On discontinuity of derivations, inducing inequivalent complete metric topologies

S.R. Patel

Department of Mathematics, C.U. Shah University
Wadhwan City, Gujarat, INDIA

srpatel.math@gmail.com

Abstract: We give an elementary method for constructing commutative Fréchet algebras with non-unique Fréchet algebra topology. The result is applied to show that the action of any non-algebraic analytic function may fail to be uniquely defined among other useful applications. We give an affirmative answer to a question of Loy (1974) for Fréchet algebras. We also obtain the uniqueness of the Fréchet algebra topology of certain Fréchet algebras with finite dimensional radicals.

Key words: Fréchet algebra of power series in infinitely many indeterminates, derivation, (in)equivalent Fréchet algebra topologies, Loy's question.

MSC (2020): Primary 46J05; Secondary 13F25, 46H40, 46J10.

1. INTRODUCTION

The objective of this article is to provide an elementary method for constructing commutative Fréchet algebras which admit two inequivalent Fréchet algebra topologies. In the Fréchet case, the *only* example of this kind so far known was that of Read [22], primarily constructed to display the breakdown of the Singer-Wermer conjecture (the commutative case), which holds in the Banach algebra case (see [24]). This example is $\mathcal{F}_\infty = \mathbb{C}[[X_0, \ldots, X_n, \ldots]]$, the algebra of all formal power series in infinitely many commuting indeterminates $X_0, \ldots, X_n, \ldots$, and has a Fréchet algebra topology τ_R with respect to which the natural derivation $\partial/\partial X_0$ is discontinuous with the range the whole algebra. The other Fréchet algebra topology on $\mathcal{F}_\infty$ is the topology τ_c of coordinatewise convergence with respect to which the derivation $\partial/\partial X_0$ is continuous. Thus Read exhibits that the situation on Fréchet algebras is significantly distinct from that on Banach algebras, and that a structure theory for Fréchet algebras acquits in a very peculiar manner from Banach algebras. For example, the distinguished Michael's problem is still stayed unsolved since

1952 (see [12] for the recent progress); Banach algebras are easily seen to be functionally continuous. In fact, the author is finalizing the manuscript on an affirmative solution to Michael's problem in which the prime idea is to view the *test case* for this problem, the commutative, semisimple Fréchet algebra $\text{Hol}(\ell^\infty)$ of all entire functions on ℓ^∞ [12, §5, §9, §10] in terms of weighted Fréchet symmetric algebra $\check{V}_W E$ over the Banach space $E = \ell^1$ (called the Dales-McClure Fréchet algebra in Section 4 below).

Further we recall that in automatic continuity theory, we would normally like to examine when and how the algebraic structure of the algebra A determines the topological structure of A, in particular, the continuity aspect (and more particularly, the uniqueness of the topology of A; see [4, 6, 12, 17, 18, 20] for more details). So it is natural to expect that the non-uniqueness of the topology would reflect some properties of the algebraic structure of A. In [14], Feldman constructed a Banach algebra with two inequivalent complete norms, in order to display the breakdown of the Wedderburn Theorem. This example is $\ell_2 \oplus \mathbb{C}$ with one norm in which ℓ_2 is dense.

It is easy to give uncountably many inequivalent Fréchet space topologies to a very familiar Fréchet spaces, namely, spaces of holomorphic functions [25]. However the same space is also a Fréchet algebra of power series, and so, it admits a unique Fréchet algebra topology [17]. Thus it is expected that the case of having inequivalent Fréchet algebra topologies by some Fréchet algebra should be rare, and the former situation should be common. We here note that the same space turns out to be a Fréchet algebra once we equip it with the zero product, and then, those uncountably many inequivalent Fréchet space topologies are now inequivalent Fréchet algebra topologies. We also remark that not every derivation on a commutative Fréchet algebra is continuous. For example, let A be an arbitrary Fréchet space with the zero product. Then it is a commutative Fréchet algebra, and every linear operator on A is evidently a derivation. But, of course, not every operator on a Fréchet space is continuous. However, we are certainly not interested in such examples equipped with the zero product.

On the contrary, as discussed above, the (dis)continuity of a derivation on a Fréchet algebra $\mathcal{F}_\infty$ has a direct connection with an (in)equivalency of the two topologies on $\mathcal{F}_\infty$. In fact, construction of a suitable topology, however, was not *simple*; it required the axiom of choice (AC), an essential development because without the AC, one normally assumes that ALL linear maps between Fréchet spaces are continuous [22]. Indeed, author shows in [19] that the algebra $\mathcal{F}_\infty$ has countably many inequivalent Fréchet algebra

topologies, followed by Read's bizarre method. Below, we take this mission one step forward, in order to give an elementary method for constructing two inequivalent Fréchet algebra topologies on a Fréchet algebra.

Loy gave a method for constructing commutative Banach algebras with two inequivalent complete norm topologies, and we implement that idea in this paper, so we shall feel free to use the terminology, conventions, a couple of arguments and proofs of theorems from [16] in order to keep our argument short for the Fréchet algebra case here. We use the discontinuity of derivations to give a Fréchet algebra A other, inequivalent, Fréchet algebra topologies. Although the Fréchet algebra topology $\tau_R + \tau_R$ of $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$ is not attainable by our way (see below), other inequivalent Fréchet algebra topologies on $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$ may be constructed [19].

This elementary method for constructing commutative Fréchet algebras with non-unique Fréchet algebra topology surprisingly also enables the demonstration of the uniqueness of the Fréchet algebra topology of certain Fréchet algebras with finite-dimensional radicals in Section 3. In the Fréchet case, though there are a few results on the uniqueness of the Fréchet algebra topology (see [4, 6, 12, 17, 18]), a class of Fréchet algebras with finite dimensional radicals for this result appears to be treated for the first time in this paper. Of course, in the Banach case, we have results on the uniqueness of the complete norm topology of Banach algebras with finite dimensional radicals (see [9, 21]); however, the hypotheses and results are very different.

