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$\otimes$-Pure model structure on the category of N-complexes

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Abstract: Let $\mathcal{G}$ be a closed symmetric monoidal concrete Grothendieck category. In this paper, we introduce a model structure on $(\mathbb{C}_N(\mathcal{G}), \mathcal{P}_{\otimes_{dw}})$ the exact category of N-complexes with the degreewise $\otimes$-pure exact structure. Our result is based on the Gillespie's Theorem by introducing two compatible cotorsion pairs on this category.

Key words: N-complexes; complete cotorsion pairs; model structure; pure derived category; pure exact structure.

MSC (2020): 18G35, 18G80, 18M05.

1. Introduction

The concept of a model category, which has been in existence for approximately fifty years, was introduced by Quillen in [27]. Quillen developed the definition of a model category to formalize the similarities between homotopy theory and homological algebra. The key examples which motivated his definition were the category of topological spaces, the category of simplicial sets, and the category of chain complexes. The fundamental problem that model categories address is the treatment of certain non-isomorphic maps (weak equivalences) that are desired to be considered as isomorphisms and since this idea of inverting weak equivalences is so central in mathematics, model categories are extremely important.

The notion of cotorsion pairs (or cotorsion theory) was invented by Salce [28] in the category of abelian groups and was rediscovered by Enochs and coauthors in the 1990's. In short, a cotorsion pair in an abelian category $\mathcal{A}$ is a pair $(\mathcal{F}, \mathcal{C})$ of classes of object of $\mathcal{A}$ each of which is the orthogonal complement of the other with respect to the Ext functor. In recent years, the investigation of cotorsion pairs has proven particularly related to the study of

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covers and envelopes, notably in the context of demonstrating the flat cover conjecture. Another application of cotorsion pairs is found in abelian model structures, as defined by Hovey. Specifically, Hovey established that a Quillen model structure in any abelian category $\mathcal{A}$ corresponds to two complete cotorsion pairs in $\mathcal{A}$ that are compatible, known as Hovey pairs. Gillespie's work further extended this concept to exact categories. Numerous examples of cotorsion pairs and their corresponding model structures on the category of complexes were introduced based on Hovey's theorem and Gillespie's work., see [16, 29, 12, 13, 14, 9, 4, 32]. One of the model structures constructed in the category of complexes of a closed symmetric monoidal Grothendieck category was introduced by Estrada, Gillespie, and Odabasi in [11]. They define the pure derived category with respect to the monoidal structure via a relative injective model structure on the category of unbounded complexes. Since the concept of N-complexes is a generalization of the ordinary complexes, it is natural to study this model structures on the category of N-complexes. The notion of N-complexes was introduced by Mayer [26] in the his study of simplicial complexes and its homological theory was studied by Kapranov and Dubois-Violette in [23, 8]. Besides their applications in theoretical physics [7, 20], the homological properties of N-complexes have become a subject of study for many authors as in [10, 17, 15, 31]. By an N-complex X, we mean a sequence $\cdots \to X^{n-1} \to X^n \to X^{n+1} \to \cdots$ such that composition of any N consecutive maps gives the zero map. In recent years, many authors have focused on N-complexes as a generalization of some concepts in the Homology like derived category and homotopy category, see [22, 17, 3, 5, 25, 24, 34, 33].

In this paper, we will introduce a model structure on $(\mathbb{C}_N(\mathcal{G}), \mathcal{P}_{\otimes_{dw}})$ the exact category of N-complexes with the degree-wise $\otimes$-pure exact structure where $\mathcal{G}$ is a concrete Grothendieck category as in subsection 2.3. Our results are based on the Gillespie's Theorem in [16] by introducing two compatible cotorsion pairs on this category. More precisely:

