

Linearity of isometries between convex Jordan curves

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Abstract

In this paper, we show that the C^1 -differentiability of the norm of a two-dimensional normed space depends only on distances between points of the unit sphere in two different ways.

As a consequence, we see that any isometry between the spheres of normed planes $\tau : S_X \rightarrow S_Y$ is linear, provided that there exist linearly independent $x, \bar{x} \in S_X$ where S_X is not differentiable and that S_X is piecewise differentiable.

We end this work by showing that the isometry $\tau : C_X \rightarrow C_Y$ is linear even if it is not an isometry between spheres: every isometry between (planar) Jordan piecewise C^1 -differentiable convex curves extends to X whenever X and Y are strictly convex and the amount of non-differentiability points of S_X and S_Y is finite and greater than 2.

1. Introduction

The study of isometries between Banach spaces led, back in the 30's, to one of the best known results in Functional Analysis, the Mazur–Ulam Theorem. This result, see [16], states that every onto isometry between two Banach spaces is affine. So, if an onto isometry preserves the origin, then the isometry is linear. Forty years later, P. Mankiewicz ([15]) proved that every onto isometry between convex bodies in two Banach spaces is also affine. The foreseeable generalisation of these results is *Every onto isometry between the spheres of two Banach spaces is linear*, and this could be ultimately generalised as *Every onto isometry between the boundaries of convex bodies of two Banach spaces is affine*, but, up to now, no-one has been able to prove or disprove this statement. This innocent-looking problem was stated in 1987 by D. Tingley ([27]), but it turns out to be way more challenging than it could seem at first glance. Tingley's Problem

has evolved to that of extending isometries between spheres to isometries between the whole spaces, and the greatest advances have been achieved when both spaces have some common structure, such as von Neumann algebras, trace class operators spaces, sums of strictly convex spaces. . . This *extension of isometries problem* has experienced a rapid development in the last few years, and there are lots of kinds of spaces where Tingley's Problem has a positive answer, see [2, 5, 6, 7, 8, 9, 10, 11, 13, 14, 17, 18, 19, 20, 21, 23, 24, 25, 26, 28].

Nevertheless, there is another way to look at this Problem. Instead of extending an isometry, one can rule out the existence of an isometry between two spheres –a trivial example: there is no isometry between the spheres of $(\mathbb{R}^2, \|\cdot\|_2)$ and $(\mathbb{R}^2, \|\cdot\|_\infty)$, say S_2 and S_∞ , because there exist $x, y, z \in S_\infty$ such that $\|x - y\|_\infty = \|x - z\|_\infty = \|y - z\|_\infty = 2$ but this cannot happen in S_2 . To the best of our knowledge, the first great achievement in this setting can be found in [13], where the authors prove that, in finite-dimensional spaces, no sphere can be isometric to a polyhedral sphere unless it is polyhedral, too. Actually, they also extend the isometry between the spheres, so the main result in [13] is

If X is finite-dimensional and polyhedral and there is an onto isometry $\tau : S_X \rightarrow S_Y$, then Y is also polyhedral and X and Y are linearly isometric.

In the same spirit, a few years later appeared this result:

If X is an inner product space and there is an onto isometry $\tau : S_X \rightarrow S_Y$, then Y is also an inner product space and X and Y are linearly isometric, see [3, 4, 18].

So, motivated by the huge advance that can be seen at a recent paper by Tarás Banach, [2], whose main result is

Every isometry $\tau : S_X \rightarrow S_Y$ between the spheres of absolutely smooth two-dimensional spaces is linear,

we began to study whether the C^2 -differentiability (that implies absolute differentiability) of some two-dimensional normed space $(X, \|\cdot\|_X)$ can be expressed in terms of $(S_X, \|\cdot\|_X)$ –unsuccessfully.

However, we have been able to determine C^1 -differentiability in two independent ways. The first way is easily seen to hold in finite-dimensional spaces, whereas we have not been able to prove whether the second one works in dimensions higher than 2 or not.

These two ways are the following:

1. Given a finite-dimensional normed space $(X, \|\cdot\|_X)$, the norm $\|\cdot\|_X$ fails to be differentiable at x if and only if there exist $\alpha, \varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ there exist $u, v \in S_X \setminus \{\pm x\}$ such that

$$\max\{\|u - x\|_X, \|v + x\|_X\} \leq \varepsilon, \quad \|u - v\|_X \leq 2 - \alpha\varepsilon.$$

2. Given $x, y, z \in S_X$ such that $\|\cdot\|_X$ is differentiable at z and $z - y = \lambda x$ for some

$\lambda > 0$, the norm $\|\cdot\|_X$ is differentiable at $x \in S_X$ if and only if

$$G(t) = \|\gamma_z(t) - y\|$$

is differentiable at 0, where $\gamma_z : \mathbb{R} \rightarrow S_X$ is an arc-length parameterization such that $\gamma_z(0) = z$.

With these two facts in mind, it is not too difficult to show the Tingley-type result in this paper:

Theorem 1.1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two-dimensional normed spaces and let $\tau : S_X \rightarrow S_Y$ be an isometry. If the amount of points in S_X where $\|\cdot\|_X$ is not differentiable is finite and greater than 3, then τ is linear.*

With the same ideas as in the proof of Theorem 1.1, we have also been able to show this result about, so to say, boundaries of convex open subsets of \mathbb{R}^2 :

Theorem 1.2. *Let $(X, \|\cdot\|_X)$ be a strictly convex normed plane such that the amount of points in S_X where $\|\cdot\|_X$ is not differentiable is finite and greater than 3. Let $C_X \subset X$ be a piecewise C^1 Jordan curve that encloses a convex set. If there is some isometry $C_X \rightarrow C_Y$ for some piecewise C^1 curve $C_Y \subset Y$, being $(Y, \|\cdot\|_Y)$ another normed plane that fulfils the same that $(X, \|\cdot\|_X)$, then τ is affine and, so, X and Y are isometric.*

1.1. Notations and background

Remark 1.3. Given some Jordan curve $C \subset \mathbb{R}^2$, we will say that C is a convex curve when it encloses a convex region.

For any convex piecewise smooth curve $C \subset \mathbb{R}^2$, it is known that there are parameterizations $\gamma : \mathbb{R} \rightarrow C$ that are smooth at t if and only if C is smooth at $\gamma(t)$ and have one-sided derivatives $\gamma'_-(t)$ and $\gamma'_+(t)$ at every other $t \in \mathbb{R}$. We will only consider these parameterizations.