In Section 4, we obtain an affirmative answer to a question of Loy from 1974, but in the Fréchet case. In [13], Domar constructed a Banach algebra with a quasinilpotent non-nilpotent radical; however, his approach was different and used an entire function argument. In particular, the question whether the construction of a Banach algebra with a quasinilpotent non-nilpotent radical using the method of the higher point derivations of infinite order is possible, is a very good question. At present, we do not know the answer. In the end, we discuss a particular example of a semisimple Fréchet algebra. Surprisingly, we note that the converse of Theorem 4.3 below does not hold.

2. Inequivalent Fréchet algebra topologies

Let A be a commutative, metrizable, locally multiplicatively convex-algebra whose topology τ may be defined by a sequence $(p_k)_{k \geq 1}$ of seminorms, and let M be a commutative Fréchet A-module, or just A-module, whose topology τ' may be defined by a sequence $(p'_k)_{k \geq 1}$ of seminorms

(assumed increasing without loss of generality), so M is a Fréchet space which is a commutative A-module such that the map $(x, m) \mapsto x \cdot m$ is continuous from $A \times M$ to M. Moreover, throughout the paper, we shall impose a condition on the seminorms p'_k of M that they are submultiplicative with respect to the module action; i.e.,

$$p'_k(x \cdot m) \leq p_k(x) p'_k(m) \qquad (k \in \mathbb{N})$$

for all $x \in A$ and all $m \in M$ (in [15], they call such an M smooth). This condition is *mild* in the sense that M will be a commutative Fréchet algebra in most of the cases below. Also, any commutative algebra A is itself a commutative A-module, and so, the Arens algebra L^ω is the best example to exhibit the necessity of this condition, since the multiplication operation in L^ω is jointly continuous, but the seminorms $\|\cdot\|_p$, $p \geq 1$, are not submultiplicative; i.e., for each $p \geq 1$, we do not have $\|fg\|_p \leq \|f\|_p \|g\|_p$ [2, §4]. Thus, the Arens algebra L^ω is not smooth [15].

For such A and M, let $H^1(A, M)$ (resp., $H^1_C(A, M)$) denote the first algebraic (resp., continuous) cohomology group (where the cochains are required to be bounded). Thus with the usual conventions $H^1(A, M)$ (resp., $H^1_C(A, M)$) is the space of (continuous) derivations of A into M, that is, linear mappings $D : A \to M$ satisfying $D(xy) = x \cdot D(y) + y \cdot D(x)$. As two simple examples, first take $M = \mathbb{C}$ with module action $x \cdot \lambda = \lambda \phi(x)$ for some multiplicative linear functional ϕ on A. If $\phi \neq 0$, then $H^1(A, \mathbb{C})$ is the point derivation space at ϕ. In the second case, we call D a derivation of A, if $M = A$.

There is a beautiful exposition of the motive for the study of derivations in [15]; for an extensive study of derivations on certain GB^*-algebras, see [26, 27, 28, 29, 30, 31]. In particular, Weigt and Zarakas studied unbounded derivations of GB^*-algebras in [30, 31]. We briefly recall that the first cohomology group measures how much the space of all (continuous) derivations of A in M differs from the space of all (continuous) inner derivations of A in M (a derivation is inner if there exist an $m \in M$ such that $D = D_m$, where $D_m(a) = am - ma$ for all $a \in A$). In particular, there are important applications of such results to cohomology theory concerning contractibility and amenability. Further, If A has an identity, then derivations of A, give rise to automorphisms of A. The set $Aut(A)$ of automorphisms of A is a subgroup of the group of all invertible operators on A. Moreover, such automorphisms are basically utilized for the proof of the well-known Singer-Wermer Theorem; see [15, §3] for the most elegant consequences in the theory of Fréchet algebras.

A couple of avatars of the Singer-Wermer conjecture in this theory such as a weaker version, a stronger version (namely, Kaplansky's conjecture), and/or an extended version, are recently discussed by author in [19]. There is also a connection between derivations and quantum physics via $Aut(A)$, which has its self-adjoint elements as the observables of the quantum system [15]. Most importantly, in view of our interest in studying discontinuous derivations, the motive for the study of unbounded derivations was provided by the problem of constructing the dynamics in statistical mechanics.

For A and M as taken in the beginning of the section, the constant

$$l_k = \sup \left\{ 1, p_k'(xm) \,:\, p_k(x) = p_k'(m) = 1 \right\} \qquad (k \in \mathbb{N}),$$

is finite (in fact, $l_k = 1$, M being smooth), and if $\overline{A}$ is the completion of A under (p_k), M is evidently an $\overline{A}$-module. Let $\mathcal{A}$ denote the vector space direct sum $\overline{A} \oplus M$ with product

$$(x, m)(y, n) = (xy, x \cdot n + y \cdot m)$$

and seminorms

$$q_k((x, m)) = p_k(x) + p_k'(m).$$

For each $D \in H^1(A, M)$, the functional

$$q_{k,D} \colon (x, m) \longmapsto p_k(x) + p_k'(D(x) - m)$$

is defined on the subalgebra $A \oplus M$ of $\mathcal{A}$ and is readily seen to be a submultiplicative seminorm thereon. If $D = 0$ the completion of $A \oplus M$ under $(q_{k,D})$ is, indeed, just $\mathcal{A}$. For arbitrary D we observe that the map

$$\theta_D \colon (x, m) \longmapsto (x, D(x) - m)$$

is an isometric isomorphism of $A \oplus M$ under $(q_{k,D})$ into $\mathcal{A}$, and so extends uniquely to a map of the completion $\mathcal{A}_D$ of $A \oplus M$ into $\mathcal{A}$. In particular, if $\iota \colon x \mapsto (x, 0)$ is the natural embedding of A into $A \oplus M$ then $q_{k,D}' \colon x \mapsto q_{k,D}(\iota(x))$ is a seminorm on A and $\theta_D \circ \iota$ extends to an isometric isomorphism of $\overline{A_D}$, the completion of A under $q_{k,D}'$, with $\overline{\mathrm{Gr}(D)}$, the closure (in $\mathcal{A}$) of the graph of D.