THEOREM 1.1. *Let $\mathcal{G}$ be as above. Then there is a model structure on the exact category $(\mathbb{C}_N(\mathcal{G}), \mathcal{P}_{\otimes_{dw}})$ where $\mathbb{C}_N(\mathcal{G})$ (resp. $\mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})$) is the class of cofibrant (resp. trivially cofibrant) objects,* $\mathrm{dg}_N \otimes$ *-PInj (resp. $(\widetilde{\otimes\text{-PInj}})_N$) is the class of fibrant (resp. trivially fibrant) objects and trivial objects are $\otimes$-pure acyclic N-complexes. we call this model structure the $\otimes$-pure injective model structure on $(\mathbb{C}_N(\mathcal{G}), \mathcal{P}_{\otimes_{dw}})$ and its homotopy category is $\mathbb{D}_{N\text{-}\otimes\text{-pur}}(\mathcal{G})$.*

The paper is organized as follows. In Section 2 we recall some generality on N-complexes and provide the background information needed through this paper such as exact category and purity. Our main result appears in Section 3 as Theorem 3.8.

2. Preliminaries

2.1. Model structures on exact category Model categories were first introduced by Quillen [27]. Let $\mathcal{C}$ be a category. A model structure on $\mathcal{C}$ is a triple $(\mathrm{Cof}, \mathrm{W}, \mathrm{Fib})$ of classes of morphisms, called *cofibrations*, *weak equivalences* and *fibrations*, respectively, such that satisfying certain axioms. Morphisms in $\mathrm{Cof}\cap\mathrm{W}$ are called *trivial cofibrations* and morphisms in $\mathrm{W}\cap\mathrm{Fib}$ are *trivial fibrations*.

The definition of model structure then was modified by some authors. The one that is commonly used nowadays is due to Hovey [21]. Hovey discovered that the existence of a model structure on any abelian category $\mathcal{A}$ is equivalent to the existence of two complete cotorsion pairs in $\mathcal{A}$ which are compatible in a precise way.

Gillespie followed [21] and focused on exact categories with model structure compatible with the exact structure. He defined cotorsion pairs in exact categories and saw that Hovey's correspondence between abelian model structures and cotorsion pairs naturally carries over to a correspondence between exact model structures and cotorsion pairs.

Recall that an exact category is a pair $(\mathcal{E}, \mathfrak{E})$ where $\mathcal{E}$ is an additive category and $\mathfrak{E}$ is a distinguished class of diagrams of the form $X \overset{i}{\to} Y \overset{d}{\to} Z$ called *conflation* (we refer to it as a short exact sequence), satisfying certain axioms which make conflations behave similar to short exact sequences in an abelian category. We refer the reader to [6] for a readable introduction to exact categories.

Before mentioning Gillespie's theorem, we recall the notion of a cotorsion pair in an exact category. Let $(\mathcal{E}, \mathfrak{E})$ be an exact category. The axioms of exact category allow us to define Yoneda Ext groups with usual properties. The abelian group $\mathrm{Ext}^1_{\mathcal{E}}(X, Y)$ is the group of equivalence classes of short exact sequences $Y \rightarrowtail Z \twoheadrightarrow X$. In particular, $\mathrm{Ext}^1_{\mathcal{E}}(X, Y) = 0$ if and only if every short exact sequence $Y \rightarrowtail Z \twoheadrightarrow X$ is isomorphic to the split exact sequence $Y \rightarrowtail Y \oplus X \twoheadrightarrow X$.

A pair $(\mathcal{F}, \mathcal{D})$ of full subcategories of $\mathcal{E}$ is called a cotorsion pair provided that

$$\mathcal{F} = {}^{\perp}\mathcal{D} \qquad \text{and} \qquad \mathcal{F}^{\perp} = \mathcal{D},$$

where $^{\perp}$ is taken with respect to the functor $\mathrm{Ext}^1_{\mathcal{E}}$. The cotorsion pair $(\mathcal{F}, \mathcal{D})$

is said to have enough projectives if for every $X \in \mathcal{E}$ there is a short exact sequence $D \rightarrowtail F \twoheadrightarrow X$ with $D \in \mathcal{D}$ and $F \in \mathcal{F}$. We say that it has enough injectives if it satisfies the dual statement. If both of these hold we say the cotorsion pair is complete. The next theorem is a result due to Hovey [21] which is described by Gillespie in the sense of exact category, see [16]. We just recall that a class of objects $\mathcal{W} \in \mathcal{E}$ is a *thick subcategory* of $\mathcal{E}$ if it is closed under direct summands and if two out of three of the terms in a short exact sequence are in $\mathcal{W}$, then so is the third.