As we will heavily use the arc-length anticlockwise parameterization of S_X beginning at some point, we will denote this curve in a special way:

If z belongs to some piecewise C^1 -differentiable, convex, Jordan curve $C \subset \mathbb{R}^2$, then $\gamma_z : \mathbb{R} \rightarrow C$ will denote the only anticlockwise parameterization of C that fulfils $\gamma_z(0) = \gamma_z(L) = z$, is injective when restricted to $[0, L)$, is L -periodic and has $\|\gamma'_{z,-}(t)\|_X = \|\gamma'_{z,+}(t)\|_X = 1$ for every $t \in \mathbb{R}$. This parameterization is also known as *the natural parameterization* of C , see [2].

Definition 1.4. *Let $(X, \|\cdot\|_X)$ be a normed space. We say that x is Birkhoff orthogonal to y , denoted as $x \perp_B y$, if $\|x + \lambda y\|_X \geq \|x\|_X$ for every $\lambda \in \mathbb{R}$. We will denote $x^\perp = \{y \in X : x \perp_B y\}$.*

The reader interested in this and related concepts may want to take a look at [1]. It is noteworthy that the main subjects of [1] are two kinds of orthogonality, one of them is obviously preserved by isometries of the sphere but we will deal with the other one.

For our particular concern, Birkhoff orthogonality is important due to the following two results.

Proposition 1.5. [12, Theorem 2.2], [1, Theorem 4.12] *For any vector x in a normed linear space X there exists a hyperplane $H \subset X$ such that $x \perp_B H$.*

Proposition 1.6. [12, Theorem 4.2], [1, Theorem 4.15] *The norm of a normed linear space X is Gâteaux differentiable at $x \in X \setminus \{0\}$ if and only if x^\perp is a hyperplane.*

As for the differentiability of finite-dimensional norms, joining [22, Theorem 25.2 and Corollary 25.5.1] we obtain:

Proposition 1.7. *Let f be a convex function on an open convex set $A \subset \mathbb{R}^d$. If f has all partial derivatives at each point of A , then $f \in C^1(A)$.*

Proposition 1.7 implies that the usual differences between the various kinds of differentiability do not exist when we deal with a convex function like $\|\cdot\|_X : \mathbb{R}^n \rightarrow \mathbb{R}$. In particular,

Lemma 1.8. *Let $(X, \|\cdot\|_X)$ be a finite-dimensional normed space. Then, the following conditions are equivalent to one another:*

- $\|\cdot\|_X$ is C^1 -differentiable.
- $\|\cdot\|_X$ is Fréchet differentiable.
- $\|\cdot\|_X$ is Gâteaux differentiable.
- S_X is a differentiable manifold.
- For each $x \in X, x \neq 0$, x^\perp is a hyperplane.
- If, in addition, X is two-dimensional, then the above conditions are equivalent to the fact that for every $x \in S_X, t \in \mathbb{R}$, the equality $\gamma'_{x,-}(t) = \gamma'_{x,+}(t)$ holds.

We will also use this Lemma that Professor Javier Alonso gifted me some years ago:

Lemma 1.9 (J. Alonso). *Let $(\mathbb{R}^2, \|\cdot\|_X)$ be a two-dimensional normed space and $x \in S_X$. If $y \in S_X$ is a side derivative of the natural parameterization of S_X at x , then $x \perp_B y$.*

Proof. We need to show that $\|x + \lambda y\|_X \geq 1$ for every $\lambda \in \mathbb{R}$, with

$$y = \lim_{t \rightarrow 0^+} \frac{\gamma_x(t) - x}{t}.$$

We have the following:

$$\begin{aligned} \|x + \lambda y\|_X &= \left\| x + \lambda \lim_{t \rightarrow 0^+} \frac{\gamma_x(t) - x}{t} \right\|_X = \lim_{t \rightarrow 0^+} \left\| x + \lambda \frac{\gamma_x(t) - x}{t} \right\|_X = \\ &= \lim_{t \rightarrow 0^+} \left\| \frac{\lambda}{t} \gamma_x(t) + \left(1 - \frac{\lambda}{t}\right) x \right\|_X \geq \lim_{t \rightarrow 0^+} \left| \left\| \frac{\lambda}{t} \gamma_x(t) \right\|_X - \left\| \left(1 - \frac{\lambda}{t}\right) x \right\|_X \right| = \\ &= \lim_{t \rightarrow 0^+} \left| \left\| \frac{\lambda}{t} \right\| - \left| 1 - \frac{\lambda}{t} \right| \right| = 1, \text{ for every } \lambda \in \mathbb{R}. \end{aligned} \quad (1)$$

□

2. Main results

We will prove that the differentiability of $\|\cdot\|_X$ at some x depends on the infinitesimal metric structure of S_X around x . Later, we will show that it can be determined by means of computations carried away far from x . Joining both facts we will arrive at our main results after some extra work.

Proposition 2.1. *Let $(X, \|\cdot\|_X)$ be a finite-dimensional space. The differentiability of $\|\cdot\|_X$ at any $x \in S_X$ depends only on the metric structure of the unit sphere $(S_X, \|\cdot\|_X)$ near x and $-x$. Namely, $\|\cdot\|_X$ fails to be differentiable at x if and only if there exist $\delta, \varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ there exist $u, v \in S_X \setminus \{\pm x\}$ such that*

$$\max\{\|u - x\|_X, \|v + x\|_X\} \leq \varepsilon, \quad \|u - v\|_X \leq 2 - \delta\varepsilon. \quad (2)$$

Proof. Let $x \in S_X$.

It is clear that if $\|\cdot\|_X$ is differentiable at x then for every $\delta, \varepsilon_0 > 0$ there is $0 < \varepsilon < \varepsilon_0$ such that (2) cannot hold for every $u, v \in S_X \setminus \{\pm x\}$.

If $\|\cdot\|_X$ is not differentiable at x , then Proposition 1.5 and Lemma 1.8 imply that x^\perp contains strictly a hyperplane, so there is some pair of independent vectors $y, z \in S_X \cap x^\perp$ such that $x \in \text{span}\{y, z\}$. We are going to show that there are $u, v \in S_X \cap \text{span}\{y, z\}$ that fulfil (2), so we may suppose that X is two-dimensional and $X = \text{span}\{y, z\}$. Taking any orientation on X , we may define γ_x . As the side derivatives of γ_x at 0 are different and fulfil $x \perp_B \gamma'_{x,-}(0)$ and $x \perp_B \gamma'_{x,+}(0)$ (Lemma 1.9), we may suppose $y = \gamma'_{x,+}(0)$, $z = \gamma'_{x,-}(0)$.

