Every non-smooth 2-dimensional Banach space has the Mazur–Ulam property

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Abstract

A Banach space X has *the Mazur–Ulam property* if any isometry from the unit sphere of X onto the unit sphere of any other Banach space Y extends to a linear isometry of the Banach spaces X,Y. A Banach space X is called *smooth* if the unit ball has a unique supporting functional at each point of the unit sphere. We prove that each non-smooth 2-dimensional Banach space has the Mazur–Ulam property.

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1. Introduction

By the classical result of Mazur and Ulam [16], every bijective isometry between Banach spaces is affine. This result essentially asserts that the metric structure of a Banach space determines its linear structure. In [14] Mankiewicz proved that every bijective isometry $f: B_X \to B_Y$ between the unit balls of two Banach spaces X,Y extends to a linear isometry of the Banach spaces. In [20] Tingley asked if the unit balls in this result of Mankiewicz can be replaced by the unit spheres. More precisely, he posed the following (still open) problem.

Problem 1.1 (Tingley, 1987). *Let $f: S_X \to S_Y$ be a bijective isometry between the unit spheres of two Banach spaces X,Y. Can f be extended to a linear isometry between the Banach spaces X,Y?*

Here for a Banach space $(X, \|\cdot\|)$ by

$$B_X = \{x \in X : \|x\| \le 1\} \quad \text{and} \quad S_X = \{x \in X : \|x\| = 1\}$$

we denote the unit ball and unit sphere of X, respectively.

Tingley's Problem 1.1 can be equivalently reformulated in terms of the Mazur–Ulam property, introduced by Cheng and Dong [6] and widely used in the literature devoted to Tingley's problem, see [3], [7], [8], [9], [11], [12], [17], [18], [19], [21].

Definition 1.2. A Banach space X is defined to have the *Mazur–Ulam property* if every isometry $f: S_X \to S_Y$ of S_X onto the unit sphere S_Y of an arbitrary Banach space Y extends to a linear isometry of the Banach spaces X,Y.

In fact, Tingley's Problem 1.1 asks whether every Banach space has the Mazur–Ulam property. There are many results on the Mazur–Ulam property in some special Banach spaces like $C(K)$, $c_0(\Gamma)$, $\ell_p(\Gamma)$, $L_p(\mu)$, see the survey [18]. By a result of Kadets and Martín [13], every polyhedral finite-dimensional Banach space has the Mazur–Ulam property.

For 2-dimensional Banach spaces this result of Kadets and Martín was improved by Cabello Sánchez who proved the following theorem in [4].

Theorem 1.3. *A 2-dimensional Banach space has the Mazur–Ulam property if is not strictly convex.*

Let us recall that a Banach space X is *strictly convex* if each convex subset of the unit sphere S_X contains at most one point.

A Banach space X is *smooth* if for every point $x \in S_X$ there exists a unique linear continuous functional $x^* : X \to \mathbb{R}$ such that $x^*(x) = 1 = \|x^*\|$. Geometrically this means that the unit ball B_X has a unique supporting hyperplane at x.

It is well-known [10, 7.23] that a reflexive Banach space X is strictly convex if and only if its dual Banach space X^* is smooth.

The main result of this paper is the following theorem, a kind of a dual version of Theorem 1.3.

Theorem 1.4. *Each non-smooth 2-dimensional Banach space has the Mazur–Ulam property.*

This theorem follows from Propositions 4.2 and 4.6, proved in Section 4. For piecewise C^1-smooth Banach spaces with more than two non-smooth points, Theorem 1.4 was proved by the second author in [5, Theorem 2.12]. In fact, many steps of the proof of Theorem 1.4 follow the lines of the proof of Theorem 2.12 in [5].

Remark 1.5. Theorems 1.4 and Proposition 4.6 (on the Mazur–Ulam property of 2-dimensional Banach spaces whose sphere contains two linearly independent special directions) are essential ingredients in the main result of the paper [2] answering the Tingley's Problem in the class of 2-dimensional Banach spaces.

In the proof of Theorem 1.4 we shall need the following helpful fact, proved by Tingley in [20].

Theorem 1.6. *If $f : S_X \to S_Y$ is a bijective isometry between unit spheres of finite-dimensional Banach spaces, then $f(-x) = -f(x)$ for all $x \in S_X$.*

2. The natural parameterization of the unit sphere of a 2-dimensional Banach space

By a 2-*based Banach space* we understand any 2-dimensional Banach space X endowed with a basis $\mathbf{e}_1, \mathbf{e}_2$.

Let X be a 2-based Banach space and $\mathbf{e}_1, \mathbf{e}_2$ be the basis of X.

The polar parameterization of the unit sphere S_X is the map

$$\mathbf{p} : \mathbb{R} \to S_X, \quad \mathbf{p} : t \mapsto \frac{\mathbf{e}^{it}}{\|\mathbf{e}^{it}\|}, \quad \text{where} \quad \mathbf{e}^{it} = \cos(t)\mathbf{e}_1 + \sin(t)\mathbf{e}_2.$$

The following properties of the polar parameterization were established in [1, §4].

Lemma 2.1. *The polar parameterization $\mathbf{p} : \mathbb{R} \to S_X$ has the following properties:*

1. $\mathbf{p}(t + \pi) = -\mathbf{p}(t)$ *for every* $t \in \mathbb{R}$*;*

2. *the function* $\mathbf{p}$ *has one-sided derivatives*

$$\mathbf{p}'_-(t) = \lim_{\varepsilon \to -0} \frac{\mathbf{p}(t+\varepsilon) - \mathbf{p}(t)}{\varepsilon} \quad \textit{and} \quad \mathbf{p}'_+(t) = \lim_{\varepsilon \to +0} \frac{\mathbf{p}(t+\varepsilon) - \mathbf{p}(t)}{\varepsilon}$$

at each point $t \in \mathbb{R}$*;*

3. *the set* $\{t \in \mathbb{R} : \mathbf{p}'_-(t) \neq \mathbf{p}'_+(t)\}$ *is at most countable.*
4. $\frac{c}{C} \cdot |\sin(\varepsilon)| \leq \|\mathbf{p}(t+\varepsilon) - \mathbf{p}(t)\| \leq \frac{2C^2}{c^2} \cdot |\varepsilon|$ *for any* $t, \varepsilon \in \mathbb{R}$*;*
5. $\frac{c}{C} \leq \min\{\|\mathbf{p}'_-(t)\|, \|\mathbf{p}'_+(t)\|\} \leq \max\{\|\mathbf{p}'_-(t)\|, \|\mathbf{p}'_+(t)\|\} \leq \frac{2C^2}{c^2}$ *for every* $t \in \mathbb{R}$*,*

where $c = \min\{\|\mathbf{e}^{it}\| : t \in \mathbb{R}\}$ *and* $C = \max\{\|\mathbf{e}^{it}\| : t \in \mathbb{R}\}$*.*

Lemma 2.1 implies that the function

$$\mathbf{s} : \mathbb{R} \to \mathbb{R}, \quad \mathbf{s} : t \mapsto \int_0^t \|\mathbf{p}'_-(u)\|\, du = \int_0^t \|\mathbf{p}'_+(u)\| du$$

is continuous and strictly increasing.

For $t \in [0, \pi]$ the value $\mathbf{s}(t)$ can be thought as the length of the curve on the sphere S_X between the points $\mathbf{p}(0)$ and $\mathbf{p}(t)$ in the Banach space X. The number

$$L = \mathbf{s}(\pi) = \int_0^\pi \|\mathbf{p}'_-(t)\| dt = \int_0^\pi \|\mathbf{p}'_+(t)\| dt$$

is called the *half-length* of the sphere S_X in X.

The image

$$\breve{S}_X = \{\mathbf{p}(t) : 0 \le t \le \pi\}$$

is called the *upper half-sphere* of the 2-based Banach space X.

Since the function $\mathbf{s}$ is continuous and increasing, there exists a unique continuous increasing function $\mathbf{t} : \mathbb{R} \to \mathbb{R}$ such that $\mathbf{s} \circ \mathbf{t}$ is the identity map of $\mathbb{R}$.

The function

$$\mathbf{r} : \mathbb{R} \to S_X,\ \mathbf{r} : s \mapsto \mathbf{p} \circ \mathbf{t}(s),$$

is called *the natural parameterization* of the sphere S_X.

The following properties of the natural parameterization were established in [1, §5].

