbb{N}$ and $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that $|D(q_j,\theta_n) - D(q_j,\theta^*)| < \epsilon$, for each $j \geq J$ and $n \geq n_0$.

Now, fix $\omega \in \Omega$; from (1), it follows that $\sup_{\theta\in\Theta} |g_n(q_j,\omega,\theta) - g_n(q,\omega,\theta)| \to 0$. From this latter, it is deduced that $q_j \in \Gamma^+_{n,\omega}$ eventually. Thus, using the same arguments as in *(i)*, $\widetilde{T}_n(q_j)(\omega)$ eventually exists and $\widetilde{T}_n(q_j)(\omega) \in C^+_{n,\omega}$. Let us denote $\widetilde{T}_n(q_j)(\omega)$ by $t_{j,\omega}$ to ease the notation; next we show that $t_{j,\omega} \to t_\omega$, as $j \to \infty$.

From (1), the convergence of $g_n(q_j,\omega,t_{j,\omega}) \to g_n(q,\omega,t_\omega)$ and $|g_n(q_j,\omega,t_{j,\omega}) - g_n(q,\omega,t_{j,\omega})| \to 0$, are deduced, so $g_n(q,\omega,t_{j,\omega}) \to g_n(q,\omega,t_\omega)$. If the sequence $\{t_{j,\omega}\}_{j\in\mathbb{N}_0}$ did not converge to t_ω, then there would exist a subsequence $\{t_{j_l,\omega}\}_{l\in\mathbb{N}} \subseteq \{t_{j,\omega}\}_{j\in\mathbb{N}}$ such that $t_{j_l,\omega} \to t^*_\omega \neq t_\omega$, as $l \to \infty$. Since $g_n(q,\omega,\cdot)$ is continuous, $g_n(q,\omega,t_{j_l\omega}) \to g_n(q,\omega,t^*_\omega)$, as $l \to \infty$. Due to all of the above, one would have $g_n(q,\omega,t_\omega) = g_n(q,\omega,t^*_\omega)$, which would contradict the uniqueness of t_ω.

If $\widetilde{T}_n(q_j)(\omega)$ exists for each $\omega \in \Omega$, the measurability of $\widetilde{T}_n(q_j)$ is immediate with the same reasoning as in *(i)*.

Finally, one obtains the measurability of $\widetilde{T}_n$ by Theorem 2 in Gowrisankaran (1972) and the facts that for each $\omega \in \Omega$, $\widetilde{T}_n(\cdot)(\omega)$ is continuous and for each $q \in \Gamma^+_{n,\omega}$, $\widetilde{T}_n(q)$ is measurable.

□

Proof of Theorem 3

Throughout this proof, all statements are made on the set $\{Z_n \to \infty\}$ and for n large enough. Let us fix $\omega \in \{Z_n \to \infty\}$ then $\Delta_n(\omega) \to \infty$, and in the remainder of this section, we consider that all the random variables are evaluated at ω, although we do not write it explicitly in order to ease the notation.

To facilitate the proof of Theorem 3, we will make use of the following lemma.

Lemma 4. *Let $q \in \widetilde{\Gamma}$, $\pi^n_D(\theta|q)$ be the D-posterior density of θ at q, $\overline{\pi}^n_D(t|q)$ be the D-posterior density function of $t = \Delta^{1/2}_{n-1}(\theta - T(q))$ at q and $R_n = \{\Delta^{1/2}_{n-1}(\theta - T(q)) :$*

$\theta \in \Theta\}$. *Under conditions of Theorem 3, for each* $t \in \mathbb{R}$,

$$\overline{\pi}_D^n(t|q) = (1 + C_n)\left(\frac{I^D(T(q))}{2\pi}\right)^{1/2} e^{-\frac{t^2 I^D(\theta_n'(t))}{2}} \cdot \left(1 + \frac{tb_1}{\Delta_{n-1}^{1/2}} + \frac{t^2 b_2}{2\Delta_{n-1}} + \frac{t^3 \pi'''(\theta_n^*(t))}{6\Delta_{n-1}^{3/2}\pi(T(q))}\right) I_{R_n}(t), \tag{2}$$

a.s. on $\{Z_n \to \infty\}$, *where* $I^D(\theta) = \ddot{D}(q,\theta)$, $b_1 = \frac{\pi'(T(q))}{\pi(T(q))}$, $b_2 = \frac{\pi''(T(q))}{\pi(T(q))}$, $\pi'(\cdot)$ *and* $\pi''(\cdot)$ *denote the first and the second derivative of the function* $\pi(\cdot)$, $\theta_n'(t)$ *and* $\theta_n^*(t)$ *are both points between* $T(q)$ *and* $T(q) + \frac{t}{\Delta_{n-1}^{1/2}}$, *for each* $n \in \mathbb{N}$, *and* $t \in \mathbb{R}$, *and* $\{C_n\}_{n\in\mathbb{N}}$ *is a sequence of real valued random variables converging to 0 a.s.*

Proof. The idea of the proof is similar to the proof of Theorem 1 in Hooker and Vidyshankar (2014) and uses arguments in Ghosh (1994) (pp. 46-47). Notice that, for each $t \in R_n$,

$$\overline{\pi}_D^n(t|q) = \frac{e^{-\Delta_{n-1}D\left(q,T(q)+\frac{t}{\Delta_{n-1}^{1/2}}\right)+\Delta_{n-1}D(q,T(q))}\pi\left(T(q)+\frac{t}{\Delta_{n-1}^{1/2}}\right)}{\displaystyle\int_{R_n} e^{-\Delta_{n-1}D\left(q,T(q)+\frac{t}{\Delta_{n-1}^{1/2}}\right)+\Delta_{n-1}D(q,T(q))}\pi\left(T(q)+\frac{t}{\Delta_{n-1}^{1/2}}\right)dt}.$$

Moreover, since $T(q) \in int(\Theta)$, one has that $\cup_{n=1}^{\infty} R_n = \mathbb{R}$. This is immediate from the fact that there exists $\eta > 0$ such that $(T(q) - \eta, T(q) + \eta) \subseteq \Theta$, consequently $(-\Delta_{n-1}^{1/2}\eta, \Delta_{n-1}^{1/2}\eta) \subseteq R_n$, and taking limit as $n \to \infty$, one obtains $\cup_{n=1}^{\infty} R_n = \mathbb{R}$.

On the one hand, using a second-order Taylor series expansion of the prior density one has that, for each $t \in R_n$,

$$\pi\left(T(q) + \frac{t}{\Delta_{n-1}^{1/2}}\right) = \pi(T(q)) + \frac{t\pi'(T(q))}{\Delta_{n-1}^{1/2}} + \frac{t^2\pi''(T(q))}{2\Delta_{n-1}} + \frac{t^3\pi'''(\theta_n^*(t))}{6\Delta_{n-1}^{3/2}}, \tag{3}$$

where $\theta_n^*(t)$ is a point between $T(q)$ and $T(q) + \frac{t}{\Delta_{n-1}^{1/2}}$. On the other hand, using a first-order Taylor series expansion of the function $D(q,\cdot)$, for each $t \in R_n$,

$$\Delta_{n-1}D\left(q, T(q) + \frac{t}{\Delta_{n-1}^{1/2}}\right) - \Delta_{n-1}D(q, T(q)) = \frac{t^2 I^D(\theta_n'(t))}{2}, \tag{4}$$

where $\theta_n'(t)$ is a point between $T(q)$ and $T(q) + \frac{t}{\Delta_{n-1}^{1/2}}$.

Let us denote

$$h_n(t) = \pi\left(T(q) + \frac{t}{\Delta_{n-1}^{1/2}}\right) e^{-\Delta_{n-1}D\left(q,T(q)+\frac{t}{\Delta_{n-1}^{1/2}}\right)+\Delta_{n-1}D(q,T(q))} I_{R_n}(t),$$

for each $n \in \mathbb{N}$, and $t \in \mathbb{R}$. Combining (3) and (4), one has

$$h_n(t) = \pi(T(q))\left(1 + \frac{tb_1}{\Delta_{n-1}^{1/2}} + \frac{t^2 b_2}{2\Delta_{n-1}} + \frac{t^3\pi'''(\theta_n^*(t))}{6\Delta_{n-1}^{3/2}\pi(T(q))}\right)e^{-\frac{t^2 I^D(\theta_n'(t))}{2}} I_{R_n}(t). \tag{5}$$

Let us define, for each $n \in \mathbb{N}$, the integrals $J_n = \int h_n(t)dt$, and

$$\begin{aligned} I_n &= \int_{R_n}\left(1 + \frac{tb_1}{\Delta_{n-1}^{1/2}} + \frac{t^2 b_2}{2\Delta_{n-1}} + \frac{t^3\pi'''(\theta_n^*(t))}{6\Delta_{n-1}^{3/2}\pi(T(q))}\right)e^{-\frac{t^2 I^D(T(q))}{2}}dt \\ &= \left(\frac{2\pi}{I^D(T(q))}\right)^{1/2}\left(1 + \frac{b_2}{2\Delta_{n-1}I^D(T(q))} + \frac{C}{6\Delta_{n-1}^{3/2}\pi(T(q))}\right) + o(1), \end{aligned}$$

where to obtain the last equality we have applied the dominated convergence theorem and that $\left(\frac{I^D(T(q))}{2\pi}\right)^{1/2} \cdot \int t^3\pi'''(\theta_n^*(t))e^{-\frac{t^2 I^D(T(q))}{2}}dt \le C$, for some constant $C > 0$, due to the boundedness of the function $\pi'''(\cdot)$. If one proves that $J_n = \pi(T(q))I_n + o(1)$, then, from all the above, one deduces (2).

To prove $J_n = \pi(T(q))I_n + o(1)$, let us fix $0 < \epsilon < I^D(T(q))$. Since $I^D(\cdot)$ is continuous at $T(q)$, there exists $\delta = \delta(\epsilon) > 0$ such that if $|\theta - T(q)| \le \delta$, then $|I^D(\theta) - I^D(T(q))| \le \epsilon$. Let us define, for each $n \in \mathbb{N}$, the set $B_n = \{t \in \mathbb{R} : |t| \le \delta\Delta_{n-1}^{1/2}\}$, and note that $J_n = J_{1n} + J_{2n}$, where $J_{1n} = \int_{B_n} h_n(t)dt$, and $J_{2n} = \int_{B_n^c} h_n(t)dt$. As a consequence, it is enough to prove that, for each $n \in \mathbb{N}$, $J_{1n} = \pi(T(q))I_n + a_n$, and $\{a_n\}_{n\in\mathbb{N}}$ and $\{J_{2n}\}_{n\in\mathbb{N}}$ are both sequences of real numbers converging to 0.

For the former, observe that $a_n = \int x_n(t)dt$, where

$$\begin{aligned} x_n(t) &= \pi(T(q))\left(1 + \frac{tb_1}{\Delta_{n-1}^{1/2}} + \frac{t^2 b_2}{2\Delta_{n-1}} + \frac{t^3\pi'''(\theta_n^*(t))}{6\Delta_{n-1}^{3/2}\pi(T(q))}\right) \\ &\quad \cdot \left(e^{-\frac{t^2 I^D(\theta_n'(t))}{2}} - e^{-\frac{t^2 I^D(T(q))}{2}}\right) I_{B_n \cap R_n}(t) \\ &\quad - \pi(T(q))\left(1 + \frac{tb_1}{\Delta_{n-1}^{1/2}} + \frac{t^2 b_2}{2\Delta_{n-1}} + \frac{t^3\pi'''(\theta_n^*(t))}{6\Delta_{n-1}^{3/2}\pi(T(q))}\right) \\ &\quad \cdot e^{-\frac{t^2 I^D(T(q))}{2}} I_{B_n^c \cap R_n}(t), \end{aligned} \tag{6}$$

consequently, to prove $a_n \to 0$ it is enough to prove that the integral of both terms in (6) converge to 0. The convergence of the first one is obtained by applying the dominated convergence theorem bearing in mind that, for each $t \in B_n \cap R_n$, $0 < I^D(T(q)) - \epsilon < I^D(\theta_n'(t)) < I^D(T(q)) + \epsilon$ (due to the fact that $\theta_n'(t)$ is a point between $T(q)$ and $T(q) + \frac{t}{\Delta_{n-1}^{1/2}}$). For the convergence of the integral of the second term, one has

$$\int \left(1 + \frac{tb_1}{\Delta_{n-1}^{1/2}} + \frac{t^2 b_2}{2\Delta_{n-1}} + \frac{t^3\pi'''(\theta_n^*(t))}{6\Delta_{n-1}^{3/2}\pi(T(q))}\right)e^{-\frac{t^2 I^D(T(q))}{2}} I_{B_n^c \cap R_n}(t)dt \le$$

$$\leq \left(\frac{2\pi}{I^D(T(q))}\right)^{1/2} \cdot \Bigg(P[Z \in B_n^c \cap R_n] + \frac{|b_1|}{\Delta_{n-1}^{1/2}} E[|Z| I_{B_n^c \cap R_n}] + \frac{|b_2|}{2\Delta_{n-1}} E[Z^2 I_{B_n^c \cap R_n}] + \frac{M}{6\Delta_{n-1}^{3/2}\pi(T(q))} E[|Z|^3 I_{B_n^c \cap R_n}] \Bigg) \to 0,$$

where $M > 0$ is an upper bound of $\pi'''(\cdot)$, and Z is a random variable following a normal distribution with mean equal to 0 and variance $I^D(T(q))^{-1}$.

The convergence $J_{2n} \to 0$ follows from the fact that $q \in \Gamma^*$. To that end, note that for the fixed $\delta > 0$, there exists $\rho > 0$ such that

$$\inf_{t \in B_n^c \cap R_n} D\left(q, T(q) + \frac{t}{\Delta_{n-1}^{1/2}}\right) - D(q, T(q)) = \inf_{\theta \in \Theta : |\theta - T(q)| > \delta} D(q, \theta) - D(q, T(q)) > \rho,$$

and as a result,

$$\begin{aligned} J_{2n} &\leq \int_{B_n^c \cap R_n} \pi\left(T(q) + \frac{t}{\Delta_{n-1}^{1/2}}\right) e^{-\rho\Delta_{n-1}} dt \\ &= \Delta_{n-1}^{1/2} e^{-\rho\Delta_{n-1}} \int_{\{\theta \in \Theta : |\theta - T(q)| > \delta\}} \pi(\theta) d\theta \to 0. \end{aligned}$$

□

Now, we use the notation and the approximation for the D-posterior density function of $t = \Delta_{n-1}^{1/2}(\theta - T(q))$ at q given in the previous lemma to prove Theorem 3. Throughout this proof, all the limits are taken as $n \to \infty$ unless specified otherwise.

(i) Observe that $\Delta_{n-1}^{1/2}(\overline{T}_n(q) - T(q)) = \int_{R_n} t\pi_D^n(t|q)dt$. Hence, to finish the proof it is enough to prove that both integrals $\int_{B_n \cap R_n} t\pi_D^n(t|q)dt$ and $\int_{B_n^c \cap R_n} t\pi_D^n(t|q)dt$ converge to 0. For the former, one has $\int_{B_n \cap R_n} t\pi_D^n(t|q)dt = (1 + c_n)\sum_{i=0}^3 I_{in}$, where $I_{in} = \int_{B_n \cap R_n} tf_{in}(t)dt$, $f_{in}(t) = \left(\frac{I^D(T(q))}{2\pi}\right)^{1/2} c_{in}(t)\ e^{-\frac{t^2 I^D(\theta_n'(t))}{2}}$, for $i = 0, 1, 2, 3$, and $c_{0n}(t) = 1$, $c_{1n}(t) = \frac{tb_1}{\Delta_{n-1}^{1/2}}$, $c_{2n}(t) = \frac{t^2 b_2}{2\Delta_{n-1}}$, and $c_{3n}(t) = \frac{t^3\pi'''(\theta_n^*(t))}{6\Delta_{n-1}^{3/2}\pi(T(q))}$. If $h(t) = \left(\frac{I^D(T(q))}{2\pi}\right)^{1/2} e^{-\frac{t^2 I^D(T(q))}{2}}$, and we prove that $I_{in} - \int_{R_n} tc_{in}(t)h(t)dt \to 0$, for $i = 0, 1, 2, 3$, then, taking into account that $\int_{R_n} tc_{in}(t)h(t)dt \to 0$, for $i = 0, 1, 2, 3$, we obtain $I_{in} \to 0$, for $i = 0, 1, 2, 3$.

To prove that $I_{in} - \int_{R_n} tc_{in}(t)h(t)dt \to 0$, for $i = 0, 1, 2, 3$, observe that, for each $i = 0, 1, 2, 3$, $tf_{in}(t)I_{B_n \cap R_n}(t) = tc_{in}(t)h(t)I_{R_n}(t) + x_n^i(t)$, with

$$x_n^i(t) = (tf_{in}(t) - tc_{in}(t)h(t))I_{B_n \cap R_n}(t) - tc_{in}(t)h(t)I_{B_n^c \cap R_n}(t). \tag{7}$$

Consequently, it is sufficient to prove that the integrals of both terms in (7) converge to 0. For the first term, similarly to Lemma 4, we apply the dominated convergence theorem.

The integrals of the second terms are bounded by absolute moments of the variable $ZI_{B_n^c \cap R_n}$, where Z follows a normal distribution with mean equal to 0 and variance equal to $I^D(T(q))^{-1}$, hence, applying again the dominated convergence theorem one has $\int tc_{in}(t)h(t)I_{B_n^c \cap R_n}(t)dt \to 0$, for $i = 0, 1, 2, 3$.

Finally, to prove that $\int_{B_n^c \cap R_n} t\overline{\pi}_D^n(t|q)dt \to 0$, making use of (2), (5), and the fact that $q \in \Gamma^*$, one obtains

$$\begin{aligned}\left| \int_{B_n^c \cap R_n} t\overline{\pi}_D^n(t|q)dt \right| &\leq \left| \pi(T(q))^{-1} \left(\frac{I^D(T(q))}{2\pi} \right)^{1/2} (1 + c_n)e^{-\Delta_{n-1}\rho} \right. \\ &\quad \left. \cdot \int_{B_n^c \cap R_n} t\pi\left(T(q) + \frac{t}{\Delta_{n-1}^{1/2}} \right) dt \right| \\ &\leq \pi(T(q))^{-1} \left(\frac{I^D(T(q))}{2\pi} \right)^{1/2} |1 + c_n| \Delta_{n-1} e^{-\Delta_{n-1}\rho} \\ &\quad \cdot \left(\int_\Theta |\theta|\pi(\theta)d\theta + |T(q)| \right) \to 0.\end{aligned}$$

For *(ii)*, let us denote $f_n(\theta) = \Delta_{n-1}^{1/2}(\theta - T(q))$ and $\widetilde{C}_n^+ = f_n(C_n^+)$. Without loss of generality, we can assume that $T(q) \in C_n^+$ for each $n \geq n_0$. Note that $f_n(\cdot)$ is a strictly increasing homeomorphism between Θ and R_n. As a consequence, $\widetilde{C}_n^+$ is a compact set and $0 \in \widetilde{C}_n^+$, for each $n \geq n_0$. For $n \geq n_0$, since $q \in \Gamma_n^+$, one obtains $\widetilde{T}_n(q) = T(q) + \frac{t_n^*}{\Delta_{n-1}^{1/2}}$, with $t_n^* = \arg\max_{t \in \widetilde{C}_n^+} \overline{\pi}_D^n(t|q) = \arg\max_{t \in R_n} \overline{\pi}_D^n(t|q) = \arg\max_{t \in \mathbb{R}} \overline{\pi}_D^n(t|q)$; hence, it suffices to prove that $t_n^* \to 0$.

As $h(t)$ is the density function of a normal distribution with mean equal to 0 and variance equal to $I^D(T(q))^{-1}$, it has a unique maximum at $t = 0$; thus, we shall prove that $\arg\max_{t \in \mathbb{R}} \overline{\pi}_D^n(t|q) \to \arg\max_{t \in \mathbb{R}} h(t)$, as $n \to \infty$, by proving that

$$\lim_{n \to \infty} \sup_{t \in \mathbb{R}} |\overline{\pi}_D^n(t|q) - h(t)| = 0. \tag{8}$$

To that end, recall that $\overline{\pi}_D^n(t|q) = \frac{h_n(t)}{J_n}$; consequently,

$$\begin{aligned}\sup_{t \in \mathbb{R}} |\overline{\pi}_D^n(t|q) - h(t)| &\leq \sup_{t \in \mathbb{R}} \left| \overline{\pi}_D^n(t|q) - \frac{h_n(t)}{\pi(T(q))} \left(\frac{I^D(T(q))}{2\pi} \right)^{1/2} \right| I_{B_n}(t) + \sup_{t \in \mathbb{R}} \overline{\pi}_D^n(t|q) I_{B_n^c}(t) \\ &\quad + \sup_{t \in \mathbb{R}} \left| \frac{h_n(t)}{\pi(T(q))} \left(\frac{I^D(T(q))}{2\pi} \right)^{1/2} - h(t) \right| I_{B_n}(t) + \sup_{t \in \mathbb{R}} h(t) I_{B_n^c}(t),\end{aligned} \tag{9}$$

thus, it is enough to prove that all the terms in (9) converge to 0, as $n \to \infty$.

For the first term in (9), one has the following inequality

$$\sup_{t \in \mathbb{R}} \left| \overline{\pi}_D^n(t|q) - \frac{h_n(t)}{\pi(T(q))} \left(\frac{I^D(T(q))}{2\pi} \right)^{1/2} \right| I_{B_n}(t) \leq$$

$$
\begin{aligned}
&\leq \left| \frac{1}{J_n} - \frac{1}{\pi(T(q))} \left(\frac{I^D(T(q))}{2\pi} \right)^{1/2} \right| \\
&\cdot \left(\pi(T(q)) + \frac{|b_1|\pi(T(q))}{\Delta_{n-1}^{1/2}} \sup_{t\in\mathbb{R}} |t| e^{-\frac{t^2(I^D(T(q))-\epsilon)}{2}} \right. \\
&+ \frac{|b_2|\pi(T(q))}{2\Delta_{n-1}} \sup_{t\in\mathbb{R}} t^2 e^{-\frac{t^2(I^D(T(q))-\epsilon)}{2}} \\
&\left. + \frac{M}{6\Delta_{n-1}^{3/2}} \sup_{t\in\mathbb{R}} |t|^3 e^{-\frac{t^2(I^D(T(q))-\epsilon)}{2}} \right),
\end{aligned}
$$

where M is an upper bound of the function $\pi'''(\cdot)$. Thus, it converges to 0 due to the fact that $\frac{1}{J_n} \to \frac{1}{\pi(T(q))} \left(\frac{I^D(T(q))}{2\pi} \right)^{1/2}$, and $|t|^i e^{-\frac{t^2(I^D(T(q))-\epsilon)}{2}}$ is a bounded function for $i = 1, 2, 3$.

Since $\pi(\cdot)$ is bounded, for the second term in (9), one has

$$
\sup_{t\in\mathbb{R}} \overline{\pi}_D^n(t|q) I_{B_n^c}(t) \leq \sup_{t\in\mathbb{R}} \frac{e^{-\rho\Delta_{n-1}}}{J_n} \pi\left(T(q) + \frac{t}{\Delta_{n-1}} \right) I_{B_n^c \cap R_n}(t) \leq \frac{e^{-\rho\Delta_{n-1}}}{J_n} M^* \to 0,
$$

where M^* denotes an upper bound of the function $\pi(\cdot)$.