Now if D is continuous, then $(q_{k,D})$ is equivalent to (q_k) on $A \oplus M$ and $(q_{k,D}')$ is equivalent to (p_k) on A. Thus $\mathcal{A}_D = \mathcal{A}$ and $\overline{A_D} = \overline{A}$. In the discontinuous case, $q_{k,D}'$ is a discontinuous seminorm on A and ι is a discontinuous isomorphism. This latter result has been used in the Fréchet case as follows:

let A be the algebra of polynomials on a fixed open neighbourhood U of the closed unit disc Δ with the compact-open topology, $M = \mathbb{C}$ with module action $p \cdot \lambda = \lambda p(1)$ for $(p, \lambda) \in A \oplus M$ and $D \colon p \mapsto p'(1)$. Here $\overline{A_D}$ is a singly generated Fréchet algebra with spectrum U and one dimensional radical (for the Banach case, see [11, 16]).

Suppose now that D is discontinuous. If M is finite dimensional, then it readily follows that $\overline{\mathrm{Gr}(D)} \bigcap \{0\} \oplus M \neq \{0\}$, and if A is complete this holds for general M by the closed graph theorem for Fréchet spaces. Thus if A is a Fréchet algebra with $H^1(A, M) \neq H^1_C(A, M)$ for some M, then A has a completion with a non-trivial nil ideal. For example, $A = \mathcal{F}_\infty$ is a Fréchet algebra under the Fréchet algebra topology τ_R imposed by Read, then $H^1(A, M) \neq H^1_C(A, M)$ for $M = \mathcal{F}_\infty$, since the derivation $\partial/\partial X_0$ is discontinuous with respect to this topology. Thus $\mathcal{F}_\infty$ has a completion with a non-trivial nil ideal. We remark that if A is a semisimple Fréchet algebra, then $H^1(A, M) = H^1_C(A, M)$ [5]. However we do not know an example of a (semisimple) non-Banach Fréchet algebra such that $H^1(A, M) = 0$ for any A-module M; the Singer-Wermer conjecture holds for commutative semisimple Banach algebras. Banach algebras with $H^1(A, M) = H^1_C(A, M)$ are discussed in [3]. Now we include one more hypothesis to these considerations to obtain the following result.

THEOREM 2.1. *Let A be a commutative Fréchet algebra, D a non-zero derivation of A into a commutative Fréchet A-module M. If D vanishes on a dense subset of A then the algebra $\overline{A_D}$ admits two inequivalent Fréchet algebra topologies.*

Proof. The proof is the same as that of Loy's Theorem 1. ∎

As a corollary, we have the following special case

COROLLARY 2.2. *Let A be the Fréchet algebra $\mathcal{F}_\infty$ under the Fréchet algebra topology τ_R and let D be the natural derivation $\partial/\partial X_0$. Then $\overline{(\mathcal{F}_\infty)}_D$ admits another Fréchet algebra topology τ_D, generated by $(q_{k,D})$, different from $\tau_R + \tau_R$, generated by (q_k).*

Proof. By [22, Theorem 2.5], $\partial/\partial X_0$ is, on $(\mathcal{F}_\infty, \tau_R)$, a discontinuous derivation which vanishes on a dense subset of $(\mathcal{F}_\infty, \tau_R)$ since $X_n \to X_0$ in $(\mathcal{F}_\infty, \tau_R)$; X_0 lies in the closure of $\mathcal{A}_0 = \mathbb{C}[[X_1, \ldots, X_n, \ldots]]$, the coefficient algebra of $\mathcal{F}_\infty = \mathbb{C}[[X_0, \ldots, X_n, \ldots]]$. Thus, by Theorem 2.1, $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$ is

also a Fréchet algebra under τ_D, generated by $(q_{k,D})$, different from $\tau_R + \tau_R$, generated by (q_k). ∎

Remark A. In view of the above corollary, [19, Theorem 3.1] provides a more general result.

In fact, we have a more general result than Theorem 2.1 as follows. Let A be a commutative Fréchet algebra, and let M be a Fréchet A-module. Set $\mathcal{U} = A \oplus M$, where $(a,x)(b,y) = (ab, a \cdot y + b \cdot x)$ for $a, b \in A$ and $x, y \in M$. Then $\mathcal{U}$ is a commutative algebra with $\mathrm{Rad}\,\mathcal{U} = \mathrm{Rad}\,A \oplus M$. Let $D \colon A \to M$ be a derivation, and set

$$q_k((a,x)) = p_k(a) + p'_k(x), \; q_{k,D}((a,x)) = p_k(a) + p'_k(D(a)-x) \;\; (a \in A, \; x \in M).$$

THEOREM 2.3. *The algebra $\mathcal{U}$ is a Fréchet algebra with respect to both (q_k) and $(q_{k,D})$. The two topologies are equivalent if and only if D is continuous.*