Lemma 2.2. *The natural parameterization $\mathbf{r} : \mathbb{R} \to S_X$ of S_X has the following properties:*

1. $\mathbf{r}(s + L) = -\mathbf{r}(s)$ *for every* $s \in \mathbb{R}$*;*
2. *the function* $\mathbf{r}$ *has one-sided derivatives*

$$\mathbf{r}'_-(s) = \lim_{\varepsilon \to -0} \frac{\mathbf{r}(s+\varepsilon) - \mathbf{r}(s)}{\varepsilon} \quad \textit{and} \quad \mathbf{r}'_+(s) = \lim_{\varepsilon \to +0} \frac{\mathbf{r}(s+\varepsilon) - \mathbf{r}(s)}{\varepsilon}$$

   at each point $s \in \mathbb{R}$*;*
3. *the set* $\{s \in \mathbb{R} : \mathbf{r}'_-(s) \neq \mathbf{r}'_+(s)\}$ *is at most countable;*
4. $\mathbf{r}$ *is non-expanding and has* $\|\mathbf{r}'_-(s)\| = \|\mathbf{r}'_+(s)\| = 1$ *for every* $s \in \mathbb{R}$*.*
5. *If* $\mathbf{r}$ *is differentiable on some open set* $U \subseteq \mathbb{R}$*, then* $\mathbf{r}$ *is continuously differentiable on* U*.*

The natural parametrization is closely related to the intrinsic metric on the half-sphere $\breve{S}_X$.

For two points $x, y \in \breve{S}_X$, the real number

$$\breve{d}(x,y) = \sup_{\varepsilon > 0} \inf \Big\{ \sum_{i=1}^{n} \|x_i - x_{i-1}\| : x_0, \dots, x_n \in \breve{S}_X,\ x_0 = x,\ x_n = y,\ \max_{1 \le i \le n} \|x_i - x_{i-1}\| < \varepsilon \Big\}$$

is called the *intrinsic distance* between the points x, y on the half-sphere $\breve{S}_X$. The following lemma can be proved by analogy with Lemma 3.1 of [1].

Lemma 2.3. *If $\mathbf{r}$ is continuously differentiable at each point $s \in (0, L)$, then the map $\mathbf{r}{\restriction}_{[0,L]}$ is an isometry of the interval $[0, L]$ onto the half-sphere $\breve{S}_X$ endowed with the intrinsic distance.*

If $\mathbf{r}$ is arbitrary, then we can prove a weaker statement.

Lemma 2.4. *For any $s \in \mathbb{R}$ and small ε we have*

$$\|\mathbf{r}(s+\varepsilon) - \mathbf{r}(s)\| = (1 + o(1)) \cdot |\varepsilon|.$$

Proof. If $\varepsilon > 0$, then $\mathbf{r}(s+\varepsilon) = \mathbf{r}(s) + \mathbf{r}'_+(s)\varepsilon + o(\varepsilon)$ and hence

$$\|\mathbf{r}(s+\varepsilon) - \mathbf{r}(s)\| = \|\mathbf{r}'_+(s) + o(1)\| \cdot |\varepsilon| = (\|\mathbf{r}'_+(s)\| + o(1)) \cdot |\varepsilon| = (1 + o(1)) \cdot |\varepsilon|.$$

By analogy we can show that $\|\mathbf{r}(s+\varepsilon) - \mathbf{r}(s)\| = (1 + o(1)) \cdot |\varepsilon|$ for small $\varepsilon < 0$. $\square$

For every parameter $s \in \mathbb{R}$ let

$$\mathbf{r}'_\pm(s) = \tfrac{1}{2}\big(\mathbf{r}'_+(s) + \mathbf{r}'_-(s)\big).$$

It is easy to see that the vectors $\mathbf{r}(s)$ and $\mathbf{r}'_\pm(s)$ are linearly independent. Then

$$\tfrac{1}{2}(\mathbf{r}'_+(s) - \mathbf{r}'_-(s)) = \jmath(s) \cdot \mathbf{r}(s) + \ddot{\jmath}(s) \cdot \mathbf{r}'_\pm(s)$$

for some real numbers $\jmath(s)$ and $\ddot{\jmath}(s)$, called the *radial* and *tangential jumps* of the derivative $\mathbf{r}'$ at s, respectively.

It follows that

$$\mathbf{r}'_+(s) = \jmath(s) \cdot \mathbf{r}(s) + (1 + \ddot{\jmath}(s)) \cdot \mathbf{r}'_\pm(s) \quad \text{and} \quad \mathbf{r}'_-(s) = -\jmath(s) \cdot \mathbf{r}(s) + (1 - \ddot{\jmath}(s)) \cdot \mathbf{r}'_\pm(s).$$

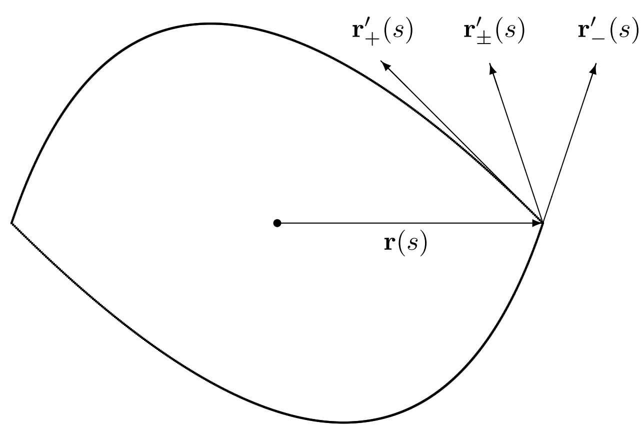
Lemma 2.5. 1. *$|\ddot{\jmath}(s)| < 1$.*

2. *$\mathbf{r}'_-(s) = \mathbf{r}'_+(s)$ iff $\jmath(s) = 0$.*
3. *$\jmath(s) \le 0$.*

Proof. 1. It is easy to see that the bases $\big(\mathbf{r}(s), \mathbf{r}'_-(s)\big)$, $\big(\mathbf{r}(s), \mathbf{r}'_+(s)\big)$, $\big(\mathbf{r}(s), \mathbf{r}'_\pm(s)\big)$ have the same orientation, which implies that for the basis $\big(\mathbf{r}(s), \mathbf{r}'_\pm(s)\big)$ the $\mathbf{r}'_\pm(s)$-coordinates $1 + \ddot{\jmath}(s)$ and $1 - \ddot{\jmath}(s)$ of the vectors $\mathbf{r}'_+(s)$ and $\mathbf{r}'_-(s)$ are positive and hence $|\ddot{\jmath}(s)| < 1$.

2. If $\mathbf{r}'_-(s) = \mathbf{r}'_+(s)$, then $0 = \frac{1}{2}(\mathbf{r}'_+(s) - \mathbf{r}'_-(s)) = \jmath(s) \cdot \mathbf{r}(s) + \ddot{\jmath}(s) \cdot \mathbf{r}'_\pm(s)$ and hence $\jmath(s) = 0$. If $\jmath(s) = 0$, then the vectors $\mathbf{r}'_+(s) = (1 + \ddot{\jmath}(s))\mathbf{r}'_\pm(s)$ and $\mathbf{r}'_-(s) = (1 - \ddot{\jmath}(s))\mathbf{r}'_\pm(s)$ are collinear and hence they are equal because they have the same norm and the bases $(\mathbf{r}(s), \mathbf{r}'_-(s))$ and $(\mathbf{r}(s), \mathbf{r}'_+(s))$ have the same orientation.