For the third term in (9), one has

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}} \left| \left(\frac{I^D(T(q))}{2\pi} \right)^{1/2} \frac{h_n(t)}{\pi(T(q))} - h(t) \right| I_{B_n}(t) \leq \\
&\leq \left(\frac{I^D(T(q))}{2\pi} \right)^{1/2} \sup_{t\in\mathbb{R}} \left| e^{-\frac{t^2 I^D(\theta_n'(t))}{2}} - e^{-\frac{t^2 I^D(T(q))}{2}} \right| I_{B_n \cap R_n}(t) \\
&+ \left(\frac{I^D(T(q))}{2\pi} \right)^{1/2} \cdot \left(\frac{|b_1|}{\Delta_{n-1}^{1/2}} \sup_{t\in\mathbb{R}} |t| e^{-\frac{t^2(I^D(T(q))-\epsilon)}{2}} \right. \\
&\left. + \frac{|b_2|}{2\Delta_{n-1}} \sup_{t\in\mathbb{R}} t^2 e^{-\frac{t^2(I^D(T(q))-\epsilon)}{2}} + \frac{M}{6\Delta_{n-1}^{3/2}\pi(T(q))} \sup_{t\in\mathbb{R}} |t|^3 e^{-\frac{t^2(I^D(T(q))-\epsilon)}{2}} \right) \\
&+ \sup_{t\in\mathbb{R}} h(t) I_{R_n^c}(t).
\end{aligned}
$$

As a result, since $\sup_{t\in\mathbb{R}} h(t) I_{R_n^c}(t) \leq h(\eta\Delta_{n-1}^{1/2}) \to 0$, $\Delta_n \to \infty$, and $|t|^i e^{-\frac{t^2(I^D(T(q))-\epsilon)}{2}}$ is a bounded function for $i = 1, 2, 3$, to establish the convergence of the third term in (9), we shall prove

$$
\sup_{t\in\mathbb{R}} \left| e^{-\frac{t^2 I^D(\theta_n'(t))}{2}} - e^{-\frac{t^2 I^D(T(q))}{2}} \right| I_{B_n \cap R_n}(t) \to 0. \tag{10}
$$

To that end, we prove that for each $0 < \epsilon < I^D(T(q))$,

$$
\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \left| e^{-\frac{t^2 I^D(\theta_n'(t))}{2}} - e^{-\frac{t^2 I^D(T(q))}{2}} \right| I_{B_n \cap R_n}(t) \leq \frac{\epsilon}{I^D(T(q)) - \epsilon}, \tag{11}
$$

thus, by taking limit as $\epsilon \to 0$, (10) follows. For each $t \in B_n \cap R_n$, let us consider the function $h_t(x) = e^{-\frac{t^2}{2}x}$, $x \geq 0$, which satisfies $h_t'(x) = -\frac{t^2}{2}e^{-\frac{t^2}{2}x}$. By the mean value theorem, for each $t \in B_n \cap R_n$,

$$\left| e^{-\frac{t^2 I^D(\theta_n'(t))}{2}} - e^{-\frac{t^2 I^D(T(q))}{2}} \right| \leq \frac{t^2}{2} e^{-\frac{t^2}{2}a_{t,n}} |I^D(\theta_n'(t)) - I^D(T(q))|,$$

for $a_{t,n}$ between $I^D(\theta_n'(t))$ and $I^D(T(q))$. Thus, $a_{t,n} > I^D(T(q)) - \epsilon$, and $e^{-\frac{t^2}{2}a_{t,n}} \leq e^{-\frac{t^2}{2}(I^D(T(q))-\epsilon)}$, where $\frac{t^2}{2}e^{-\frac{t^2}{2}(I^D(T(q))-\epsilon)}$ is a function bounded by $\frac{1}{I^D(T(q))-\epsilon}$. As a consequence,

$$\sup_{t \in B_n \cap R_n} \left| e^{-\frac{t^2 I^D(\theta_n'(t))}{2}} - e^{-\frac{t^2 I^D(T(q))}{2}} \right| \leq \frac{\epsilon}{I^D(T(q)) - \epsilon},$$

and (11) follows.

For the forth term in (9), one has $\sup_{t \in \mathbb{R}} h(t) I_{B_n^c}(t) = h(\delta \Delta_{n-1}^{1/2}) \to 0$.

From (8), one has that $|\overline{\pi}_D^n(t_n^*|q) - h(t_n^*)| \to 0$, and $\overline{\pi}_D^n(t_n^*|q) \to h(0)$. The former is immediate from the inequality $|\overline{\pi}_D^n(t_n^*|q) - h(t_n^*)| \leq \sup_{t \in \mathbb{R}} |\overline{\pi}_D^n(t|q) - h(t)|$, and the latter follows from $|\overline{\pi}_D^n(t_n^*|q) - h(0)| \leq \sup_{t \in \mathbb{R}} |\overline{\pi}_D^n(t|q) - h(t)|$; now, it is straightforward that $h(t_n^*) \to h(0)$.

If $\{t_n^*\}_{n \in \mathbb{N}}$ does not converge to 0, then there exists $\varepsilon > 0$, such that for each $N_0 \in \mathbb{N}$, $|t_N^*| > \varepsilon$, for some $N > N_0$. Consequently, $|h(t_N^*) - h(0)| > h(0) - h(\varepsilon) > 0$. Therefore, for $\nu = h(0) - h(\varepsilon)$, and each $N_0 \in \mathbb{N}$, there exists $N > N_0$ satisfying $|h(t_N^*) - h(0)| > \nu$, which contradicts $h(t_n^*) \to h(0)$. This completes the proof of the theorem.

□

2 Proof of asymptotic properties of the EDAP and MDAP estimators

For simplicity, we will assume that $P[Z_n \to \infty] = 1$. Moreover, unless specified otherwise, we will assume that all the limits are taken as $n \to \infty$ and we shall keep the same notation throughout this section. Let us fix $\omega \in \{Z_n \to \infty\}$ and henceforth, in order to lighten the notation, we will consider that all the random variables are evaluated at ω.

Proof of Theorem 4.2

In order to prove this theorem, we are to make use of next lemma.

Lemma 5. *Under the hypotheses of Theorem 4.2, let*

$$w_n(t) = \pi\left(\hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}}\right) e^{-\Delta_{n-1}\left(D\left(\hat{p}_n, \hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}}\right) - D(\hat{p}_n, \hat{\theta}_n^D)\right)}$$

$$- \pi(\theta_p) e^{-\frac{1}{2}t^2 I^D(\theta_p)}, \quad t \in \mathbb{R}.$$

Then, as $n \to \infty$*,*

(i) $\int |w_n(t)| dt \to 0$ *a.s. on* $\{Z_n \to \infty\}$.

(ii) $\int |t||w_n(t)| dt \to 0$ *a.s. on* $\{Z_n \to \infty\}$.

Proof. Let us denote $R_n = \{\Delta_{n-1}^{1/2}(\theta - \hat{\theta}_n^D) : \theta \in \Theta\}$, $n \in \mathbb{N}$. Since $\theta_p \in int(\Theta)$, by (9) in Assumption 3, one has that $\hat{\theta}_n^D$ is eventually in $int(\Theta)$ and, as a result, in an analogous manner to that in Lemma 4, one obtains $\cup_{n=1}^{\infty} R_n = \mathbb{R}$.

(i) Observe that $w_n(t) I_{R_n^c}(t) = -\pi(\theta_p) e^{-\frac{1}{2}t^2 I^D(\theta_p)} I_{R_n^c}(t)$, consequently, using the dominated convergence theorem, since $I_{R_n^c}(t) \to 0$, $|w_n(t) I_{R_n^c}(t)| \leq \pi(\theta_p) e^{-\frac{1}{2}t^2 I^D(\theta_p)}$, for each $t \in \mathbb{R}$, and $\int e^{-\frac{1}{2}t^2 I^D(\theta_p)} dt < \infty$, one has that $\int_{R_n^c} w_n(t) dt \to 0$.

On the other hand, for each fixed $t \in \mathbb{R}$, using a first-order Taylor series expansion, one has

$$D\left(\hat{p}_n, \hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}}\right) = D(\hat{p}_n, \hat{\theta}_n^D) + \frac{t}{\Delta_{n-1}^{1/2}} \dot{D}(\hat{p}_n, \hat{\theta}_n^D) + \frac{1}{2}\left(\frac{t}{\Delta_{n-1}^{1/2}}\right)^2 \ddot{D}(\hat{p}_n, \theta_n'(t)),$$

where $\theta_n'(t)$ is a real number between $\hat{\theta}_n^D$ and $\hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}}$. Observe that $\theta_n'(t) \in \Theta$ if and only if $t \in R_n$; hence, for each $t \in \mathbb{R}$ fixed, there exists $n_0 = n_0(t)$ such that $\theta_n'(t) \in \Theta$ for each $n \geq n_0$. Thus,

$$w_n(t) = \pi\left(\hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}}\right) e^{-\frac{t^2}{2} I_n^D(\theta_n'(t))} - \pi(\theta_p) e^{-\frac{t^2}{2} I^D(\theta_p)}.$$

For each $t \in \mathbb{R}$, by (9) in Assumption 3, one has that $\{\theta_n'(t)\}_{n \in \mathbb{N}}$ is a sequence that converges to θ_p, and by (b) in Assumption 3, $I_n^D(\theta_n'(t)) \to I^D(\theta_p)$. Moreover, the continuity of $\pi(\cdot)$ and the fact that $\hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}} \to \theta_p$, for each $t \in \mathbb{R}$ fixed, guarantee that $\pi\left(\hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}}\right) \to \pi(\theta_p)$; hence, $w_n(t) \to 0$, for each $t \in \mathbb{R}$.

Let $0 < \epsilon < I^D(\theta_p)$. Since $\pi(\cdot)$ is continuous, there exists $\delta_1 = \delta_1(\epsilon)$ such that if $|\theta - \theta_p| \leq \delta_1$, then $|\pi(\theta) - \pi(\theta_p)| \leq \epsilon$. By (b) in Assumption 3, one also has that there exist $n_1 = n_1(\epsilon)$, and $\delta_2 = \delta_2(\epsilon)$ such that if $|\theta - \theta_p| \leq \delta_2$, and $n \geq n_1$, then $|I_n^D(\theta) - I^D(\theta_p)| \leq \epsilon$. In addition, we can safely assume that $\delta_1 = \delta_2$. On the other hand, from (9) in Assumption 3, the existence of $n_2 = n_2(\delta) \in \mathbb{N}$ such that $|\hat{\theta}_n^D - \theta_p| \leq \delta_1/2$, for each $n \geq n_2$, is guaranteed.

Let us fix $\delta = \delta_1/2$, $N_0 = \max(n_1, n_2) = N_0(\delta_1, \epsilon)$, and define the sets $B_n = \{t \in \mathbb{R} : |t| \leq \delta \Delta_{n-1}^{1/2}\}$, $n \in \mathbb{N}$. Firstly, we shall prove $\int_{B_n \cap R_n} |w_n(t)| dt \to 0$.

Note that for each $t \in B_n \cap R_n$, $n \geq N_0$, $\left|\pi\left(\hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}}\right) - \pi(\theta_p)\right| \leq \epsilon$, and $|I_n^D(\theta'_n(t)) - I^D(\theta_p)| \leq \epsilon$. Consequently, for each $n \geq N_0$,

$$
\begin{aligned}
|w_n(t)| I_{B_n \cap R_n}(t) &\leq \left(\pi\left(\hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}}\right) e^{-\frac{t^2}{2} I_n^D(\theta'_n(t))} + \pi(\theta_p) e^{-\frac{t^2}{2} I^D(\theta_p)}\right) \cdot I_{B_n \cap R_n}(t) \\
&\leq 2\left(\pi(\theta_p) + \epsilon\right) e^{-\frac{t^2}{2}\left(I^D(\theta_p) - \epsilon\right)},
\end{aligned}
$$

with $\int e^{-\frac{t^2}{2}\left(I^D(\theta_p)-\epsilon\right)} dt = \left(\frac{2\pi}{I^D(\theta_p)-\epsilon}\right)^{1/2} < \infty$.

Since $w_n(t) \to 0$, in particular, $|w_n(t)| I_{B_n \cap R_n}(t) \to 0$, and by applying the dominated convergence theorem one obtains $\int_{B_n \cap R_n} |w_n(t)| dt \to 0$.

Now, we shall prove $\int_{B_n^c \cap R_n} |w_n(t)| dt \to 0$. From the previous Taylor series expansion, for $t \in \mathbb{R}$, one has

$$
D\left(\hat{p}_n, \hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}}\right) - D(\hat{p}_n, \hat{\theta}_n^D) = \frac{I_n^D(\theta'_n(t))}{2} \frac{t^2}{\Delta_{n-1}}.
$$

Bearing in mind that $p \in \Gamma^*$, (a) and (9) in Assumption 3, and the fact that $\hat{p}_n \to p$ in l_1, the existence of some $\rho > 0$ and $N_* \in \mathbb{N}$ satisfying

$$
\inf_{t \in B_n^c \cap R_n} D\left(\hat{p}_n, \hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}}\right) - D(\hat{p}_n, \hat{\theta}_n^D) > \rho, \tag{12}
$$

for $n \geq N_*$ is guaranteed, and as a consequence, for each $t \in B_n^c \cap R_n$, $n \geq N_*$, $I_n^D(\theta'_n(t)) > \frac{2}{t^2}\rho\Delta_{n-1}$, therefore $e^{-\frac{t^2}{2} I_n^D(\theta'_n(t))} \leq e^{-\rho\Delta_{n-1}}$. Hence, for each $n \geq N_*$, one has

$$
\begin{aligned}
\int_{B_n^c \cap R_n} |w_n(t)| dt &\leq e^{-\Delta_{n-1}\rho} \int_{B_n^c \cap R_n} \pi\left(\hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}}\right) dt + \pi(\theta_p) \int_{B_n^c \cap R_n} e^{-\frac{1}{2}t^2 I^D(\theta_p)} dt \\
&\leq e^{-\Delta_{n-1}\rho} \Delta_{n-1}^{1/2} + \pi(\theta_p) \left(\frac{2\pi}{I^D(\theta_p)}\right)^{1/2} P[|Z| > \delta \Delta_{n-1}^{1/2}],
\end{aligned}
$$

where Z follows a normal distribution with mean equal to 0 and variance equal to $I^D(\theta_p)^{-1}$; therefore, $\int_{B_n^c \cap R_n} |w_n(t)| dt \to 0$.

(ii) Firstly, note that for each $t \in \mathbb{R}$ fixed, $|t||w_n(t)| \to 0$. As done before, $\int_{R_n^c} |t||w_n(t)| dt \to 0$, thus, it is enough to prove that $\int_{B_n \cap R_n} |t||w_n(t)| dt \to 0$, and $\int_{B_n^c \cap R_n} |t||w_n(t)| dt \to 0$.

On the one hand, for each $n \geq N_0$,

$$
|t||w_n(t)| I_{B_n \cap R_n}(t) \leq 2|t| \left(\pi(\theta_p) + \epsilon\right) e^{-\frac{t^2}{2}\left(I^D(\theta_p) - \epsilon\right)},
$$

and $\int_{B_n \cap R_n} |t||w_n(t)|dt \to 0$ follows from dominated convergence theorem and bearing in mind that
$$\int |t|e^{-\frac{t^2}{2}\left(I^D(\theta_p)-\epsilon\right)}dt = \frac{2}{I^D(\theta_p)-\epsilon}.$$

On the other hand, for each $n \geq N_*$, one has

$$\begin{aligned}\int_{B_n^c \cap R_n} |t||w_n(t)|dt &\leq e^{-\Delta_{n-1}\rho}\int_{B_n^c \cap R_n} |t|\pi\left(\hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}}\right)dt \\ &\quad + \pi(\theta_p)\int_{B_n^c \cap R_n} |t|e^{-\frac{1}{2}t^2 I^D(\theta_p)}dt \\ &\leq e^{-\Delta_{n-1}\rho}\Delta_{n-1}^{1/2}\int_\Theta |\theta|\pi\left(\theta\right)d\theta + e^{-\Delta_{n-1}\rho}\Delta_{n-1}|\hat{\theta}_n^D| \\ &\quad + \pi(\theta_p)\left(\frac{2\pi}{I^D(\theta_p)}\right)^{1/2} E[|Z|I_{B_n^c}],\end{aligned}$$

where Z follows a normal distribution with mean equal to 0 and variance equal to $I^D(\theta_p)^{-1}$; therefore, $\int_{B_n^c \cap R_n} |w_n(t)|dt \to 0$. □

Now, we show the proof of Theorem 4.2.

(i) From the fact that $\hat{p}_n \to p$ in l_1, (a) in Assumption 3 and the continuity of $D(q,\cdot)$ on Θ for each $q \in \Gamma$, one has that $\sup_{\theta\in\Theta}|D(\hat{p}_n,\theta) - D(p,\theta)| \to 0$, hence $D(\hat{p}_n,\hat{\theta}_n^D) \to D(p,\theta_p)$.

Recall that $\dot{D}(\hat{p}_n,\hat{\theta}_n^D) = \dot{D}(p,\theta_p) = 0$, and $I_n^D(\hat{\theta}_n^D) \to I^D(\theta_p)$.

For each $t \in R_n$, we can write

$$\overline{\pi}_D^n(t|\hat{p}_n) = K_n^{-1}\pi\left(\hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}}\right)e^{-\Delta_{n-1}\left(D\left(\hat{p}_n,\hat{\theta}_n^D+\frac{t}{\Delta_{n-1}^{1/2}}\right)-D\left(\hat{p}_n,\hat{\theta}_n^D\right)\right)},$$

where $K_n = \displaystyle\int_{R_n}\pi\left(\hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}}\right)e^{-\Delta_{n-1}\left(D\left(\hat{p}_n,\hat{\theta}_n^D+\frac{t}{\Delta_{n-1}^{1/2}}\right)-D(\hat{p}_n,\hat{\theta}_n^D)\right)}dt$, and $\overline{\pi}_D^n(t|\hat{p}_n) = 0$, for $t \in R_n^c$. As a consequence,

$$\begin{aligned}\int\left|\overline{\pi}_D^n(t|\hat{p}_n) - \left(\frac{I^D(\theta_p)}{2\pi}\right)^{1/2} e^{-\frac{t^2}{2}I^D(\theta_p)}\right|dt &= \int_{R_n^c}\left(\frac{I^D(\theta_p)}{2\pi}\right)^{1/2} e^{-\frac{t^2}{2}I^D(\theta_p)}dt \\ &+ K_n^{-1}\int_{R_n}\left|\pi\left(\hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}}\right)e^{-\Delta_{n-1}\left(D\left(\hat{p}_n,\hat{\theta}_n^D+\frac{t}{\Delta_{n-1}^{1/2}}\right)-D(\hat{p}_n,\hat{\theta}_n^D)\right)}\right. \\ &\left. - K_n\left(\frac{I^D(\theta_p)}{2\pi}\right)^{1/2} e^{-\frac{t^2}{2}I^D(\theta_p)}\right|dt\end{aligned}$$

$$\leq \int_{R_n^c} \left(\frac{I^D(\theta_p)}{2\pi}\right)^{1/2} e^{-\frac{t^2}{2}I^D(\theta_p)}dt + K_n^{-1}(I_{1n}+I_{2n}),$$

where the first integral converges to 0 by applying the dominated convergence theorem, and

$$\begin{aligned} I_{1n} &= \int_{R_n} |w_n(t)|dt, \\ I_{2n} &= \int_{R_n} \left|\pi(\theta_p)e^{-\frac{1}{2}t^2 I^D(\theta_p)} - K_n \left(\frac{I^D(\theta_p)}{2\pi}\right)^{1/2} e^{-\frac{t^2}{2}I^D(\theta_p)}\right|dt. \end{aligned}$$

In the proof of Lemma 5, we showed that $K_n \to \pi(\theta_p)\left(\frac{2\pi}{I^D(\theta_p)}\right)^{1/2} \neq 0$; hence, it is enough to prove that $I_{in} \to 0$, for $i = 1, 2$. From Lemma 5, we also have that $I_{1n} \to 0$, and $\left|\pi(\theta_p) - K_n \left(\frac{I^D(\theta_p)}{2\pi}\right)^{1/2}\right| \displaystyle\int_{R_n} e^{-\frac{t^2}{2}I^D(\theta_p)}dt \to 0$.

(ii) We have

$$\begin{aligned} &\int |t| \left|\overline{\pi}_D^n(t|\hat{p}_n) - \left(\frac{I^D(\theta_p)}{2\pi}\right)^{1/2} e^{-\frac{t^2}{2}I^D(\theta_p)}\right|dt = \\ &\quad = \int_{R_n^c} |t| \left(\frac{I^D(\theta_p)}{2\pi}\right)^{1/2} e^{-\frac{t^2}{2}I^D(\theta_p)}dt \\ &\quad + K_n^{-1}\int_{R_n} |t| \left|\pi\left(\hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}}\right) e^{-\Delta_{n-1}\left(D\left(\hat{p}_n, \hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}}\right) - D(\hat{p}_n, \hat{\theta}_n^D)\right)}\right. \\ &\quad \left. - K_n \left(\frac{I^D(\theta_p)}{2\pi}\right)^{1/2} e^{-\frac{t^2}{2}I^D(\theta_p)}\right|dt \\ &\quad \leq \int_{R_n^c} |t| \left(\frac{I^D(\theta_p)}{2\pi}\right)^{1/2} e^{-\frac{t^2}{2}I^D(\theta_p)}dt + K_n^{-1}(\bar{I}_{1n} + \bar{I}_{2n}), \end{aligned}$$

where again the first integral converges to 0 by applying the dominated convergence theorem, and

$$\begin{aligned} \bar{I}_{1n} &= \int_{R_n} |t||w_n(t)|dt \\ \bar{I}_{2n} &= \int_{R_n} |t| \left|\pi(\theta_p)e^{-\frac{1}{2}t^2 I^D(\theta_p)} - K_n \left(\frac{I^D(\theta_p)}{2\pi}\right)^{1/2} e^{-\frac{t^2}{2}I^D(\theta_p)}\right|dt. \end{aligned}$$

From Lemma 5 *(ii)*, we also have that $\bar{I}_{1n} \to 0$, and bearing in mind that $K_n \to \pi(\theta_p)\left(\frac{2\pi}{I^D(\theta_p)}\right)^{1/2} \neq 0$, we obtain

$$\bar{I}_{2n} = \left|\pi(\theta_p) - K_n \left(\frac{I^D(\theta_p)}{2\pi}\right)^{1/2}\right| \int_{R_n} |t| e^{-\frac{t^2}{2}I^D(\theta_p)}dt \to 0.$$

(iii) Note that

$$\int \left| \overline{\pi}^n_D(t|\hat{p}_n) - \left(\frac{I^D_n(\hat{\theta}^D_n)}{2\pi}\right)^{1/2} e^{-\frac{1}{2}t^2 I^D_n(\hat{\theta}^D_n)} \right| dt \leq$$
$$\leq \int \left| \overline{\pi}^n_D(t|\hat{p}_n) - \left(\frac{I^D(\theta_p)}{2\pi}\right)^{1/2} e^{-\frac{1}{2}t^2 I^D(\theta_p)} \right| dt$$
$$+ \int \left| \left(\frac{I^D(\theta_p)}{2\pi}\right)^{1/2} e^{-\frac{1}{2}t^2 I^D(\theta_p)} - \left(\frac{I^D_n(\hat{\theta}^D_n)}{2\pi}\right)^{1/2} e^{-\frac{1}{2}t^2 I^D_n(\hat{\theta}^D_n)} \right| dt$$

From *(i)*, we have that the first integral converges to 0. Since $I^D_n(\hat{\theta}^D_n) \to I^D(\theta_p)$, using Glick's theorem (see Devroye and Györfi (1985), p.10), we also have that the second integral converges to 0.