Taking into account that $x \perp_B y$ if and only if $x \perp_B -y$, we may suppose that there exist $\lambda, \mu > 0$ such that $x = -\lambda y + \mu z$. Consider the basis $\mathcal{B} = \{x, y\}$. Taking coordinates with respect to \mathcal{B} , we have $z = (z_1, z_2)$ and $z_1 = 1/\mu, z_2 = \lambda/\mu > 0$. By the very definition of Birkhoff orthogonality, $x \perp_B y$ implies $\|(1, t)\|_X = \|x + ty\|_X \geq 1$ and $-x \perp_B z$ implies $\|(-1 + tz_1/z_2, t)\|_X = \|-x + tz/z_2\|_X \geq 1$ for every $t \in \mathbb{R}$. It is clear that, moreover, $\|(\alpha, t)\|_X \geq \alpha$ and $\|(-\alpha + tz_1/z_2, t)\|_X \geq \alpha$ for every $\alpha > 0$, so we have

$$B_X \subset \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha \leq 1, \beta z_1/z_2 \leq \alpha + 1\}.$$

Furthermore, B_X contains the convex hull of $\{(-1, 0), (0, 1), (1, 0)\}$. On the one hand, this means that $\|\cdot\|_X \leq \|\cdot\|_1$. On the other hand, this implies that for each $t \in]0, 1[$ the line $\{(\alpha, t) : \alpha \in \mathbb{R}\}$ intersects with S_X at exactly two points $u = (a^+(t), t), v = (a^-(t), t)$, with

$$-1 + tz_1/z_2 \leq a^-(t) \leq -1 + t, \quad 1 - t \leq a^+(t) \leq 1.$$

Thus, we obtain $\|u - v\|_X = \|(a^+(t) - a^-(t), 0)\|_X \leq 2 - tz_1/z_2$ and

$$\|u - x\|_X = \|(a^+(t) - 1, t)\|_X \leq |a^+(t) - 1| + |t| = 1 - a^+(t) + t \leq 2t,$$

$$\|v + x\|_X = \|(a^-(t) + 1, t)\|_X \leq |a^-(t) + 1| + |t| = 1 + a^-(t) + t \leq 2t.$$

The last three inequalities end the proof, taking $\varepsilon = 2t$ and $\delta = z_1/(2z_2)$. \square

Corollary 2.2. *Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be finite-dimensional normed spaces whose spheres are isometric. Then, $\|\cdot\|_Y$ is differentiable if and only if $\|\cdot\|_X$ is also differentiable.*

Proof. It is straightforward from Proposition 2.1 and the fact that every onto isometry between finite-dimensional spheres preserves antipodes ([27, Theorem, p. 377]). \square

Corollary 2.3. *Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be two-dimensional normed spaces and $\tau : S_X \rightarrow S_Y$ a surjective isometry between their spheres. Then, either both spaces are piecewise C^1 -differentiable or none of them is. Moreover, if $\|\cdot\|_X$ is piecewise C^1 then it is C^1 -differentiable at x if and only if $\|\cdot\|_Y$ is C^1 -differentiable at $\tau(x)$.*

Remark 2.4. To avoid confusion, we will use the notation $] \alpha, \beta [$ to denote the open interval whose endpoints are α and β . Thus, (α, β) will always be a vector in \mathbb{R}^2 .

Proposition 2.5. *Let $\|\cdot\|_X$ be a strictly convex norm defined on $X = \mathbb{R}^2$. Consider $x, y, z \in S_X$, and $\lambda \in]0, 2[$ such that $z = y + \lambda x$ and $\|\cdot\|_X$ is differentiable at z . In these conditions, $\|\cdot\|_X$ is differentiable at x if and only if*

$$G(t) = \|\gamma_z(t) - y\|_X$$

is differentiable at $t = 0$. In particular, the differentiability at x depends on the metric at y and around z .

Proof. It is clear that $\|\cdot\|_X$ is differentiable at x and z , then G is differentiable at 0 because it is the composition of differentiable functions.

Suppose, on the other hand, that $\|\cdot\|_X$ is not differentiable at x , i.e., $\gamma'_{x,+}(0) \neq \gamma'_{x,-}(0)$. For the sake of clarity, we will consider the basis $\mathcal{B} = \{-\gamma'_{x,-}(0), x\}$ so the position of x, y, z is like in Figure 1, i.e., $x = (0, 1)$ and $z - y = (0, \lambda)$.

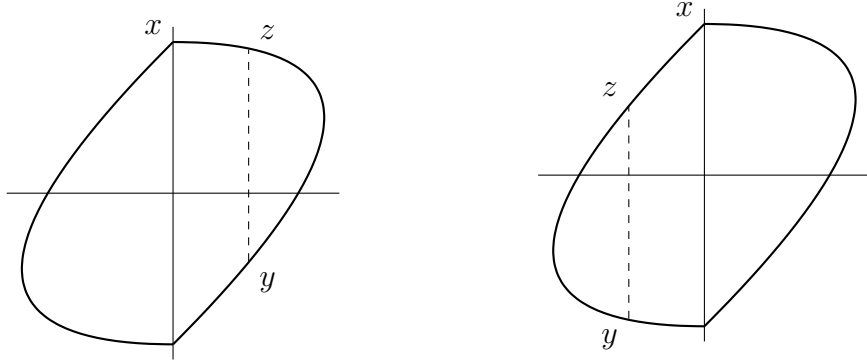


Figure 1: With the basis \mathcal{B} , y and z have the same first coordinate and S_X arrives at x horizontally.

Let us show that G is not differentiable at 0. To this end, let z'_1 and z'_2 be the coordinates of $\gamma'_z(0)$ in the basis \mathcal{B} . With these assumptions, z'_1 is negative (as in the figure). As $G(0) = \lambda$, we have

$$G'_-(0) = \lim_{\varepsilon \rightarrow 0^-} \frac{\|\gamma_z(\varepsilon) - y\|_X - \lambda}{\varepsilon}.$$

As S_X is differentiable at z ,

$$\|\gamma_z(\varepsilon) - (z + \varepsilon\gamma'_z(0))\|_X = o(\varepsilon),$$

so we have

$$\begin{aligned} G'_-(0) &= \lim_{\varepsilon \rightarrow 0^-} \frac{\|z + \varepsilon\gamma'_z(0) - y\|_X - \lambda}{\varepsilon} = \lim_{\varepsilon \rightarrow 0^-} \frac{\|\lambda x + \varepsilon\gamma'_z(0)\|_X - \lambda}{\varepsilon} = \\ &= \lim_{\varepsilon \rightarrow 0^-} \frac{\|(0, \lambda) + \varepsilon(z'_1, z'_2)\|_X - \lambda}{\varepsilon} = z'_2, \end{aligned} \quad (3)$$

where the last equality holds because $(0, \lambda) + \varepsilon(z'_1, z'_2)$ lies in the first quadrant and is close to $(0, \lambda)$. In this situation,

$$\|(0, \lambda) + \varepsilon(z'_1, z'_2)\|_X = \|(0, \lambda + \varepsilon z'_2)\|_X + o(\varepsilon) = \lambda + \varepsilon z'_2 + o(\varepsilon).$$

So, $G'_-(0) = z'_2$.