3. The inequality $\jmath(s) \le 0$ follows from the convexity of the ball B_X, see the following picture.



□

In the following lemma (which can be considered as a quantitative version of Proposition 2.5 in [5]) we use the standard function $\mathrm{sign} : \mathbb{R} \to \{-1, 0, 1\}$ defined by the formula

$$\mathrm{sign}(\varepsilon) = \begin{cases} 1 & \text{if } \varepsilon > 0; \\ 0 & \text{if } \varepsilon = 0; \\ -1 & \text{if } \varepsilon < 0. \end{cases}$$

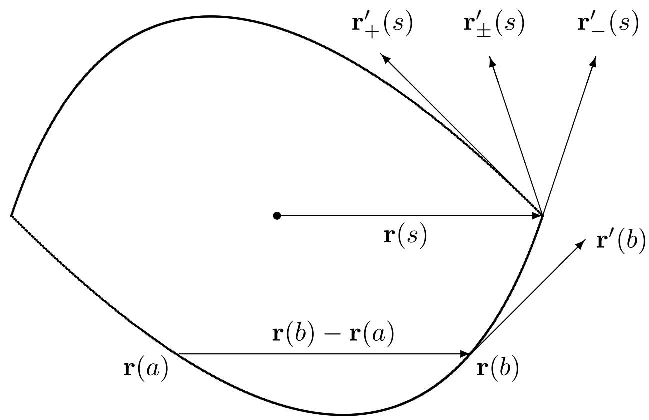
Lemma 2.6. *Let $a, b, s \in \mathbb{R}$ be such that $\mathbf{r}$ is differentiable at b and*

$$0 \neq \mathbf{r}(b) - \mathbf{r}(a) = \|\mathbf{r}(b) - \mathbf{r}(a)\| \cdot \mathbf{r}(s).$$

Let $\mathbf{r}'(b) = x \cdot \mathbf{r}(s) + y \cdot \mathbf{r}'_{\pm}(s)$ for some $x, y \in \mathbb{R}$ with $y \neq 0$. Then $y > 0$ and for any small ε we have the asymptotic formula

$$\|\mathbf{r}(b+\varepsilon) - \mathbf{r}(a)\| = \|\mathbf{r}(b) - \mathbf{r}(a)\| + x \cdot \varepsilon - \frac{\operatorname{sign}(\varepsilon) \cdot \jmath(s) \cdot y}{1 + \operatorname{sign}(\varepsilon) \cdot \ddot{\jmath}(s)} \cdot \varepsilon + o(\varepsilon).$$

Proof. The positivity of y follows from $0 \neq \mathbf{r}(b) - \mathbf{r}(a) = \|\mathbf{r}(b) - \mathbf{r}(a)\| \cdot \mathbf{r}(s)$ and the same orientation of the bases $(\mathbf{r}(s), \mathbf{r}'_{\pm}(s))$ and $(\mathbf{r}(s), \mathbf{r}'(b))$, see the following picture.



Since $\mathbf{r}(b) - \mathbf{r}(a) = \|\mathbf{r}(b) - \mathbf{r}(a)\| \cdot \mathbf{r}(s)$, for a small ε there exists a small δ such that

$$\mathbf{r}(b+\varepsilon) - \mathbf{r}(a) = \|\mathbf{r}(b+\varepsilon) - \mathbf{r}(a)\| \cdot \mathbf{r}(s+\delta).$$

By the differentiability of $\mathbf{r}$ at b, we obtain

$$\begin{aligned}\mathbf{r}(b+\varepsilon) - \mathbf{r}(a) = \mathbf{r}(b) + \mathbf{r}'(b)\varepsilon + o(\varepsilon) - \mathbf{r}(a) = \|\mathbf{r}(b) - \mathbf{r}(a)\| \cdot \mathbf{r}(s) + (x \cdot \mathbf{r}(s) + y \cdot \mathbf{r}'_{\pm}(s)) \cdot \varepsilon + o(\varepsilon) = \\ = \big(\|\mathbf{r}(b) - \mathbf{r}(a)\| + x\varepsilon + o(\varepsilon)\big) \cdot \mathbf{r}(s) + (y\varepsilon + o(\varepsilon)) \cdot \mathbf{r}'_{\pm}(s).\end{aligned}$$

If $\delta > 0$, then

$$\mathbf{r}(s+\delta) = \mathbf{r}(s) + \mathbf{r}'_+(s)\delta + o(\delta) = (1 + (\jmath(s) + o(1))\delta) \cdot \mathbf{r}(s) + (1 + \ddot{\jmath}(s) + o(1))\delta \cdot \mathbf{r}'_{\pm}(s).$$

If $\delta < 0$, then

$$\mathbf{r}(s+\delta) = \mathbf{r}(s) + \mathbf{r}'_-(s)\delta + o(\delta) = (1 - (\jmath(s) + o(1))\delta) \cdot \mathbf{r}(s) + (1 - \ddot{\jmath}(s) + o(1))\delta \cdot \mathbf{r}'_{\pm}(s).$$

In both cases we obtain

$$\mathbf{r}(s+\delta) = (1 + (\operatorname{sign}(\delta) \cdot \jmath(s) + o(1))\delta) \cdot \mathbf{r}(s) + (1 + \operatorname{sign}(\delta) \cdot \ddot{\jmath}(s) + o(1))\delta \cdot \mathbf{r}'_{\pm}(s).$$

It follows that

$$
\begin{aligned}
&(\|\mathbf{r}(b)-\mathbf{r}(a)\|+x\varepsilon+o(\varepsilon))\cdot\mathbf{r}(s)+(y\varepsilon+o(\varepsilon))\cdot\mathbf{r}'_{\pm}(s)=\mathbf{r}(b+\varepsilon)-\mathbf{r}(a)=\\
&=\|\mathbf{r}(b+\varepsilon)-\mathbf{r}(a)\|\cdot\mathbf{r}(s+\delta)=\\
&=\|\mathbf{r}(b+\varepsilon)-\mathbf{r}(a)\|\cdot\big(1+(\mathrm{sign}(\delta)\cdot\dot{\jmath}(s)+o(1))\delta)\cdot\mathbf{r}(s)+(1+\mathrm{sign}(\delta)\cdot\ddot{\jmath}(s)+o(1))\delta\cdot\mathbf{r}'_{\pm}(s)\big).
\end{aligned}
$$

Writing this equation in coordinates, we obtain two equations:

$$
\|\mathbf{r}(b+\varepsilon)-\mathbf{r}(a)\|\cdot\big(1+(\mathrm{sign}(\delta)\cdot\dot{\jmath}(s)+o(1))\delta\big)=\|\mathbf{r}(b)-\mathbf{r}(a)\|+x\cdot\varepsilon+o(\varepsilon) \tag{1}
$$

and

$$
\|\mathbf{r}(b+\varepsilon)-\mathbf{r}(a)\|\cdot(1+\mathrm{sign}(\delta)\cdot\ddot{\jmath}(s)+o(1))\cdot\delta=y\cdot\varepsilon+o(\varepsilon). \tag{2}
$$

The equation (2) implies

$$
\delta=\frac{(y+o(1))\varepsilon}{\|\mathbf{r}(b+\varepsilon)-\mathbf{r}(a)\|(1+\mathrm{sign}(\delta)\cdot\ddot{\jmath}(s))}=\frac{(y+o(1))\varepsilon}{\|\mathbf{r}(b)-\mathbf{r}(a)\|\cdot(1+\mathrm{sign}(\delta)\cdot\ddot{\jmath}(s))}.
$$

Since $y>0$ and $|\ddot{\jmath}(s)|<1$, this implies

$$
\mathrm{sign}(\delta)=\mathrm{sign}(\varepsilon)+o(1).
$$

After substitution of δ into the equation (1), we obtain

$$
\begin{aligned}
&\|\mathbf{r}(b+\varepsilon)-\mathbf{r}(a)\|=\frac{\|\mathbf{r}(b)-\mathbf{r}(a)\|+x\cdot\varepsilon+o(\varepsilon)}{1+(\mathrm{sign}(\delta)\cdot\dot{\jmath}(s)+o(1))\delta}=\\
&=\big(\|\mathbf{r}(b)-\mathbf{r}(a)\|+x\cdot\varepsilon+o(\varepsilon)\big)\cdot\big(1-(\mathrm{sign}(\delta)\cdot\dot{\jmath}(s)+o(1))\delta\big)=\\
&=\big(\|\mathbf{r}(b)-\mathbf{r}(a)\|+x\cdot\varepsilon+o(\varepsilon)\big)\cdot\Big(1-\frac{(\mathrm{sign}(\varepsilon)\cdot\dot{\jmath}(s)+o(1))(y+o(1))\cdot\varepsilon}{\|\mathbf{r}(b)-\mathbf{r}(a)\|(1+\mathrm{sign}(\varepsilon)\cdot\ddot{\jmath}(s))}\Big)=\\
&=\|\mathbf{r}(b)-\mathbf{r}(a)\|+x\cdot\varepsilon-\frac{\mathrm{sign}(\varepsilon)\cdot\dot{\jmath}(s)\cdot y}{1+\mathrm{sign}(\varepsilon)\cdot\ddot{\jmath}(s)}\cdot\varepsilon+o(\varepsilon).
\end{aligned}
$$

□

Lemmas 2.6 and 2.5 imply the following corollary.