□

Proof of Theorem 4.3

Let $t = \Delta^{1/2}_{n-1}(\theta - \hat{\theta}^D_n)$ and $\overline{\pi}^n_D(t|\hat{p}_n)$ its D-posterior density function at $\hat{p}_n$.

(i) Bearing in mind that

$$\theta^{*D}_n = \int_\Theta \theta \pi^n_D(\theta|\hat{p}_n)d\theta = \int_{R_n} \left(\hat{\theta}^D_n + \frac{t}{\Delta^{1/2}_{n-1}}\right) \overline{\pi}^n_D(t|\hat{p}_n)dt,$$

and Theorem 4.2 *(ii)*, we obtain

$$\Delta^{1/2}_{n-1}(\theta^{*D}_n - \hat{\theta}^D_n) = \int_{R_n} t\overline{\pi}^n_D(t|\hat{p}_n)dt \to \left(\frac{I^D(\theta_p)}{2\pi}\right)^{1/2} \int te^{-\frac{1}{2}t^2 I^D(\theta_p)}dt = 0,$$

a.s. on $\{Z_n \to \infty\}$.

(ii) Note that $\Delta^{1/2}_{n-1}(\theta^{*D}_n - \theta_p) = \Delta^{1/2}_{n-1}(\theta^{*D}_n - \hat{\theta}^D_n) + \Delta^{1/2}_{n-1}(\hat{\theta}^D_n - \theta_p)$. From (10) in Assumption 3 and *(i)*, making use of Slutsky's theorem, one has

$$\Delta^{1/2}_{n-1}(\theta^{*D}_n - \theta_p) \xrightarrow[n\to\infty]{d} N(0, I^D(\theta_p)^{-1}).$$

□

Proof of Theorem 4.4

The proof is analogous to that in Theorem 3 *(ii)*. Let $R_n = \{\Delta^{1/2}_{n-1}(\theta - \hat{\theta}^D_n) : \theta \in \Theta\}$. With analogous arguments to those in Lemma 4, one can prove that for each $t \in \mathbb{R}$,

$$\overline{\pi}^n_D(t|\hat{p}_n) = (1 + C_n)\left(\frac{I^D_n(\hat{\theta}^D_n)}{2\pi}\right)^{1/2} e^{-\frac{t^2 I^D_n({\theta'}_n(t))}{2}}\cdot$$
$$\cdot\left(1 + \frac{tb_{1n}}{\Delta^{1/2}_{n-1}} + \frac{t^2 b_{2n}}{2\Delta_{n-1}} + \frac{t^3\pi'''(\theta^*_n(t))}{6\Delta^{3/2}_{n-1}\pi(\hat{\theta}^D_n))}\right) I_{R_n}(t),$$

a.s. on $\{Z_n \to \infty\}$, where $b_{1n} = \frac{\pi'(\hat{\theta}_n^D)}{\pi(\hat{\theta}_n^D)}$, $b_{2n} = \frac{\pi''(\hat{\theta}_n^D)}{\pi(\hat{\theta}_n^D)}$, $\theta'_n(t)$ and $\theta^*_n(t)$ are both points between $\hat{\theta}_n^D$ and $\hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}}$, for each $n \in \mathbb{N}$, and $t \in \mathbb{R}$, and $\{C_n\}_{n\in\mathbb{N}}$ is a sequence of real valued random variables converging to 0 a.s.

Moreover, the following inequality holds

$$\begin{aligned}\sup_{t\in\mathbb{R}} |\overline{\pi}_D^n(t|\hat{p}_n) - \varphi(t;\theta_p)| \leq \sup_{t\in\mathbb{R}} &\left|\overline{\pi}_D^n(t|\hat{p}_n) - \frac{h_n(t)}{\pi(\hat{\theta}_n^D)}\left(\frac{I_n^D(\hat{\theta}_n^D)}{2\pi}\right)^{1/2}\right| I_{B_n}(t) \\ &+ \sup_{t\in\mathbb{R}} \overline{\pi}_D^n(t|\hat{p}_n) I_{B_n^c}(t) \\ &+ \sup_{t\in\mathbb{R}} \left|\left(\frac{I_n^D(\hat{\theta}_n^D)}{2\pi}\right)^{1/2} \frac{h_n(t)}{\pi(\hat{\theta}_n^D)} - \varphi(t;\theta_p)\right| I_{B_n}(t) \\ &+ \sup_{t\in\mathbb{R}} \varphi(t;\theta_p) I_{B_n^c}(t), \end{aligned} \tag{13}$$

where $h_n(t) = \pi\left(\hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}}\right) e^{-\Delta_{n-1} D\left(\hat{p}_n, \hat{\theta}_n^D + \frac{t}{\Delta_{n-1}^{1/2}}\right) + \Delta_{n-1} D\left(\hat{p}_n, \hat{\theta}_n^D\right)} I_{R_n}(t)$, for each $n \in \mathbb{N}$ and $t \in \mathbb{R}$.

Following the same arguments as in Theorem 3 *(ii)*, one can verify that all the terms in (13) converge to 0 using that $\Delta_n \to \infty$, $\pi(\hat{\theta}_n^D) \to \pi(\theta_p)$, $I_n^D(\hat{\theta}_n^D) \to I^D(\theta_p)$, $J_n = \int h_n(t)dt$, $n \in \mathbb{N}$, satisfies $\frac{1}{J_n} - \frac{1}{\pi(\hat{\theta}_n^D)}\left(\frac{I_n^D(\hat{\theta}_n^D)}{2\pi}\right)^{1/2} \to 0$, $\pi(\cdot)$ is bounded, $\sup_{t\in\mathbb{R}} e^{-\frac{t^2 I^D(\theta_p)}{2}} = 1$, and $|t|^i e^{-\frac{t^2(I^D(\theta_p)-\epsilon)}{2}}$ is a bounded function for $i = 1, 2, 3$.

□

Proof of Theorem 4.5

The proof is analogous to that in Theorem 3 *(ii)*.

To prove *(i)*, observe that there exists $n_0 \in \mathbb{N}$ such that $\widetilde{T}_n(\hat{p}_n)$ exists and is unique for $n \geq n_0$. Since $\hat{p}_n \in \Gamma_n^+$ for $n \geq n_0$, one has $\widetilde{T}_n(\hat{p}_n) = \hat{\theta}_n^D + \frac{t_n^*}{\Delta_{n-1}^{1/2}}$, with $t_n^* = \arg\max_{t\in R_n} \overline{\pi}_D^n(t|\hat{p}_n) = \arg\max_{t\in\mathbb{R}} \overline{\pi}_D^n(t|\hat{p}_n)$; hence, it is sufficient to prove that $t_n^* \to 0$.

Note that, since $\varphi(t;\theta_p)$ is the density function of a normal distribution with mean equal to 0 and variance equal to $I^D(\theta_p)^{-1}$, it has a unique maximum at $t = 0$; thus, we shall prove that $\arg\max_{t\in\mathbb{R}} \overline{\pi}_D^n(t|\hat{p}_n) \to \arg\max_{t\in\mathbb{R}} \varphi(t;\theta_p)$.

By Theorem 4.4, one has that $|\overline{\pi}_D^n(t_n^*|\hat{p}_n) - \varphi(t_n^*;\theta_p)| \to 0$, and $|\overline{\pi}_D^n(t_n^*|\hat{p}_n) - \varphi(0;\theta_p)| \to 0$. Now, it is straightforward that $|\varphi(t_n^*;\theta_p) - \varphi(0;\theta_p)| \to 0$, and as a result, $t_n^* \to 0$.

(ii) is straightforward, using Slutsky's theorem, by (c) in Assumption 3, *(i)* and

$$\Delta_{n-1}^{1/2}(\theta_n^{+D} - \theta_p) = \Delta_{n-1}^{1/2}(\theta_n^{+D} - \hat{\theta}_n^D) + \Delta_{n-1}^{1/2}(\hat{\theta}_n^D - \theta_p).$$

□

3 Proof of robustness properties of the EDAP and MDAP estimators

As it was done in Section 2, we will assume that $P[Z_n \to \infty] = 1$ and we shall keep the same notation throughout this section. Let us fix $\omega \in \{Z_n \to \infty\}$ and henceforth, in order to lighten the notation, we will consider that all the random variables are evaluated at ω.

Proof of Theorem 5.2

First of all, note that for each $L \in \mathbb{N}_0$ and $n \in \mathbb{N}$, the influence function satisfies

$$IF(L, \overline{T}_n, p) = \frac{\partial}{\partial \alpha}\left(\frac{\int_\Theta \theta e^{-\Delta_{n-1} D(p(\theta_0,\alpha,L),\theta)}\pi(\theta)d\theta}{\int_\Theta e^{-\Delta_{n-1} D(p(\theta_0,\alpha,L),\theta)}\pi(\theta)d\theta}\right)_{|\alpha=0}.$$

Let us denote $M = \sup_{\delta\in[-1,\infty)} |G(\delta)| < \infty$, $\overline{M} = \sup_{\delta\in[-1,\infty)} |G'(\delta)| < \infty$, and $F_{\theta_0}(\theta,\alpha,L) = e^{-\Delta_{n-1}D(p(\theta_0,\alpha,L),\theta)}\pi(\theta)$, for each $\theta \in \Theta$, $\alpha \in (0,1)$, $L \in \mathbb{N}_0$. Observe that, $F_{\theta_0}(\theta,\alpha,L) \geq e^{-\Delta_{n-1}M}\pi(\theta)$, and as a consequence,

$$\int_\Theta F_{\theta_0}(\theta,\alpha,L)d\theta \geq e^{-\Delta_{n-1}M}\int_\Theta \pi(\theta)d\theta = e^{-\Delta_{n-1}M},$$

moreover,

$$\begin{aligned}\left|\frac{\partial}{\partial\alpha}\left(\frac{\theta F_{\theta_0}(\theta,\alpha,L)}{\int_\Theta F_{\theta_0}(\theta,\alpha,L)d\theta}\right)\right| &= \left|\frac{\theta\frac{\partial}{\partial\alpha}F_{\theta_0}(\theta,\alpha,L)}{\int_\Theta F_{\theta_0}(\theta,\alpha,L)d\theta} - \frac{\theta F_{\theta_0}(\theta,\alpha,L)\frac{\partial}{\partial\alpha}\left(\int_\Theta F_{\theta_0}(\theta,\alpha,L)d\theta\right)}{\left(\int_\Theta F_{\theta_0}(\theta,\alpha,L)d\theta\right)^2}\right| \\ &\leq 4\overline{M}\Delta_{n-1}|\theta|\pi(\theta)e^{2\Delta_{n-1}M}.\end{aligned}$$

Let us denote $C_L(\boldsymbol{p}_{\theta_0},\theta) = \frac{\partial}{\partial\alpha}D(p(\theta_0,\alpha,L),\theta)_{|\alpha=0}$, and the expectation with respect the D-posterior of θ given the probability distribution $p = \boldsymbol{p}_{\theta_0}$ by $E_{\pi^n_D(\theta|p)}[\cdot]$. One obtains

$$\begin{aligned}IF(L,\overline{T}_n,p) &= \Delta_{n-1}\frac{\int_\Theta -\theta C_L(\boldsymbol{p}_{\theta_0},\theta)e^{-\Delta_{n-1}D(p,\theta)}\pi(\theta)d\theta}{\int_\Theta e^{-\Delta_{n-1}D(p,\theta)}\pi(\theta)d\theta} \\ &\quad + \Delta_{n-1}\frac{\int_\Theta C_L(\boldsymbol{p}_{\theta_0},\theta)e^{-\Delta_{n-1}D(p,\theta)}\pi(\theta)d\theta}{\left(\int_\Theta e^{-\Delta_{n-1}D(p,\theta)}\pi(\theta)d\theta\right)^2}\cdot\int_\Theta \theta e^{-\Delta_{n-1}D(p,\theta)}\pi(\theta)d\theta \\ &= -\Delta_{n-1}E_{\pi^n_D(\theta|p)}\left[\theta C_L(\boldsymbol{p}_{\theta_0},\theta)\right] + \Delta_{n-1}E_{\pi^n_D(\theta|p)}[\theta]\,E_{\pi^n_D(\theta|p)}\left[C_L(\boldsymbol{p}_{\theta_0},\theta)\right].\end{aligned}$$

Thus, it is enough to prove that, for each $n \in \mathbb{N}$, $|E_{\pi^n_D(\theta|p)}\left[C_L(\boldsymbol{p}_{\theta_0},\theta)\right]| < \infty$, and $|E_{\pi^n_D(\theta|p)}\left[\theta C_L(\boldsymbol{p}_{\theta_0},\theta)\right]| < \infty$. For the former, one has

$$|E_{\pi^n_D(\theta|p)}\left[C_L(\boldsymbol{p}_{\theta_0},\theta)\right]| \leq \frac{\int_\Theta |C_L(\boldsymbol{p}_{\theta_0},\theta)|e^{-\Delta_{n-1}r}\pi(\theta)d\theta}{\int_\Theta e^{-\Delta_{n-1}R}\pi(\theta)d\theta} \leq 2\overline{M}e^{\Delta_{n-1}(R-r)} < \infty,$$

where $\inf_{\theta\in\Theta} D(p,\theta) = r \geq 0$, and $\sup_{\theta\in\Theta} D(p,\theta) = R \leq M < \infty$. Analogously,

$$|E_{\pi_D^n(\theta|p)}\left[\theta C_L(\boldsymbol{p}_{\theta_0},\theta)\right]| \leq \frac{\int_\Theta |\theta||C_L(\boldsymbol{p}_{\theta_0},\theta)|e^{-\Delta_{n-1}r}\pi(\theta)d\theta}{\int_\Theta e^{-\Delta_{n-1}R}\pi(\theta)d\theta}$$
$$\leq 2\overline{M}e^{\Delta_{n-1}(R-r)}\int_\Theta |\theta|\pi(\theta)d\theta < \infty,$$

hence, the results follows.

□

Proof of Theorem 5.3

To prove this theorem and some of the following results, we will make use of the following lemma.

Lemma 6. *Let $\alpha \in (0,1)$ and consider a family of probability distributions $\{\bar{q}_L\}_{L\in\mathbb{N}_0}$ satisfying (a) and (b) in Assumption 5, and suppose that (b) in Assumption 4 and (c) in Assumption 5 hold. Then, for any $\delta > 0$:*

(i) There exist $L_1 \in \mathbb{N}_0$, and $M_1 > 0$ such that

$$\sup_{|\theta|>M_1}\sup_{L>L_1} |D((1-\alpha)p + \alpha\bar{q}_L,\theta) - D(\alpha\bar{q}_L,\theta)| < \delta.$$

(ii) For all $M_2 > 0$, there exists $L_2 \in \mathbb{N}_0$ such that

$$\sup_{|\theta|<M_2}\sup_{L>L_2} |D((1-\alpha)p + \alpha\bar{q}_L,\theta) - D((1-\alpha)p,\theta)| < \delta.$$

The results also hold for the Hellinger distance.

The proof is analogous to that for Lemma 1 in Hooker and Vidyshankar (2014) and it is omitted.

Now, we shall prove Theorem 5.3.

(i) We follow the same steps as in the proof of Corollary 2 in Hooker and Vidyshankar (2014). Since $G(\cdot)$ is strictly convex, $\inf_{\theta\in\Theta,q\in\Gamma} D(q,\theta) = r \geq 0$, and under the assumptions, $\sup_{\theta\in\Theta,q\in\Gamma} D(q,\theta) = R < \infty$. For each $\theta \in \Theta$, $\alpha \in [0,1]$, and $q \in \Gamma$, one has that $e^{-\Delta_{n-1}R} \leq e^{-\Delta_{n-1}D((1-\alpha)p+\alpha q,\theta)} \leq e^{-\Delta_{n-1}r}$, and as a consequence,

$$e^{-\Delta_{n-1}(R-r)}\int_\Theta |\theta|\pi(\theta)d\theta \leq |\overline{T}_n((1-\alpha)p+\alpha q)| \leq e^{-\Delta_{n-1}(r-R)}\int_\Theta |\theta|\pi(\theta)d\theta,$$

hence, the result yields.

(ii) Throughout this proof, all the limits are taken as $L \to \infty$ unless specified otherwise. Let us fix $\alpha \in (0,1)$, and assume that there is a breakdown at level α, that is, $|\theta_L| \to \infty$, with $\theta_L = \arg\max_{\theta\in\Theta} \pi_D^n(\theta|(1-\alpha)p + \alpha\overline{q}_L)$.

From the definition of θ_L,

$$\pi_D^n(\theta_0|(1-\alpha)p + \alpha\overline{q}_L) \leq \pi_D^n(\theta_L|(1-\alpha)p + \alpha\overline{q}_L). \tag{14}$$

By Lemma 6 *(i)* and *(ii)* one obtains, respectively,

$$\lim_{L\to\infty} D((1-\alpha)p + \alpha\overline{q}_L, \theta_L) = \lim_{L\to\infty} D(\alpha\overline{q}_L, \theta_L), \tag{15}$$

$$\lim_{L\to\infty} D((1-\alpha)p + \alpha\overline{q}_L, \theta) = D((1-\alpha)p, \theta), \quad \text{for each } \theta \in \Theta, \tag{16}$$

then, taking limit as $L \to \infty$ in (14) we obtain a contradiction, and as a consequence, there is no breakdown at level α, for each $\alpha \in (0,1)$. Indeed, by (16) and the dominated convergence theorem one has

$$\lim_{L\to\infty} \int_\Theta e^{-\Delta_{n-1} D((1-\alpha)p+\alpha\overline{q}_L,\theta)}\pi(\theta)d\theta = \int_\Theta e^{-\Delta_{n-1} D((1-\alpha)p,\theta)}\pi(\theta)d\theta,$$

thus, for the first term in (14), since $D((1-\alpha)p, \theta_0) = G(-\alpha)$, we have the following limit

$$\lim_{L\to\infty} \pi_D^n(\theta_0|(1-\alpha)p + \alpha\overline{q}_L) = \frac{e^{-\Delta_{n-1}G(-\alpha)}\pi(\theta_0)}{\int_\Theta e^{-\Delta_{n-1}D((1-\alpha)p,\theta)}\pi(\theta)d\theta} > 0.$$

For the right hand side in (14), by Jensen's inequality, one has $D(\alpha\overline{q}_L, \theta) \geq G(\alpha - 1)$, and $D((1-\alpha)p, \theta) \geq G(-\alpha)$, for each $\theta \in \Theta$. Consequently, using (15),

$$\begin{aligned}\lim_{L\to\infty} \pi_D^n(\theta_L|(1-\alpha)p + \alpha\overline{q}_L) &= \frac{\lim_{L\to\infty} e^{-\Delta_{n-1}D(\alpha\overline{q}_L,\theta_L)}\pi(\theta_L)}{\int_\Theta e^{-\Delta_{n-1}D((1-\alpha)p,\theta)}\pi(\theta)d\theta} \\ &\leq \frac{e^{-\Delta_{n-1}G(\alpha-1)}}{\int_\Theta e^{-\Delta_{n-1}D((1-\alpha)p,\theta)}\pi(\theta)d\theta} \lim_{L\to\infty} \pi(\theta_L) = 0,\end{aligned}$$

where we have applied that $\lim_{|\theta|\to\infty} \pi(\theta) = 0$.

□

Proof of Theorem 5.4

In order to prove this theorem, we will make use of the following lemma.

Lemma 7. *Let us fix $\alpha \in (0,1)$. Under conditions of Lemma 6, if there exists $\delta > 0$ such that*

$$\inf_{L\in\mathbb{N}_0} \inf_{\theta\in\Theta} D(\alpha\overline{q}_L, \theta) > \inf_{\theta\in\Theta} D((1-\alpha)p, \theta) + \delta,$$

then, there exists $L_0 \in \mathbb{N}_0$ such that for every $\lambda \in (0, \delta)$, there is $M^* > 0$ satisfying

$$\inf_{L>L_0} \inf_{|\theta|>M^*} (D((1-\alpha)p + \alpha\overline{q}_L, \theta) - D\left((1-\alpha)p + \alpha\overline{q}_L, T\left((1-\alpha)p + \alpha\overline{q}_L\right)\right)) \geq \lambda.$$

The proof is similar to that for Lemma 2 in Hooker and Vidyshankar (2014) and it is omitted.

We shall prove Theorem 5.4.

(i) To that end, let us denote

$$B_1 = \{\alpha \in (0,1) : b(\alpha, T, p) < \infty\},$$
$$B_2 = \{\alpha \in (0,1) : \limsup_{n\to\infty} b(\alpha, \overline{T}_n, p) < \infty\}.$$

We shall prove that $B_1 = B_2$ and, as a result, we obtain $B(T,p) = B(\{\overline{T}_n\}_{n\in\mathbb{N}}, p)$.

Let us assume that $\alpha \in B_1^c$; then, we can find a sequence of probability distributions $\{\bar{q}_L\}_{L\in\mathbb{N}}$ such that $|T((1-\alpha)p+\alpha\bar{q}_L)| \to \infty$, as $L \to \infty$. By Theorem 3 *(i)*, one deduces that for each $L \in \mathbb{N}$,

$$\forall \epsilon > 0,\ \exists k_L = k_L(\epsilon) \in \mathbb{N} :\ |\overline{T}_n((1-\alpha)p + \alpha\bar{q}_L) - T((1-\alpha)p + \alpha\bar{q}_L)| < \epsilon,$$

for $n \geq k_L$ and with $k_L \to \infty$, as $L \to \infty$. Let $M > 0$, and let us fix $\epsilon = 1$, and $k_L = k_L(1)$, $L \in \mathbb{N}$. From the convergence $|T((1-\alpha)p + \alpha\bar{q}_L)| \to \infty$, as $L \to \infty$, one has

$$\exists L_0 \in \mathbb{N} :\ |T((1-\alpha)p + \alpha\bar{q}_L)| > M + 1, \quad \forall L \geq L_0.$$

Now, by using the triangle inequality, for $L \geq L_0$ and $n \geq k_L$,

$$\begin{aligned} |\overline{T}_n((1-\alpha)p + \alpha\bar{q}_L)| &\geq |T((1-\alpha)p + \alpha\bar{q}_L)| \\ &\quad - |\overline{T}_n((1-\alpha)p + \alpha\bar{q}_L) - T((1-\alpha)p + \alpha\bar{q}_L)| > M. \end{aligned}$$

From all the above, and the fact that $\overline{T}_n(p) \to \theta_0$, as $n \to \infty$, one has that $\limsup_{n\to\infty} b(\alpha, \overline{T}_n, p) = \infty$, and $\alpha \in B_2^c$.

Now, we shall prove that if $\alpha \in B_1$, then $\alpha \in B_2$. To that end, we prove that for each $n \in \mathbb{N}$, and any sequence of contaminating distributions $\{\bar{q}_L\}_{L\in\mathbb{N}}$ in $\overline{\Gamma}$, there exists $L_0 \in \mathbb{N}$ such that $|\overline{T}_n((1-\alpha)p + \alpha\bar{q}_L)| \leq M_n$, if $L \geq L_0$, where $\overline{\Gamma}$ denotes the family of contaminating distributions satisfying (a) and (b) in Assumption 5, and $M_n \to M$, for some $M > 0$.