THEOREM 2.1. ([16, THEOREM 3.3]) *Let $(\mathcal{E}, \mathfrak{E})$ be an exact category with an exact model structure. Let $\mathcal{C}$ be the class of cofibrant objects, $\mathcal{F}$ be the class of fibrant objects and $\mathcal{W}$ be the class of trivial objects. Then $\mathcal{W}$ is a thick subcategory of $\mathcal{E}$ and both $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are complete cotorsion pairs in $\mathcal{A}$. If we further assume that $(\mathcal{E}, \mathfrak{E})$ is weakly idempotent complete then the converse holds. That is, given two compatible cotorsion pairs $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$, each complete and with $\mathcal{W}$ a thick subcategory, then there is an exact model structure on $\mathcal{E}$ where $\mathcal{C}$ is the class of cofibrant objects, $\mathcal{F}$ is the class of fibrant objects and $\mathcal{W}$ is the class of trivial objects.*

2.2. THE CATEGORY OF N-COMPLEXES ON EXACT CATEGORY Let $(\mathcal{E}, \mathfrak{E})$ be an efficient exact category. We fix a positive integer $N \geq 2$. An N-complex is a diagram

$$\cdots \xrightarrow{d_{\mathbf{X}}^{i-1}} X^i \xrightarrow{d_{\mathbf{X}}^{i}} X^{i+1} \xrightarrow{d_{\mathbf{X}}^{i+1}} \cdots$$

with $X^i \in \mathcal{E}$ and morphisms $d_{\mathbf{X}}^i \in \mathrm{Hom}_{\mathcal{E}}(X^i, X^{i+1})$ satisfying $d^N = 0$. That is, composing any N-consecutive maps gives 0. A morphism between N-complexes is a commutative diagram

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{d_{\mathbf{X}}^{i-1}} & X^i & \xrightarrow{d_{\mathbf{X}}^{i}} & X^{i+1} & \xrightarrow{d_{\mathbf{X}}^{i+1}} & \cdots \\
 & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} & & \\
\cdots & \xrightarrow{d_{\mathbf{Y}}^{i-1}} & Y^i & \xrightarrow{d_{\mathbf{Y}}^{i}} & Y^{i+1} & \xrightarrow{d_{\mathbf{Y}}^{i+1}} & \cdots
\end{array}$$

We denote by $\mathbb{C}_N(\mathcal{E})$ the category of unbounded N-complexes on $(\mathcal{E}, \mathfrak{E})$. For any object M of $\mathcal{E}$ and any j and $1 \leq i \leq N$, let

$$D_i^j(M): \quad \cdots \longrightarrow 0 \longrightarrow X^{j-i+1} \xrightarrow{d_{\mathbf{X}}^{j-i+1}} \cdots \xrightarrow{d_{\mathbf{X}}^{j-2}} X^{j-1} \xrightarrow{d_{\mathbf{X}}^{j-1}} X^j \longrightarrow 0 \longrightarrow \cdots$$

be an N-complex satisfying $X^n = M$ and $d^n_{\mathbf{X}} = 1_M$ for all $(j-i+1 \le n \le j)$. For $0 \le r < N$ and $i \in \mathbb{Z}$, we define

$$d^i_{\mathbf{X},\{r\}} := d^{i+r-1}_{\mathbf{X}} \cdots d^i_{\mathbf{X}}.$$

In this notation $d^i_{\{1\}} = d^i$ and $d^i_{\{0\}} = 1_{X_i}$.