Corollary 2.7. *Let $a,b,s\in\mathbb{R}$ be real numbers such that the map $\mathbf{r}$ is differentiable at b, $\mathbf{r}'(b)\notin\mathbb{R}\cdot\mathbf{r}(s)$ and $0\ne\mathbf{r}(b)-\mathbf{r}(a)=\|\mathbf{r}(b)-\mathbf{r}(a)\|\cdot\mathbf{r}(s)$. The map $\mathbf{r}$ is differentiable at the point s if and only if the map*

$$
\nu:\mathbb{R}\to\mathbb{R},\quad \nu:\varepsilon\mapsto\|\mathbf{r}(b+\varepsilon)-\mathbf{r}(a)\|,
$$

is differentiable at zero.

3. Recognizing smooth points on the unit sphere

A point x of the unit sphere of a Banach space X is defined to be *smooth* if there exists a unique linear continuous functional $x^*:X\to\mathbb{R}$ such that $x^*(x)=1=\|x^*\|$.

The following result of Cabello Sánchez [5, 2.1] shows that smooth points on the unit sphere can be recognized by measurements of distances on the sphere.

Lemma 3.1. *A point $p \in S_X$ is non-smooth if and only if there exists positive real numbers δ and ε_0 such that for every positive $\varepsilon < \varepsilon_0$ there are points $x, y \in S_X$ such that*

$$\max\{\|x-p\|, \|y+p\|\} < \varepsilon \quad \text{and} \quad \|x-y\| < 2 - \delta\varepsilon.$$

Another smoothness criterion is given by the following lemma.

Lemma 3.2. *Let X be a strictly convex 2-dimensional Banach space and $a, b, c \in S_X$ be points such that $\mathbf{r}$ is differentiable at b and $0 \neq b-a = \|b-a\| \cdot c$. The point c is not smooth if and only if there exist positive real numbers δ and ε_0 such that for every positive $\varepsilon < \varepsilon_0$ there are two distinct points $x, y \in S_X$ such that $\|x-b\| = \varepsilon = \|y-b\|$ and $\|x-a\| + \|y-a\| > 2 \cdot \|b-a\| + \delta \cdot \varepsilon$.*

Proof. Assume that the point b is smooth and fix any basis $\mathbf{e}_1, \mathbf{e}_2$ for the space X such that $\mathbf{e}_1 = c$. Let $\mathbf{r} : \mathbb{R} \to S_X$ be the natural parameterization of the 2-based Banach space $(X, \mathbf{e}_1, \mathbf{e}_2)$. Then $\mathbf{r}(0) = \mathbf{e}_1 = c$. Find real numbers α, β such that $a = \mathbf{r}(\alpha)$ and $b = \mathbf{r}(\beta)$. Since the point b is smooth, the function $\mathbf{r}$ is differentiable at β. Write the derivative $\mathbf{r}'(\beta)$ as $x \cdot \mathbf{r}(0) + y \cdot \mathbf{r}'_\pm(0)$ for some real numbers x, y. Since X is strictly convex and $0 \neq b - a = \|b-a\| \cdot c$, the vector $\mathbf{r}'(\beta)$ is not parallel to the vector c and hence $y \neq 0$. By Lemma 2.6, $y > 0$.

For every $\varepsilon \in [0, 2]$ let

$$\varepsilon_+ = \min\{s \in [0, +\infty) : \|\mathbf{r}(\beta+s) - \mathbf{r}(\beta)\| = \varepsilon\}$$

and

$$\varepsilon_- = \max\{s \in (-\infty, 0] : \|\mathbf{r}(\beta+s) - \mathbf{r}(\beta)\| = \varepsilon\}.$$

By Lemma 2.4,

$$\varepsilon_+ = \varepsilon + o(\varepsilon) = \varepsilon + o(\varepsilon_+) \quad \text{and} \quad \varepsilon_- = -\varepsilon + o(\varepsilon) = -\varepsilon + o(\varepsilon_-).$$

If the point c is not smooth, then $\mathbf{r}'_-(0) \neq \mathbf{r}'_+(0)$ and $\jmath(0) < 0$ by Lemma 2.5. Then the number

$$\delta = -\frac{\jmath(0) \cdot y}{1 - \ddot{\jmath}(0)^2}$$

is positive.

By Lemma 2.6, for a small $\varepsilon > 0$ we have

$$\begin{aligned}
&\|\mathbf{r}(\beta+\varepsilon_+) - \mathbf{r}(\alpha)\| + \|\mathbf{r}(\beta+\varepsilon_-) - \mathbf{r}(\alpha)\| = \\
&= \|b-a\| + x \cdot \varepsilon_+ - \frac{\jmath(0) \cdot y \cdot |\varepsilon_+|}{1 + \ddot{\jmath}(0)} + o(\varepsilon) + \|b-a\| + x \cdot \varepsilon_- - \frac{\jmath(0) \cdot y \cdot |\varepsilon_-|}{1 - \ddot{\jmath}(0)} + o(\varepsilon) = \\
&= 2 \cdot \|b-a\| + x(\varepsilon_+ + \varepsilon_-) - \frac{\jmath(0) \cdot y \cdot |\varepsilon|(1+o(1))}{1 + \ddot{\jmath}(0)} - \frac{\jmath(0) \cdot y \cdot |\varepsilon|(1+o(1))}{1 - \ddot{\jmath}(0)} + o(\varepsilon) = \\
&= 2 \cdot \|b-a\| + o(\varepsilon) + 2\delta|\varepsilon| + o(\varepsilon) = 2 \cdot \|b-a\| + (2\delta + o(1))|\varepsilon|.
\end{aligned}$$

and hence there exists $\varepsilon_0 > 0$ such that for any positive $\varepsilon < \varepsilon_0$ and the points $x = \mathbf{r}(\beta + \varepsilon_+)$ and $y = \mathbf{r}(\beta + \varepsilon_-)$ we have

$$\|x-a\| + \|y-a\| = \|\mathbf{r}(\beta+\varepsilon_+) - \mathbf{r}(\alpha)\| + \|\mathbf{r}(\beta+\varepsilon_-) - \mathbf{r}(\alpha)\| = 2 \cdot \|b-a\| + (2\delta + o(1))|\varepsilon| > 2 \cdot \|b-a\| + \delta|\varepsilon|.$$

The choice of ε_+ and ε_- guarantees that $\|x-a\| = \varepsilon = \|y-a\|$. This completes the proof of the "only if" part of the lemma.

To prove the "if" part, assume that there exist positive δ and ε_0 such that for any positive $\varepsilon < \varepsilon_0$ there exist distinct points $x, y \in S_X$ such that $\|x-b\| = \varepsilon = \|y-b\|$ and

$$\|x-a\| + \|y-a\| > 2\|b-a\| + \delta \cdot \varepsilon.$$

By Monotonicity Lemma [15, §3.5], we can assume that ε_0 is so small that for any positive $\varepsilon < \varepsilon_0$ the set $\{x \in S_X : \|x-b\| = \varepsilon\}$ coincides with the doubleton $\{\mathbf{r}(\beta+\varepsilon_+), \mathbf{r}(\beta+\varepsilon_-)\}$. Assuming that the point c is smooth, we conclude that $\mathbf{r}'_-(0) = \mathbf{r}'_+(0)$ and hence $\jmath(0) = 0 = \ddot{\jmath}(0)$. By Lemma 2.6,

$$\|\mathbf{r}(\beta+\varepsilon_+) - a\| + \|\mathbf{r}(\beta+\varepsilon_-) - a\| = 2 \cdot \|b-a\| + x \cdot (\varepsilon_+ + \varepsilon_-) + o(\varepsilon) = 2 \cdot \|b-a\| + o(\varepsilon).$$

Replacing ε_0 by a smaller positive number, we can assume that

$$\|\mathbf{r}(\beta+\varepsilon_+) - a\| + \|\mathbf{r}(\beta+\varepsilon_-) - a\| = 2 \cdot \|b-a\| + o(\varepsilon) < 2 \cdot \|b-a\| + \delta \cdot |\varepsilon|$$

for all positive $\varepsilon < \varepsilon_0$. But this contradicts our assumption. This contradiction shows that the point c is not smooth. □

4. Special directions on the unit sphere

Definition 4.1. Let X be a Banach space. A point $x \in S_X$ is called *special* if for any bijective isometry $f : S_X \to S_Y$ to the unit sphere of a Banach space Y and any points $y, z \in S_X$ with $y - z = \|y-z\| \cdot x$ we have

$$f(y) - f(z) = \|f(y) - f(z)\| \cdot f(x) = \|y-z\| \cdot f(x).$$

Proposition 4.2. *Every non-smooth point of the unit sphere of a strictly convex 2-dimensional Banach space is special.*

Proof. Let $c \in S_X$ be a non-smooth point of the sphere S_X. Since the Banach space X is strictly convex, the set $c^\perp = \{x \in S_X : \{x\} = S_X \cap (x + \mathbb{R}c)\}$ contains exactly two points. The complement $S_X \setminus c^\perp$ has exactly two connected components. Let A be the connected component of $S_X \setminus c^\perp$ containing the point c. It follows that $S_X = (-A) \cup c^\perp \cup A$.