The choice of the basis has nothing to do with the value of $G'_-(0)$. Had we computed $G'_+(0)$ by using the basis $\bar{\mathcal{B}} = \{-\gamma'_{x,+}(0), x\}$, we would have arrived at $G'_+(0) = \bar{z}'_2$, where (\bar{z}'_1, \bar{z}'_2) are the coordinates of $\gamma'_z(0)$ with respect to $\bar{\mathcal{B}}$. What we need to see is that $z'_2 \neq \bar{z}'_2$. So, consider the linear automorphism of \mathbb{R}^2 given by $T(a, b) = (\bar{a}, \bar{b})$ when

$$a\gamma'_{x,-}(0) + bx = \bar{a}\gamma'_{x,+}(0) + \bar{b}x.$$

If we had $\bar{z}'_2 = z_2$, then T and the map $(a, b) \mapsto (\bar{z}'_1 a/z_1, b)$ would agree at $(0, 1)$ and (z_1, z_2) . Both maps are linear and these vectors form a basis, so they must be the same map. This readily implies that S_X is differentiable at x , a contradiction that ends the proof. \square

Remark 2.6. Consider \mathbb{R}^2 endowed with the hexagonal norm $\|\cdot\|_X$ defined as

$$\|(a, b)\|_X = \begin{cases} \max\{|a|, |b|\} & \text{if } ab \geq 0, \\ |a| + |b| & \text{if } ab < 0 \end{cases}$$

The norm $\|\cdot\|_X$ is not differentiable at $x = (0, 1)$, but if we take $y = (1, 1/3)$ and $z = (1, 2/3)$ then $\|\gamma_z(t) - y\|_X = 1/3 + t$ for $t \in [-1/3, 1/3]$, so $\|\gamma_z(t) - y\|_X$ is differentiable at $t = 0$. So, if $\|\cdot\|_X$ is not strictly convex then Proposition 2.5 does not need to hold.

Questions 2.7. Can Proposition 2.5 be generalised to finite-dimensional spaces or arbitrary dimension?

Is Proposition 2.5 true if we replace differentiability by C^2 -differentiability?

2.8. As we will often need to refer to differentiability and non-differentiability points, for the sake of readability we will denote $D(C_X)$ (resp., $\text{ND}(C_X)$) the sets where a curve C_X is differentiable (resp., where it is not differentiable) so we will write $x \in D(C_X)$ (resp., $x \in \text{ND}(C_X)$) instead of x is a *differentiability* (resp., *non-differentiability*) point of C_X .

Before we proceed with our main results, we need these technical Lemmas:

Lemma 2.9. *Let $C \subset \mathbb{R}^2$ be a Jordan curve that encloses a convex region and suppose that the extreme points are all different –that is, there are some $c^1 = (c_1^1, c_2^1)$, $c^2 = (c_1^2, c_2^2)$, $c^3 = (c_1^3, c_2^3)$, $c^4 = (c_1^4, c_2^4) \in C$ such that*

$$c_1^1 = \min\{c_1 : (c_1, c_2) \in C\}, c_2^2 = \min\{c_2 : (c_1, c_2) \in C\};$$

$$c_1^3 = \max\{c_1 : (c_1, c_2) \in C\}, c_2^4 = \max\{c_2 : (c_1, c_2) \in C\};$$

$$c_1^1 < \min\{c_1^2, c_1^3, c_1^4\}, c_2^2 < \min\{c_2^1, c_2^3, c_2^4\}, c_1^3 > \max\{c_1^1, c_1^2, c_1^4\}, c_2^4 > \max\{c_2^1, c_2^2, c_2^3\}.$$

Then, for any $x = (x_1, x_2) \in \mathbb{R}^2$ there are $u = (u_1, u_2), v = (v_1, v_2), w = (w_1, w_2) \in C$, $t \neq 0$ such that

$$u - v = tx, \quad w_1 = u_1 \quad \text{and} \quad w_2 = v_2.$$

Proof. If $x = 0$ then we may take any $u \in C$ and $w = v = u, t = 1$. So, we only need to show that the result holds for $x \neq 0$. It is clear that we may suppose $\|x\|_\infty = 1$, so we will show that for every $x \in S_\infty$ there exist t, u, v, w as in the statement. If $x = (1, 0)$ we just need to find $u, v \in C$ that belong to the same horizontal line (in this case, $w = u$) and if $x = (0, 1)$ it suffices to find $u, v \in C$ in the same vertical line and take $w = v$, so we may suppose $x_1 x_2 \neq 0$. Suppose that $x_1, x_2 > 0$, the other cases are analogous.

As C is convex, there is exactly one point or segment at the undermost end of C . Suppose it is just one point, say $w^0 = c^2$, and analyse what happens when we move along the curve anticlockwise until we reach the rightmost point or segment of C , suppose again that it is a singleton, say $w^1 = c^3$. As $w^1 \neq w^0$, we may consider some anticlockwise parameterization of C that has $\gamma(0) = w^0$ and $\gamma(1) = w^1$, we will denote $w^s = \gamma(s)$.

It is clear that, for any $s \in]0, 1[$, w^s is the undermost point of the intersection of C with the vertical line where it lies. Denote u^s the uppermost point of this intersection. Analogously, w^s is the rightmost intersection of C with the horizontal line where it lies, we will denote its leftmost point as v^s . What we need to see is that for every proportion x_1/x_2 there is some w^s such that $(w_1^s - v_1^s)/(u_2^s - w_2^s) = x_1/x_2$.

But the map $s \in]0, 1[\mapsto (w_1^s - v_1^s)/(u_2^s - w_2^s)$ is continuous and its limits are 0 at 0 and ∞ at 1. So, at some $s \in]0, 1[$ we get the desired equality.