Definition 2.2. A morphism $f : \mathbf{X} \longrightarrow \mathbf{Y}$ of N-complexes is called null-homotopic if there exists $s^i \in \mathrm{Hom}_{\mathcal{E}}(X^i, Y^{i-N+1})$ such that

$$f^i = \sum_{j=0}^{N-1} d^{i-(N-1-j)}_{\mathbf{Y},\{N-1-j\}} s^{i+j} d^i_{\mathbf{X},\{j\}}.$$

We denote by $\mathbb{K}_N(\mathcal{E})$ the homotopy category of unbounded N-complexes on $(\mathcal{E}, \mathfrak{E})$.

Definition 2.3. For $\mathbf{X} = (X^i, d^i) \in \mathbb{C}_N(\mathcal{E})$, we define suspension functor $\Sigma : \mathbb{K}_N(\mathcal{E}) \longrightarrow \mathbb{K}_N(\mathcal{E})$ as follows:

$$(\Sigma\mathbf{X})^m = \coprod\nolimits_{i=m+1}^{m+N-1} X^i, \qquad (\Sigma^{-1}\mathbf{X})^m = \coprod\nolimits_{i=m-1}^{m-N+1} X^i,$$

$$d^m_{\Sigma\mathbf{X}} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ -d^{m+1}_{\{N-1\}} & -d^{m+2}_{\{N-2\}} & \cdots & \cdots & \cdots & -d^{m+N-1} \end{bmatrix},$$

$$d^m_{\Sigma^{-1}\mathbf{X}} = \begin{bmatrix} -d^{m-1} & 1 & 0 & \cdots & \cdots & 0 \\ -d^{m-1}_{\{2\}} & 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ -d^{m-1}_{\{N-2\}} & 0 & \cdots & \cdots & 0 & 1 \\ -d^{m-1}_{\{N-1\}} & 0 & \cdots & \cdots & \cdot & 0 \end{bmatrix}.$$

Let $\mathfrak{E}_{dw}(\mathcal{E})$ be the collection of conflations in $\mathbb{C}_N(\mathcal{E})$ with split short exact sequences in each degree. In the same manner in [22] it can be shown that $(\mathbb{C}_N(\mathcal{E}), \mathfrak{E}_{dw}(\mathcal{E}))$ is a Frobenius category and its stable category is the homotopy category $\mathbb{K}_N(\mathcal{E})$ of $\mathcal{E}$. So $\mathbb{K}_N(\mathcal{E})$ together with this suspension functor is a triangulated category, see [22, Theorem 2.6].

Recall that $\mathrm{Ext}^1_{\mathbb{C}_N(\mathcal{E})}(X, Y)$ is the group of (equivalence classes) of short exact sequences $\mathbf{Y} \rightarrowtail \mathbf{Z} \twoheadrightarrow \mathbf{X}$. We let $\mathrm{Ext}^1_{dw}(X, Y)$ be the subgroup of $\mathrm{Ext}^1_{\mathbb{C}_N(\mathcal{E})}(X, Y)$ consisting of those short exact sequences which are split in each degree.

LEMMA 2.4. *For N-complex* $\mathbf{X}$ *and* $\mathbf{Y}$*, we have*

$$\mathrm{Ext}^1_{dw}(\mathbf{Y}, \mathbf{X}) \cong \mathrm{Hom}_{\mathbb{K}_N(\mathcal{E})}(\mathbf{Y}, \Sigma\mathbf{X}).$$

