

# A precision on the concept of strict convexity in non-Archimedean analysis

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ABSTRACT. We prove that the only non-Archimedean strictly convex spaces are the zero space and the one-dimensional linear space over  $\mathbb{Z}/3\mathbb{Z}$ , with any of its trivial norms.

## 1. Introduction

In order to find a non-Archimedean version of the Mazur-Ulam theorem, M. Moslehian and G. Sadeghi introduced in [2] the class of non-Archimedean strictly convex spaces.

Later on, A. Kubzdela observed that the only non-Archimedean strictly convex space over a field with a non-trivial valuation is the zero space ([1, Theorem 2]).

In this note, we prove:

**PROPOSITION 2.3** *The only non-Archimedean strictly convex spaces are the zero space and the one-dimensional linear space over  $\mathbb{Z}/3\mathbb{Z}$ , with any of its trivial norms.*

## 2. Non-Archimedean strictly convex spaces

Firstly, recall that on a non-Archimedean normed space  $X$ , “any triangle is isosceles”; that is to say, for any  $x, y \in X$ ,

$$\|y\| < \|x\| \quad \Rightarrow \quad \|x + y\| = \|x\| .$$

**DEFINITION 2.1** ([2]). *A non-Archimedean normed space  $X$  over a field  $\mathbb{K}$  is strictly convex if*

(SC1)  $\|2\| = 1$ .

(SC2) *for any pair of vectors  $x, y \in X$ ,  $\|x\| = \|y\| = \|x + y\|$  implies  $x = y$ .*

Observe that (SC2) may also be rephrased by saying that there are no equilateral triangles; that is to say, that for any pair of distinct vectors  $x \neq y \in X$ ,

$$\|y\| = \|x\| \quad \Rightarrow \quad \|x + y\| < \|x\| .$$

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EXAMPLE 2.2. If  $X$  is a one-dimensional normed linear space over a finite field, then its norm is trivial; i.e., there exists  $a \in (0, \infty)$  such that  $\|x\| = a$  for every nonzero vector  $x \in X$ .

The one-dimensional linear space over  $\mathbb{Z}/3\mathbb{Z}$ , with any of its possible trivial norms, is a strictly convex space, in the sense of the Definition above.

PROPOSITION 2.3. *If  $X$  is a non-zero strictly convex space, then it is linearly isometric to a one-dimensional normed space over  $\mathbb{Z}/3\mathbb{Z}$ .*

PROOF. First of all, observe that a strictly convex space can only occur in characteristic 3: for any vector  $x \in X$ ,

$$\|2x\| = \|x\| = \|-x\|;$$

as  $2x + (-x) = x$ , condition (SC2) implies that  $2x = -x$ ; that is to say,  $3x = 0$  for any vector  $x$ , and we conclude that  $3 = 0$  in  $\mathbb{K}$ .

Now suppose there are two non-zero vectors  $x, y \in X$  such that  $y \neq \pm x$  and we will arrive to a contradiction.

Condition (SC1) implies that the characteristic of  $\mathbb{K}$  is not 2, and, hence,  $x + y \neq x - y$ . Without loss of generality we may also assume that

$$\|y\| \leq \|x\| \quad \text{and} \quad \|x - y\| \leq \|x + y\|.$$

If  $\|y\| < \|x\|$ , then  $x + y$  and  $x - y$  are distinct vectors with the same norm,  $\|x + y\| = \|x\| = \|x - y\|$ , and whose sum maintains the norm

$$\|(x + y) + (x - y)\| = \|2x\| = \|x\| = \|x + y\|,$$

in contradiction with (SC2).

If  $\|y\| = \|x\|$ , then (SC2) implies the absurd chains of inequalities

$$\begin{aligned} \|x + y\| &< \|x\| = \|2x\| = \|(x + y) + (x - y)\| \leq \max\{\|x + y\|, \|x - y\|\}, \\ \|x - y\| &< \|x\| = \|2x\| = \|(x + y) + (x - y)\| \leq \max\{\|x + y\|, \|x - y\|\}. \end{aligned}$$

□

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