Let $\theta : A \to -A$ be the function assigning to each point $x \in A$ the unique point $y \in -A$ such that $x - y = \|x-y\| \cdot c$. It is clear that the function θ is injective and $\theta(c) = -c$.

Now take any bijective isometry $f : S_X \to S_Y$ of S_X onto the unit sphere of a Banach space Y and consider the continuous function

$$g : A \to S_Y, \ g : x \mapsto \frac{f(x) - f(\theta(x))}{\|f(x) - f(\theta(x))\|} = \frac{f(x) - f(\theta(x))}{\|x - \theta(x)\|}.$$

Tingley's Theorem 1.6 implies $g(c) = \frac{f(c)-f(-c)}{\|f(c)-f(-c)\|} = f(c)$. To prove that the point c is special, it suffices to check that the function g is constant. To derive a contradiction, assume that the function g is not constant. Then the image $g(A)$ of A is uncountable, being a path-connected set that contains more than one point.

Let Λ_X, Λ_Y be the sets of non-smooth points on the spheres S_X and S_Y, respectively. By Lemma 2.2, the sets Λ_X, Λ_Y are at most countable and so is the set $\Lambda_Y \cup g(\Lambda_X)$. Since the set $g(A)$ is uncountable, there exists a point $b \in A$ such that $g(b) \notin \Lambda_Y \cup g(A \cap \Lambda_X)$, which means that b is a smooth point of S_X and $g(b)$ is a smooth point of S_Y. Let $a = \theta(b) \in -A$. The definition of the map θ ensures that $b - a = \|b-a\| \cdot c$.

Since the point c is not smooth, we can apply Lemma 3.2 and find positive δ and ε_0 such that for every positive $\varepsilon < \varepsilon_0$ there exist two distinct points $x, y \in S_X$ such that $\|x - b\| = \varepsilon = \|y - b\|$ and $\|x - a\| + \|y - a\| > 2\|b - a\| + \delta\varepsilon$.

Since $f : S_X \to S_Y$ is an isometry and b is a smooth point of the sphere S_X, its image $f(b)$ is a smooth point of the sphere S_Y, according to Lemma 3.1. Observe that

$$0 \neq f(b) - f(a) = f(b) - f(\theta(b)) = \|b - \theta(b)\| \cdot g(b) = \|b - a\| \cdot g(b) = \|f(b) - f(a)\| \cdot g(b).$$

Since the point $g(b) \in S_Y$ is smooth, we can apply Lemma 3.2 and find a positive $\varepsilon < \varepsilon_0$ such that for any distinct points $u, v \in S_Y$ with $\|u - f(b)\| = \varepsilon = \|v - f(b)\|$ we have $\|u - f(a)\| + \|v - f(a)\| \not> 2 \cdot \|f(b) - f(a)\| + \delta\varepsilon$.

By the choice of δ and ε_0, there exist points $x, y \in S_X$ such that $\|x - b\| = \varepsilon = \|y - b\|$ and $\|x - b\| + \|y - b\| > 2 \cdot \|b - a\| + \delta\varepsilon$. Since $f : S_X \to S_Y$ is an isometry, for the points $u = f(x)$ and $v = f(y)$, we obtain

$$\|f(x) - f(b)\| = \|x - b\| = \varepsilon = \|y - b\| = \|f(y) - f(b)\|$$

and

$$\|f(x) - f(a)\| + \|f(y) - f(a)\| = \|x - a\| + \|y - a\| > 2 \cdot \|b - a\| + \delta\varepsilon = 2 \cdot \|f(b) - f(a)\| + \delta\varepsilon,$$

which contradicts the choice of ε. □

Definition 4.3. Let $x, y \in S_X$ be two points on the unit sphere of a 2-dimensional Banach space X. The pair (x, y) is called

- *singular* if $\{x, -x\} \cap \{y, -y\} \neq \emptyset$ or there exists a point $z \in S_X$ such that $S_X \cap (z + \mathbb{R}x) = \{z\} = S_X \cap (z + \mathbb{R}y)$;
- *regular* if it is not singular.

The following lemma is a variant of Lemma 2.9 in [5].

Lemma 4.4. *Let $u, v \in S_X$ be two points on the unit sphere of a strictly convex 2-dimensional Banach space X. If the pair (u, v) is regular, then for any point $w \in S_X$ there exist points $x, y, z \in S_X$ such that*

$$x \in z + \mathbb{R}u, \quad y \in z + \mathbb{R}v \quad \text{and} \quad 0 \neq x - y \in \mathbb{R}w.$$

Proof. By the strict convexity of X, for every $z \in S_X$ there exist unique points $\vec{u}(z), \vec{v}(z) \in S_X$ such that $\{z, \vec{u}(z)\} = S_X \cap (z + \mathbb{R}u)$ and $\{z, \vec{v}(z)\} = S_X \cap (z + \mathbb{R}v)$. It is easy to see that the function

$$\varphi : S_X \to X, \quad \varphi : z \mapsto \vec{u}(z) - \vec{v}(z),$$

is continuous. Since the pair (u, v) is regular, $\varphi(z) \neq 0$ for any $z \in S_X$. Then the function

$$\psi : S_X \to S_X, \quad \psi : z \mapsto \frac{\varphi(z)}{\|\varphi(z)\|} = \frac{\vec{u}(z) - \vec{v}(z)}{\|\vec{u}(z) - \vec{v}(z)\|}$$

is well-defined and continuous.

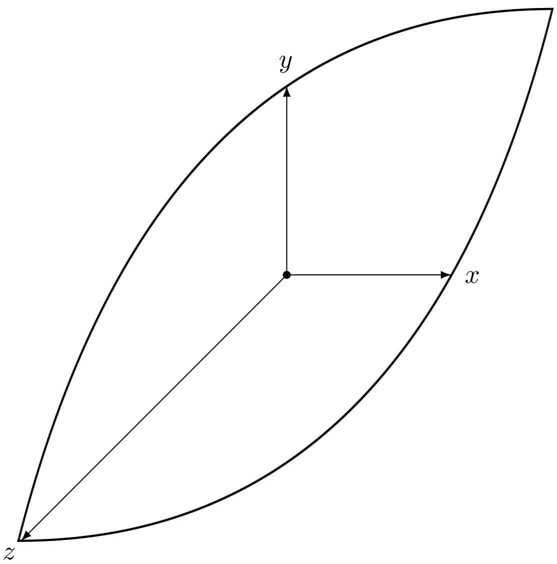
By the central symmetry of S_X, for every point $z \in S_X$ we have $\vec{u}(-z) = -\vec{u}(z)$ and $\vec{v}(-z) = -\vec{v}(z)$, which implies $\psi(-z) = -\psi(z)$. Therefore, the image $\psi(S_X)$ of S_X contains two opposite points. By the connectedness of $\psi(S_X)$, for any $w \in S_X$ there exists $z \in S_X$ such that

$$w = \psi(z) = \frac{\vec{u}(z) - \vec{v}(z)}{\|\vec{u}(z) - \vec{v}(z)\|}.$$

Then the points $x = \vec{u}(z)$, $y = \vec{v}(z)$ and z have the required properties. □

Lemma 4.5. *If the unit sphere of a strictly convex* 2*-dimensional Banach space X contains two linearly independent special points, then S_X contains two special points x, y such that the pair (x, y) is regular.*

Proof. Let $x, y \in S_X$ be two linearly independent special points on the unit sphere of X. If the pair (x, y) is regular, then we are done. So, assume that (x, y) is singular. Then there exists a point $z \in S_X$ such that $S_X \cap (z + \mathbb{R}x) = \{z\} = S_X \cap (z + \mathbb{R}y)$. It follows that z is a non-smooth point of the sphere S_X. By Proposition 4.2, the point z is special. Taking into account that the Banach space X is strictly convex, it can be shown that the pairs (x, z) and (y, z) are regular.