Proof. The proof is exactly similar to [3, Lemma 2.4]. ∎

Let $\mathcal{A}$ be an abelian category and $\mathbf{X}$ be an N-complex of objects of $\mathcal{A}$ as follows:

$$\cdots \xrightarrow{d_{\mathbf{X}}^{i-1}} X^i \xrightarrow{d_{\mathbf{X}}^{i}} X^{i+1} \xrightarrow{d_{\mathbf{X}}^{i+1}} \cdots$$

we define

$$\mathrm{Z}^i_r(\mathbf{X}) := \mathrm{Ker}(d_{\mathbf{X}}^{i+r-1} \cdots d_{\mathbf{X}}^{i}), \qquad \mathrm{B}^i_r(\mathbf{X}) := \mathrm{Im}((d_{\mathbf{X}}^{i-1} \cdots d_{\mathbf{X}}^{i-r}),$$

$$\mathrm{C}^i_r(\mathbf{X}) := \mathrm{Coker}((d_{\mathbf{X}}^{i-1} \cdots d_{\mathbf{X}}^{i-r}), \qquad \mathrm{H}^i_r(\mathbf{X}) := \mathrm{Z}^i_r(\mathbf{X})/\mathrm{B}^i_{N-r}(\mathbf{X}).$$

Therefore in each degree, we have $N-1$ cycle and clearly $\mathrm{Z}^n_N(\mathbf{X}) = X^n$.

DEFINITION 2.5. Let $\mathbf{X} \in \mathbb{C}_N(\mathcal{A})$. We say $\mathbf{X}$ is N-acyclic if $\mathrm{H}^i_r(\mathbf{X}) = 0$ for each $i \in \mathbb{Z}$ and all $r = 1, 2, \ldots, N-1$. We denote the full subcategory of $\mathbb{C}_N(\mathcal{A})$ consisting of N-acyclic complexes by $\mathbb{C}_{N\text{-ac}}(\mathcal{A})$.

Remark 2.6. An N-complex $\mathbf{X}$ is N-acyclic if and only if there exists some r with $1 \le r \le N-1$ such that $\mathrm{H}^i_r(\mathbf{X}) = 0$ for each integer i, see [23].

Remark 2.7. By using [22, Proposition 3.2 (2)], it is easy to see that whenever $\mathbf{X}$ is an N-acyclic complex then $\Sigma\mathbf{X}$ and $\Sigma^{-1}\mathbf{X}$ are N-acyclic complexes.

We also have the following lemma:

LEMMA 2.8. *For an object $M \in \mathcal{A}$, $i \in \mathbb{Z}$, $1 \leq r \leq N-1$, and $\mathbf{X}, \mathbf{Y} \in \mathbb{C}_N(\mathcal{A})$ we have the following isomorphisms:*

$$\mathrm{Hom}_{\mathbb{C}_N(\mathcal{A})}(D_r^i(M), \mathbf{Y}) \cong \mathrm{Hom}_{\mathcal{A}}(M, \mathrm{Z}_r^i(\mathbf{Y})).$$

Proof. See [15, Section 4] or [33, Lemma 2.2] for more details. ∎

2.3. PURITY Let $\mathcal{G}$ be a closed symmetric monoidal Grothendieck category endowed with a faithful functor $U : \mathcal{G} \to \mathbf{Set}$, where $\mathbf{Set}$ denotes the category of sets. By abuse of notation, we write $x \in G$ instead of $x \in U(G)$, for any object G in $\mathcal{G}$. Analogously, $|G|$ will denote the cardinality of $U(G)$. We will also assume that there exists an infinite regular cardinal λ such that for each $G \in \mathcal{G}$ and any set $S \subseteq G$ with $|S| < \lambda$, there is a subobject $X \subseteq G$ such that $S \subseteq X \subseteq G$ and $|X| < \lambda$.