If instead of one point there is a segment at the bottom of C then we take w^0 as the leftmost point of this segment, if there is one segment at the rightmost end of C then w^1 is at the top of the segment and everything goes undisturbed. \square

Lemma 2.10. *Consider \mathbb{R}^2 endowed with the norm $\|(\lambda, \mu)\|_1 = |\lambda| + |\mu|$. Let $C \subset \mathbb{R}^2$ be a convex Jordan curve that does not fulfil the conditions of Lemma 2.9 because some point, say c , is extreme in two directions. For each $a \in C$, consider the sequence $(a_n)_n \subset C$ defined as $a_1 = a$ and, for $n \geq 1$, a_{n+1} is the closest point from c that shares some coordinate with a_n . In these conditions, $(a_n)_n \rightarrow c$ unless a is the strict extreme in the two other directions, in which case $a_n = a, \forall n \in \mathbb{N}$.*

2.11. If there are two possible choices for a given a_{n+1} then we choose the point lying in the same vertical line as a_n .

Proof. If the conditions in the statement are fulfilled, then it is clear that $(a_n)_n$ has some accumulation point because C is compact and the accumulation point must be its limit because $(a_n)_n$ is monotonic in both coordinates. The only possible limit is c , so we are done. \square

Theorem 2.12. *Let $(X, \|\cdot\|_X)$ be a two-dimensional normed space whose unit sphere S_X is piecewise C^1 -differentiable and has at least two points $x \neq \pm\bar{x}$, with $x, \bar{x} \in \text{ND}(S_X)$. For any normed plane $(Y, \|\cdot\|_Y)$, every isometry $\tau : S_X \rightarrow S_Y$ is linear.*

Proof. If X is not strictly convex then the result holds by [4, Corollary 3.8], so we may suppose that $\|\cdot\|_X$ is strictly convex.

Suppose there are linearly independent $x, \bar{x} \in \text{ND}(S_X)$, consider the basis $\mathcal{B}_X = \{x, \bar{x}\}$, and let $\tau : S_X \rightarrow S_Y$ be an onto isometry. The only point in S_X at distance 2 from x is $-x$, so it is obvious that $\tau(-x) = -\tau(x)$ and we obtain that $\tau(x)$ and $\tau(\bar{x})$ are linearly independent so we may consider the basis $\mathcal{B}_Y = \{\tau(x), \tau(\bar{x})\}$. Taking coordinates with respect to \mathcal{B}_X and \mathcal{B}_Y , we have $x = (1, 0)_X, \tau(x) = (1, 0)_Y$, so both x and $-x$ lie on the same horizontal line and $\tau(x)$ and $\tau(-x)$ do, too. We are going to see that this happens to $\tau(u), \tau(v) \in S_Y$ for any couple $u = (u_1, u_2), v = (v_1, v_2) \in S_X$ such that $u_1 > v_1$ and $v_2 = u_2$, i.e., such that $u - v = \lambda x$ for some $\lambda > 0$ —observe that, actually, $\lambda = \|u - v\|_X \in]0, 2]$. Let ${}^\perp x$ be the only point in S_X such that ${}^\perp x \perp_B x$ and whose second coordinate is negative (${}^\perp x$ is unique because $\|\cdot\|_X$ is strictly convex, see [1, Theorem 4.15]). We will denote as C the (relative) interior of the arc of S_X that joins ${}^\perp x$ with $-{}^\perp x$ and contains x , observe that $S_X = C \cup -C \cup \{\pm {}^\perp x\}$. Consider

$$\mathcal{C}_x = \{u \in C : \tau(u) - \tau(v) = \lambda \tau(x) \text{ if } u - v = \lambda x, \lambda \in]0, 2]\}$$

We are going to show that $\mathcal{C}_x = C$, so we will have the equivalence

$$u - v = \lambda x \Leftrightarrow \tau(u) - \tau(v) = \lambda \tau(x).$$

If $(u_n)_n \rightarrow u \in C$ and $u_n \in \mathcal{C}_x$ for every $n \in \mathbb{N}$, then consider the corresponding sequences $(v_n)_n, (\lambda_n)_n$. We have, for every $n \in \mathbb{N}$,

$$u_n - v_n = \lambda_n x \quad \text{and} \quad \tau(u_n) - \tau(v_n) = \lambda_n \tau(x). \quad (4)$$

It is clear that both $(v_n)_n, (\lambda_n)_n$ must converge and that

$$u - v = \lambda x \quad \text{and} \quad \tau(u) - \tau(v) = \lambda \tau(x),$$

where $\lambda = \lim(\lambda_n), v = \lim(v_n)$. So, $u \in \mathcal{C}_x$ and \mathcal{C}_x is, therefore, closed in C .

Suppose now that $(u_n)_n \rightarrow u$ and $u \in \mathcal{C}_x$. Take λ, v and $(\lambda_n)_n, (v_n)_n$ such that $u - v = \lambda x, u_n - v_n = \lambda_n x$. As there are finitely many points in $\text{ND}(S_X)$, we may suppose that none of them is u_n or v_n . In this situation, Proposition 2.1 implies that S_Y is differentiable at $\tau(u_n)$ and so, Proposition 2.5 implies that $\tau(u_n) - \tau(v_n) = \lambda_n y_n$, with $y_n \in \text{ND}(S_Y)$. As $\text{ND}(S_Y)$ is finite, there is some y that appears infinitely many times in $(y_n)_n$, so passing to a subsequence we may suppose that $\tau(u_n) - \tau(v_n) = \lambda_n y$ for every $n \in \mathbb{N}$. Of course, $\lim(\tau(u_n))_n = \tau(u)$, $\lim(\tau(v_n))_n = \tau(v)$ and $\lim(\lambda_n)_n = \lambda$, so

$$\lambda y = \lim(\tau(u_n) - \tau(v_n))_n = \tau(u) - \tau(v) = \lambda \tau(x),$$

we obtain that $y = \tau(x)$ and this means that \mathcal{C}_x is open.

We have seen that \mathcal{C}_x is non-empty –because $x \in \mathcal{C}_x^-$, closed and open, so the connectedness of C shows that $\mathcal{C}_x = C$.

So, $u - v = \lambda x$ implies $\tau(u) - \tau(v) = \lambda\tau(x)$. Of course, the same applies to \bar{x} , so what we actually have is that $u - v = \lambda x + \mu\bar{x}$ implies $\tau(u) - \tau(v) = \lambda\tau(x) + \mu\tau(\bar{x})$ whenever there exists $w \in S_X$ such that either $w = v + \mu\bar{x} = u - \lambda x$ or $w = v + \lambda x = u - \mu\bar{x}$.