Proof. Certainly $(\mathcal{U}, (q_k))$ is a Fréchet algebra and $q_{k,D}$ is a seminorm on $\mathcal{U}$ for each $k \in \mathbb{N}$. For $(a,x), (b,y) \in \mathcal{U}$, we have

$$\begin{aligned}
q_{k,D}((a,x)(b,y)) &= p_k(ab) + p'_k(a \cdot (D(b) - y) + b \cdot (D(a) - x)) \\
&\leq (p_k(a) + p'_k(D(a) - x))(p_k(b) + p'_k(D(b) - y)) \\
&= q_{k,D}((a,x))q_{k,D}((b,y)),
\end{aligned}$$

and so $q_{k,D}$ is a submultiplicative seminorm on $\mathcal{U}$ for each $k \in \mathbb{N}$. We now show that $(\mathcal{U}, (q_{k,D}))$ is a Fréchet algebra. Let $((a_n, x_n))$ be a Cauchy sequence in $(\mathcal{U}, (q_{k,D}))$. Then (a_n) and $(D(a_n) - x_n)$ are Cauchy sequences in $(A, (p_k))$ and $(M, (p'_k))$, respectively. Since A and M are Fréchet spaces, there exists $a \in A$ and $x \in M$ such that $a_n \to a$ and $D(a_n) - x_n \to x$. Then $(a_n, x_n) \to (a, D(a) - x))$ in $(\mathcal{U}, (q_{k,D}))$ and so $(\mathcal{U}, (q_{k,D}))$ is a Fréchet algebra.

Suppose that D is continuous. Then, for each $m \in \mathbb{N}$, there exists $n(m) \in \mathbb{N}$ and a constant $c_m > 0$ such that

$$\begin{aligned}
q_{m,D}((a,x)) &\leq p_m(a) + c_m p_{n(m)}(a) + p'_m(x) \\
&\leq (1 + c_m)q_{n(m)}((a,x)) \qquad ((a,x) \in \mathcal{U}),
\end{aligned}$$

and so the two topologies are equivalent, by the open mapping theorem for Fréchet spaces.

Conversely, suppose that the two topologies are equivalent on the algebra $\mathcal{U}$. Then, for each $m \in \mathbb{N}$, there exists $n(m) \in \mathbb{N}$ and a constant $c_m > 0$ such that $q_{m,D}((a,x)) \leq c_m q_{n(m)}((a,x))$ $((a,x) \in \mathcal{U})$. Hence

$$p'_m(D(a)) \leq q_{m,D}((a,0)) \leq c_m\, q_{n(m)}((a,0)) = c_m\, p_{n(m)}(a) \qquad (a \in A),$$

and so D is continuous.

Alternatively, we remark from the above discussion that θ_D is an isometric isomorphism of $(\mathcal{U},(q_k))$ onto $(\mathcal{U},(q_{k,D}))$, and so, the two topologies are equivalent whenever D is continuous, and vice-versa. ∎

As corollaries, we have the following results.

COROLLARY 2.4. *There is a commutative algebra with a one-dimensional radical which is a Fréchet algebra with respect to two inequivalent Fréchet algebra topologies.*

Proof. Let A be a Fréchet function algebra with a discontinuous point derivation D at a continuous character ϕ. Then $\mathbb{C}$ is a Fréchet A-module with respect to the operation $(f,z) \mapsto \phi(f)z$, $A \times \mathbb{C} \to \mathbb{C}$, and so we are in a situation where Theorem 2.3 applies: $\mathcal{U} = A \oplus \mathbb{C}$ is a Fréchet algebra with respect to the product

$$(f,z)(g,w) = \big(fg, \phi(f)w + \phi(g)z\big)$$

and each of the topologies generated by (q_k) and $(q_{k,D})$, respectively, where $q_k((f,z)) = p_k(f) + \mid z \mid$ and $q_{k,D}((f,z)) = p_k(f) + \mid D(f) - z \mid$.

As above, $\mathrm{Rad}\,\mathcal{U} = \{0\} \oplus \mathbb{C}$, and so $\mathrm{Rad}\,\mathcal{U}$ is one dimensional. ∎

Remark B. As discussed in Section 1, Feldman's example in the Banach case has a one dimensional radical. In the above corollary, an example in the Fréchet case appears to be treated for the first time in this paper. The significance of such an example in the Fréchet case is also explained at the end of this section. Also, we need to consider a discontinuous point derivation D at a *continuous character* in the above proof, because, as is well-known, every character on a commutative Banach algebra is continuous, but this is not known in the Fréchet case, and is the most prestigious Michael's problem.

COROLLARY 2.5. *Let A be the Fréchet algebra $\mathcal{F}_\infty$ under the Fréchet algebra topology τ_c and let D be the natural derivation $\partial/\partial X_0$. Then the two topologies τ_D, generated by $(q_{k,D})$, and $\tau_c + \tau_c$, generated by (q_k), are equivalent on the algebra $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$ if and only if D is continuous.*

Proof. Clearly, $\partial/\partial X_0$ is a continuous derivation on $(\mathcal{F}_\infty, \tau_c)$. Thus, by Theorem 2.3, the two topologies τ_D, generated by $(q_{k,D})$, and $\tau_c + \tau_c$, generated by (q_k), are equivalent on the algebra $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$. Converse is trivial. ∎

Remark C. As discussed in Section 1, we construct two examples of Fréchet algebras with countably many inequivalent Fréchet algebra topologies. The first example is the algebra $\mathcal{F}_\infty$, and the second example is the algebra $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$ [19].

As an application of this method, let A be the algebra $\mathrm{Hol}(U)$ of holomorphic functions on the open unit disc U, with the compact-open topology. Let $\mathcal{O}$ be the algebra of functions holomorphic in a neighbourhood of $\overline{U}$, and let $\psi \colon A \to \mathcal{O}$ be the monomorphism $\psi x(\lambda) = x(\lambda/2)$, $x \in A$, $|\lambda| < 2$. Finally let M be a Banach space, T an endomorphism of M with norm at most 1. Then, following Loy's argument on p. 412, M is an A-module. In particular, we may consider the situation developed in [7] but in the Fréchet case, where $M = C[0,1]$ and T is the operator of indefinite integration. Non-zero linear mappings $\beta \colon \mathcal{O} \to M$ are constructed to vanish on polynomials and satisfy

$$\beta(fg) = f(T)\beta(g) + g(T)\beta(f) \qquad (f, g \in \mathcal{O}).$$

Letting $D = \beta\psi$ we have that $D \colon A \to M$ is a derivation vanishing on polynomials. Since $D \neq 0$, it is necessarily discontinuous (which answers a question of [3] in the Fréchet case).