On the one hand, since $\alpha \in B_1$, $\sup_{\bar{q}\in\overline{\Gamma}} |T((1-\alpha)p + \alpha\bar{q})| < \infty$.

Let us fix $\lambda \in (0, \delta)$. By Lemma 7, one can find $L_0 \in \mathbb{N}_0$ and $M^* > 0$ satisfying

$$D((1-\alpha)p + \alpha\bar{q}_L, \theta) \geq D\left((1-\alpha)p + \alpha\bar{q}_L, T\left((1-\alpha)p + \alpha\bar{q}_L\right)\right) + \lambda,$$

for $L \geq L_0$, and $|\theta| \geq M^*$, consequently,

$$e^{-\Delta_{n-1} D((1-\alpha)p+\alpha\bar{q}_L,\theta)} \leq e^{-\Delta_{n-1} D((1-\alpha)p+\alpha\bar{q}_L, T((1-\alpha)p+\alpha\bar{q}_L)) - \Delta_{n-1}\lambda},$$

for $L \geq L_0$, and $|\theta| \geq M^*$.

Moreover, since $\alpha \in B_1$, one can deduced, using the mean value theorem, condition (A4) in González et al. (2017) and Assumption *(c)* (or alternatively condition (A6) in González et al. (2017) for the Hellinger distance), that for the fixed λ, there exists $\epsilon > 0$ such that if $|\theta - T((1-\alpha)p + \alpha\bar{q}_L)| < \epsilon$, then

$$D((1-\alpha)p + \alpha\bar{q}_L, \theta) < D((1-\alpha)p + \alpha\bar{q}_L, T((1-\alpha)p + \alpha\bar{q}_L)) + \lambda/2,$$

for any $L \in N_0$, and hence

$$e^{-\Delta_{n-1}D((1-\alpha)p+\alpha\bar{q}_L,\theta)} > e^{-\Delta_{n-1}D((1-\alpha)p+\alpha\bar{q}_L,T((1-\alpha)p+\alpha\bar{q}_L))-\Delta_{n-1}\lambda/2}.$$

Let us denote $B = \{\theta \in \Theta : |\theta| < M^*\}$ and $K(x) = \left(\int_{x-\epsilon}^{x+\epsilon} \pi(\theta)d\theta\right)^{-1} \int_{M^*}^{\infty} |\theta|\pi(\theta)d\theta$. Then, for $L \geq L_0$,

$$\begin{aligned}
|\overline{T}_n((1-\alpha)p + \alpha\bar{q}_L)| &= \int_B |\theta|\pi_D^n(\theta|(1-\alpha)p + \alpha\bar{q}_L)d\theta + \int_{B^c} |\theta|\pi_D^n(\theta|(1-\alpha)p + \alpha\bar{q}_L)d\theta \\
&\leq M^* + \frac{\int_{B^c} |\theta| e^{-\Delta_{n-1}D((1-\alpha)p+\alpha\bar{q}_L,\theta)}\pi(\theta)d\theta}{\int_{T((1-\alpha)p+\alpha\bar{q}_L)-\epsilon}^{T((1-\alpha)p+\alpha\bar{q}_L)+\epsilon} e^{-\Delta_{n-1}D((1-\alpha)p+\alpha\bar{q}_L,\theta)}\pi(\theta)d\theta} \\
&\leq \frac{e^{-\Delta_{n-1}D((1-\alpha)p+\alpha\bar{q}_L,T((1-\alpha)p+\alpha\bar{q}_L))-\Delta_{n-1}\lambda}\int_{B^c}|\theta|\pi(\theta)d\theta}{e^{-\Delta_{n-1}D((1-\alpha)p+\alpha\bar{q}_L,T((1-\alpha)p+\alpha\bar{q}_L))-\Delta_{n-1}\lambda/2}\int_{T((1-\alpha)p+\alpha\bar{q}_L)-\epsilon}^{T((1-\alpha)p+\alpha\bar{q}_L)+\epsilon}\pi(\theta)d\theta} + M^* \\
&\leq M^* + e^{-\Delta_{n-1}\lambda/2}K^*,
\end{aligned}$$

where $K^* = \sup\{K(t) : |t| \leq \sup_{L^* > L_0} |T((1-\alpha)p + \alpha\bar{q}_{L^*})|\}$. Note that $K^* < \infty$ due to the fact that $\alpha \in B_1$. Thus, $M^* + e^{-\Delta_{n-1}\lambda/2}K^* \to M^*$, as $n \to \infty$.

(ii) Let us write $\theta^* = \arg\max_{\theta\in\Theta} \pi(\theta)$, $\overline{g}_n(q,\theta) = \Delta_{n-1}D(q,\theta) - \log(\pi(\theta^*))$, and

$$\widetilde{B}_2 = \{\alpha \in (0,1) : \limsup_{n\to\infty} b(\alpha, \widetilde{T}_n, p) < \infty\},$$

and, as done before, we shall prove that $B_1 = \widetilde{B}_2$ to obtain $B(T,p) = B(\{\widetilde{T}_n\}_{n\in\mathbb{N}}, p)$.

Assume that $\alpha \in B_1$, and observe that for each $q \in \Gamma$, $g_n(q,\theta) \geq \overline{g}_n(q,\theta)$, for each $\theta \in \Theta$, and $g_n(q,\theta^*) = \overline{g}_n(q,\theta^*)$, hence, for $n \in \mathbb{N}$ sufficiently large $|\widetilde{T}_n(q) - \theta^*| \leq |T(q) - \theta^*|$. As a consequence,

$$\begin{aligned}
|\widetilde{T}_n((1-\alpha)p + \alpha\bar{q}) - \widetilde{T}_n(p)| &\leq |\widetilde{T}_n((1-\alpha)p + \alpha\bar{q}) - \theta^*| + |\theta^* - \widetilde{T}_n(p)| \\
&\leq |T((1-\alpha)p + \alpha\bar{q}) - T(p)| + |T(p) - \theta^*| + |\theta^* - T(p)|,
\end{aligned}$$

and $b(\alpha, \widetilde{T}_n, p) \leq b(\alpha, T, p) + 2|T(p) - \theta^*|$; thus, $\alpha \in \widetilde{B}_2$.

The fact that if $\alpha \in B_1^c$, then $\alpha \in \widetilde{B}_2^c$, can be proved in an identical way to that in *(i)*.

□

Proof of Theorem 5.5

(i) is immediate using the same arguments as in the proof of Theorem 5.3 *(ii)*.

(ii) Let us denote

$$J = \int_\Theta e^{-\Delta_{n-1} D((1-\alpha)p,\theta)} \pi(\theta) d\theta,$$
$$J_L = \int_\Theta e^{-\Delta_{n-1} D((1-\alpha)p+\alpha\bar{q}_L,\theta)} \pi(\theta) d\theta, \qquad \text{for each } L \in \mathbb{N}_0.$$

With this notation, for each $\theta \in \Theta$, one has

$$\begin{aligned}\left|\pi_D^n(\theta|(1-\alpha)p+\alpha\bar{q}_L) - \pi_D^n(\theta|(1-\alpha)p)\right| &= \frac{\pi(\theta)}{J\cdot J_L} \\ &\quad \cdot \left| J e^{-\Delta_{n-1} D((1-\alpha)p+\alpha\bar{q}_L,\theta)} - J_L e^{-\Delta_{n-1} D((1-\alpha)p,\theta)} \right| \\ &\le \frac{\pi(\theta)}{J_L} \left| e^{-\Delta_{n-1} D((1-\alpha)p+\alpha\bar{q}_L,\theta)} - e^{-\Delta_{n-1} D((1-\alpha)p,\theta)} \right| \\ &\quad + \frac{|J-J_L|}{J\cdot J_L} e^{-\Delta_{n-1} D((1-\alpha)p,\theta)} \pi(\theta).\end{aligned}$$

Observe that $J_L \to J$, as $L \to \infty$, and using Jensen's inequality, one has that $D((1-\alpha)p,\theta) \ge G(-\alpha)$, and consequently $\int_\Theta e^{-\Delta_{n-1} D((1-\alpha)p,\theta)} \pi(\theta) d\theta \le e^{-\Delta_{n-1} G(-\alpha)}$; hence, it suffices to prove that $I_L \to 0$, as $L \to \infty$, where

$$I_L = \int_\Theta \left| e^{-\Delta_{n-1} D((1-\alpha)p+\alpha\bar{q}_L,\theta)} - e^{-\Delta_{n-1} D((1-\alpha)p,\theta)} \right| \pi(\theta) d\theta.$$

Let us consider the function $f : x \in [c,\infty) \to f(x) = e^{-\Delta_{n-1} x}$, with $c = \min(0, G(-\alpha))$, and let us fix $\epsilon > 0$. Since $f(\cdot)$ is uniformly continuous, there exists $\delta = \delta(\epsilon)$ such that if $|x-y| < \delta$, then $|e^{-\Delta_{n-1}x} - e^{-\Delta_{n-1}y}| < \epsilon$. Given $M > 0$, by Lemma 6 *(ii)*, one has that there exists $L_0 = L_0(M,\delta)$ such that

$$|D((1-\alpha)p+\alpha\bar{q}_L,\theta) - D((1-\alpha)p,\theta)| < \delta, \quad \forall \theta \in \Theta : |\theta| < M, \forall L > L_0.$$

Let us write

$$I_{L,1}^{(M)} = \int_{\{|\theta|<M\}} \left| e^{-\Delta_{n-1} D((1-\alpha)p+\alpha\bar{q}_L,\theta)} - e^{-\Delta_{n-1} D((1-\alpha)p,\theta)} \right| \pi(\theta) d\theta,$$
$$I_{L,2}^{(M)} = \int_{\{|\theta|\ge M\}} \left| e^{-\Delta_{n-1} D((1-\alpha)p+\alpha\bar{q}_L,\theta)} - e^{-\Delta_{n-1} D((1-\alpha)p,\theta)} \right| \pi(\theta) d\theta,$$

thus, $I_L = I_{L,1}^{(M)} + I_{L,2}^{(M)}$. On the one hand, $I_{L,1}^{(M)} \le \epsilon \int_{\{|\theta|<M\}} \pi(\theta) d\theta \le \epsilon$, for each $L > L_0$. On the other hand, $I_{L,2}^{(M)} \le \left(1 + e^{-\Delta_{n-1} G(-\alpha)}\right) \int_{\{|\theta|\ge M\}} \pi(\theta) d\theta$, for each $L \in \mathbb{N}_0$. Combining all the above, one has that for any $M > 0$, and $\epsilon > 0$,

$$\limsup_{L\to\infty} I_L \le \epsilon + \left(1 + e^{-\Delta_{n-1} G(-\alpha)}\right) \int_{\{|\theta|\ge M\}} \pi(\theta) d\theta.$$

Thus, since $\lim_{M\to\infty} \int_{\{|\theta|\ge M\}} \pi(\theta) d\theta = 0$, one obtains $\limsup_{L\to\infty} I_L \le \epsilon$ and taking limit as $\epsilon \to 0$, one has that $\lim_{L\to\infty} I_L = 0$.

□

4 Additional details of the example based on a real data set

4.1 Additional figures and tables

The contour plots of the D-posterior density functions of (q_0, q_1) at $\hat{p}_n$ for experiments 1 and 2 are represented in Figures 1 and 2.

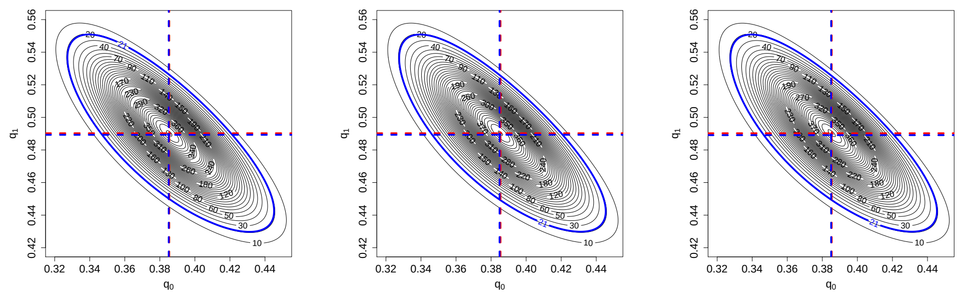

Figure 1: Contour plot for the D-posterior density of (q_0, q_1) of the first experiment, together with the EDAP estimate (intersection of blue dashed lines) and the MDAP estimate (intersection of red dashed lines). Blue contour line represents the 95% HPD region. Left: HD. Centre: NED. Right: KL.

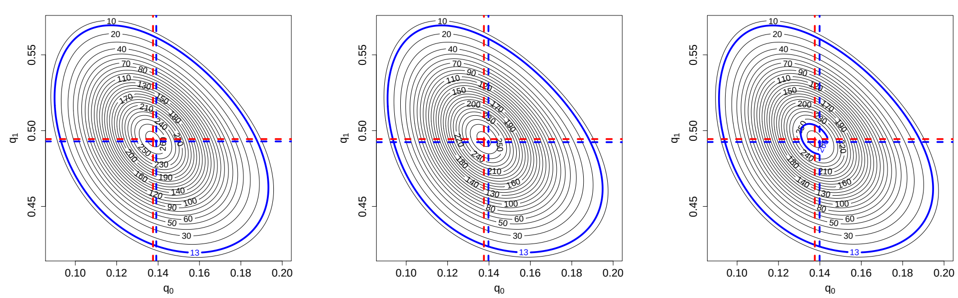

Figure 2: Contour plot for the D-posterior density of (q_0, q_1) of the second experiment, together with the EDAP estimate (intersection of blue dashed lines) and the MDAP estimate (intersection of red dashed lines). Blue contour line represents the 95% HPD region. Left: HD. Centre: NED. Right: KL.

In Table 1, an estimation of the probability that $m > 1$, with respect to the D-posterior distribution at $\hat{p}_n$, is provided for the two experiments and the different disparity measures.

	Experiment 1	Experiment 2
HD	0.3424	0.4234
NED	0.3382	0.4177
KL	0.3380	0.4174

Table 1: Probability that $m > 1$, with respect to the D-posterior distribution at $\hat{p}_n$, for HD, NED, and KL.

4.2 Sensitivity analysis

A brief description of the sensitivity analysis carried out for the example on oligodendrocyte cell populations is presented in this subsection in Tables 2, 3, 4 and 5.

5 Additional details of the simulated example

5.1 Simulated data

The data for the simulated example are provided in Tables 6 and 7. Recall that in this example, the initial number of individuals is $Z_0 = 1$, the reproduction law is a geometric distribution with parameter $\theta_0 = 0.3$ and which is contaminated by outliers, which can happen at the point 11 with probability 0.15. The control variables $\phi_n(k)$ has Poisson distributions with mean λk, for each $k, n \in \mathbb{N}_0$.

5.2 Additional figures

The figures described in Section 6.2 of the paper are gathered in this subsection.

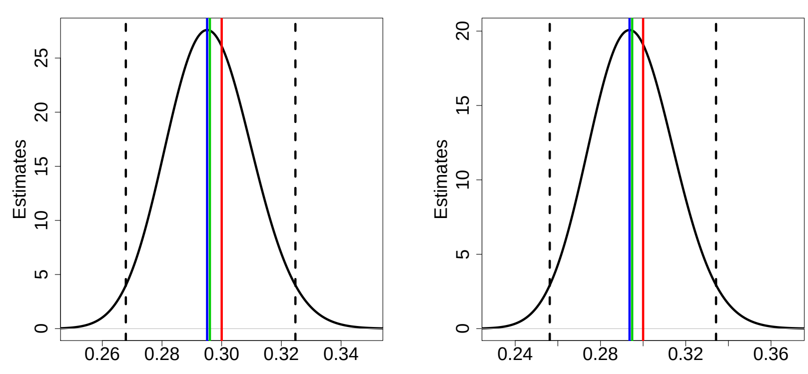

Figure 3: Estimate of the D-posterior density of θ given the sample z^*_{45}, together with EDAP (blue line) and MDAP (green line), HPD interval (dashed line) and true value of θ_0 (red line) for the HD (left) and the NED (right).

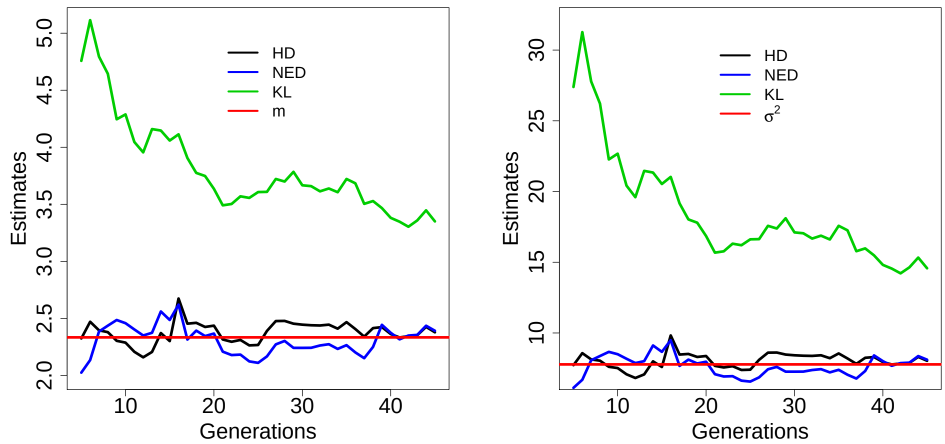

Figure 4: Evolution of the estimates of m (left) and σ^2 (right) based on the EDAP estimates of θ_0 considering the HD (black), the NED (blue) and the KL (green). Horizontal red lines represent the true value of the corresponding parameter.

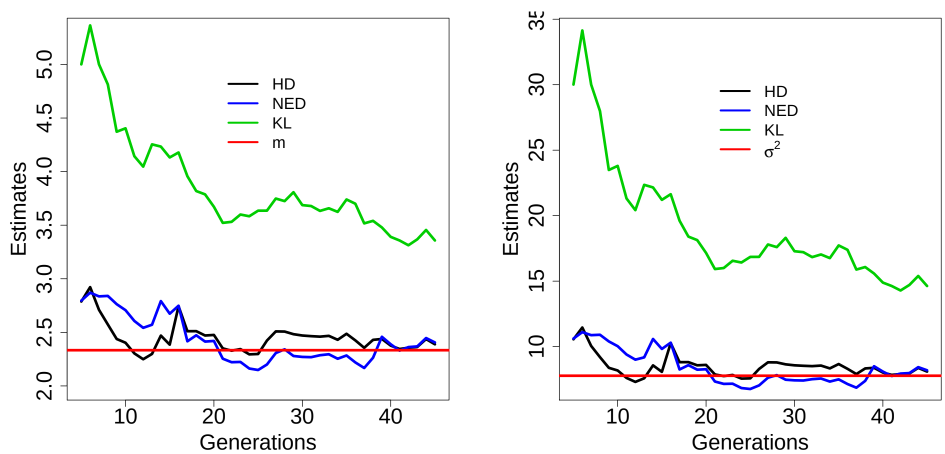

Figure 5: Evolution of the estimates of m (left) and σ^2 (right) based on the MDAP estimates of θ_0 considering the HD (black), the NED (blue) and the KL (green). Horizontal red lines represent the true value of the corresponding parameter.

5.3 Influence functions figure for EDAP estimator at p

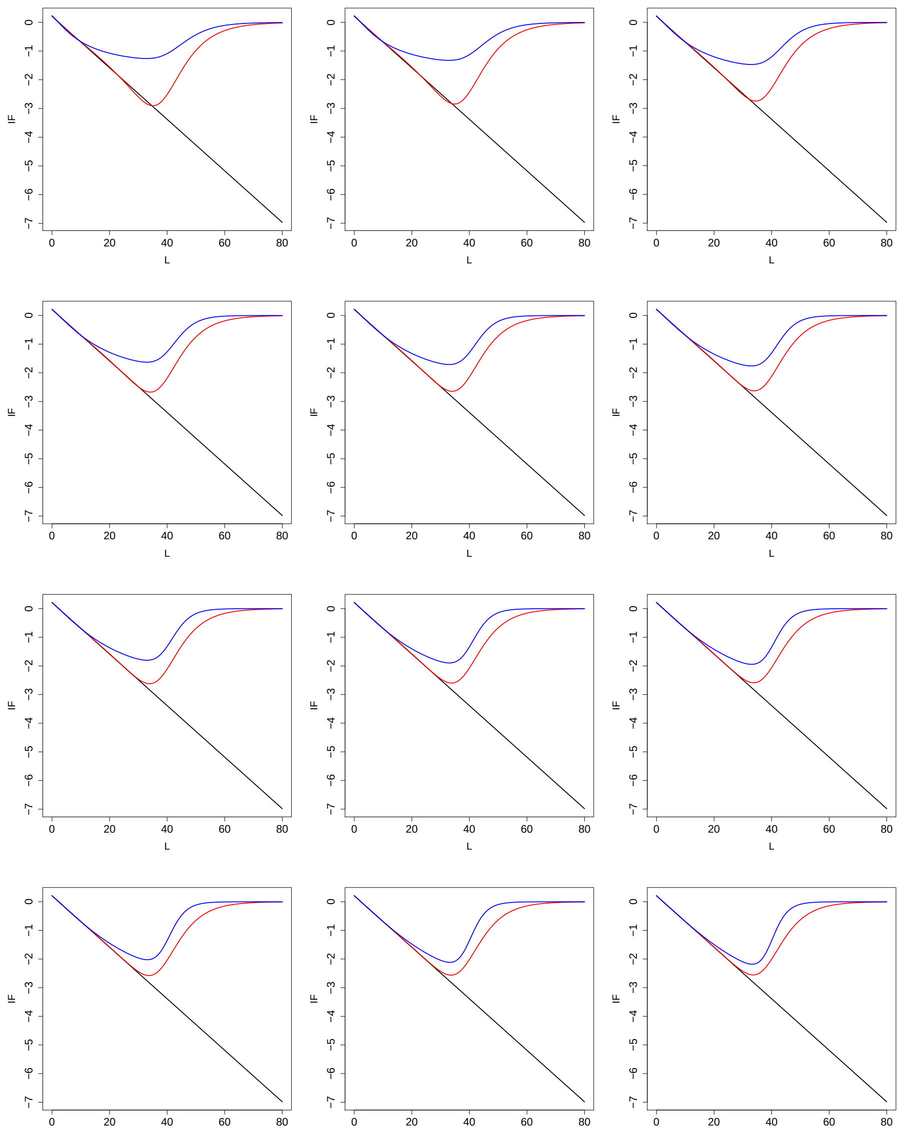

Figure 6: Evolution of influence functions for EDAP estimator at p based on KL (black line), HD (red line) and NED (blue line), $IF(L, \bar{T}_n, p)$, with $p = \boldsymbol{p}_{0.3}$ for the sample z_n^*, where $n = 10, 13, 16, 19, 22, 25, 28, 31, 34, 37, 40, 43$, with $\Delta_9 = 29$, $\Delta_{12} = 34$, $\Delta_{15} = 48$, $\Delta_{18} = 69$, $\Delta_{21} = 87$, $\Delta_{24} = 95$, $\Delta_{27} = 111$, $\Delta_{30} = 133$, $\Delta_{33} = 148$, $\Delta_{36} = 190$, $\Delta_{39} = 233$, $\Delta_{42} = 270$, from left to right and from top to bottom.