Now we have two options. If we are in the hypotheses of Lemma 2.9, then w exists for every possible direction, and from the fact that τ is an isometry, we get

$$\|\lambda\tau(x) + \mu\tau(\bar{x})\|_Y = \|\lambda x + \mu\bar{x}\|_X.$$

As we have taken coordinates with respect to $\{x, \bar{x}\}$ and $\{\tau(x), \tau(\bar{x})\}$, we get $\|(\lambda, \mu)\|_Y = \|(\lambda, \mu)\|_X$. This means that in these coordinates we have $\|\cdot\|_Y = \|\cdot\|_X$, so [4, Theorem 2.3] implies that τ is linear.

If we cannot apply Lemma 2.9, then there is some $c \in S_X$ that is strictly extremal in two different directions. So, we may apply Lemma 2.10 to show that, given any $a \neq \pm c \in S_X$, the sequence $(\tau(a_n))_n$ is the same as the sequence $((\tau(a))_n)_n$, i.e., the sequence originated in $\tau(a)$. This implies that for every $n \in \mathbb{N}$, and with the coordinates taken again with respect to $\{x, \bar{x}\}$ and $\{\tau(x), \tau(\bar{x})\}$ we have $\tau(a_n) - \tau(a) = a_n - a$. As $(a_n)_n \rightarrow c$, this means that for every $a \in C$, we have $\tau(a) - \tau(c) = a - c$. From here we readily see that τ is linear in S_X and this completes the proof. \square

Remark 2.13. If the only non-differentiability points in S_X are $\pm x$, then it is clear from the previous proof that $u - v = \lambda x$ implies $\tau(u) - \tau(v) = \lambda\tau(x)$, but we have not been able to infer from here that τ must be linear. Taking any basis $\mathcal{B}_X = \{x, \bar{x}\}$ and considering $\mathcal{B}_y = \{\tau(x), \tau(\bar{x})\}$ we have, in coordinates, $\tau(\alpha, \beta) - \tau(\alpha', \beta) = (\alpha - \alpha', 0)$ for every $(\alpha, \beta), (\alpha', \beta) \in S_X$ but we have not been able to deduce anything for points with different second coordinates.

Theorem 2.14 (Mankiewicz Property). *Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be strictly convex normed planes such that both $\text{ND}(S_X)$ and $\text{ND}(S_Y)$ are finite and contain more than three points. Let $C_X \subset X$ be a piecewise C^1 Jordan curve that encloses a convex set. If there is some isometry $C_X \rightarrow C_Y$ for some piecewise C^1 curve $C_Y \subset Y$, then τ is affine and X and Y are isometric.*

Proof. First of all, we hasten to remark that in strictly convex spaces, if three points z^1, z^2, z^3 fulfil $\|z^1 - z^3\| = \|z^1 - z^2\| + \|z^2 - z^3\|$, then z^2 belongs to the segment whose endpoints are z^1 and z^3 , we will denote this segment as $[z^1, z^3]$. With this, it is not hard to see that a curve C encloses a convex region if and only if for every triple of collinear points $z^1, z^2, z^3 \in C$, the curve C contains the segment $[z^1, z^3]$. So, our hypotheses imply that C_Y is convex, too.

The first we need to show is that Corollary 2.3 and Proposition 2.5 still apply in this situation, i.e., that if $u^0, v^0 \in C_X$ fulfil that $(u^0 - v^0)/\|u^0 - v^0\|_X \in \text{ND}(S_X)$, then for every $u, v \in C_X$ we have the equivalence $u - v = \lambda(u^0 - v^0)$ if and only if $\tau(u) - \tau(v) = \lambda(\tau(u^0) - \tau(v^0))$.

For the equivalent of Corollary 2.3, we have to make do without $-x$, but the only thing really useful of having $-x$ was that $x \in \text{D}(S_X)$ if and only if $-x \in \text{D}(S_X)$. In any case, we are going to show that $x \in \text{D}(C_X)$ is equivalent to $\tau(x) \in \text{D}(C_Y)$. As the proof is going to be quite different, we will denote the point as a instead of x .

Let $a \in C_X$ and let us analyse the set

$$\text{NDif}(a) = \{b \in C_X : \|\gamma_a(t) - b\|_X \text{ is not differentiable at } t = 0\},$$

observe that one has $a \in \text{NDif}(a)$ for every $a \in C_X$.

If $a \in \text{D}(C_X)$, then it is clear that $a \neq b \in \text{NDif}(a)$ implies $(a-b)/\|a-b\|_X \in \text{ND}(S_X)$ no matter whether $b \in \text{D}(C_X)$ or not. As there are only finitely many points in $\text{ND}(S_X)$, say $\text{ND}(S_X) = \{x^1, \dots, x^m\}$, when $b \in \text{NDif}(a)$ one has $b - a = \|b - a\|_X x^i$ for some $i \in \{1, \dots, m\}$ –we are considering as unrelated points x^i and $-x^i$.

Claim 1. For any $a \in \text{D}(C_X)$, $\text{NDif}(a)$ contains, at most, one segment and finitely many isolated points. If it contains one segment, then one of its endpoints is a .

Proof. We need to show that for every $i \in \{1, \dots, m\}$, there is at most one point in $\text{NDif}(a)$ that can be written as $b - a = \|b - a\|_X x^i$ unless there is a segment that fulfils it. Indeed, if b^1, b^2 fulfil $b^1 - a = \|b^1 - a\|_X x^i$ and $b^2 - a = \|b^2 - a\|_X x^i$ with $\|b^1 - a\|_X < \|b^2 - a\|_X$, then b^1 lies in the interior of the segment $[a, b^2]$ –i.e., the closed segment whose endpoints are a and b^2 . As $a, b^1, b^2 \in C_X$ and C_X encloses a convex region, the segment $[a, b^2]$ is included in C_X and it is obvious that there is only one segment included in C_X that has a as its endpoint –recall that C_X is differentiable at a . If a is interior to some segment, then no more segments can arrive to a and it is clear that $\|\gamma_a(t) - b\|_X$ is differentiable at 0 for any point in the same segment. \square

If we have, instead, $a \in \text{ND}(C_X)$, then $\text{NDif}(a)$ includes every $b \in C_X$ such that $(a-b)/\|a-b\|_X \in \text{D}(S_X)$. Indeed, let $x = (a-b)/\|a-b\|_X \in \text{D}(S_X)$ and consider \bar{x} as any of the two opposite vectors in S_X such that $x \perp_B \bar{x}$ –i.e., $\bar{x} = \pm \gamma'_x(0) \in S_X$. With the basis $\mathcal{B}_X = \{\bar{x}, x\}$, the sphere S_X and the line $\{(\lambda, 1) : \lambda \in \mathbb{R}\}$ are tangent. This implies that, for every $\mu \in]-1, 1[$, the line $\{(\lambda, \mu) : \lambda \in \mathbb{R}\}$ meets S_X in two points, say $b = (b_1, b_2), c = (c_1, c_2)$, and the first coordinates of these points have different sign. Moreover, as $\|\cdot\|_X$ is strictly convex, $\{(\lambda, 1) : \lambda \in \mathbb{R}\} \cap S_X = \{x\}$. Both these facts will be important later.