Given an infinite regular cardinal κ. Recall that an object $X \in \mathcal{G}$ is called κ-presentable if the functor $\mathrm{Hom}_{\mathcal{G}}(X, -) : \mathcal{G} \to \mathbf{Ab}$ preserves κ-filtered colimits. An object $X \in \mathcal{G}$ is called κ-generated whenever $\mathrm{Hom}_{\mathcal{G}}(X, -)$ preserves κ-filtered colimits of monomorphisms. By our assumption, it is easy to see that

$$|X| < \lambda \quad \Longleftrightarrow \quad X \text{ is } \lambda\text{-presentable} \quad \Longleftrightarrow \quad X \text{ is } \lambda\text{-generated.}$$

DEFINITION 2.9. Let $\mathcal{F}$ be a class of objects of $\mathcal{G}$. Then $\mathcal{F}$ is called deconstructible if there exists a set $\mathcal{S} \subseteq \mathcal{G}$ of objects such that $\mathcal{F} = \text{Filt-}\mathcal{S}$ where Filt-$\mathcal{S}$ is the class of all $\mathcal{S}$-filtered objects in $\mathcal{G}$

DEFINITION 2.10. An exact category $\mathcal{E}$ is of *Grothendieck type* if $\mathcal{E}$ is efficient and deconstructible in itself.

DEFINITION 2.11. A morphism $f : X \to Y$ in $\mathcal{G}$ is called λ-pure if for any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y, \end{array}$$

where A and B are λ-presentable, there is a morphism $h : B \to X$ such that $i = h \circ g$.

According to the [1, 2.29, p. 86] every λ-pure morphism in $\mathcal{G}$ is a monomorphism.

Remark 2.12. ([1, 2.30, p. 86]) In $\mathcal{G}$, a morphism is λ-pure monomorphism if and only if it is a λ-directed colimit of split monomorphisms.

THEOREM 2.13. ([1, THEOREM 2.33]) *There exists arbitrary large regular cardinal $\kappa > \lambda$ such that every κ-presentable subobject X of A in $\mathcal{G}$ is contained in a λ-pure subobject X' of A, where X' is κ-presentable.*

DEFINITION 2.14. A monomorphism $f : X \to Y$ is called $\otimes$-pure if $f \otimes C$ is monomorphism for all $C \in \mathcal{G}$.

By Remark 2.12 and the fact that $\otimes$ preserves λ-colimits, it is clear that every λ-pure morphism is $\otimes$-pure monomorphism. We denote the proper class of λ-pure (respect, $\otimes$-pure) short exact sequences in $\mathcal{G}$ by $\mathcal{P}$ (respect, $\mathcal{P}_{\otimes}$). Therefore we have the containment $\mathcal{P} \subseteq \mathcal{P}_{\otimes}$.

Remark 2.15. It is also straightforward to check that $\mathcal{G}$ with the exact structure $\mathcal{P}_{\otimes}$ is an efficient exact category.

3. $\otimes$-PURE INJECTIVE MODEL STRUCTURE ON $\mathbb{C}_N(\mathcal{G})$

In this section we will introduce a model structure on $(\mathbb{C}_N(\mathcal{G}), \mathcal{P}_{\otimes_{dw}})$ the exact category of N-complexes with the degree-wise $\otimes$-pure exact structure where $\mathcal{G}$ is a concrete Grothendieck category as in subsection 2.3. Our results are based on [16, Theorem 3.3] by introducing two compatible cotorsion pairs on this category. First, we start by defining some new classes in $(\mathbb{C}_N(\mathcal{G}), \mathcal{P}_{\otimes_{dw}})$.

DEFINITION 3.1. An N-complex X in $\mathbb{C}_N(\mathcal{G})$ is called $\otimes$-acyclic if it is acyclic in $(\mathbb{C}_N(\mathcal{G}), \mathcal{P}_{\otimes_{dw}})$. In fact, each sequence $0 \to \mathrm{Z}_r^n(\mathbf{X}) \to X^n \to \mathrm{Z}_{N-r}^{n+r}(\mathbf{X}) \to 0$ is $\otimes$-pure exact for $n \in \mathbb{Z}$ and $1 \le r \le N-1$, or equivalently, $\mathbf{X} \otimes C$ is N-acyclic for all $C \in \mathcal{G}$. We denote by $\mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})$ the class of all $\otimes$-acyclic N-complexes.