We also remark that A and D here satisfy the hypothesis of Theorem 2.1 since polynomials are dense in A. Thus $\overline{A_D}$ has two inequivalent Fréchet algebra topologies with infinite dimensional radical (by the construction of β). Indeed algebras with finite dimensional radical have unique functional calculus by the Fréchet analogue of [7, Theorem 1] so that the argument above shows that any derivation of A into a finite dimensional A-module vanishing on polynomials must be zero. In fact, such a derivation is continuous, since A is an algebra of class $\mathcal{B}$ (see [10]) with $A \oplus M$ a strongly decomposable Fréchet algebra with finite dimensional radical M, and so $\theta_D\iota$ is continuous by Theorem 3.1 below.

Remark D. The results established by Dales in [7] have appropriate analogues in the Fréchet case.

Using the present example we can show that the exponential function in such an algebra is not independent of the Fréchet algebra topology (see Loy's argument on p. 413 for details).

We recall that the first discontinuous functional calculus map was constructed by Allan (see [1, Theorem 8]). The algebras satisfying this theorem are $L^1_{\mathrm{loc}}(\mathbb{R}^+)$, $L^1(\mathbb{R}^+, W)$ and $C^\infty(\mathbb{R}^+)$. It follows from proof of the theorem that the above condition is sufficient on an algebra A for the existence of a discontinuous functional calculus homomorphism. However, the algebra $\mathcal{U} = \mathrm{Hol}(U) \oplus \mathbb{C}$ shows that it is not a necessary condition. In fact, it is shown in [1, Theorem 8] that there is an incomplete metrizable topology on $\mathrm{Hol}(U)$ which dominates the compact-open topology, and the completion of $\mathrm{Hol}(U)$ in such a topology has a nilpotent radical, as discussed before. Such a result is impossible for $\mathcal{F} = \mathbb{C}[[X]]$ (since X is locally nilpotent) and $C(U)$ (no adequate theory of point derivations).

3. Uniqueness of the Fréchet algebra topology

In the above incidents the ideal adjoined was consistently nilpotent of index 2. Now we view how to get more general ideals.

Let A and M be as in Section 2, and let M be also a commutative Fréchet algebra. Let $D = \{D_1, \ldots, D_r\}$ be a higher derivation of rank r of A into M. In parallel to $\mathcal{A}$ let $\mathcal{A}_r$ denote the vector space $\overline{A} \oplus M^r$ with product

$$
\left(x, \{m_i\}\right)\left(y, \{n_i\}\right) = \left(xy, \left\{xn_i + ym_i + \sum_{j=1}^{i-1} m_j n_{i-j}\right\}\right)
$$

and seminorms

$$
q_k\left(\left(x, \{m_i\}\right)\right) = p_k(x) + \sum_{i=1}^{r} p_k'(m_i).
$$

We also have the seminorms $q_{k,D}$ on $A \oplus M^r$,

$$
q_{k,D}\left(\left(x, \{m_i\}\right)\right) = p_k(x) + \sum_{i=1}^{r} p_k'(D_i(x) - m_i),
$$

the isomorphism $\theta_D\colon \left(x, \{m_i\}\right) \mapsto \left(x, \{D_i(x) - m_i\}\right)$ and the completion $\overline{A_D}$ of A under $q_{k,D}'\colon x \mapsto q_{k,D}((\iota(x)))$. We consider a specific case below.

Let A be the algebra of polynomials on a fixed open neighbourhood U of the closed unit disc Δ, with seminorms p_k generating the compact-open topology, $M = \mathbb{C}$ with module action as before and a higher point derivation $D = \{D_1, \ldots, D_r\}$ of rank r given by $D_i p = \frac{p^{(i)}(1)}{i!}$, so that

$$
q_{k,D}{}'(p) = p_k(p) + \sum_{i=1}^{r} \frac{\left|p^{(i)}(1)\right|}{i!}.
$$

Then, Loy's arguments for the Banach case can also be applied to the Fréchet case to see that $\overline{A_D}$ has a radical which is nilpotent of index r.

We remark that in this example $\mathcal{A}_r$, and hence $\overline{A_D}$, is a strongly decomposable Fréchet algebra of class $\mathcal{B}$ with finite dimensional radical and so has a unique Fréchet algebra topology by Corollary 3.3 below. We note that Section 5 of [10] could be extended in the Fréchet case (up to Proposition 5.5). In particular, we have the following

THEOREM 3.1. *Let B be an algebra of class $\mathcal{B}$ and let A be a strongly decomposable Fréchet algebra with finite dimensional radical R. Then any homomorphism $\theta \colon B \to A$ is continuous.*

PROPOSITION 3.2. *Let A be an algebra of class $\mathcal{B}$ with finite dimensional radical R. Then any decomposition of A is a strong decomposition.*

Then, as a corollary to Theorem 3.1, we have the following result.