5.4 Different types/amounts of contamination in the mixture model for gross errors at a point.

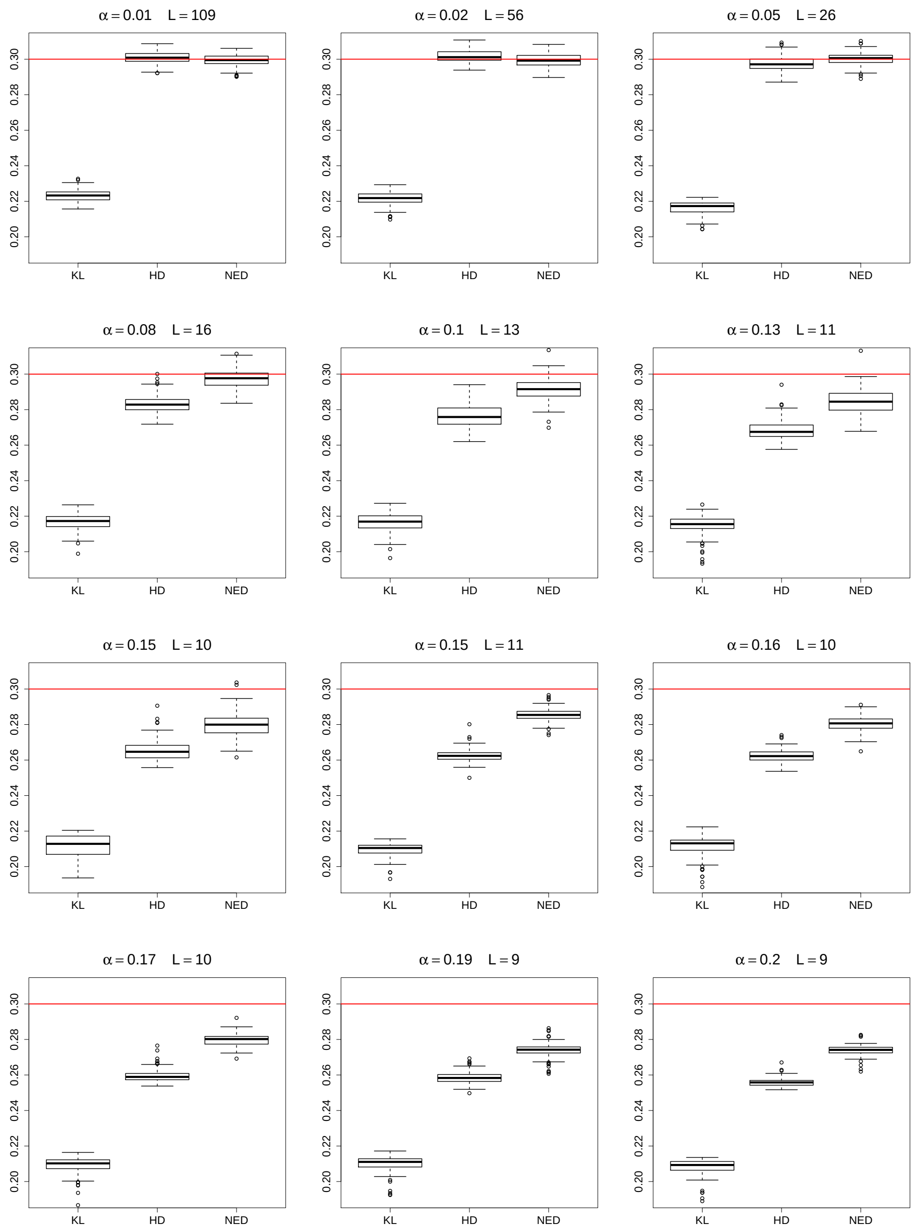

Figure 7: Box-plots of the EDAP estimates in the last generation based on 100 simulations of gross error contaminated models until generation 45 for the KL, HD and NED. Offspring distribution: geometric with parameter 0.3 and Poisson control distribution with parameter $0.3k$.

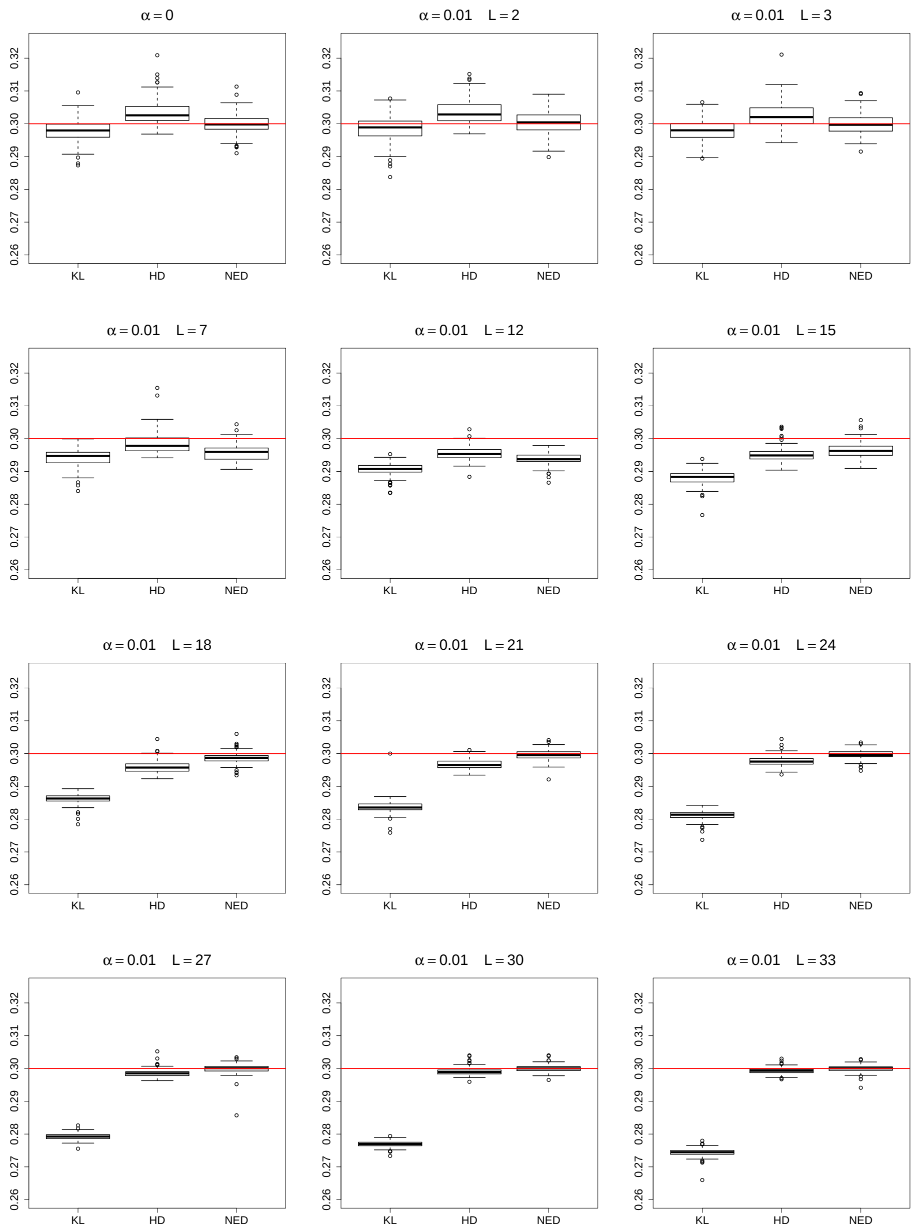

Figure 8: Box-plots of the EDAP estimates based on 45 generations and 100 simulations of gross error contaminated models. Offspring distribution: geometric with parameter 0.3 and Poisson control distribution with parameter $0.45k$.

5.5 Sensitivity analysis for the simulated example

A summary of the sensitivity analysis performed for the simulated example is provided in this section. For sake of brevity, we show the results for the generations 25 (in Tables 8, 9 and 10), 35 (in Tables 11, 12 and 13) and 45 (in Tables 14, 15 and 16). Recall that $\theta_0 = 0.3$. The maximum differences between the estimates and θ_0 are 0.0174, 0.0161, and 0.065, for $n = 25, 35$ and 45, respectively.

			HD			NED			KL		
α_1	α_2	α_3	p_0	p_1	p_2	p_0	p_1	p_2	p_0	p_1	p_2
1	1	1	0.3852	0.4892	0.1257	0.3853	0.4889	0.1259	0.3850	0.4891	0.1259
0.5	0.5	0.5	0.3852	0.4900	0.1248	0.3848	0.4901	0.1251	0.3854	0.4893	0.1253
1.9268	2.4512	0.6220	0.3852	0.4907	0.1241	0.3852	0.4903	0.1244	0.3853	0.4905	0.1242
1	1	8	0.4007	0.4563	0.1430	0.3827	0.4625	0.1548	0.3756	0.4713	0.1531
1	2	7	0.3746	0.4843	0.1411	0.3818	0.4875	0.1307	0.3714	0.4885	0.1401
1	3	6	0.3762	0.4888	0.1349	0.3786	0.4835	0.1379	0.3764	0.4849	0.1387
1	4	5	0.3792	0.4885	0.1324	0.3786	0.4880	0.1333	0.3786	0.4879	0.1335
1	5	4	0.3777	0.4918	0.1305	0.3779	0.4903	0.1317	0.3780	0.4911	0.1309
1	6	3	0.3792	0.4930	0.1278	0.3782	0.4932	0.1286	0.3788	0.4928	0.1284
1	7	2	0.3787	0.4956	0.1256	0.3784	0.4953	0.1263	0.3787	0.4952	0.1261
1	8	1	0.3786	0.4976	0.1238	0.3783	0.4980	0.1236	0.3784	0.4978	0.1238
1	1	8	0.4092	0.4393	0.1515	0.3836	0.4615	0.1549	0.3490	0.4790	0.1720
2	1	7	0.3801	0.4850	0.1350	0.3735	0.4871	0.1394	0.3780	0.4847	0.1373
3	1	6	0.3848	0.4803	0.1348	0.3817	0.4798	0.1384	0.3805	0.4854	0.1340
4	1	5	0.3866	0.4808	0.1326	0.3847	0.4801	0.1352	0.3851	0.4828	0.1321
5	1	4	0.3880	0.4820	0.1301	0.3886	0.4807	0.1308	0.3876	0.4810	0.1314
6	1	3	0.3907	0.4817	0.1276	0.3909	0.4810	0.1281	0.3903	0.4816	0.1281
7	1	2	0.3937	0.4811	0.1252	0.3938	0.4807	0.1255	0.3929	0.4809	0.1262
8	1	1	0.3955	0.4812	0.1233	0.3964	0.4797	0.1239	0.3939	0.4811	0.1250
1	8	1	0.3783	0.4983	0.1234	0.3788	0.4977	0.1235	0.3783	0.4979	0.1237
2	7	1	0.3810	0.4956	0.1234	0.3807	0.4954	0.1239	0.3808	0.4955	0.1237
3	6	1	0.3835	0.4930	0.1235	0.3835	0.4927	0.1238	0.3832	0.4930	0.1238
4	5	1	0.3858	0.4907	0.1234	0.3857	0.4905	0.1238	0.3859	0.4903	0.1238
5	4	1	0.3883	0.4884	0.1233	0.3880	0.4882	0.1238	0.3883	0.4879	0.1238
6	3	1	0.3907	0.4860	0.1234	0.3907	0.4857	0.1236	0.3906	0.4856	0.1238
7	2	1	0.3927	0.4837	0.1236	0.3932	0.4829	0.1239	0.3925	0.4837	0.1238
8	1	1	0.3952	0.4805	0.1242	0.3962	0.4801	0.1237	0.3950	0.4816	0.1234

Table 2: Sensitivity analysis for EDAP estimators in experiment 1.

			HD			NED			KL		
α_1	α_2	α_3	p_0	p_1	p_2	p_0	p_1	p_2	p_0	p_1	p_2
1	1	1	0.3860	0.4900	0.1240	0.3860	0.4900	0.1240	0.3860	0.4900	0.1240
0.5	0.5	0.5	0.3850	0.4910	0.1240	0.3850	0.4910	0.1240	0.3850	0.4910	0.1240
1.9268	2.4512	0.6220	0.3860	0.4910	0.1230	0.3860	0.4910	0.1230	0.3860	0.4910	0.1230
1	1	8	0.3790	0.4820	0.1390	0.3790	0.4820	0.1390	0.3790	0.4820	0.1390
1	2	7	0.3790	0.4850	0.1360	0.3790	0.4840	0.1370	0.3790	0.4840	0.1370
1	3	6	0.3790	0.4870	0.1340	0.3790	0.4870	0.1340	0.3790	0.4870	0.1340
1	4	5	0.3790	0.4890	0.1320	0.3790	0.4890	0.1320	0.3790	0.4890	0.1320
1	5	4	0.3790	0.4920	0.1290	0.3790	0.4920	0.1290	0.3790	0.4920	0.1290
1	6	3	0.3790	0.4940	0.1270	0.3790	0.4940	0.1270	0.3790	0.4940	0.1270
1	7	2	0.3790	0.4960	0.1250	0.3790	0.4960	0.1250	0.3790	0.4960	0.1250
1	8	1	0.3790	0.4990	0.1220	0.3790	0.4990	0.1220	0.3790	0.4990	0.1220
1	1	8	0.3790	0.4820	0.1390	0.3790	0.4820	0.1390	0.3790	0.4820	0.1390
2	1	7	0.3820	0.4820	0.1360	0.3810	0.4820	0.1370	0.3810	0.4820	0.1370
3	1	6	0.3840	0.4820	0.1340	0.3840	0.4820	0.1340	0.3840	0.4820	0.1340
4	1	5	0.3860	0.4820	0.1320	0.3860	0.4820	0.1320	0.3860	0.4820	0.1320
5	1	4	0.3890	0.4820	0.1290	0.3880	0.4820	0.1300	0.3880	0.4820	0.1300
6	1	3	0.3910	0.4820	0.1270	0.3910	0.4820	0.1270	0.3910	0.4820	0.1270
7	1	2	0.3930	0.4820	0.1250	0.3930	0.4820	0.1250	0.3930	0.4820	0.1250
8	1	1	0.3960	0.4820	0.1220	0.3960	0.4820	0.1220	0.3960	0.4820	0.1220
1	8	1	0.3790	0.4990	0.1220	0.3790	0.4990	0.1220	0.3790	0.4990	0.1220
2	7	1	0.3810	0.4970	0.1220	0.3810	0.4970	0.1220	0.3810	0.4970	0.1220
3	6	1	0.3840	0.4940	0.1220	0.3840	0.4940	0.1220	0.3840	0.4940	0.1220
4	5	1	0.3860	0.4920	0.1220	0.3860	0.4920	0.1220	0.3860	0.4920	0.1220
5	4	1	0.3890	0.4890	0.1220	0.3890	0.4890	0.1220	0.3890	0.4890	0.1220
6	3	1	0.3910	0.4870	0.1220	0.3910	0.4870	0.1220	0.3910	0.4870	0.1220
7	2	1	0.3930	0.4850	0.1220	0.3930	0.4850	0.1220	0.3930	0.4850	0.1220
8	1	1	0.3960	0.4820	0.1220	0.3960	0.4820	0.1220	0.3960	0.4820	0.1220

Table 3: Sensitivity analysis for MDAP estimators in experiment 1.

			HD			NED			KL		
α_1	α_2	α_3	p_0	p_1	p_2	p_0	p_1	p_2	p_0	p_1	p_2
1	1	1	0.1389	0.4933	0.3678	0.1397	0.4925	0.3679	0.1397	0.4926	0.3677
0.5	0.5	0.5	0.1384	0.4934	0.3682	0.1387	0.4932	0.3681	0.1388	0.4934	0.3678
0.6877	2.4721	1.8401	0.1371	0.4947	0.3682	0.1374	0.4945	0.3681	0.1376	0.4942	0.3683
1	1	8	0.1348	0.4820	0.3832	0.1368	0.4798	0.3834	0.1368	0.4813	0.3819
1	2	7	0.1357	0.4845	0.3798	0.1361	0.4839	0.3800	0.1362	0.4838	0.3800
1	3	6	0.1356	0.4879	0.3765	0.1361	0.4877	0.3762	0.1362	0.4875	0.3763
1	4	5	0.1358	0.4912	0.3730	0.1362	0.4912	0.3727	0.1363	0.4910	0.3728
1	5	4	0.1358	0.4950	0.3692	0.1362	0.4946	0.3692	0.1362	0.4945	0.3693
1	6	3	0.1357	0.4986	0.3657	0.1362	0.4983	0.3655	0.1363	0.4981	0.3656
1	7	2	0.1356	0.5024	0.3620	0.1361	0.5020	0.3619	0.1363	0.5014	0.3623
1	8	1	0.1357	0.5059	0.3584	0.1358	0.5057	0.3585	0.1363	0.5056	0.3581
1	1	8	0.1358	0.4795	0.3846	0.1367	0.4821	0.3811	0.1359	0.4810	0.3831
2	1	7	0.1396	0.4803	0.3802	0.1400	0.4807	0.3792	0.1399	0.4796	0.3805
3	1	6	0.1424	0.4807	0.3769	0.1434	0.4809	0.3757	0.1440	0.4802	0.3758
4	1	5	0.1468	0.4797	0.3734	0.1473	0.4803	0.3724	0.1471	0.4791	0.3738
5	1	4	0.1511	0.4797	0.3692	0.1497	0.4793	0.3709	0.1510	0.4793	0.3696
6	1	3	0.1532	0.4826	0.3643	0.1546	0.4831	0.3624	0.1531	0.4799	0.3670
7	1	2	0.1715	0.4705	0.3580	0.1606	0.4872	0.3522	0.1603	0.4705	0.3692
8	1	1	0.1923	0.4419	0.3658	0.1757	0.4387	0.3856	0.1647	0.4607	0.3746
1	8	1	0.1355	0.5060	0.3585	0.1362	0.5056	0.3583	0.1367	0.5055	0.3578
2	7	1	0.1392	0.5021	0.3588	0.1399	0.5018	0.3583	0.1400	0.5019	0.3581
3	6	1	0.1426	0.4986	0.3588	0.1432	0.4985	0.3583	0.1435	0.4978	0.3587
4	5	1	0.1464	0.4950	0.3586	0.1469	0.4954	0.3577	0.1467	0.4940	0.3594
5	4	1	0.1505	0.4915	0.3580	0.1509	0.4911	0.3580	0.1508	0.4911	0.3581
6	3	1	0.1539	0.4891	0.3570	0.1527	0.4888	0.3585	0.1537	0.4881	0.3582
7	2	1	0.1521	0.4837	0.3642	0.1549	0.4841	0.3610	0.1570	0.4835	0.3595
8	1	1	0.1465	0.4443	0.4092	0.1940	0.4602	0.3458	0.1776	0.4258	0.3966

Table 4: Sensitivity analysis for EDAP estimators in experiment 2.

			HD			NED			KL		
α_1	α_2	α_3	p_0	p_1	p_2	p_0	p_1	p_2	p_0	p_1	p_2
1	1	1	0.1380	0.4940	0.3680	0.1380	0.4940	0.3680	0.1380	0.4940	0.3680
0.5	0.5	0.5	0.1360	0.4960	0.3680	0.1360	0.4960	0.3680	0.1360	0.4960	0.3680
0.6877	2.4721	1.8401	0.1350	0.4960	0.3690	0.1350	0.4960	0.3690	0.1350	0.4960	0.3690
1	1	8	0.1340	0.4820	0.3840	0.1340	0.4820	0.3840	0.1340	0.4820	0.3840
1	2	7	0.1340	0.4860	0.3800	0.1340	0.4860	0.3800	0.1340	0.4860	0.3800
1	3	6	0.1340	0.4890	0.3770	0.1340	0.4890	0.3770	0.1340	0.4890	0.3770
1	4	5	0.1340	0.4930	0.3730	0.1340	0.4930	0.3730	0.1340	0.4930	0.3730
1	5	4	0.1340	0.4960	0.3700	0.1340	0.4960	0.3700	0.1340	0.4960	0.3700
1	6	3	0.1340	0.5	0.3660	0.1340	0.5	0.3660	0.1340	0.5	0.3660
1	7	2	0.1340	0.5040	0.3620	0.1340	0.5040	0.3620	0.1340	0.5040	0.3620
1	8	1	0.1340	0.5070	0.3590	0.1340	0.5070	0.3590	0.1340	0.5070	0.3590
1	1	8	0.1340	0.4820	0.3840	0.1340	0.4820	0.3840	0.1340	0.4820	0.3840
2	1	7	0.1380	0.4820	0.3800	0.1380	0.4820	0.3800	0.1380	0.4820	0.3800
3	1	6	0.1410	0.4820	0.3770	0.1410	0.4820	0.3770	0.1410	0.4820	0.3770
4	1	5	0.1450	0.4820	0.3730	0.1450	0.4820	0.3730	0.1450	0.4820	0.3730
5	1	4	0.1480	0.4820	0.3700	0.1490	0.4820	0.3690	0.1490	0.4820	0.3690
6	1	3	0.1520	0.4820	0.3660	0.1520	0.4820	0.3660	0.1520	0.4820	0.3660
7	1	2	0.1550	0.4820	0.3630	0.1560	0.4820	0.3620	0.1560	0.4820	0.3620
8	1	1	0.1590	0.4820	0.3590	0.1590	0.4820	0.3590	0.1590	0.4820	0.3590
1	8	1	0.1340	0.5070	0.3590	0.1340	0.5070	0.3590	0.1340	0.5070	0.3590
2	7	1	0.1380	0.5030	0.3590	0.1380	0.5030	0.3590	0.1380	0.5030	0.3590
3	6	1	0.1410	0.5	0.3590	0.1410	0.5	0.3590	0.1410	0.5	0.3590
4	5	1	0.1450	0.4960	0.3590	0.1450	0.4960	0.3590	0.1450	0.4960	0.3590
5	4	1	0.1480	0.4930	0.3590	0.1480	0.4930	0.3590	0.1480	0.4930	0.3590
6	3	1	0.1520	0.4890	0.3590	0.1520	0.4890	0.3590	0.1520	0.4890	0.3590
7	2	1	0.1550	0.4860	0.3590	0.1560	0.4850	0.3590	0.1560	0.4850	0.3590
8	1	1	0.1590	0.4820	0.3590	0.1590	0.4820	0.3590	0.1590	0.4820	0.3590

Table 5: Sensitivity analysis for MDAP estimators in experiment 2.

n	Z_n	$\phi_n(Z_n)$	$Z_n(0)$	$Z_n(1)$	$Z_n(2)$	$Z_n(3)$	$Z_n(4)$	$Z_n(5)$	$Z_n(6)$	$Z_n(7)$	$Z_n(8)$	$Z_n(9)$	$Z_n(10)$	$Z_n(11)$	$Z_n(12)$	$Z_n(13)$	$Z_n(14)$
0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
1	8	4	1	0	0	1	0	0	0	0	1	0	0	1	0	0	0
2	22	5	2	1	0	0	1	0	0	0	0	0	0	1	0	0	0
3	16	3	0	0	0	0	0	0	1	0	0	0	0	2	0	0	0
4	28	3	2	0	0	0	1	0	0	0	0	0	0	0	0	0	0
5	4	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
6	11	2	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0
7	4	3	0	1	1	0	0	0	0	0	1	0	0	0	0	0	0
8	11	4	1	0	1	2	0	0	0	0	0	0	0	0	0	0	0
9	8	3	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0
10	14	3	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0
11	5	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
12	1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
13	11	3	1	0	1	0	0	0	0	0	0	0	1	0	0	0	0
14	12	5	1	2	0	0	1	0	0	0	0	0	0	1	0	0	0
15	17	6	0	2	1	0	0	1	0	1	0	0	0	1	0	0	0
16	27	10	3	3	2	0	0	0	0	0	0	0	0	2	0	0	0
17	29	6	1	2	1	0	1	0	0	1	0	0	0	0	0	0	0
18	15	5	1	1	1	1	0	0	0	0	0	0	0	1	0	0	0
19	17	6	1	1	2	0	1	1	0	0	0	0	0	0	0	0	0
20	14	8	2	4	1	0	0	0	0	0	0	0	0	1	0	0	0
21	17	4	1	0	2	0	0	0	0	0	0	0	0	1	0	0	0
22	15	2	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0

Table 6: Simulated data from the initial generation to the 22-th generation.

n	Z_n	$\phi_n(Z_n)$	$Z_n(0)$	$Z_n(1)$	$Z_n(2)$	$Z_n(3)$	$Z_n(4)$	$Z_n(5)$	$Z_n(6)$	$Z_n(7)$	$Z_n(8)$	$Z_n(9)$	$Z_n(10)$	$Z_n(11)$	$Z_n(12)$	$Z_n(13)$	$Z_n(14)$
23	13	4	2	0	1	0	0	0	0	0	0	0	0	1	0	0	0
24	13	2	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0
25	12	3	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0
26	11	4	0	0	0	1	1	0	0	0	1	0	0	1	0	0	0
27	26	9	2	2	2	0	1	0	0	0	0	0	1	1	0	0	0
28	31	7	2	1	1	0	0	0	0	0	0	0	0	3	0	0	0
29	36	9	3	2	0	1	2	0	1	0	0	0	0	0	0	0	0
30	19	6	1	1	1	1	1	0	0	0	0	0	0	1	0	0	0
31	21	5	1	0	1	2	1	0	0	0	0	0	0	0	0	0	0
32	12	4	1	0	0	1	1	0	0	0	0	0	0	1	0	0	0
33	18	6	2	0	3	0	0	0	0	0	0	0	0	1	0	0	0
34	17	8	1	0	1	1	1	1	0	0	0	0	0	3	0	0	0
35	47	16	6	2	2	2	0	0	0	0	1	0	0	3	0	0	0
36	53	18	8	1	4	3	0	0	1	0	1	0	0	0	0	0	0
37	32	12	1	2	1	1	4	0	1	1	0	0	0	1	0	0	0
38	47	16	7	3	1	0	2	1	0	0	0	0	0	1	0	0	1
39	43	15	5	4	3	1	0	0	0	0	1	0	0	1	0	0	0
40	32	14	5	3	0	3	0	1	0	0	0	0	0	2	0	0	0
41	39	13	4	2	3	0	1	2	0	0	0	0	1	0	0	0	0
42	32	10	3	1	2	0	0	0	0	0	0	0	1	3	0	0	0
43	48	18	4	2	2	0	0	2	3	0	1	0	0	4	0	0	0
44	86	20	7	3	6	2	1	0	0	0	0	0	0	0	0	0	1
45	39	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·

Table 7: Simulated data from the 23-th generation to the 45-th generation.