Denote a'_-, a'_+ the (different) side derivatives of C_X in a . In the basis $\mathcal{B}_X = \{\bar{x}, x\}$, we have $a'_- = (a'_{-,1}, a'_{-,2}), a'_+ = (a'_{+,1}, a'_{+,2})$. The speed of growing of $\|\gamma_a(t) - b\|_X$ as we

are arriving at a in the direction of a'_- is $a'_{-,2}$ and the speed of growing in the direction of a'_+ is $a'_{+,2}$. This can be seen as in Proposition 2.5 or by thinking this situation as if we had partial derivatives: $\|\gamma_a(t) - b\|_X$ would grow at speed 1 if $\gamma'_a(0) = x = (0, 1)$ and the speed would be 0 if $\gamma'_a(0) = \bar{x} = (1, 0)$. For any linear combination (a_1, a_2) we have speed a_2 . So, we need to show that $a'_{-,2} \neq a'_{+,2}$. As C_X encloses a convex region, the signs of $a'_{-,1}$ and $a'_{+,1}$ are the same –maybe one of them is zero. So, if $a'_{-,2} = a'_{+,2}$, then we would have two points in S_X with the same second coordinate in the same quadrant, but we have just seen that this cannot happen.

This means that $\text{NDif}(a)$ contains every point in C_X but, at most, two segments that include a and finitely many other points. In particular, there exist some open $U \subset C_X$ such that $U \subset \text{NDif}(a)$ and $U \cup \{a\}$ is not contained in a metric segment.

Gathering all these facts, we obtain that the metric structure of $\text{NDif}(a)$ determines the differentiability of C_X at a –and this implies that $\tau(a) \in \text{D}(C_Y)$ if and only if $a \in \text{D}(C_X)$.

As for the analogous of Proposition 2.5, we need to show that $(u - v)/\|u - v\|_X \in \text{ND}(S_X)$ implies $(\tau(u) - \tau(v))/\|\tau(u) - \tau(v)\|_Y \in \text{ND}(S_Y)$.

So, let $x = (u - v)/\|u - v\|_X \in \text{ND}(S_X)$ and consider, as in the proof of Proposition 2.5, the basis $\mathcal{B} = \{-\gamma'_{x,-}(0), x\}$. We have u and v in the same vertical line and u is over v . In coordinates, $u_1 = v_1, u_2 > v_2$.

Suppose that $u \in \text{D}(C_X)$ and that there is no segment in C_X that contains u and v . Then, the map

$$G(t) = \|\gamma_u(t) - v\|_X$$

is not differentiable at $t = 0$, the proof is the same as the one in Proposition 2.5.

In the proof of Theorem 2.12, we defined C as the interior of one of the arcs that join ${}^\perp x$ and $-{}^\perp x$. The analogous way to define this is by taking C as the interior of the arc that joins the lowermost point or segment in C_X with its uppermost point or segment passing through the right part of C_X –so, C does not include any of its endpoints.

Later, we defined

$$\mathcal{C}_x = \{u \in C : \tau(u) - \tau(v) = \lambda\tau(x) \text{ if } u - v = \lambda x, \lambda > 0\},$$

but now we do not have $\tau(x)$, so we need to modify the definition of the set \mathcal{C}_x . For this, there is an equivalent way to state $u - v = \lambda x$ and $\tau(u) - \tau(v) = \lambda\tau(x)$. We can take $u^0 \in C$, $v^0 \in C_X$ such that $u^0 - v^0 = \lambda_0 x$ with $\lambda_0 > 0$, eliminate the condition $u - v = \lambda x$ by writing $u - \lambda x$ instead of v and define our new subset as any of the following equivalent ways:

$$\mathcal{C}_x = \{u \in C : \tau(u) - \tau(u - \lambda x) = \lambda(\tau(u^0) - \tau(v^0))/\lambda_0, \lambda > 0\},$$

$$\mathcal{C}_x = \{u \in C : \tau(u) - \tau(u - \lambda x) = \lambda(\tau(u^0) - \tau(u^0 - \lambda_0 x))/\lambda_0, \lambda > 0\}.$$

Now, the proof of Theorem 2.12 shows that \mathcal{C}_x is open and closed in C . Again, \mathcal{C}_x is not empty because $u^0 \in \mathcal{C}_x$, so $\mathcal{C}_x = C$. It is clear that if for every u in C_X such that there is exactly one $v \in C_X$ such that $u - v = \|u - v\|_X x$ one has $\tau(u) - \tau(v) = \lambda(\tau(u^0) - \tau(v^0))$ for some $\lambda > 0$, we have the same when u belongs to a segment whose direction is x . So, denoting $y = (\tau(u^0) - \tau(v^0))/\|\tau(u^0) - \tau(v^0)\|_Y$ we have $u - v = \|u - v\|_X x$ if and only if $\tau(u) - \tau(v) = \|\tau(u) - \tau(v)\|_Y y$.

If we consider $\bar{x} \in \text{ND}(S_X)$, $\bar{x} \neq \pm x$ and $\bar{u}^0, \bar{v}^0 \in C_X$ such that $\bar{u}^0 - \bar{v}^0 = \|\bar{u}^0 - \bar{v}^0\|_X \bar{x}$, then the same argument as before shows that $\bar{u} - \bar{v} = \|\bar{u} - \bar{v}\|_X \bar{x}$ is equivalent to $\tau(\bar{u}) - \tau(\bar{v}) = \|\tau(\bar{u}) - \tau(\bar{v})\|_Y \bar{y}$, with $\bar{y} = (\tau(\bar{u}^0) - \tau(\bar{v}^0))/\|\tau(\bar{u}^0) - \tau(\bar{v}^0)\|_Y$.

With this, if we consider the bases $\{x, \bar{x}\}$ and $\{y, \bar{y}\}$, we have the equivalences

$$u - v = (\|u - v\|_X, 0) \text{ if and only if } \tau(u) - \tau(v) = (\|u - v\|_X, 0),$$

$$u - v = (0, \|u - v\|_X) \text{ if and only if } \tau(u) - \tau(v) = (0, \|u - v\|_X).$$

Now we have two options: if we can apply Lemma 2.10 then the remainder of the proof goes as the last part of the proof of Theorem 2.12. Otherwise, we can apply Lemma 2.9 to obtain that, in the bases $\{x, \bar{x}\}$ and $\{y, \bar{y}\}$, we have $\|\cdot\|_X = \|\cdot\|_Y$. It remains to show that τ is affine.