□

The following proposition generalizes Theorem 2.12 in [5] (and uses the same idea of the proof).

Proposition 4.6. *A* 2*-dimensional Banach space X has the Mazur–Ulam property if its sphere contains two linearly independent special points.*

Proof. If X is not strictly convex, then X has the Mazur–Ulam property by Theorem 1.3. So, we assume that X is strictly convex. Let $u, v \in S_X$ be two linearly independent special points on S_X. By Lemma 4.5, we can additionally assume that the pair (u, v) is regular.

Let $f : S_X \to S_Y$ be a bijective isometry of S_X onto the unit sphere S_Y of an arbitrary Banach space Y. Let $L : X \to Y$ be a unique linear operator such that $L(u) = f(u)$ and $L(v) = f(v)$. We claim that $L{\restriction}S_X = f$. Fix any point $w \in S_X$ and using Lemma 4.4, find points $x, y, z \in S_X$ such that $x = z + au$, $y = z + bv$ and $0 \neq x - y = cw$ for some real numbers a, b, c. Replacing x, y, z by $-x, -y, -z$, if necessary, we can assume that $c > 0$. Observe that $|a| = \|x - z\|$. If $a > 0$, then by the special property of u, the equality $x = z + au = z + \|x - z\| \cdot u$ implies $f(x) = f(z) + \|x - z\| \cdot f(u) = f(z) + a \cdot f(u)$. If $a < 0$, then the equality $x = z + au = z - \|x - z\| u$ implies $z = x + \|x - z\| \cdot u$. By the special property of u, we have $f(z) = f(x) + \|x - z\| \cdot f(u) = f(x) - a \cdot f(u)$ and hence $f(x) = f(z) + a \cdot f(u)$. In both cases we obtain $f(x) = f(z) + a \cdot f(u)$. By analogy we can show that $f(y) = f(z) + b \cdot f(v)$. Then

$$f(x) - f(y) = (f(z) + a \cdot f(u)) - (f(z) + b \cdot f(v)) = a \cdot f(u) - b \cdot f(v) = a \cdot L(u) - b \cdot L(v).$$

It follows from $x - y = c \cdot w = \|x - y\| \cdot w$ that

$$L(w) = \frac{L(x) - L(y)}{\|x - y\|} = \frac{L(z + au) - L(z + bv)}{\|x - y\|} = \frac{a{\cdot}L(u) - b{\cdot}L(v)}{\|x - y\|} = \frac{f(x) - f(y)}{\|x - y\|}$$

and finally

$$\|L(w)\| = \frac{\|f(x) - f(y)\|}{\|x - y\|} = 1.$$

Therefore, the linear operator $L : X \to Y$ is an isometry.

Consider the isometry $g = L^{-1} \circ f : S_X \to S_X$ and observe that $g(u) = u$ and $g(v) = v$. By Theorem 1.6, $g(-u) = -g(u) = -u$ and $g(-v) = -g(v) = -v$. Since the space X is strictly convex, for any distinct points $x, y \in X$ and positive real numbers a, b the intersection $(x + aS_X) \cap (y + bS_X)$ contains at most two distinct points, see Monotonicity Lemma in [15, §3.5]. This fact can be used to show (cf. [4, 2.3]) that each point $x \in S_X$ is the unique point of the intersection

$$(u + \|x - u\|S_X) \cap (-u + \|x + u\|S_X) \cap (v + \|x - v\|S_X) \cap (-v + \|x + v\|S_X)$$

which implies that $g(x) = x$ and hence $f = L{\restriction}_{S_X}$. $\square$

5. Banach spaces with exactly two non-smooth points on the unit sphere

Let X be a strictly convex 2-dimensional Banach space whose unit sphere contains exactly two non-smooth points. Let $\mathbf{e}_1 \in S_X$ be one of these non-smooth points. Take any vector $\mathbf{e}_2 \in X$ which is linearly independent with $\mathbf{e}_1$ and consider the natural parametrization $\mathbf{r} : \mathbb{R} \to S_X$ of the 2-based Banach space $(X, \mathbf{e}_1, \mathbf{e}_2)$. For this parameterization we have $\mathbf{r}(0) = \mathbf{e}_1$ and $\mathbf{r}'_-(0) \neq \mathbf{r}'_+(0)$ as $\mathbf{e}_1 = \mathbf{r}(0)$ is a non-smooth point of the unit sphere. Replacing the vector $\mathbf{e}_2$ by $\mathbf{r}'_\pm(0) = \frac{1}{2}(\mathbf{r}'_-(0) + \mathbf{r}'_+(0))$, we can assume that $\mathbf{e}_2 = \mathbf{r}'_\pm(0)$.

We recall that

$$\tfrac{1}{2}(\mathbf{r}'_+(0) - \mathbf{r}'_-(0)) = \jmath(0) \cdot \mathbf{r}(0) + \ddot{\jmath}(0) \cdot \mathbf{r}'_\pm(0) = \jmath(0) \cdot \mathbf{e}_1 + \ddot{\jmath}(0) \cdot \mathbf{e}_2$$

and the numbers $\jmath(0), \ddot{\jmath}(0)$ are called *radial* and *tangential jumps* of the derivative $\mathbf{r}'$ at zero. By Lemma 2.5, $|\ddot{\jmath}(0)| < 1$. We claim that those jumps are determined by the metric of the unit sphere.

For every point $x \in S_X$ let $\overline{x}$ be the unique point of the sphere such that $\{x, \overline{x}\} = S_X \cap (x + \mathbb{R}\mathbf{e}_1)$. The uniqueness of $\overline{x}$ follows from the strict convexity of X.

Lemma 5.1.

$$\lim_{\varepsilon \to +0} \frac{\|\mathbf{r}(\varepsilon) - \mathbf{r}(0)\|}{\|\overline{\mathbf{r}(\varepsilon)} + \mathbf{r}(0)\|} = \frac{1 - \ddot{\jmath}(0)}{1 + \ddot{\jmath}(0)} \quad \textit{and} \quad \lim_{\varepsilon \to +0} \frac{\|\mathbf{r}(\varepsilon) - \overline{\mathbf{r}(\varepsilon)}\| - 2}{2\varepsilon} = \frac{\jmath(0)}{1 - \ddot{\jmath}(0)}.$$

Proof. For a small positive ε, find a positive δ such that $\overline{\mathbf{r}(\varepsilon)} = -\mathbf{r}(-\delta)$. Observe that

$$\begin{aligned}\mathbf{r}(\varepsilon) = \mathbf{r}(0) + \mathbf{r}'_+(0)\varepsilon + o(\varepsilon) = \mathbf{e}_1 + \big(\jmath(0)\mathbf{e}_1 + (1 + \ddot{\jmath}(0))\mathbf{e}_2\big) \cdot \varepsilon + o(\varepsilon) = \\ = (1 + \jmath(0)\varepsilon + o(\varepsilon))\mathbf{e}_1 + (1 + \ddot{\jmath}(0) + o(1))\varepsilon\mathbf{e}_2\end{aligned}$$

and

$$\begin{aligned}\mathbf{r}(-\delta) = \mathbf{r}(0) - \mathbf{r}'_-(0)\delta + o(\delta) = \mathbf{e}_1 - \big(-\jmath(0)\mathbf{e}_1 + (1 - \ddot{\jmath}(0))\mathbf{e}_2\big)\delta + o(\delta) = \\ = (1 + \jmath(0)\delta + o(\delta))\mathbf{e}_1 - (1 - \ddot{\jmath}(0) + o(1))\delta\mathbf{e}_2.\end{aligned}$$

The equality $-\mathbf{r}(-\delta) = \overline{\mathbf{r}(\varepsilon)}$ implies $(1 + \ddot{\jmath}(0) + o(1))\varepsilon = (1 - \ddot{\jmath}(0) + o(1))\delta$ and

$$\frac{\varepsilon}{\delta} = \frac{1 - \ddot{\jmath}(0) + o(1)}{1 + \ddot{\jmath}(0) + o(1)}.$$