COROLLARY 3.3. *If A is a decomposable Fréchet algebra of class $\mathcal{B}$ with finite dimensional radical R, then A has a unique Fréchet algebra topology.*

In contrast to this we have the following

THEOREM 3.4. *Let A be a commutative Fréchet algebra, $D = \{D_1, \ldots, D_r\}$ a higher point derivation of rank r of A into $\mathbb{C}$ such that $\{D_1, \ldots, D_r\}$ is a set of discontinuous functionals. Then $\overline{A_D}$ admits two inequivalent Fréchet algebra topologies and has nilpotent elements of index r.*

Proof. We first follow an analogous argument preceding to Theorem 2.3 in this case (i.e., for A and $M = \mathbb{C}$, the product and the seminorms as taken in the beginning of the section). Now apply an alternative approach from the proof of Theorem 2.3. Hence the proof. ∎

4. ANSWER TO LOY'S QUESTION IN THE FRÉCHET CASE

In 1974, Loy raised the question of whether quasinilpotent non-nilpotent radicals are obtainable in some analogous fashion. We now answer this question for a more general case of Fréchet algebras as follows. We remark that the Jacobson radical in a commutative Fréchet algebra A may also be defined as the set of quasinilpotent elements, that is, $\operatorname{Rad} A = \{x \in A : r(x) = 0\}$,

where $r(x) = \sup_{k\in\mathbb{N}} r_k(x_k)$, $x = (x_k) \in A = \varprojlim A_k$, the Arens-Michael representation of A [17].

First, we consider the Banach algebra situation. So let A be a commutative normed algebra, M an A-module which is also a commutative Banach algebra, $D = \{D_1, D_2, \dots\}$ a higher derivation of infinite order of A into M, so that for $x, y \in A$ and $s \in \mathbb{N}$,

$$D_s(xy) = xD_s(y) + yD_s(x) + \sum_{i=1}^{s-1} D_i(x)D_{s-i}(y).$$

In parallel to $\mathcal{A}_r$ let $\mathcal{A}_\infty$ denote the vector space $\overline{A} \oplus M^\infty$ with product

$$\big(x, \{m_i\}\big)\,\big(y, \{n_i\}\big) = \left(xy, \left\{xn_i + ym_i + \sum_{j=1}^{i-1} m_j n_{i-j}\right\}\right)$$

and metric

$$d\big((x, \{m_i\}), 0\big) = \|x\| + \sum_{i=1}^{\infty} \frac{2^{-i}\|m_i\|'}{1 + \|m_i\|'}.$$

We also have the metric d_D on $A \oplus M^\infty$,

$$d\big((x, \{m_i\}), 0\big)_D = \|x\| + \sum_{i=1}^{\infty} \frac{2^{-i}\|D_i(x) - m_i\|'}{1 + \|D_i(x) - m_i\|'},$$

the isomorphism $\theta_D\colon \big(x, \{m_i\}\big) \to \big(x, \{D_i(x) - m_i\}\big)$ and the completion $\overline{A_D}$ of A under the metric $x \mapsto d(\iota(x), 0)_D$. We consider a specific case below.

Let A be the algebra of polynomials on the closed unit disc Δ, with the uniform norm $\|\cdot\|_\infty$, $M = \mathbb{C}$ with module action as before and a higher point derivation $D = \{D_1, D_2, \dots\}$ of infinite order given by $D_i p = \frac{p^{(i)}(1)}{i!}$, so $M^\infty = \mathbb{C}_0[[X]]$, with powers in M^∞ move to the right due to the convolution product (and thus, they would eventually be zero in $M^r, r \in \mathbb{N}$); of course, continuity of multiplication is apparent if one thinks of the usual coordinate-wise convergence topology τ_c on $\mathbb{C}[[X]]$, which is equivalent to d. We wish to show that given a sequence (α_i) of complex numbers, there exists a sequence (p_n) in A with $\|p_n\|_\infty \to 0$ and $D_i(p_n) \to \alpha_i$ for each i. By Loy's arguments on p. 415, for each fixed $k \in \mathbb{N}$, there is a sequence (p_n^k) in A such that $\|p_n^k\|_\infty \to 0$ and $D_i(p_n^k) \to \alpha_i$ for $i = 1, \dots, k$. For each $k \in \mathbb{N}$, choose a polynomial $p_k = p_n^k$ from the sequence (p_n^k) by taking n sufficiently large such that $\|p_k\|_\infty < \frac{1}{k}$ and $|D_i(p_k) - \alpha_i| < \frac{1}{k}$ for $i = 1, \dots, k$. Then we have a sequence (p_k) such that $\|p_k\|_\infty < \frac{1}{k}$ and for $i \in \mathbb{N}$, for any $k > i$, $|D_i(p_k) - \alpha_i| < \frac{1}{k}$. Thus $\|p_k\|_\infty \to 0$ and for every $i \in \mathbb{N}$, $D_i(p_k) \to \alpha_i$. The existence of the required

sequence (p_n) is now clear. Now let Θ denote the extension by continuity of the isomorphism $\theta_D \iota$ to $\overline{A_D}$. Then if $(x, \{\alpha_i\}) \in A \oplus M^\infty$ we have

$$(x, \{\alpha_i\}) = (x, \{D_i x\}) + (0, \{\alpha_i - D_i x\}) \in \Theta(\overline{A_D}),$$

and since Θ is an isometry and $\overline{A_D}$ is complete, it follows that Θ maps $\overline{A_D}$ onto $\mathcal{A}_\infty = \overline{A} \oplus M^\infty$ which contains the radical $\{0\} \oplus M^\infty$ (note that, by an analogous argument, preceding to Theorem 2.3, $\operatorname{Rad} \mathcal{A}_\infty = \operatorname{Rad} \overline{A} \oplus M^\infty = \{0\} \oplus M^\infty$). Thus $\overline{A_D}$ has a radical which has quasinilpotent non-nilpotent elements. We do not know whether $\overline{A_D}$ has a unique Fréchet algebra topology. However we have the following result whose proof we omit (see [8, 2.2.46 (ii)] for details on the Dales-McClure Banach algebra, and to know about the two inequivalent Fréchet algebra topologies on $\overline{A_D}$, follow either Theorem 3.4 or Theorem 2.3).