We will also consider the following classes of $\mathbb{C}_N(\mathcal{G})$:

DEFINITION 3.2. let $\mathbb{C}_N(\mathcal{G})$ be as above. We define:

(1) The class of $\widetilde{(\otimes\text{-PInj})}_N$ consisting of all $\mathbf{X} \in \mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})$ such that $\mathrm{Z}_r^i(\mathbf{X}) \in \otimes$-PInj for all r, i, where $\otimes$-PInj is the class of all injective object with respect to exact structure $\mathcal{P}_{\otimes}$ in $\mathcal{G}$.

(2) The class of $\mathrm{dg}_N \otimes$-PInj, consisting of all $\mathbf{X} \in \mathbb{C}_N(\otimes\text{-PInj})$ such that $\mathrm{Hom}_{\mathbb{K}_N(\mathcal{G})}(\mathbf{E}, \mathbf{X}) = 0$ whenever $\mathbf{E} \in \mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})$.

PROPOSITION 3.3. *The pair* $(\mathbb{C}_N(\mathcal{G}), \widetilde{(\otimes\text{-}PInj)}_N)$ *is a complete cotorsion pair in the exact category* $(\mathbb{C}_N(\mathcal{G}), \mathcal{P}_{\otimes_{dw}})$. *Moreover,*

$$\widetilde{(\otimes\text{-}PInj)}_N = \mathrm{dg}_N \otimes\text{-}PInj \cap \mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G}).$$

Proof. First of all, the category $(\mathbb{C}_N(\mathcal{G}), \mathcal{P}_{\otimes_{dw}})$ is of a Grothendieck type. Indeed, $\mathbb{C}_N(\mathcal{G})$ is a Grothendieck category and any λ-pure subobject give us a degree-wise λ-pure monomorphism, hence degree-wise $\otimes$-pure monomorphism as well. Note that, colimits in $\mathbb{C}_N(\mathcal{G})$ are computed pointwise. We will show that $\mathbb{C}_N(\mathcal{G}) = \text{Filt-}\mathbb{C}_N(\mathcal{G})^\kappa$, where $\mathbb{C}_N(\mathcal{G})^\kappa$ is the class of all κ-presentable objects in $\mathbb{C}_N(\mathcal{G})$ for a regular cardinal $\kappa > \lambda$. First of all, note that if $\mathbf{X} \subseteq \mathbf{X}' \subseteq \mathbf{Y}$ is such that $\mathbf{X} \subseteq \mathbf{Y}$ and $\mathbf{X}'/\mathbf{X} \subseteq \mathbf{Y}/\mathbf{X}$ are λ-pure-monic in $\mathbb{C}_N(\mathcal{G})$ then $\mathbf{X}' \subseteq \mathbf{Y}$ is also λ-pure monic. Now let $\mathbf{X}$ be an N-complex. By Theorem 2.13 there exist a cardinal $\kappa_1 > \lambda$ such that for $0 \subseteq \mathbf{X}$ there exists κ_1-presentable $\mathbf{X}_1$ such that $\mathbf{X}_1 \subseteq \mathbf{X}$ is λ-pure. Then set $\mathbf{Y} = \mathbf{X}/\mathbf{X}_1$. Similarly, by applying Theorem 2.13 there is a regular cardinal $\kappa_2 > \lambda$ and $\mathbf{X}_2/\mathbf{X}_1$ such that $\mathbf{X}_2/\mathbf{X}_1 \subseteq \mathbf{Y}$ is λ-pure monic and $\mathbf{X}_2/\mathbf{X}_1$ is κ_2-presentable. According to the above fact we can say that $\mathbf{X}_2 \subseteq \mathbf{X}$ is also λ-pure monic. Whit this procedure and by utilizing the fact that every λ-pure monomorphsim is $\otimes$-pure monomorphism we can establish the existence of a large cardinal κ in which the filtration is constructed in $\mathbb{C}_N(\mathcal{G})$ with the $\otimes$-pure exact structure. So if we prove that $\widetilde{(\otimes\text{-PInj})}_N$ is equal to all injective objects in the exact category $(\mathbb{C}_N(\mathcal{G}), \mathcal{P}_{\otimes_{dw}})$ then by [29, Corollary 5.9] we get that $(\mathbb{C}_N(\mathcal{G}), \widetilde{(\otimes\text{-PInj})}_N)$ is a complete cotorsion pair in the exact category $(\mathbb{C}_N(\mathcal{G}), \mathcal{P}_{\otimes_{dw}})$. To this end, it is easy to check that $\mathbf{X}$ is in $\widetilde{(\otimes\text{-PInj})}_N$ if and only if $\mathrm{Z}^n_1(\mathbf{X}) \in \otimes$-PInj and $\mathbf{X} \cong \prod_{n\in\mathbb{Z}} D^{n+N-1}_N(\mathrm{Z}^n_1(\mathbf{X}))$ see [33, Corollary 3.5]. Moreover, in a similar manner of [17, Theorem 3.3] it can be said that $\widetilde{(\otimes\text{-PInj})}_N$ is precisely the class of all contractible N-complexes with the pure injective component, so clearly