Then

$$\lim_{\varepsilon\to+0}\frac{\|\mathbf{r}(\varepsilon)-\mathbf{r}(0)\|}{\|\overline{\mathbf{r}(\varepsilon)}+\mathbf{r}(0)\|}=\lim_{\varepsilon\to+0}\frac{\|(\mathbf{r}'_+(0)+o(1))\varepsilon)\|}{\|-\mathbf{r}(-\delta)+\mathbf{r}(0)\|}=\lim_{\varepsilon\to+0}\frac{\|\mathbf{r}'_+(0){+}o(1)\|\cdot|\varepsilon|}{\|(\mathbf{r}'_-(0)+o(1))\delta\|}=\lim_{\varepsilon\to+0}\frac{|\varepsilon|}{|\delta|}=\frac{1-\ddot{\jmath}(0)}{1+\ddot{\jmath}(0)}.$$

On the other hand,

$$\begin{aligned}&\|\mathbf{r}(\varepsilon)-\overline{\mathbf{r}(\varepsilon)}\|=\|\mathbf{r}(\varepsilon)+\mathbf{r}(-\delta)\|=\\&=\|(1+\jmath(0)\varepsilon+o(\varepsilon))\mathbf{e}_1+(1+\ddot{\jmath}(0)+o(1))\varepsilon\mathbf{e}_2+(1+\jmath(0)\delta+o(\delta))\mathbf{e}_1-(1-\ddot{\jmath}(0)+o(1))\delta\mathbf{e}_2\|=\\&=\|(2+\jmath(0)(\varepsilon+\delta)+o(\varepsilon+\delta))\mathbf{e}_1\|=2+(\jmath(0)+o(1))\varepsilon\Big(1+\frac{1+\ddot{\jmath}(0)+o(1)}{1-\ddot{\jmath}(0)+o(1)}\Big)=2+2\varepsilon\frac{\jmath(0)+o(1)}{1-\ddot{\jmath}(0)}\end{aligned}$$

and hence

$$\lim_{\varepsilon\to+0}\frac{\|\mathbf{r}(\varepsilon)-\overline{\mathbf{r}(\varepsilon)}\|-2}{2\varepsilon}=\frac{\jmath(0)}{1-\ddot{\jmath}(0)}.$$

□

Lemma 5.2. *Let $s, \bar{s} \in \mathbb{R}$ be two real numbers such that $0 \neq \mathbf{r}(s) - \mathbf{r}(\bar{s}) = \|\mathbf{r}(s) - \mathbf{r}(\bar{s})\| \cdot \mathbf{e}_1$ and $\mathbf{r}(s) \notin \{\mathbf{e}_1, -\mathbf{e}_1\}$. Let $\mathbf{r}'(s) = x\mathbf{e}_1 + y\mathbf{e}_2$ and $\mathbf{r}'(\bar{s}) = \overline{x}\mathbf{e}_1 + \overline{y}\mathbf{e}_2$ for some real numbers $x, y, \overline{x}, \overline{y}$. For a small real number ε let $\bar{\varepsilon}$ be the unique small real number such that $\mathbf{r}(\bar{s}+\bar{\varepsilon}) = \overline{\mathbf{r}(s+\varepsilon)}$. Then*

1. $y > 0$ *and* $\overline{y} < 0$;
2. $\lim\limits_{\varepsilon\to 0} \dfrac{\|\mathbf{r}(\bar{s}+\bar{\varepsilon}) - \mathbf{r}(\bar{s})\|}{\|\mathbf{r}(s+\varepsilon) - \mathbf{r}(s)\|} = -\dfrac{y}{\overline{y}}$;
3. $\lim\limits_{\varepsilon\to 0} \dfrac{\|\mathbf{r}(s+\varepsilon) - \mathbf{r}(\bar{s}+\bar{\varepsilon})\| - \|\mathbf{r}(s) - \mathbf{r}(\bar{s})\|}{\varepsilon} = x - \overline{x} \cdot \dfrac{y}{\overline{y}}$;
4. $\lim\limits_{\varepsilon\to +0} \dfrac{\|\mathbf{r}(s+\varepsilon) - \mathbf{r}(\bar{s})\| - \|\mathbf{r}(s) - \mathbf{r}(\bar{s})\|}{\varepsilon} = x - \dfrac{\dot{\jmath}(0)\cdot y}{1+\ddot{\jmath}(0)}$;
5. $\lim\limits_{\varepsilon\to -0} \dfrac{\|\mathbf{r}(s+\varepsilon) - \mathbf{r}(\bar{s})\| - \|\mathbf{r}(s) - \mathbf{r}(\bar{s})\|}{\varepsilon} = x + \dfrac{\dot{\jmath}(0)\cdot y}{1-\ddot{\jmath}(0)}$;
6. *The numbers $x, y, \overline{x}, \overline{y}$ are uniquely determined by the equations* (2)–(5).

Proof. It follows from $0 \neq \mathbf{r}(s) - \mathbf{r}(\bar{s}) = \|\mathbf{r}(s) - \mathbf{r}(\bar{s})\| \cdot \mathbf{e}_1$ that $y > 0$ and $\overline{y} < 0$.

For a small number ε we have

$$\mathbf{r}(s+\varepsilon) - \mathbf{r}(s) = (\mathbf{r}'(s) + o(1))\varepsilon = (x + o(1))\varepsilon\mathbf{e}_1 + (y + o(1))\varepsilon\mathbf{e}_2$$

and

$$\mathbf{r}(\bar{s}+\bar{\varepsilon}) - \mathbf{r}(\bar{s}) = \mathbf{r}'(\bar{s})\bar{\varepsilon} + o(\bar{\varepsilon}) = (\overline{x} + o(1))\bar{\varepsilon}\mathbf{e}_1 + (\overline{y} + o(1))\bar{\varepsilon}\mathbf{e}_2.$$

The equality $\overline{\mathbf{r}(s+\varepsilon)} = \mathbf{r}(\bar{s}+\bar{\varepsilon})$ implies

$$(y + o(1))\varepsilon = (\overline{y} + o(1))\bar{\varepsilon}$$

and then

$$\lim_{\varepsilon\to 0} \frac{\|\mathbf{r}(\bar{s}+\bar{\varepsilon}) - \mathbf{r}(\bar{s})\|}{\|\mathbf{r}(s+\varepsilon) - \mathbf{r}(s)\|} = \lim_{\varepsilon\to 0} \frac{\|\mathbf{r}'(\bar{s}) + o(1)\| \cdot |\bar{\varepsilon}|}{\|\mathbf{r}'(s) + o(1)\| \cdot |\varepsilon|} = \lim_{\varepsilon\to 0} \frac{|\bar{\varepsilon}|}{|\varepsilon|} = \lim_{\varepsilon\to 0} \frac{|y + o(1)|}{|\overline{y} + o(1)|} = \frac{|y|}{|\overline{y}|} = -\frac{y}{\overline{y}}.$$

Also

$$\begin{aligned}
&\mathbf{r}(s+\varepsilon) - \mathbf{r}(\bar{s}+\bar{\varepsilon}) = \mathbf{r}(s) + \mathbf{r}'(s)\varepsilon + o(\varepsilon) - \big(\mathbf{r}(\bar{s}) + \mathbf{r}'(\bar{s})\bar{\varepsilon} + o(\bar{\varepsilon})\big) = \\
&= (\mathbf{r}(s) - \mathbf{r}(\bar{s})) + (x + o(1))\varepsilon\mathbf{e}_1 + (y + o(1))\varepsilon\mathbf{e}_2 - (\overline{x} + o(1))\bar{\varepsilon}\mathbf{e}_1 - (\overline{y} + o(1))\bar{\varepsilon}\mathbf{e}_2 = \\
&= \|\mathbf{r}(s) - \mathbf{r}(\bar{s})\|\mathbf{e}_1 + (x\varepsilon - \overline{x}\bar{\varepsilon} + o(\varepsilon + \bar{\varepsilon})) \cdot \mathbf{e}_1 = \\
&= \Big(\|\mathbf{r}(s) - \mathbf{r}(\bar{s})\| + (x - \overline{x} \cdot \frac{y}{\overline{y}} + o(1))\varepsilon\Big) \cdot \mathbf{e}_1
\end{aligned}$$

and hence

$$\lim_{\varepsilon\to 0} \frac{\|\mathbf{r}(s+\varepsilon) - \mathbf{r}(\bar{s}+\bar{\varepsilon})\| - \|\mathbf{r}(s) - \mathbf{r}(\bar{s})\|}{\varepsilon} = x - \overline{x} \cdot \frac{y}{\overline{y}}.$$

By Lemma 2.6,

$$\|\mathbf{r}(s+\varepsilon) - \mathbf{r}(\bar{s})\| = \|\mathbf{r}(s) - \mathbf{r}(\bar{s})\| + x\varepsilon - \frac{\operatorname{sign}(\varepsilon)\cdot\dot{\jmath}(0)\cdot y}{1 + \operatorname{sign}(\varepsilon)\ddot{\jmath}(0)}\varepsilon + o(\varepsilon)$$

and hence

$$\lim_{\varepsilon\to +0} \frac{\|\mathbf{r}(s+\varepsilon) - \mathbf{r}(\bar{s})\| - \|\mathbf{r}(s) - \mathbf{r}(\bar{s})\|}{\varepsilon} = x - \frac{\dot{\jmath}(0)\cdot y}{1+\ddot{\jmath}(0)}$$

and

$$\lim_{\varepsilon\to -0}\frac{\|\mathbf{r}(s+\varepsilon)-\mathbf{r}(\bar{s})\|-\|\mathbf{r}(s)-\mathbf{r}(\bar{s})\|}{\varepsilon}=x+\frac{\dot{\jmath}(0)\cdot y}{1-\ddot{\jmath}(0)}.$$

Therefore, the items (1)–(5) of Lemma 5.2 are proved.