THEOREM 4.1. *Let A be the Dales-McClure Banach algebra, $D = (D_i)$ a totally discontinuous higher point derivation of infinite order at a character ϕ. Then $\overline{A_D}$ admits two inequivalent Fréchet algebra topologies and has quasinilpotent non-nilpotent elements.*

Remark E. Since Θ above is an isometric isomorphism, $\overline{A_D}$ is always a Fréchet algebra, and it can never be a Banach algebra, even if A is a normed (resp., Banach) algebra. This happens because $D = \{D_1, D_2, \dots\}$ is a totally discontinuous higher point derivation of infinite order (with $M = \mathbb{C}$), and so, by [12, Theorem 11.2], we have an epimorphism onto $M^\infty = \mathcal{F}X$. In other words, Loy's question cannot have a solution in the Banach case. However, we do not know whether Loy's question can have a solution in the Banach case otherwise (cf. a question raised in Introduction).

Now we can consider the Fréchet case by combining the situations given in Section 3 and in the Banach case above. We consider a specific case below.

Let A be the algebra of polynomials on a fixed open neighbourhood U of the closed unit disc Δ, with seminorms p_k generating the compact-open topology, $M = \mathbb{C}$ with module action as before and a higher point derivation D of infinite order as above, so that

$$q'_{k,D}(p) = p_k(p) + \sum_{i=1}^\infty \frac{\left|p^{(i)}(1)\right|}{i!}.$$

Following the arguments given in the Banach case above, we see that $\overline{A_D}$ has a radical which has quasinilpotent non-nilpotent elements. We do not know

whether $\overline{A_D}$ has a unique Fréchet algebra topology. However we have the following result whose proof we omit. We note that the Dales-McClure Fréchet algebra can be constructed along the lines of the Dales-McClure Banach algebra by replacing a weight ω on $\mathbb{Z}^+$ by an increasing sequence $W = (\omega_k)$ of weights on $\mathbb{Z}^+$. Thus, $\check{V}_W E = \bigcap_{k=1}^{\infty} \check{V}_{\omega_k} E$, is an analogue of the Beurling-Fréchet algebra (see [4, Example 1.2]).

THEOREM 4.2. *Let A be the Dales-McClure Fréchet algebra, $D = (D_i)$ a totally discontinuous higher point derivation of infinite order at a continuous character ϕ. Then $\overline{A_D}$ admits two inequivalent Fréchet algebra topologies and has quasinilpotent non-nilpotent elements.*

We now obtain some special results when A is the algebra of polynomials on a fixed open neighbourhood U of the closed unit disc Δ. Let (M_k) be a sequence of positive reals such that

$$\frac{M_k}{k!} \geq \frac{M_i}{i!} \frac{M_{k-i}}{(k-i)!}$$

for $1 \leq i < k$ so that, in particular, the sequence $k \mapsto (\frac{k!}{M_k})^{\frac{1}{k}}$ is monotonic decreasing, with limit $\gamma \geq 0$. Consider the seminorms q_k on A given by

$$q_k(p) = p_k(p) + \sum_{i=1}^{\infty} \frac{\left| p^{(i)}(1) \right|}{M_i}$$

and let $\overline{A_\infty}$ be the completion of A under (q_k). Then we have the following result (see Loy's Theorem 3 for details in the Banach case).

THEOREM 4.3. *$\overline{A_\infty}$ is semisimple if $\gamma > 0$. For each $k \in \mathbb{N}$, (A, q_k) is natural if and only if $\gamma = 0$.*

Proof. Let $\overline{A_\infty} = \varprojlim \left(\overline{A_\infty} \right)_k$ be the Arens-Michael representation of $\overline{A_\infty}$, then for each $k \in \mathbb{N}$, $\left(\overline{A_\infty} \right)_k$ satisfies Loy's Theorem 3 (and hence, the use of (M_k) is implicit and necessary for the Fréchet case). Thus, for each $k \in \mathbb{N}$, $\left(\overline{A_\infty} \right)_k$ is semisimple if $\gamma > 0$, and so, $\overline{A_\infty}$ is also a semisimple Fréchet algebra. Moreover, for each $k \in \mathbb{N}$, (A, q_k) is natural if and only if $\gamma = 0$. ∎

Remark F. In this case, the inverse limit algebra $(A, (q_k))$ is also natural. However we note that for the inverse limit algebra $(A, (q_k))$ to be natural, then it is not necessary that each (A, q_k) is natural. We exhibit this interesting case

as follows. For each fixed $k \in \mathbb{N}$, let $\{M_i^k\}$ be a sequence of positive reals with properties as given above. Consider the seminorms q_k' on A given by

$$q_k'(p) = p_k(p) + \sum_{i=1}^{\infty} \frac{|p^{(i)}(1)|}{M_i^k}$$

and let $\overline{A_\infty}$ be the completion of A under (q_k'). We select the sequence $\{M_i^k\}$ of sequences of positive reals such that the limit $\gamma_k > 0$ for each fixed k, but $\gamma_k \to 0$ as $k \to \infty$, that is, $\gamma = \lim \gamma_k = 0$. Thus, by Loy's Theorem 3, each (A, q_k') is not natural and the completion $(\overline{A_\infty})_k$ is semisimple. Then $\overline{A_\infty} = \varprojlim (\overline{A_\infty})_k$ is a semisimple Fréchet algebra. This shows that the converse of Theorem 4.3 does not hold. Also the inverse limit algebra $(A, (q_k'))$ is natural. To exhibit the above situation, take $M_i^k = k^i i!$ so that $\gamma_k = \frac{1}{k}$ with $\lim \gamma_k = 0$. On the other hand, Rolewicz in [23] constructed an example of a semisimple Fréchet algebra which is not an inverse limit of semisimple Banach algebras. So it is possible to have $\overline{A_\infty} = \varprojlim (\overline{A_\infty})_k$ a semisimple Fréchet algebra such that each $(\overline{A_\infty})_k$ is not a semisimple Banach algebra, and so, by Loy's Theorem 3, $\gamma_k = 0$ for each k (with $\gamma = \lim \gamma_k = 0$) which implies that each (A, q_k') is natural and so is the inverse limit algebra $(A, (q_k'))$.