ρ	β	Prior mean	Prior variance	θ_{25}^{*HD}	θ_{25}^{+HD}	θ_{25}^{*NED}	θ_{25}^{+NED}
0.1	0.1	0.5	0.2083	0.3056	0.3027	0.3056	0.3027
0.1	0.5	0.1667	0.0868	0.3052	0.3023	0.3052	0.3023
0.1	1	0.0909	0.0394	0.3047	0.3017	0.3047	0.3017
0.1	1.5	0.0625	0.0225	0.3041	0.3012	0.3041	0.3012
0.1	2	0.0476	0.0146	0.3036	0.3007	0.3036	0.3007
0.1	2.5	0.0385	0.0103	0.3031	0.3002	0.3031	0.3002
0.1	3	0.0323	0.0076	0.3026	0.2997	0.3026	0.2997
0.1	3.5	0.0278	0.0059	0.3021	0.2992	0.3021	0.2992
0.1	4	0.0244	0.0047	0.3016	0.2988	0.3016	0.2988
0.1	4.5	0.0217	0.0038	0.3011	0.2982	0.3011	0.2982
0.1	5	0.0196	0.0032	0.3006	0.2978	0.3006	0.2978
0.5	0.1	0.8333	0.0868	0.3066	0.3036	0.3066	0.3036
0.5	0.5	0.5	0.1250	0.3061	0.3032	0.3061	0.3032
0.5	1	0.3333	0.0889	0.3056	0.3027	0.3056	0.3027
0.5	1.5	0.2500	0.0625	0.3051	0.3022	0.3051	0.3022
0.5	2	0.2	0.0457	0.3046	0.3017	0.3046	0.3017
0.5	2.5	0.1667	0.0347	0.3040	0.3012	0.3040	0.3012
0.5	3	0.1429	0.0272	0.3035	0.3007	0.3035	0.3007
0.5	3.5	0.1250	0.0219	0.3030	0.3002	0.3030	0.3002
0.5	4	0.1111	0.0180	0.3025	0.2997	0.3025	0.2997
0.5	4.5	0.1	0.0150	0.3020	0.2992	0.3020	0.2992
0.5	5	0.0909	0.0127	0.3015	0.2987	0.3015	0.2987
1	0.1	0.9091	0.0394	0.3077	0.3048	0.3077	0.3048
1	0.5	0.6667	0.0889	0.3073	0.3044	0.3073	0.3044
1	1	0.5	0.0833	0.3068	0.3039	0.3068	0.3039
1	1.5	0.4000	0.0686	0.3063	0.3033	0.3063	0.3033
1	2	0.3333	0.0556	0.3057	0.3028	0.3057	0.3028
1	2.5	0.2857	0.0454	0.3052	0.3023	0.3052	0.3023
1	3	0.2500	0.0375	0.3047	0.3018	0.3047	0.3018
1	3.5	0.2222	0.0314	0.3042	0.3013	0.3042	0.3013
1	4	0.2	0.0267	0.3037	0.3008	0.3037	0.3008
1	4.5	0.1818	0.0229	0.3031	0.3003	0.3031	0.3003
1	5	0.1667	0.0198	0.3026	0.2998	0.3026	0.2998
1.5	0.1	0.9375	0.0225	0.3089	0.3060	0.3089	0.3060
1.5	0.5	0.7500	0.0625	0.3085	0.3056	0.3085	0.3056
1.5	1	0.6000	0.0686	0.3080	0.3051	0.3080	0.3051
1.5	1.5	0.5	0.0625	0.3074	0.3045	0.3074	0.3045
1.5	2	0.4286	0.0544	0.3069	0.3040	0.3069	0.3040
1.5	2.5	0.3750	0.0469	0.3064	0.3035	0.3064	0.3035
1.5	3	0.3333	0.0404	0.3059	0.3030	0.3059	0.3030
1.5	3.5	0.3000	0.0350	0.3053	0.3025	0.3053	0.3025
1.5	4	0.2727	0.0305	0.3048	0.3020	0.3048	0.3020
1.5	4.5	0.2500	0.0268	0.3043	0.3015	0.3043	0.3015
1.5	5	0.2308	0.0237	0.3038	0.3010	0.3038	0.3010

Table 8: Sensitivity analysis for generation $n = 25$.

ρ	β	Prior mean	Prior variance	θ_{25}^{*HD}	θ_{25}^{+HD}	θ_{25}^{*NED}	θ_{25}^{+NED}
2	0.1	0.9524	0.0146	0.3101	0.3072	0.3101	0.3072
2	0.5	0.8000	0.0457	0.3097	0.3068	0.3097	0.3068
2	1	0.6667	0.0556	0.3092	0.3062	0.3092	0.3062
2	1.5	0.5714	0.0544	0.3086	0.3057	0.3086	0.3057
2	2	0.5	0.0500	0.3081	0.3052	0.3081	0.3052
2	2.5	0.4444	0.0449	0.3076	0.3047	0.3076	0.3047
2	3	0.4000	0.0400	0.3070	0.3042	0.3070	0.3042
2	3.5	0.3636	0.0356	0.3065	0.3036	0.3065	0.3036
2	4	0.3333	0.0317	0.3060	0.3031	0.3060	0.3031
2	4.5	0.3077	0.0284	0.3055	0.3026	0.3055	0.3026
2	5	0.2857	0.0255	0.3050	0.3022	0.3050	0.3022
2.5	0.1	0.9615	0.0103	0.3113	0.3084	0.3113	0.3084
2.5	0.5	0.8333	0.0347	0.3109	0.3080	0.3109	0.3080
2.5	1	0.7143	0.0454	0.3104	0.3074	0.3104	0.3074
2.5	1.5	0.6250	0.0469	0.3098	0.3069	0.3098	0.3069
2.5	2	0.5556	0.0449	0.3093	0.3064	0.3093	0.3064
2.5	2.5	0.5	0.0417	0.3088	0.3059	0.3088	0.3059
2.5	3	0.4545	0.0381	0.3082	0.3053	0.3082	0.3053
2.5	3.5	0.4167	0.0347	0.3077	0.3048	0.3077	0.3048
2.5	4	0.3846	0.0316	0.3072	0.3043	0.3072	0.3043
2.5	4.5	0.3571	0.0287	0.3066	0.3038	0.3066	0.3038
2.5	5	0.3333	0.0261	0.3061	0.3033	0.3061	0.3033
3	0.1	0.9677	0.0076	0.3126	0.3096	0.3126	0.3096
3	0.5	0.8571	0.0272	0.3121	0.3091	0.3121	0.3091
3	1	0.7500	0.0375	0.3116	0.3086	0.3116	0.3086
3	1.5	0.6667	0.0404	0.3110	0.3081	0.3110	0.3081
3	2	0.6000	0.0400	0.3105	0.3075	0.3105	0.3075
3	2.5	0.5455	0.0381	0.3099	0.3070	0.3099	0.3070
3	3	0.5	0.0357	0.3094	0.3065	0.3094	0.3065
3	3.5	0.4615	0.0331	0.3089	0.3060	0.3089	0.3060
3	4	0.4286	0.0306	0.3083	0.3055	0.3083	0.3055
3	4.5	0.4000	0.0282	0.3078	0.3050	0.3078	0.3050
3	5	0.3750	0.0260	0.3073	0.3045	0.3073	0.3045
3.5	0.1	0.9722	0.0059	0.3138	0.3108	0.3138	0.3108
3.5	0.5	0.8750	0.0219	0.3133	0.3103	0.3133	0.3103
3.5	1	0.7778	0.0314	0.3128	0.3098	0.3128	0.3098
3.5	1.5	0.7000	0.0350	0.3122	0.3093	0.3122	0.3093
3.5	2	0.6364	0.0356	0.3117	0.3087	0.3117	0.3087
3.5	2.5	0.5833	0.0347	0.3111	0.3082	0.3111	0.3082
3.5	3	0.5385	0.0331	0.3106	0.3077	0.3106	0.3077
3.5	3.5	0.5	0.0312	0.3101	0.3072	0.3101	0.3072
3.5	4	0.4667	0.0293	0.3095	0.3066	0.3095	0.3066
3.5	4.5	0.4375	0.0273	0.3090	0.3061	0.3090	0.3061
3.5	5	0.4118	0.0255	0.3085	0.3056	0.3085	0.3056

Table 9: Sensitivity analysis for generation $n = 25$ (continuation).

ρ	β	Prior mean	Prior variance	θ_{25}^{*HD}	θ_{25}^{+HD}	θ_{25}^{*NED}	θ_{25}^{+NED}
4	0.1	0.9756	0.0047	0.3150	0.3120	0.3150	0.3120
4	0.5	0.8889	0.0180	0.3145	0.3115	0.3145	0.3115
4	1	0.8000	0.0267	0.3140	0.3110	0.3140	0.3110
4	1.5	0.7273	0.0305	0.3134	0.3104	0.3134	0.3104
4	2	0.6667	0.0317	0.3129	0.3099	0.3129	0.3099
4	2.5	0.6154	0.0316	0.3123	0.3094	0.3123	0.3094
4	3	0.5714	0.0306	0.3118	0.3089	0.3118	0.3089
4	3.5	0.5333	0.0293	0.3112	0.3083	0.3112	0.3083
4	4	0.5	0.0278	0.3107	0.3078	0.3107	0.3078
4	4.5	0.4706	0.0262	0.3102	0.3073	0.3102	0.3073
4	5	0.4444	0.0247	0.3096	0.3068	0.3096	0.3068
4.5	0.1	0.9783	0.0038	0.3162	0.3132	0.3162	0.3132
4.5	0.5	0.9000	0.0150	0.3157	0.3127	0.3157	0.3127
4.5	1	0.8182	0.0229	0.3152	0.3122	0.3152	0.3122
4.5	1.5	0.7500	0.0268	0.3146	0.3116	0.3146	0.3116
4.5	2	0.6923	0.0284	0.3141	0.3111	0.3141	0.3111
4.5	2.5	0.6429	0.0287	0.3135	0.3106	0.3135	0.3106
4.5	3	0.6000	0.0282	0.3130	0.3100	0.3130	0.3100
4.5	3.5	0.5625	0.0273	0.3124	0.3095	0.3124	0.3095
4.5	4	0.5294	0.0262	0.3119	0.3090	0.3119	0.3090
4.5	4.5	0.5	0.0250	0.3113	0.3085	0.3113	0.3085
4.5	5	0.4737	0.0237	0.3108	0.3079	0.3108	0.3079
5	0.1	0.9804	0.0032	0.3174	0.3144	0.3174	0.3144
5	0.5	0.9091	0.0127	0.3169	0.3139	0.3169	0.3139
5	1	0.8333	0.0198	0.3164	0.3134	0.3164	0.3134
5	1.5	0.7692	0.0237	0.3158	0.3128	0.3158	0.3128
5	2	0.7143	0.0255	0.3153	0.3123	0.3153	0.3123
5	2.5	0.6667	0.0261	0.3147	0.3118	0.3147	0.3118
5	3	0.6250	0.0260	0.3141	0.3112	0.3141	0.3112
5	3.5	0.5882	0.0255	0.3136	0.3107	0.3136	0.3107
5	4	0.5556	0.0247	0.3131	0.3102	0.3131	0.3102
5	4.5	0.5263	0.0237	0.3125	0.3096	0.3125	0.3096
5	5	0.5	0.0227	0.3120	0.3091	0.3120	0.3091

Table 10: Sensitivity analysis for generation $n = 25$ (continuation).

ρ	β	Prior mean	Prior variance	θ_{35}^{*HD}	θ_{35}^{+HD}	θ_{35}^{*NED}	θ_{35}^{+NED}
0.1	0.1	0.5	0.2083	0.2881	0.2865	0.2881	0.2865
0.1	0.5	0.1667	0.0868	0.2879	0.2863	0.2879	0.2863
0.1	1	0.0909	0.0394	0.2876	0.2860	0.2876	0.2860
0.1	1.5	0.0625	0.0225	0.2874	0.2857	0.2874	0.2857
0.1	2	0.0476	0.0146	0.2871	0.2855	0.2871	0.2855
0.1	2.5	0.0385	0.0103	0.2868	0.2852	0.2868	0.2852
0.1	3	0.0323	0.0076	0.2866	0.2849	0.2866	0.2849
0.1	3.5	0.0278	0.0059	0.2863	0.2847	0.2863	0.2847
0.1	4	0.0244	0.0047	0.2860	0.2844	0.2860	0.2844
0.1	4.5	0.0217	0.0038	0.2858	0.2842	0.2858	0.2842
0.1	5	0.0196	0.0032	0.2855	0.2839	0.2855	0.2839
0.5	0.1	0.8333	0.0868	0.2887	0.2870	0.2887	0.2870
0.5	0.5	0.5	0.1250	0.2884	0.2868	0.2884	0.2868
0.5	1	0.3333	0.0889	0.2882	0.2865	0.2882	0.2865
0.5	1.5	0.2500	0.0625	0.2879	0.2863	0.2879	0.2863
0.5	2	0.2	0.0457	0.2876	0.2860	0.2876	0.2860
0.5	2.5	0.1667	0.0347	0.2874	0.2857	0.2874	0.2857
0.5	3	0.1429	0.0272	0.2871	0.2855	0.2871	0.2855
0.5	3.5	0.1250	0.0219	0.2868	0.2852	0.2868	0.2852
0.5	4	0.1111	0.0180	0.2866	0.2849	0.2866	0.2849
0.5	4.5	0.1	0.0150	0.2863	0.2847	0.2863	0.2847
0.5	5	0.0909	0.0127	0.2860	0.2844	0.2860	0.2844
1	0.1	0.9091	0.0394	0.2893	0.2877	0.2893	0.2877
1	0.5	0.6667	0.0889	0.2891	0.2874	0.2891	0.2874
1	1	0.5	0.0833	0.2888	0.2872	0.2888	0.2872
1	1.5	0.4000	0.0686	0.2886	0.2869	0.2886	0.2869
1	2	0.3333	0.0556	0.2883	0.2866	0.2883	0.2866
1	2.5	0.2857	0.0454	0.2880	0.2864	0.2880	0.2864
1	3	0.2500	0.0375	0.2878	0.2861	0.2878	0.2861
1	3.5	0.2222	0.0314	0.2875	0.2859	0.2875	0.2859
1	4	0.2	0.0267	0.2872	0.2856	0.2872	0.2856
1	4.5	0.1818	0.0229	0.2870	0.2853	0.2870	0.2853
1	5	0.1667	0.0198	0.2867	0.2851	0.2867	0.2851
1.5	0.1	0.9375	0.0225	0.2900	0.2883	0.2900	0.2883
1.5	0.5	0.7500	0.0625	0.2898	0.2881	0.2898	0.2881
1.5	1	0.6000	0.0686	0.2895	0.2878	0.2895	0.2878
1.5	1.5	0.5	0.0625	0.2892	0.2876	0.2892	0.2876
1.5	2	0.4286	0.0544	0.2890	0.2873	0.2890	0.2873
1.5	2.5	0.3750	0.0469	0.2887	0.2870	0.2887	0.2870
1.5	3	0.3333	0.0404	0.2884	0.2868	0.2884	0.2868
1.5	3.5	0.3000	0.0350	0.2881	0.2865	0.2881	0.2865
1.5	4	0.2727	0.0305	0.2879	0.2863	0.2879	0.2863
1.5	4.5	0.2500	0.0268	0.2876	0.2860	0.2876	0.2860
1.5	5	0.2308	0.0237	0.2873	0.2857	0.2873	0.2857

Table 11: Sensitivity analysis for generation $n = 35$.

ρ	β	Prior mean	Prior variance	θ_{35}^{*HD}	θ_{35}^{+HD}	θ_{35}^{*NED}	θ_{35}^{+NED}
2	0.1	0.9524	0.0146	0.2907	0.2890	0.2907	0.2890
2	0.5	0.8000	0.0457	0.2904	0.2888	0.2904	0.2888
2	1	0.6667	0.0556	0.2902	0.2885	0.2902	0.2885
2	1.5	0.5714	0.0544	0.2899	0.2882	0.2899	0.2882
2	2	0.5	0.0500	0.2896	0.2880	0.2896	0.2880
2	2.5	0.4444	0.0449	0.2893	0.2877	0.2893	0.2877
2	3	0.4000	0.0400	0.2891	0.2874	0.2891	0.2874
2	3.5	0.3636	0.0356	0.2888	0.2872	0.2888	0.2872
2	4	0.3333	0.0317	0.2885	0.2869	0.2885	0.2869
2	4.5	0.3077	0.0284	0.2883	0.2866	0.2883	0.2866
2	5	0.2857	0.0255	0.2880	0.2864	0.2880	0.2864
2.5	0.1	0.9615	0.0103	0.2913	0.2896	0.2913	0.2896
2.5	0.5	0.8333	0.0347	0.2911	0.2894	0.2911	0.2894
2.5	1	0.7143	0.0454	0.2908	0.2892	0.2908	0.2892
2.5	1.5	0.6250	0.0469	0.2906	0.2889	0.2906	0.2889
2.5	2	0.5556	0.0449	0.2903	0.2886	0.2903	0.2886
2.5	2.5	0.5	0.0417	0.2900	0.2884	0.2900	0.2884
2.5	3	0.4545	0.0381	0.2897	0.2881	0.2897	0.2881
2.5	3.5	0.4167	0.0347	0.2895	0.2878	0.2895	0.2878
2.5	4	0.3846	0.0316	0.2892	0.2876	0.2892	0.2876
2.5	4.5	0.3571	0.0287	0.2889	0.2873	0.2889	0.2873
2.5	5	0.3333	0.0261	0.2887	0.2870	0.2887	0.2870
3	0.1	0.9677	0.0076	0.2920	0.2903	0.2920	0.2903
3	0.5	0.8571	0.0272	0.2918	0.2901	0.2918	0.2901
3	1	0.7500	0.0375	0.2915	0.2898	0.2915	0.2898
3	1.5	0.6667	0.0404	0.2912	0.2896	0.2912	0.2896
3	2	0.6000	0.0400	0.2909	0.2893	0.2909	0.2893
3	2.5	0.5455	0.0381	0.2907	0.2890	0.2907	0.2890
3	3	0.5	0.0357	0.2904	0.2888	0.2904	0.2888
3	3.5	0.4615	0.0331	0.2901	0.2885	0.2901	0.2885
3	4	0.4286	0.0306	0.2899	0.2882	0.2899	0.2882
3	4.5	0.4000	0.0282	0.2896	0.2879	0.2896	0.2879
3	5	0.3750	0.0260	0.2893	0.2877	0.2893	0.2877
3.5	0.1	0.9722	0.0059	0.2927	0.2910	0.2927	0.2910
3.5	0.5	0.8750	0.0219	0.2924	0.2908	0.2924	0.2908
3.5	1	0.7778	0.0314	0.2922	0.2905	0.2922	0.2905
3.5	1.5	0.7000	0.0350	0.2919	0.2902	0.2919	0.2902
3.5	2	0.6364	0.0356	0.2916	0.2900	0.2916	0.2900
3.5	2.5	0.5833	0.0347	0.2913	0.2897	0.2913	0.2897
3.5	3	0.5385	0.0331	0.2911	0.2894	0.2911	0.2894
3.5	3.5	0.5	0.0312	0.2908	0.2891	0.2908	0.2891
3.5	4	0.4667	0.0293	0.2905	0.2889	0.2905	0.2889
3.5	4.5	0.4375	0.0273	0.2902	0.2886	0.2902	0.2886
3.5	5	0.4118	0.0255	0.2900	0.2883	0.2900	0.2883

Table 12: Sensitivity analysis for generation $n = 35$ (continuation).