The equations (4),(5) determine the numbers x, y uniquely because

$$\begin{vmatrix} 1 & -\frac{\dot{\jmath}(0)}{1+\ddot{\jmath}(0)} \\ 1 & \frac{\dot{\jmath}(0)}{1-\ddot{\jmath}(0)} \end{vmatrix}=\frac{\dot{\jmath}(0)}{1-\ddot{\jmath}(0)}+\frac{\dot{\jmath}(0)}{1+\ddot{\jmath}(0)}=\frac{2\cdot\dot{\jmath}(0)}{1-\ddot{\jmath}(0)^2}\neq 0.$$

The equation (2) allows us to find $\overline{y}$ and then $\overline{x}$ can be found from the equation (3). □

6. Proof of Theorem 1.4

Given any non-smooth 2-dimensional Banach space, we should prove that X has the Mazur–Ulam property. If X is not strictly convex, then X has the Mazur–Ulam property by Theorem 1.3. If the sphere S_X contains more than two non-smooth points, then X has the Mazur–Ulam property by Propositions 4.2 and 4.6. So, we assume that X is strictly convex and S_X contains exactly two non-smooth points. Let $\mathbf{e}_1$ be one of them. Then $-\mathbf{e}_1$ is the other non-smooth point of X.

Take any vector $\mathbf{e}_2 \in X\setminus(\mathbb{R}\cdot\mathbf{e}_1)$ and consider the natural parameterization $\mathbf{r}:\mathbb{R}\to X$ of the 2-based Banach space $(X,\mathbf{e}_1,\mathbf{e}_2)$. Since $\mathbf{e}_1$ is a non-smooth point of S_X, the one-sided derivatives $\mathbf{r}'_-(0)$ and $\mathbf{r}'_+(0)$ are distinct. Replacing the vector $\mathbf{e}_2$ by $\mathbf{r}'_\pm(0)$, we can assume that $\mathbf{e}_2=\mathbf{r}'_\pm(0)$. Let $L=\min\{s\in[0,\infty):\mathbf{r}(s)=-\mathbf{e}_1\}$ be the half-length of the sphere S_X, and $\breve{S}_X=\mathbf{r}([0,L])$ be the upper half-sphere of X. By Lemmas 2.2(5) and 2.3, the restriction $\mathbf{r}{\restriction}_{[0,L]}:[0,L]\to\breve{S}_X$ is an isometry of $[0,L]$ onto the half-sphere $\breve{S}_X$ endowed with the intrinsic metric.

To show that the space X has the Mazur–Ulam property, fix any bijective isometry $f:S_X\to S_Y$ of S_X onto the unit sphere of an arbitrary Banach space Y. It is clear that the space Y is 2-dimensional. Lemma 3.1 implies that $\tilde{\mathbf{e}}_1=f(\mathbf{e}_1)$ and $-\tilde{\mathbf{e}}_1$ are unique non-smooth points of the sphere S_Y. Repeating the above argument, we can find a vector $\tilde{\mathbf{e}}_2\in Y\setminus(\mathbb{R}\cdot\tilde{\mathbf{e}}_1)$ such that for the natural parameterization $\tilde{\mathbf{r}}:\mathbb{R}\to Y$ of the 2-based Banach space $(Y,\tilde{\mathbf{e}}_1,\tilde{\mathbf{e}}_2)$ we have $\tilde{\mathbf{r}}'_\pm(0)=\tilde{\mathbf{e}}_2$. Replacing $\tilde{\mathbf{e}}_2$ by $-\tilde{\mathbf{e}}_2$, if necessary, we can additionally assume that $\tilde{\mathbf{r}}(\varepsilon)\in f(\breve{S}_X)$ for small positive numbers ε.

Then $f(\breve{S}_X)$ coincides with the half-sphere $\breve{S}_Y$ of the 2-based Banach space $(Y,\tilde{\mathbf{e_1}},\tilde{\mathbf{e}}_2)$. Since $\breve{S}_X$ is isometric to $\breve{S}_Y$, the spheres S_X and S_Y have the same half-length. By Lemma 2.3, the restriction $\tilde{\mathbf{r}}{\restriction}_{[0,L]}:[0,L]\to\breve{S}_Y$ is an isometry of $[0,L]$ onto the half-sphere $\breve{S}_Y$ endowed with the intrinsic metric. The isometry $f{\restriction}_{\breve{S}_X}:\breve{S}_X\to\breve{S}_Y$ remains an isometry with respect to the intrinsic metrics on the half-spheres. Then $(\tilde{\mathbf{r}}{\restriction}_{[0,L]})^{-1}\circ f\circ\mathbf{r}{\restriction}_{[0,L]}$ is an isometry of $[0,L]$ which does not move zero and hence is the identity map of $[0,L]$. Consequently, $\tilde{\mathbf{r}}(s)=f\circ\mathbf{r}(s)$ for all $s\in[0,L]$. Using this fact and Lemma 2.2(1), we can show that $\tilde{\mathbf{r}}(s)=f\circ\mathbf{r}(s)$ for all $s\in\mathbb{R}$.

Let $I:X\to Y$ be the linear operator such that $I(\mathbf{e}_1)=\tilde{\mathbf{e}}_1$ and $I(\mathbf{e}_2)=\tilde{\mathbf{e}}_2$. Applying Proposition 4.2 and Theorem 1.6 and 5.1, we can show that the spheres S_X and S_Y have the same radial and tangential jumps $\dot{\jmath}(0)$ and $\ddot{\jmath}(0)$. Lemma 5.2 implies that $I(\underline{\mathbf{r}'(s)})=\tilde{\mathbf{r}}'(s)$ for every $s\in(0,L)$ with $\mathbf{r}(s)\neq\overline{\mathbf{r}(s)}$. Since X is strictly convex, the set $\{s\in[0,L]:\mathbf{r}(s)=\overline{\mathbf{r}(s)}\}$ is a singleton. Now the continuity of the functions $\mathbf{r}'$ and $\tilde{\mathbf{r}}'$ on $(0,L)$ implies that $I(\mathbf{r}'(s))=\tilde{\mathbf{r}}'(s)$ for all $s\in(0,L)$. Since $\mathbf{r}$ is continuously differentiable on $[0,L]$, for every $s\in[0,L]$ we have $\mathbf{r}(s)=\mathbf{r}(0)+\int_0^s\mathbf{r}'(t)\,dt$ and hence

$$I(\mathbf{r}(s))=I(\mathbf{r}(0))+\int_0^s I(\mathbf{r}'(t))\,dt=\tilde{\mathbf{r}}(0)+\int_0^s\tilde{\mathbf{r}}'(t)dt=\tilde{\mathbf{r}}(s)=f\circ\mathbf{r}(s).$$

By Theorem 1.6 and Lemma 2.2(1),

$$I(\mathbf{r}(s+L)) = I(-\mathbf{r}(s)) = -I(\mathbf{r}(s)) = -f(\mathbf{r}(s)) = f(-\mathbf{r}(s)) = f(\mathbf{r}(s+L))$$

for every $s \in [0, L]$ and hence $I \circ \mathbf{r}\restriction_{[L,2L]} = \tilde{\mathbf{r}}\restriction_{[L,2L]}$. Therefore, I is a linear operator extending the isometry f. The equality $I(S_X) = S_Y$ implies $I(B_X) = B_Y$, which means that I is a linear isometry of the Banach spaces X, Y.

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