REFERENCES

[1] G.R. ALLAN, Fréchet algebras and formal power series, *Studia Math.* **119** (1996), 271–278.

[2] R. ARENS, The space L^ω and convex topological rings, *Bull. Amer. Math. Soc.* **52** (1946), 931–935.

[3] W.G. BADE, P.C. CURTIS JR., The continuity of derivations of Banach algebras, *J. Functional Analysis* **16** (1974), 372–387.

[4] S.J. BHATT, S.R. PATEL, On Fréchet algebras of power series, *Bull. Austral. Math. Soc.* **66** (2002), 135–148.

[5] R.L. CARPENTER, Continuity of derivations on F-algebras, *Amer. J. Math.* **93** (1971), 500–502.

[6] R.L. CARPENTER, Uniqueness of topology for commutative semisimple F-algebras, *Proc. Amer. Math. Soc.* **29** (1971), 113–117.

[7] H.G. DALES, The uniqueness of the functional calculus, *Proc. London Math. Soc. (3)* **27** (1973), 638–648.

[8] H.G. DALES, "Banach Algebras and Automatic Continuity", London Math. Soc. Monogr. (N.S.) 24, Clarendon Press, Oxford, 2000.

[9] H.G. DALES, R.J. LOY, Uniqueness of the norm topology for Banach algebras with finite-dimensional radical, *Proc. London Math. Soc. (3)* **74** (1997), 633–661.

[10] H.G. DALES, J.P. MCCLURE, Continuity of homomorphisms into certain commutative Banach algebras, *Proc. London Math. Soc. (3)* **26** (1973), 69–81.

[11] H.G. DALES, J.P. MCCLURE, Completion of normed algebras of polynomials, *J. Austral. Math. Soc. Ser. A* **20** (1975), 504–510.

[12] H.G. DALES, S.R. PATEL, C.J. READ, Fréchet algebras of power series, in: "Banach Algebras 2009", Banach Center Publi. 91, Polish Acad. Sci., Warsaw, 2010, 123–158.

[13] Y. DOMAR, A Banach algebra related to the disk algebra, in: "Radical Banach Algebras and Automatic Continuity 1981", Lecture Notes in Math. 975, Springer-Verlag, Berlin-New York, 1983, 309–311.

[14] C. FELDMAN, The Wedderburn principal theorem in Banach algebras, *Proc. Amer. Math. Soc.* **2** (1951), 771–777.

[15] M. FRAGOULOPOULOU, M. WEIGT, I. ZARAKAS, Derivations of locally convex *-algebras, *Extracta Math.* **26** (2011), 45–60.

[16] R.J. LOY, Commutative Banach algebras with non-unique complete norm topology, *Bull. Austral. Math. Soc.* **10** (1974), 409–420.

[17] S.R. PATEL, Fréchet algebras, formal power series, and automatic continuity, *Studia Math.* **187** (2008), 125–136.

[18] S.R. PATEL, Uniqueness of the Fréchet algebra topology on certain Fréchet algebras, *Studia Math.* **234** (2016), 31–47.

[19] S.R. PATEL, On avatars of the Singer-Wermer conjecture in Fréchet algebras, in: "Proceedings of the ICTAA 2021", Mathematics Studies 8, Tartu, 2021, 121–137.

[20] S.R. PATEL, On functional analytic approach for Gleason's problem in the theory of SCV, *Banach J. Math. Anal.* **16** (2022), article no. 47, 17 pp.

[21] H.L. PHAM, Automatic continuity for Banach algebras with finite-dimensional radicals, *J. Austral. Math. Soc.* **81** (2006), 279–295.

[22] C.J. READ, Derivations with large separating subspace, *Proc. Amer. Math. Soc.* **130** (2002), 3671–3677.

[23] S. ROLEWICZ, Example of semisimple m-convex B_0-algebra, which is not a projective limit of semisimple B-algebras, *Bull. Acad. Polon. Sci. Sér. Math. Astronom. Phys.* **11** (1963), $459-462$.

[24] M.P. THOMAS, The image of a derivation is contained in the radical, *Ann. of Math.* **128** (1988), $435-460$.

[25] D. VOGT, Non-natural topologies on spaces of holomorphic functions, *Ann. Polon. Math.* **108** (2013), $215-217$.

[26] M. WEIGT, I. ZARAKAS, A note on derivations of Pro-C^*-algebras into complete locally convex bimodules, *New Zealand J. Math.* **41** (2011), $143-152$.

[27] M. WEIGT, I. ZARAKAS, Derivations of generalized B*-algebras, *Extracta Math.* **28** (2013), $77-94$.

[28] M. WEIGT, I. ZARAKAS, Spatiality of derivations of Fréchet GB*-algebras, *Studia Math.* **231** (2015), $219-239$.

[29] M. WEIGT, I. ZARAKAS, Derivations of Fréchet nuclear GB*-algebras, *Bull. Austral. Math. Soc.* **92** (2015), $290-301$.

[30] M. WEIGT, I. ZARAKAS, Unbounded derivations of GB*-algebras, in Oper. Theory Adv. Appl., 247, Birkhäuser/Springer, Cham, 2015, $69-82$.

[31] M. WEIGT, I. ZARAKAS, On domains of unbounded derivations of generalized B*-algebras, *Banach J. Math. Anal.* **12** (2018), $873-908$.