ρ	β	Prior mean	Prior variance	θ_{35}^{*HD}	θ_{35}^{+HD}	θ_{35}^{*NED}	θ_{35}^{+NED}
4	0.1	0.9756	0.0047	0.2933	0.2917	0.2933	0.2917
4	0.5	0.8889	0.0180	0.2931	0.2914	0.2931	0.2914
4	1	0.8000	0.0267	0.2928	0.2912	0.2928	0.2912
4	1.5	0.7273	0.0305	0.2926	0.2909	0.2926	0.2909
4	2	0.6667	0.0317	0.2923	0.2906	0.2923	0.2906
4	2.5	0.6154	0.0316	0.2920	0.2903	0.2920	0.2903
4	3	0.5714	0.0306	0.2917	0.2901	0.2917	0.2901
4	3.5	0.5333	0.0293	0.2915	0.2898	0.2915	0.2898
4	4	0.5	0.0278	0.2912	0.2895	0.2912	0.2895
4	4.5	0.4706	0.0262	0.2909	0.2893	0.2909	0.2893
4	5	0.4444	0.0247	0.2906	0.2890	0.2906	0.2890
4.5	0.1	0.9783	0.0038	0.2940	0.2923	0.2940	0.2923
4.5	0.5	0.9000	0.0150	0.2938	0.2921	0.2938	0.2921
4.5	1	0.8182	0.0229	0.2935	0.2918	0.2935	0.2918
4.5	1.5	0.7500	0.0268	0.2932	0.2916	0.2932	0.2916
4.5	2	0.6923	0.0284	0.2929	0.2913	0.2929	0.2913
4.5	2.5	0.6429	0.0287	0.2927	0.2910	0.2927	0.2910
4.5	3	0.6000	0.0282	0.2924	0.2907	0.2924	0.2907
4.5	3.5	0.5625	0.0273	0.2921	0.2905	0.2921	0.2905
4.5	4	0.5294	0.0262	0.2918	0.2902	0.2918	0.2902
4.5	4.5	0.5	0.0250	0.2916	0.2899	0.2916	0.2899
4.5	5	0.4737	0.0237	0.2913	0.2897	0.2913	0.2897
5	0.1	0.9804	0.0032	0.2947	0.2930	0.2947	0.2930
5	0.5	0.9091	0.0127	0.2945	0.2928	0.2945	0.2928
5	1	0.8333	0.0198	0.2942	0.2925	0.2942	0.2925
5	1.5	0.7692	0.0237	0.2939	0.2922	0.2939	0.2922
5	2	0.7143	0.0255	0.2936	0.2919	0.2936	0.2919
5	2.5	0.6667	0.0261	0.2933	0.2917	0.2933	0.2917
5	3	0.6250	0.0260	0.2931	0.2914	0.2931	0.2914
5	3.5	0.5882	0.0255	0.2928	0.2911	0.2928	0.2911
5	4	0.5556	0.0247	0.2925	0.2909	0.2925	0.2909
5	4.5	0.5263	0.0237	0.2922	0.2906	0.2922	0.2906
5	5	0.5	0.0227	0.2920	0.2903	0.2920	0.2903

Table 13: Sensitivity analysis for generation $n = 35$ (continuation).

ρ	β	Prior mean	Prior variance	θ_{45}^{*HD}	θ_{45}^{+HD}	θ_{45}^{*NED}	θ_{45}^{+NED}
0.1	0.1	0.5	0.2083	0.2959	0.2950	0.2959	0.2950
0.1	0.5	0.1667	0.0868	0.2958	0.2948	0.2958	0.2948
0.1	1	0.0909	0.0394	0.2956	0.2947	0.2956	0.2947
0.1	1.5	0.0625	0.0225	0.2955	0.2946	0.2955	0.2946
0.1	2	0.0476	0.0146	0.2953	0.2944	0.2953	0.2944
0.1	2.5	0.0385	0.0103	0.2952	0.2943	0.2952	0.2943
0.1	3	0.0323	0.0076	0.2950	0.2941	0.2950	0.2941
0.1	3.5	0.0278	0.0059	0.2949	0.2940	0.2949	0.2940
0.1	4	0.0244	0.0047	0.2947	0.2938	0.2947	0.2938
0.1	4.5	0.0217	0.0038	0.2946	0.2937	0.2946	0.2937
0.1	5	0.0196	0.0032	0.2944	0.2935	0.2944	0.2935
0.5	0.1	0.8333	0.0868	0.2962	0.2953	0.2962	0.2953
0.5	0.5	0.5	0.1250	0.2960	0.2951	0.2960	0.2951
0.5	1	0.3333	0.0889	0.2959	0.2950	0.2959	0.2950
0.5	1.5	0.2500	0.0625	0.2957	0.2948	0.2957	0.2948
0.5	2	0.2	0.0457	0.2956	0.2947	0.2956	0.2947
0.5	2.5	0.1667	0.0347	0.2954	0.2946	0.2954	0.2946
0.5	3	0.1429	0.0272	0.2953	0.2944	0.2953	0.2944
0.5	3.5	0.1250	0.0219	0.2952	0.2942	0.2952	0.2942
0.5	4	0.1111	0.0180	0.2950	0.2941	0.2950	0.2941
0.5	4.5	0.1	0.0150	0.2949	0.2940	0.2949	0.2940
0.5	5	0.0909	0.0127	0.2947	0.2938	0.2947	0.2938
1	0.1	0.9091	0.0394	0.2965	0.2956	0.2965	0.2956
1	0.5	0.6667	0.0889	0.2964	0.2955	0.2964	0.2955
1	1	0.5	0.0833	0.2962	0.2953	0.2962	0.2953
1	1.5	0.4000	0.0686	0.2961	0.2952	0.2961	0.2952
1	2	0.3333	0.0556	0.2959	0.2950	0.2959	0.2950
1	2.5	0.2857	0.0454	0.2958	0.2949	0.2958	0.2949
1	3	0.2500	0.0375	0.2957	0.2948	0.2957	0.2948
1	3.5	0.2222	0.0314	0.2955	0.2946	0.2955	0.2946
1	4	0.2	0.0267	0.2954	0.2945	0.2954	0.2945
1	4.5	0.1818	0.0229	0.2952	0.2943	0.2952	0.2943
1	5	0.1667	0.0198	0.2951	0.2942	0.2951	0.2942
1.5	0.1	0.9375	0.0225	0.2969	0.2960	0.2969	0.2960
1.5	0.5	0.7500	0.0625	0.2968	0.2958	0.2968	0.2958
1.5	1	0.6000	0.0686	0.2966	0.2957	0.2966	0.2957
1.5	1.5	0.5	0.0625	0.2965	0.2956	0.2965	0.2956
1.5	2	0.4286	0.0544	0.2963	0.2954	0.2963	0.2954
1.5	2.5	0.3750	0.0469	0.2962	0.2953	0.2962	0.2953
1.5	3	0.3333	0.0404	0.2960	0.2951	0.2960	0.2951
1.5	3.5	0.3000	0.0350	0.2959	0.2950	0.2959	0.2950
1.5	4	0.2727	0.0305	0.2957	0.2948	0.2957	0.2948
1.5	4.5	0.2500	0.0268	0.2956	0.2947	0.2956	0.2947
1.5	5	0.2308	0.0237	0.2954	0.2945	0.2954	0.2945

Table 14: Sensitivity analysis for generation $n = 45$.

ρ	β	Prior mean	Prior variance	θ_{45}^{*HD}	θ_{45}^{+HD}	θ_{45}^{*NED}	θ_{45}^{+NED}
2	0.1	0.9524	0.0146	0.2972	0.2963	0.2972	0.2963
2	0.5	0.8000	0.0457	0.2971	0.2962	0.2971	0.2962
2	1	0.6667	0.0556	0.2970	0.2961	0.2970	0.2961
2	1.5	0.5714	0.0544	0.2968	0.2959	0.2968	0.2959
2	2	0.5	0.0500	0.2967	0.2958	0.2967	0.2958
2	2.5	0.4444	0.0449	0.2965	0.2956	0.2965	0.2956
2	3	0.4000	0.0400	0.2964	0.2955	0.2964	0.2955
2	3.5	0.3636	0.0356	0.2962	0.2953	0.2962	0.2953
2	4	0.3333	0.0317	0.2961	0.2952	0.2961	0.2952
2	4.5	0.3077	0.0284	0.2959	0.2950	0.2959	0.2950
2	5	0.2857	0.0255	0.2958	0.2949	0.2958	0.2949
2.5	0.1	0.9615	0.0103	0.2976	0.2967	0.2976	0.2967
2.5	0.5	0.8333	0.0347	0.2975	0.2966	0.2975	0.2966
2.5	1	0.7143	0.0454	0.2973	0.2964	0.2973	0.2964
2.5	1.5	0.6250	0.0469	0.2972	0.2963	0.2972	0.2963
2.5	2	0.5556	0.0449	0.2970	0.2961	0.2970	0.2961
2.5	2.5	0.5	0.0417	0.2969	0.2960	0.2969	0.2960
2.5	3	0.4545	0.0381	0.2967	0.2958	0.2967	0.2958
2.5	3.5	0.4167	0.0347	0.2966	0.2957	0.2966	0.2957
2.5	4	0.3846	0.0316	0.2964	0.2955	0.2964	0.2955
2.5	4.5	0.3571	0.0287	0.2963	0.2954	0.2963	0.2954
2.5	5	0.3333	0.0261	0.2961	0.2952	0.2961	0.2952
3	0.1	0.9677	0.0076	0.2979	0.2970	0.2979	0.2970
3	0.5	0.8571	0.0272	0.2978	0.2969	0.2978	0.2969
3	1	0.7500	0.0375	0.2977	0.2968	0.2977	0.2968
3	1.5	0.6667	0.0404	0.2975	0.2966	0.2975	0.2966
3	2	0.6000	0.0400	0.2974	0.2965	0.2974	0.2965
3	2.5	0.5455	0.0381	0.2972	0.2963	0.2972	0.2963
3	3	0.5	0.0357	0.2971	0.2962	0.2971	0.2962
3	3.5	0.4615	0.0331	0.2969	0.2960	0.2969	0.2960
3	4	0.4286	0.0306	0.2968	0.2959	0.2968	0.2959
3	4.5	0.4000	0.0282	0.2966	0.2957	0.2966	0.2957
3	5	0.3750	0.0260	0.2965	0.2956	0.2965	0.2956
3.5	0.1	0.9722	0.0059	0.2983	0.2974	0.2983	0.2974
3.5	0.5	0.8750	0.0219	0.2982	0.2973	0.2982	0.2973
3.5	1	0.7778	0.0314	0.2980	0.2971	0.2980	0.2971
3.5	1.5	0.7000	0.0350	0.2979	0.2970	0.2979	0.2970
3.5	2	0.6364	0.0356	0.2977	0.2968	0.2977	0.2968
3.5	2.5	0.5833	0.0347	0.2976	0.2967	0.2976	0.2967
3.5	3	0.5385	0.0331	0.2974	0.2965	0.2974	0.2965
3.5	3.5	0.5	0.0312	0.2973	0.2964	0.2973	0.2964
3.5	4	0.4667	0.0293	0.2971	0.2962	0.2971	0.2962
3.5	4.5	0.4375	0.0273	0.2970	0.2961	0.2970	0.2961
3.5	5	0.4118	0.0255	0.2968	0.2959	0.2968	0.2959

Table 15: Sensitivity analysis for generation $n = 45$ (continuation).

ρ	β	Prior mean	Prior variance	θ_{45}^{*HD}	θ_{45}^{+HD}	θ_{45}^{*NED}	θ_{45}^{+NED}
4	0.1	0.9756	0.0047	0.2987	0.2977	0.2987	0.2977
4	0.5	0.8889	0.0180	0.2985	0.2976	0.2985	0.2976
4	1	0.8000	0.0267	0.2984	0.2975	0.2984	0.2975
4	1.5	0.7273	0.0305	0.2982	0.2973	0.2982	0.2973
4	2	0.6667	0.0317	0.2981	0.2972	0.2981	0.2972
4	2.5	0.6154	0.0316	0.2979	0.2970	0.2979	0.2970
4	3	0.5714	0.0306	0.2978	0.2969	0.2978	0.2969
4	3.5	0.5333	0.0293	0.2976	0.2967	0.2976	0.2967
4	4	0.5	0.0278	0.2975	0.2966	0.2975	0.2966
4	4.5	0.4706	0.0262	0.2973	0.2964	0.2973	0.2964
4	5	0.4444	0.0247	0.2972	0.2963	0.2972	0.2963
4.5	0.1	0.9783	0.0038	0.2990	0.2981	0.2990	0.2981
4.5	0.5	0.9000	0.0150	0.2989	0.2980	0.2989	0.2980
4.5	1	0.8182	0.0229	0.2987	0.2978	0.2987	0.2978
4.5	1.5	0.7500	0.0268	0.2986	0.2977	0.2986	0.2977
4.5	2	0.6923	0.0284	0.2984	0.2975	0.2984	0.2975
4.5	2.5	0.6429	0.0287	0.2983	0.2974	0.2983	0.2974
4.5	3	0.6000	0.0282	0.2981	0.2972	0.2981	0.2972
4.5	3.5	0.5625	0.0273	0.2980	0.2971	0.2980	0.2971
4.5	4	0.5294	0.0262	0.2978	0.2969	0.2978	0.2969
4.5	4.5	0.5	0.0250	0.2977	0.2968	0.2977	0.2968
4.5	5	0.4737	0.0237	0.2975	0.2966	0.2975	0.2966
5	0.1	0.9804	0.0032	0.2994	0.2984	0.2994	0.2984
5	0.5	0.9091	0.0127	0.2992	0.2983	0.2992	0.2983
5	1	0.8333	0.0198	0.2991	0.2982	0.2991	0.2982
5	1.5	0.7692	0.0237	0.2989	0.2980	0.2989	0.2980
5	2	0.7143	0.0255	0.2988	0.2979	0.2988	0.2979
5	2.5	0.6667	0.0261	0.2986	0.2977	0.2986	0.2977
5	3	0.6250	0.0260	0.2985	0.2976	0.2985	0.2976
5	3.5	0.5882	0.0255	0.2983	0.2974	0.2983	0.2974
5	4	0.5556	0.0247	0.2982	0.2973	0.2982	0.2973
5	4.5	0.5263	0.0237	0.2980	0.2971	0.2980	0.2971
5	5	0.5	0.0227	0.2979	0.2970	0.2979	0.2970

Table 16: Sensitivity analysis for generation $n = 45$ (continuation).

6 Simulation programmes

In this Section, we gather the simulations programmes implemented for the methodology proposed in this paper. We present the programmes for the two particular examples we have described, while the implementation of this method in a general case is a question that we aimed at tackling in a near future.

6.1 Simulation programmes of Example 6.1 - Oligodendrocytes

Computation of the non-parametric MLEs

The function `MLE()` provides a matrix with four columns corresponding to the MLEs of the offspring distribution and the probability of non-emigration given the number of emigrants, `em`; the total number of individuals, `tot`; the number of progenitors of type T_1 without any offspring, `p.nooff`; the number of progenitors of type T_1 with two offspring of type T_1, `p.2off`; and the number of progenitors of type T_1 giving rise to one offspring of type T_2, `p.t2`. The first three columns of the output matrix are the MLEs of p_0, p_1 and p_2, respectively and the last one is the MLE of the parameter γ.

```
MLE<-function(em,p.nooff,p.2off,p.t2,tot)
{
  delta<-tot-em
  p<-cbind(p.nooff,p.2off,p.t2)/delta
  gamma<-delta/tot
  cbind(p,gamma)
}
```

Computation of the EDAP and MDAP estimators

The function `rho_HD()` computes the Hellinger distance between two distributions `q`, with finite support, and `param` defined on a subset of the non-negative integers, with the Hellinger distance defined in such a way that the corresponding function $G(\cdot)$ satisfies $G'(0) = 0$ and $G''(0) = 1$.

```
rho_HD<-function(q,param)
{
  4*(1-sum(sqrt(q*param)))
}
```

Analogously, the function `rho_NED()` computes the negative exponential disparity between two distributions `q`, with finite support, and `param` defined on a subset of the non-negative integers, with the negative exponential disparity having its function $G(\cdot)$ satisfying $G'(0) = 0$ and $G''(0) = 1$.

```
rho_NED<-function(q,param)
{
  k<-length(q)
  k1<-(1:k)[(param!=0)][q[(1:k)[(param!=0)]]!=0]
  k2<-(1:k)[(param!=0)][q[(1:k)[(param!=0)]]==0]
  delta<-q[k1]/param[k1]-1
  sum((exp(-delta)-1+delta)*param[k1])+(exp(1)-2)*sum(param[k2])
```

```
}
```

The function `rho_KL()` determines the Kullback-Leibler divergence between two distributions `q`, with finite support, and `param` defined on a subset of the non-negative integers, where the corresponding function $G(\cdot)$ satisfies $G'(0) = 0$ and $G''(0) = 1$.

```
rho_KL<-function(q,param)
{
  k<-q!=0
  sum(q[k]*(log(q[k])-log(param[k])))-1
}
```

The function `exp_rho_HD()` provides the exponential function involved in (13) given the total number of progenitors, `phi`, and the probability distributions `q` and `param` and taking the Hellinger distance as a disparity measure.

```
exp_rho_HD<-function(phi,q,param)
{
  exp(-phi*rho_HD(q,param))
}
```

Their counterpart functions using the negative exponential disparity and the Kullback-Leibler divergence are the following ones:

```
exp_rho_NED<-function(phi,q,param)
{
 exp(-phi*rho_NED(q,param))
}

exp_rho_KL<-function(phi,q,param)
{
  if (min(param)==0)
  {
    x<-0
  } else {
    x<-exp(-phi*rho_KL(q,param))
  }
}
```

The function `mean_exp_rho_HD()` is used in the approximation of the first and the second integrals - depending on whether `i` $= 1$ or `i` $= 2$ - by the Monte Carlo integration method in the definition of the EDAP estimators based on the Hellinger distance in (14) provided the total number of progenitors, `phi`; the non-parametric MLE of the offspring distribution, `q`; and the probability distribution `param`.

```
mean_exp_rho_HD<-function(phi,q,param,i)
{
  param[i]*exp_rho_HD(phi,q,param)
}
```

Similarly, we define analogous functions for the negative exponential disparity and the Kullback-Leibler divergence.

```
mean_exp_rho_NED<-function(phi,q,param,i)
{
 param[i]*exp_rho_NED(phi,q,param)
}

mean_exp_rho_KL<-function(phi,q,param,i)
{
 param[i]*exp_rho_KL(phi,q,param)
}
```

The function `g_HD()` gives the value at the point `y` of the function in the numerator of (13) given the total number of progenitors, `phi`, the probability distribution `p.est` on $\{(0,0),(1,0),(0,2)\}$, with a Dirichlet distribution with parameter `alph` as a prior distribution on $\{(0,0),(1,0),(0,2)\}$ and the Hellinger distance as a disparity measure. In this function, to compute the value of the distribution function of a Dirichlet distribution we make use of the function `ddirichlet()` of the package `gtools` (see Warnes et al. (2018)).

```
g_HD<-function(phi,p.est,y,alph)
{
  exp_rho_HD(phi,p.est,y)*ddirichlet(y,alpha=alph)
}
```

In an analogous manner, we define the functions for the negative exponential disparity and the Kullback-Leibler divergence.

```
g_NED<-function(phi,p.est,y,alph)
{
  exp_rho_NED(phi,p.est,y)*ddirichlet(y,alpha=alph)
}

g_KL<-function(phi,p.est,y,alph)
{
  exp_rho_KL(phi,p.est,y)*ddirichlet(y,alpha=alph)
}
```

In order to compute the MDAP estimates we make use of the function `nloptr()` of the package `nloptr` (see Ypma et al. (2018) for details)). This function enables us to minimize a function subject to certain constraints. To that end, we need to transform our maximization problem into a minimization problem and to define the functions that we would like to minimize and the constraints in such a way that the inputs of both functions are the exactly the same.

The following functions give the opposite of the value at the point `y` of the function in the numerator of (13) given the total number of progenitors, `phi`, the probability distribution `p.est` on $\{(0,0),(1,0),(0,2)\}$ and with a Dirichlet distribution with parameter `alph` as a prior distribution on $\{(0,0),(1,0),(0,2)\}$, for the Hellinger distance, the negative exponential disparity and the Kullback-Leibler divergence, respectively.

```
g_max_HD<-function(y,alph,phi,p.est)
{
  -g_HD(phi,p.est,y,alph)
}

g_max_NED<-function(y,alph,phi,p.est)
{
  -g_NED(phi,p.est,y,alph)
}

g_max_KL<-function(y,alph,phi,p.est)
{
  -g_KL(phi,p.est,y,alph)
}
```

The following function is used to define the constraints in the optimization problem when computing the MDAP estimator. We remark that to use the function `nloptr()`, we need to define a function $h(\cdot)$ such that the constraint in the optimization problem is $h(y) = 0$, for a vector y. Thus, in our case since the vector y must be a proper probability distribution, the following function is the implementation of $h(y) = \sum_i y_i - 1$. Moreover, the function needs to have the same inputs as the function we would like to minimize, and thus, we need to provide the total number of progenitors, `phi`, the probability distribution `p.est` on $\{(0,0),(1,0),(0,2)\}$ and the parameter of the Dirichlet distribution on $\{(0,0),(1,0),(0,2)\}$.

```
eval_cons<-function(y,alph,phi,p.est)
{
  sum(y)-1
}
```

The function `edap.mdap_HD()` provides the EDAP and MDAP estimates based on the Hellinger distance given the total number of progenitors, `Delta`; a probability distribution on $\{(0,0),(1,0),(0,2)\}$, `p.est`; and the parameter of the Dirichlet distribution, `alpha`. The integral in the definition of the EDAP estimators in (14) are approximated by the Monte Carlo integration using a sample of `N.int` points. The values `xini` and `tol` are the initial value and the tolerance level needed to compute the MDAP estimator. The output of this function is a list of the EDAP estimates, `edap`, the MDAP estimates, `mdap`, and the maximum of the function in the numerator of (13), `max.Dpost`.

```
edap.mdap_HD<-function(Delta,p.est,N.int,alpha,xini,tol,...)
{
 sam.dist<-rdirichlet(N.int,alpha)
 f.int<-mean(apply(sam.dist,1,exp_rho_HD,phi=Delta,q=p.est))
 edap.p0<-mean(apply(sam.dist,1,mean_exp_rho_HD,phi=Delta,q=p.est,i=1))
     /f.int
 edap.p1<-mean(apply(sam.dist,1,mean_exp_rho_HD,phi=Delta,q=p.est,i=2))
     /f.int
 x<-nloptr(x0=xini,eval_f=g_max_HD,lb=c(0,0,0),ub=c(1,1,1),eval_g_eq
     =eval_cons,opts=list("algorithm"="NLOPT_GN_ISRES","xtol_rel"=tol),
     alph=alpha,phi=Delta,p.est=p.est)
 list(edap=c(edap.p0,edap.p1),mdap=x$solution[c(1,2)],max.Dpost=
     (-x$objective))
}
```

Similar functions are defined for the negative exponential disparity and the Kullback-Leibler divergence.

```
edap.mdap_NED<-function(Delta,p.est,N.int,alpha,xini,tol,...)
{
 sam.dist<-rdirichlet(N.int,alpha)
 f.int<-mean(apply(sam.dist,1,exp_rho_NED,phi=Delta,q=p.est))
 edap.p0<-mean(apply(sam.dist,1,mean_exp_rho_NED,phi=Delta,q=p.est,i=1))
     /f.int
 edap.p1<-mean(apply(sam.dist,1,mean_exp_rho_NED,phi=Delta,q=p.est,i=2))
     /f.int
 x<-nloptr(x0=xini,eval_f=g_max_NED,lb=c(0,0,0),ub=c(1,1,1),eval_g_eq
     =eval_cons,opts=list("algorithm"="NLOPT_GN_ISRES","xtol_rel"=tol),
     alph=alpha,phi=Delta,p.est=p.est)
 list(edap=c(edap.p0,edap.p1),mdap=x$solution[c(1,2)],max.Dpost
     =(-x$objective))
}

edap.mdap_KL<-function(Delta,p.est,N.int,N.sim,alpha,xini,tol,...)
```

```
{
 sam.dist<-rdirichlet(N.int,alpha)
 f.int<-mean(apply(sam.dist,1,exp_rho_KL,phi=Delta,q=p.est))
 edap.p0<-mean(apply(sam.dist,1,mean_exp_rho_KL,phi=Delta,q=p.est,i=1))
     /f.int
 edap.p1<-mean(apply(sam.dist,1,mean_exp_rho_KL,phi=Delta,q=p.est,i=2))
     /f.int
 x<-nloptr(x0=xini,eval_f=g_max_KL,lb=c(0.01,0.01,0.01),ub=c(1,1,1),
     eval_g_eq=eval_cons,opts=list("algorithm"="NLOPT_GN_ISRES",
     "xtol_rel"=tol),alph=alpha,phi=Delta,p.est=p.est)
 list(edap=c(edap.p0,edap.p1),mdap=x$solution[c(1,2)],max.Dpost
     =(-x$objective))
}
```