Now, we may suppose that $Y = X$ and we need to show that $\tau(a) - \tau(b) = a - b$ for every $a, b \in C_X$.

In what follows we suppose that C_X has no horizontal nor vertical segment. An analogous idea gives a proof for the other cases. Observe that if a and b belong to the same horizontal or vertical segment, then we have $\tau(a) - \tau(b) = a - b$. We will denote by W, S, E and N respectively the leftmost, the undermost, rightmost and uppermost points in C_X . We will also denote as SW, SE, NE and NW the closed arcs that join each pair of consecutive extremal points.

We will denote $\mathcal{E}_a = \{b \in C_X : \tau(a) - \tau(b) = a - b\}$. So, we need to show that $\mathcal{E}_a = C_X$ for some (every) $a \in C_X$.

For any $a \in C_X$, denote $a^0 = a$; $a^1 \in C_X$ is the other point that lies in the same horizontal line as a^0 , $a^2 \in C_X$ lies in the same vertical line as a^1 and so on. Furthermore, let $a^{-1} \in C_X$ be the point that lies in the same vertical line as a^0 , $a^{-2} \in C_X$ is in the same horizontal line as a^{-1} . . . This *bi-infinite sequence* may hit some extremal point and, so to say, get stuck –but this changes nothing. Moreover, if C_X has some vertical or horizontal symmetry, then $a^4 = a^0 = a^{-4}$. Still, everything goes fine. It is clear that, given $n, m \in \mathbb{Z}$, one has $\tau(a^n) - \tau(a^m) = a^n - a^m$. The continuity of the isometry is enough to ensure

$$\tau(\lim(a^{n_k})) - \tau(\lim(a^{m_k})) = \lim(a^{n_k}) - \lim(a^{m_k})$$

for any convergent subsequences $(a^{n_k}), (a^{m_k})$. So, $\tau(c^1) - \tau(c^2) = c^1 - c^2$ whenever $c^1, c^2 \in \overline{\{a^n : n \in \mathbb{Z}\}}$, i.e., $\{a^n : n \in \mathbb{Z}\} \subset \mathcal{E}_a$.

To end the proof we need three more facts.

Claim 2. For $a, b \in C_X$, if there are $i_1, i_2, i_3, i_4, j_1, j_2, j_3, j_4 \in \mathbb{Z}$ such that each $a^{i_k} - b^{j_k}$ lies in the k -th quadrant, then $b \in \mathcal{E}_a$ or, equivalently, $\mathcal{E}_b = \mathcal{E}_a$.

Proof. We may suppose, after composition with some translation, that $\tau(a) = a$. If $\tau(a) - \tau(b) \neq a - b$, then there is some $v \neq 0$ such that $\tau(b^n) = b^n + v$ for every $n \in \mathbb{Z}$. As $\|\cdot\|_X$ is strictly convex, there is a half-plane H such that $\|u - v\|_X > \|u\|_X$ whenever $u \in H$. Namely, with $w = {}^\perp v$, $H = \{\alpha v + \beta w : \alpha \in]-\infty, 0], \beta \in \mathbb{R}\}$. In particular, there is a whole quadrant included in H , so for some $k \in \{1, 2, 3, 4\}$ one has

$$\|\tau(a^{i_k}) - \tau(b^{j_k})\|_X = \|a^{i_k} - b^{j_k} - v\|_X > \|a^{i_k} - b^{j_k}\|_X,$$

a contradiction with the fact that τ is an isometry. \square

Claim 3. For every $a \in C_X$, the intersection of $\{a^n : n \in \mathbb{Z}\}$ with every arc SW, SE, NE and NW is nonempty.

Proof. Let us see that $SW \cap \{a^n : n \in \mathbb{Z}\} \neq \emptyset$, the other cases follow by symmetry. As C_X is convex, some point $b = (b_1, b_2)$ belongs to the arc SW if and only if there is no point $c = (c_1, c_2) \in C_X$ such that $c_1 \leq b_1, c_2 \leq b_2$ and $c \neq b$. Given $a \in C_X \setminus SW$, either a^1 or a^{-1} –maybe both– has a coordinate that is smaller than that of a , say $a_1^1 < a_1$. If $a^1 \notin SW$, then $a_2^2 < a_2$ and so on. If there is no n such that $a^n \in SW$, then the sequence has an accumulation point, but this accumulation point must be the limit of the sequence because the sequence is bounded and nonincreasing in both coordinates. So, we are in the conditions of Lemma 2.10, a contradiction. \square

Claim 4. For $S = (S_1, S_2)$ –the undermost point of C_X – we have either $SW \subset \mathcal{E}_S$ or $SE \subset \mathcal{E}_S$.

Proof. Let $b = (S_1, b_2) \in C_X$ be the other point with the same first coordinate as S . It is clear that $b \in \mathcal{E}_S$. Given $c^1 = (c_1^1, c_2^1) \in C_X$ with $c_2^1 < b_2$, take $c^2 = (c_1^1, c_2^2) \in C_X$ with, say, $c_1^1 < c_1^2$. We have $c^2 \in \mathcal{E}_{c^1}$ and moreover $b - c^1, b - c^2, S - c^2, S - c^1$ are, respectively in the first, second, third and fourth quadrants. Claim 2 implies that $c^1, c^2 \in \mathcal{E}_S$ so, for every $c = (c_1, c_2) \in C_X$ such that $c_2 \leq b_2$ one has $c \in \mathcal{E}_S$. As C_X encloses a convex region, $E_1 < b_1 < W_1$ implies $b_2 \geq \min\{E_2, W_2\}$. Now we may suppose $E_2 \leq b_2$, that implies $u_2 \leq b_2$ for every $(u_1, u_2) \in SE$, so $SE \subset \mathcal{E}_S$. \square

Now we just need to use Claims 3 and 4 to see that $\mathcal{E}_S = C_X$, so we have finished the proof. \square

Remark 2.15. After Theorem 2.12, [2, Theorem 1.5] and [4, Corollary 3.8], the only possibility for the existence of a nonlinear isometry between two-dimensional spheres is that both of them are strictly convex and one of the following holds:

- Both spheres are C^1 -differentiable and at least one of them is not absolutely smooth.
- None of the spheres is piecewise differentiable, i.e., there are infinitely many points of non-differentiability in each sphere.
- Both spheres have exactly two points of non-differentiability, say, x and $-x$.

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