Computation of D-posterior density function

The function `post.dens()` provides an estimate of the D-posterior density function for the three disparity measures considered given the total number of progenitors, `Delta`; a probability distribution on $\{(0,0),(1,0),(0,2)\}$, `p.est`; and the parameter of the Dirichlet distribution, `alpha`. The integral in the denominator of (13) is approximated by the Monte Carlo integration using a sample of `N.int` points drawn from a Dirichlet distribution using the function `rdirichlet()` of the package `gtools` (see Warnes et al. (2018)). The output of this function is a list of an approximation of the joint HD-posterior density function- using a grid of `N.sim` cells in each direction -, `denHD`; an approximation of the joint NED-posterior density function, `denNED`; an approximation of the joint KL-posterior density function, `denKL`; and the points where the three posterior density functions were approximated, `param`.

```
post.dens<-function(Delta,p.est,N.int,N.sim,alpha)
{
 sam.dist<-rdirichlet(N.int,alpha)
 p0<-seq(0,1,length.out=N.sim)
 param<-matrix(0,nrow=N.sim^2,ncol=3)
 param[,1]<-as.vector(sapply(p0,rep,times=N.sim))
 param[,2]<-rep(p0,N.sim)
 param[,3]<-1-param[,2]-param[,1]
 param<-param[apply(param[,1:2],1,sum)<=1,]
 den_HD<-apply(param,1,g_HD,phi=Delta,p.est=p.est,alph=alpha)/
           mean(apply(sam.dist,1,exp_rho_HD,phi=Delta,q=p.est))
 den_NED<-apply(param,1,g_NED,phi=Delta,p.est=p.est,alph=alpha)/
           mean(apply(sam.dist,1,exp_rho_NED,phi=Delta,q=p.est))
 den_KL<-apply(param,1,g_KL,phi=Delta,p.est=p.est,alph=alpha)/
           mean(apply(sam.dist,1,exp_rho_KL,phi=Delta,q=p.est))
 list(x=param,denHD=den_HD,denNED=den_NED,denKL=den_KL)
}
```

Computation of the probability of having supercritical reproduction of type T_1

The function p_m() provides the probability that the mean number of offspring of type T_1 is greater than one for the Hellinger distance, the negative exponential disparity and the Kullback-Leibler divergence given the number of progenitors, Delta; the non-parametric MLE estimator of the offspring distribution, p.est; and a Dirichlet distribution with parameter alpha as the prior distribution. The integrals involved in the calculations are approximated by the Monte Carlo integration method using a sample of N.int points.

```
p_m<-function(Delta,p.est,N.int,alpha,...)
{
  sam.dist<-rdirichlet(N.int,alpha)
  sam.dist.m<-sam.dist[sam.dist[,2]>=0.5,]
  x1<-sum(apply(sam.dist.m,1,exp_rho_HD,phi=Delta,q=p.est))/
       sum(apply(sam.dist,1,exp_rho_HD,phi=Delta,q=p.est))
  x2<-sum(apply(sam.dist.m,1,exp_rho_NED,phi=Delta,q=p.est))/
       sum(apply(sam.dist,1,exp_rho_NED,phi=Delta,q=p.est))
  x3<-sum(apply(sam.dist.m,1,exp_rho_KL,phi=Delta,q=p.est))/
       sum(apply(sam.dist,1,exp_rho_KL,phi=Delta,q=p.est))
  pr.superc<-c(x1,x2,x3)
}
```

Computation of the HDP regions

The function hpd_region() provides the HDP region of the parameter θ with a credibility level of cred given the total number of progenitors, Delta; the non-parametric MLE of the offspring distribution, p.est; and using a Dirichlet distribution with parameter alpha as the prior distribution. The HPD region is estimated as the region of the parameter space delimited by the contour line of the estimated D-posterior density within which the parameter value lies with a probability of at least cred. The corresponding contour line is approximated up to an error of tol and to that end, the D-posterior density in (13) is estimated in a grid of N.int points in each direction and using the Monte Carlo integration method to estimate the integral in the denominator. The additional argument Dp.inf is the level of the initial contour line, which is updated in each iteration until obtaining the desired contour line. The output of this function is a list of objects, where ini_samp is the sample of the aforesaid Dirichlet distribution used to approximate the D-posterior density functions, ini_denHD is the estimation of the HD-posterior density function at those points, sam_HD is the set of points of the sample ini_sam which fall within the HPD region for the Hellinger distance and HD.hpd is the level of the desired contour line of the HD-posterior density. The function also provides the counterparts of ini_denHD, sam_HD and HD.hpd for the negative exponential disparity (NED) and the Kullback-Leibler divergence (KL).

```
hpd_region<-function(Delta,p.est,N.int,alpha,cred,tol=0.1,Dp.inf=0)
```

```
{
 sam.dist<-rdirichlet(N.int,alpha)
 f.int.HD<-mean(apply(sam.dist,1,exp_rho_HD,phi=Delta,q=p.est))
 f.int.NED<-mean(apply(sam.dist,1,exp_rho_NED,phi=Delta,q=p.est))
 f.int.KL<-mean(apply(sam.dist,1,exp_rho_KL,phi=Delta,q=p.est))
 den.HD<-apply(sam.dist,1,g_HD,phi=Delta,p.est=p.est,alph=alpha)
   /f.int.HD
 den.NED<-apply(sam.dist,1,g_NED,phi=Delta,p.est=p.est,alph=alpha)
   /f.int.NED
 den.KL<-apply(sam.dist,1,g_KL,phi=Delta,p.est=p.est,alph=alpha)
   /f.int.KL
 x.HD<-Dp.inf
 x.NED<-Dp.inf
 x.KL<-Dp.inf
 selec.HD<-den.HD>=x.HD
 selec.NED<-den.NED>=x.NED
 selec.KL<-den.KL>=x.KL
 sam.dist.HD<-sam.dist[selec.HD,]
 sam.dist.NED<-sam.dist[selec.NED,]
 sam.dist.KL<-sam.dist[selec.KL,]
 probHD<-sum(apply(sam.dist.HD,1,exp_rho_HD,phi=Delta,q=p.est))/sum(
   apply(sam.dist,1,exp_rho_HD,phi=Delta,q=p.est))
 probNED<-sum(apply(sam.dist.NED,1,exp_rho_NED,phi=Delta,q=p.est))/
   sum(apply(sam.dist,1,exp_rho_NED,phi=Delta,q=p.est))
 probKL<-sum(apply(sam.dist.KL,1,exp_rho_KL,phi=Delta,q=p.est))/sum(
   apply(sam.dist,1,exp_rho_KL,phi=Delta,q=p.est))
 while(probHD>=cred)
 {
   x.HD<-x.HD+tol
   selec.HD<-den.HD>=x.HD
   sam.dist.HD<-sam.dist[selec.HD,]
   probHD<-sum(apply(sam.dist.HD,1,exp_rho_HD,phi=Delta,q=p.est))/sum(
     apply(sam.dist,1,exp_rho_HD,phi=Delta,q=p.est))
 }
 while(probNED>=cred)
 {
   x.NED<-x.NED+tol
   selec.NED<-den.NED>=x.NED
   sam.dist.NED<-sam.dist[selec.NED,]
   probNED<-sum(apply(sam.dist.NED,1,exp_rho_NED,phi=Delta,q=p.est))/
     sum(apply(sam.dist,1,exp_rho_NED,phi=Delta,q=p.est))
 }
 while(probKL>=cred)
 {
   x.KL<-x.KL+tol
   selec.KL<-den.KL>=x.KL
```

```
   sam.dist.KL<-sam.dist[selec.KL,]
   probKL<-sum(apply(sam.dist.KL,1,exp_rho_KL,phi=Delta,q=p.est))/sum(
     apply(sam.dist,1,exp_rho_KL,phi=Delta,q=p.est))
 }
 x.HD<-x.HD-tol
 selec.HD<-den.HD>=x.HD
 sam.dist.HD<-sam.dist[selec.HD,]
 x.NED<-x.NED-tol
 selec.NED<-den.NED>=x.NED
 sam.dist.NED<-sam.dist[selec.NED,]
 x.KL<-x.KL-tol
 selec.KL<-den.KL>=x.KL
 sam.dist.KL<-sam.dist[selec.KL,]
 list(ini_denHD=den.HD,ini_denNED=den.NED,ini_denKL=den.KL,ini_sam=
   sam.dist,sam_HD=sam.dist.HD,sam_NED=sam.dist.NED,sam_KL=sam.dist.
   KL,HD.hpd=x.HD,NED.hpd=x.NED,KL.hpd=x.KL)
}
```

6.2 Simulation programmes of Example 6.2 - Simulated example

Simulation of CBPs

The function `rgeomc()` generates `x` random numbers from a geometric distribution with parameter `prob` and contaminated with a mixture model for gross errors at the point `L` with probability `alpha`.

```
rgeomc<-function(x,prob,L,alpha)
{
  u<-runif(x)
  c(rep(L,sum(u<=alpha)),rgeom(sum(u>alpha),prob))
}
```

The function `CBP()` generates `max.gen` generations of a CBP starting with `z0` individuals and whose offspring distribution is a Geometric distribution with parameter `prob` which is contaminated with a mixture model for gross errors at the point `L` with probability `alpha`. The control variables in a generation with population size equal to `k` are Poisson distribution with parameter `k*lambda`. The output of this function is a list with the population sizes, `z`, the whole family tree, `x`, and the number of progenitors in each generation, `control`.

```
CBP<-function(z0,max.gen,prob,lambda,L,alpha)
{
  z<-z0
```

```
  control<-rpois(1,z*lambda)
  offs<-rgeomc(control,prob,L,alpha)
  x<-rbind(offs)
  z<-c(z,sum(offs))
  for (j in 2:max.gen)
  {
    im.control<-rpois(1,z[j]*lambda)
    control<-c(control,im.control)
    offs<-rgeomc(im.control,prob,L,alpha)
    if(ncol(x)<im.control)
    {
      x<-cbind(x,matrix(0,nrow(x),im.control-ncol(x)))
    }
    else
    {
      offs<-c(offs,rep(0,ncol(x)-im.control))
    }
    x<-rbind(x,offs)
    z<-c(z,sum(offs))
  }
  list(z=z,x=x,control=control)
}
```

Computation of the non-parametric MLE

The function `estim()` calculates the non-parametric MLE of the offspring distribution and the offspring mean given the population sizes, `z`, the entire family tree, `tree`, and the number of progenitors in each generation, `phi`.

```
estim<-function(z,tree,phi)
{
  zjk<-numeric()
  for(i in 1:max(tree)) zjk<-cbind(zjk,apply(tree==i,1,sum))
  zj0<-phi-apply(zjk,1,sum)
  zjk<-cbind(zj0,zjk)
  estp<-apply(zjk,2,cumsum)/cumsum(phi)
  estm<-cumsum(z[-1])/cumsum(phi)
  list(p=estp,m=estm)
}
```

Computation of the EDAP and MDAP estimators and D-posterior density function

The function `rho_HD_geom()` calculates the Hellinger distance between a probability distribution `q` and a geometric distribution with parameter `prob` with the Hellinger distance defined in such a way that the function $G(\cdot)$ satisfies $G'(0) = 0$ and $G''(0) = 1$.

```
rho_HD_geom<-function(q,prob)
{
  k<-length(q)
  4*(1-sum(sqrt(q*dgeom(0:(k-1),prob))))
}
```

The function `rho_NED_geom()` gives the negative exponential disparity between a probability distribution `q` and a geometric distribution with parameter `prob` with the negative exponential disparity defined in such a way that the function $G(\cdot)$ satisfies $G'(0) = 0$ and $G''(0) = 1$.

```
rho_NED_geom<-function(q,prob)
{
 k<-length(q)
 k1<-((0:(k-1))[(dgeom(0:(k-1),prob)!=0)])[q!=0]
 k2<-((0:(k-1))[(dgeom(0:(k-1),prob)!=0)])[q==0]
 delta<-(q[k1+1]/dgeom(k1,prob))-1
 sum(dgeom(k1,prob)*(exp(-delta)-1))+(sum(dgeom(k2,prob))+pgeom(max(k1)
      +1,prob,lower.tail=F))*(exp(1)-1)
}
```

The function `rho_KL_geom()` provides the Kullback-Leibler divergence between a probability distribution `q` and a geometric distribution with parameter `prob` with the Kullback-Leibler divergence defined in such a way that the function $G(\cdot)$ satisfies $G'(0) = 0$ and $G''(0) = 1$.

```
rho_KL_geom<-function(q,prob)
{
 k<-length(q)
 coord<-q!=0
 sum(q[coord]*(log(q[coord])-log(dgeom((0:(k-1))[coord],prob))))
}
```

The function `g_HD()` provides the value at `y` of the function in the integrand of equation (5) for the Hellinger distance given the total number of progenitors, `phi`; the non-parametric MLE, `p.est`; and the parameters of the beta prior distribution, `sh1` and `sh2`.

```
g_HD<-function(phi,p.est,y,sh1,sh2)
{
```

```
  exp(-phi*rho_HD_geom(p.est,y))*dbeta(y,shape1=sh1,shape2=sh2)
}
```

We also defined analogous functions for the negative exponential disparity and the Kullback-Leibler divergence.

```
g_NED<-function(phi,p.est,y,sh1,sh2)
{
  exp(-phi*rho_NED_geom(p.est,y))*dbeta(y,shape1=sh1,shape2=sh2)
}

g_KL<-function(phi,p.est,y,sh1,sh2)
{
  exp(-phi*rho_KL_geom(p.est,y))*dbeta(y,shape1=sh1,shape2=sh2)
}
```

As we did for the implementation of our methodology in Example 1 using the function `nloptr()`, we transform the problem of maximizing the D-posterior density function to determine the MDAP estimator into a minimization problem. To that end, we define the following three functions for the Hellinger distance, the negative exponential disparity and the Kullback-Leibler divergence, respectively.

```
g_max_HD<-function(y,phi,p.est,sh1,sh2)
{
  -g_HD(phi,p.est,y,sh1,sh2)
}

g_max_NED<-function(y,phi,p.est,sh1,sh2)
{
  -g_NED(phi,p.est,y,sh1,sh2)
}

g_max_KL<-function(y,phi,p.est,sh1,sh2)
{
  -g_KL(phi,p.est,y,sh1,sh2)
}
```

The function `posterior.d_HD_geom()` provides the estimation of the HD-posterior density, `den`, at `N.sim` points equally distributed in the interval [`lo`,`up`], `x`, together with the EDAP and MDAP estimators, `eap` and `map`, respectively, for the Hellinger distance and considering the geometric distributions as the parametric family. To that end, the total number of progenitors, `Delta`, the non-parametric MLE of the offspring distribution, `p.est`, and the parameters of the beta distribution, `sh1` and `sh2`, used as the prior distribution are provided. The integrals in the definition of the D-posterior density function and the EDAP estimator are approximated using the function `cotes()` of the package `pracma` (see Borchers (2019)). Moreover, an initial value, `xini`, and a tolerance level, `tol`, are also given as inputs in order to compute the MDAP estimators.

```
posterior.d_HD_geom<-function(Delta,p.est,N.sim,lo=.001,up=.999,sh1,sh2,
            xini=0.5,tol,...)
{
 theta<-seq(lo,up,len=N.sim)
 f.den<-function(phi,p.est,theta,sh1,sh2)
 {
  sapply(theta,g_HD,phi=phi,p.est=p.est,sh1=sh1,sh2=sh2)
 }
 mean.post<-function(phi,p.est,theta,sh1,sh2)
 {
  g.mean<-function(phi,p.est,y,sh1,sh2) y*g_HD(phi,p.est,y,sh1,sh2)
  sapply(theta,g.mean,phi=phi,p.est=p.est,sh1=sh1,sh2=sh2)
 }
 f.int<-cotes(f.den,a=lo,b=up,n=1000,nodes=7,phi=Delta,p.est=p.est,
            sh1=sh1,sh2=sh2)
 mean.int<-cotes(mean.post,a=lo,b=up,n=1000,nodes=7,phi=Delta,
            p.est=p.est,sh1=sh1,sh2=sh2)/f.int
 mdap<-nloptr(x0=xini,eval_f=g_max_HD,lb=0.0001,ub=0.9999,opts=list(
            "algorithm"="NLOPT_LN_COBYLA","xtol_rel"=tol),phi=Delta,
            p.est=p.est,sh1=sh1,sh2=sh2)$solution
 den<-f.den(Delta,p.est,theta,sh1,sh2)/f.int
 list(x=theta, den=den, eap=mean.int, map=mdap)
}
```

We also implemented the counterpart functions for the negative exponential disparity and the Kullback-Leibler divergence.

```
posterior.d_NED_geom<-function(Delta,p.est,N.sim,lo=.001,up=.999,sh1,sh2,
             xini=0.5,tol,...)
{
 theta<-seq(lo,up,len=N.sim)
 f.den<-function(phi,p.est,theta,sh1,sh2)
 {
   sapply(theta,g_NED,phi=phi,p.est=p.est,sh1=sh1,sh2=sh2)
 }
 mean.post<-function(phi,p.est,theta,sh1,sh2)
 {
   g.mean<-function(phi,p.est,y,sh1,sh2) y*g_NED(phi,p.est,y,sh1,sh2)
   sapply(theta,g.mean,phi=phi,p.est=p.est,sh1=sh1,sh2=sh2)
 }
 f.int<-cotes(f.den, a=lo, b=up, n=1000, nodes=7, phi=Delta,p.est=p.est,
```

```
             sh1=sh1,sh2=sh2)
 mean.int<-cotes(mean.post, a=lo, b=up, n=1000, nodes=7, phi=Delta,
             p.est=p.est,sh1=sh1,sh2=sh2)/f.int
 mdap<-nloptr(x0=xini,eval_f=g_max_NED,lb=0.0001,ub=0.9999,opts=list(
             "algorithm"="NLOPT_LN_COBYLA","xtol_rel"=tol),phi=Delta,
             p.est=p.est,sh1=sh1,sh2=sh2)$solution
 den<-f.den(Delta,p.est,theta,sh1,sh2)/f.int
 list(x=theta,den=den, eap=mean.int, map=mdap)
}

posterior.d_KL_geom<-function(Delta,p.est,N.sim,lo=.001,up=.999,sh1,sh2,
             xini=0.5,tol,...)
{
 theta<-seq(lo,up,len=N.sim)
 f.den<-function(phi,p.est,theta,sh1,sh2)
 {
   sapply(theta,g_KL,phi=phi,p.est=p.est,sh1=sh1,sh2=sh2)
 }
 mean.post<-function(phi,p.est,theta,sh1,sh2)
 {
   g.mean<-function(phi,p.est,y,sh1,sh2) y*g_KL(phi,p.est,y,sh1,sh2)
   sapply(theta,g.mean,phi=phi,p.est=p.est,sh1=sh1,sh2=sh2)
 }
 f.int<-cotes(f.den, a=lo, b=up, n=1000, nodes=7, phi=Delta,p.est=p.est,
             sh1=sh1,sh2=sh2)
 mean.int<-cotes(mean.post, a=lo, b=up, n=1000, nodes=7, phi=Delta,
             p.est=p.est,sh1=sh1,sh2=sh2)/f.int
 mdap<-nloptr(x0=xini,eval_f=g_max_KL,lb=0.0001,ub=0.9999,opts=list(
             "algorithm"="NLOPT_LN_COBYLA","xtol_rel"=tol),phi=Delta,
             p.est=p.est,sh1=sh1,sh2=sh2)$solution
 den<-f.den(Delta,p.est,theta,sh1,sh2)/f.int
 list(x=theta, den=den, eap=mean.int, map=mdap)
}
```

Computation of the HPD intervals

The function `hpd()` enables to obtain an approximation of the lower and upper limits of the HPD intervals - with a credibility level of `1-alpha` - of certain parameter given a sample `theta` and the estimation of the posterior density at those points, `den`. The accuracy of the estimation is controlled with the argument `accuracy`.

```
hpd<-function(theta,den,alpha=.05,accuracy=1e-3)
{
  x<-max(theta[(theta[2]-theta[1])*cumsum(den)<=alpha/2])
```

```
  y<-min(theta[(theta[2]-theta[1])*cumsum(den)>=1-alpha/2])
  i<-sum(theta<=x)
  j<-sum(theta<=y)
  if(den[i]>den[j])
  {
    while(den[i]-den[j]>accuracy)
    {
      i<-i-1
      j<-j-1
    }
    CI<-list(qt=c(x,y),d=c(den[j],den[i]),hp=c(theta[i],theta[j]))
  }
  else
  {
    while(den[j]-den[i]>accuracy)
    {
      i<-i+1
      j<-j+1
    }
    CI<-list(qt=c(x,y),d=c(den[i],den[j]),hp=c(theta[i],theta[j]))
  }
  CI
}
```

References

Borchers, H. W. (2019). *pracma: Practical Numerical Math Functions*. R package version 2.2.5.
URL https://CRAN.R-project.org/package=pracma 59

Cheng, A. and Vidyashankar, A. N. (2006). "Minimum Hellinger distance estimation for randomized play the winner design." *Journal of Statistical Planning and Inference*, 136(6): 1875–1910. 2

Debreu, G. (1967). "Integration of correspondences." In *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability, Volume 2: Contributions to Probability Theory, Part 1*, 351–372. University of California Press. 5

Devroye, L. and Györfi, L. (1985). *Nonparametric Density Estimation: the L_1 View*. New York: John Willey and Sons. 16

Ghosh, J. K. (1994). "Higher Order Asymptotics." *NSF-CBMS Regional Conference Series in Probability and Statistics*, 4: i–111. 6

González, M., Minuesa, C., and del Puerto, I. (2017). "Minimum disparity estimation in controlled branching process." *Electronic Journal of Statistics*, 11(1): 295–325. 2, 3, 22

Gowrisankaran, K. (1972). "Measurability of functions in product spaces." *Proceedings of the American Mathematical Society*, 31: 485–488. 4, 5

Hooker, G. and Vidyshankar, A. N. (2014). "Bayesian model robustness via disparities." *Test*, 23: 556–584. 6, 19, 21

Royden, H. L. (1988). *Real Analysis*. Macmillan Publisher Company. 4

Wagner, D. H. (1977). "Survey of Measurable Selection Theorems." *SIAM Journal on Control and Optimization*, 15(5): 859–903. 2

Warnes, G. R., Bolker, B., and Lumley, T. (2018). *gtools: Various R Programming Tools*. R package version 3.8.1.
URL `https://CRAN.R-project.org/package=gtools` 49, 52

Ypma, J., Borchers, H. W., and Eddelbuettek, D. (2018). *Package 'nloptr'*. R package version 1.2.1.
URL `https://cran.r-project.org/web/packages/nloptr/nloptr.pdf` 50