$$\widetilde{(\otimes\text{-PInj})}_N \subseteq \mathrm{dg}_N \otimes\text{-PInj} \cap \mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G}).$$

Conversely, let $\mathbf{X} \in \mathrm{dg}_N \otimes\text{-PInj} \cap \mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})$. By assumption, $id_{\mathbf{X}} : \mathbf{X} \to \mathbf{X}$ is null homotopic. So $\mathbf{X}$ is a contractible N-complexes with each term is in $\otimes$-PInj, hence $\mathbf{X} \in \widetilde{(\otimes\text{-PInj})}_N$. ∎

We need the following lemma:

Lemma 3.4. *Let* $\mathbf{X}$ *be a* $\otimes$*-acyclic* N*-complex and* $\mathbf{X}'$ *be an* N*-subcomplex of* $\mathbf{X}$*. If* $\mathbf{X}'$ *is* N*-acyclic and* $\mathrm{Z}_r^n(\mathbf{X}') \subseteq \mathrm{Z}_r^n(\mathbf{X})$ *is* $\otimes$*-pure for each* $n \in \mathbb{Z}$*, then* $\mathbf{X}'$ *is* $\otimes$*-acyclic and* $X'^n \subseteq X^n$ *is* $\otimes$*-pure for each* $n \in \mathbb{Z}$*.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Z}_r^n(\mathbf{X}') & \xrightarrow{i'_n} & X'^n & \longrightarrow & \mathrm{Z}_{N-r}^{n+r}(\mathbf{X}') & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle z^n} & & \downarrow{\scriptstyle \imath^n} & & \downarrow{\scriptstyle z^{n+r}} & & \\ 0 & \longrightarrow & \mathrm{Z}_r^n(\mathbf{X}) & \xrightarrow{i_n} & X^n & \longrightarrow & \mathrm{Z}_{N-r}^{n+r}(\mathbf{X}) & \longrightarrow & 0\,. \end{array}$$

Since i_n and z^n are $\otimes$-pure monomorphism, hence we can say that $\imath^n \circ i'_n$ is also $\otimes$-pure monomorphism. This gives that $\mathrm{Z}_r^n(\mathbf{X}') \subseteq X'^n$ is a $\otimes$-pure monomorphism. So the above sequence is $\otimes$-pure and therefore $\mathbf{X}'$ is $\otimes$-acyclic. By using five lemma we can say that $\imath^n$ is also a $\otimes$-pure monomorphism. ∎

Proposition 3.5. *The class* $\mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})$ *is deconstructible.*

Proof. We show that there is a regular cardinal κ such that $\mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G}) = \text{Filt-}\mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})^{\kappa}$. First of all, $\mathbb{C}_{N\text{-}\otimes\ text-ac}(\mathcal{G})$ is closed under colimits. Indeed, if $\{C_i; \varphi_{ij} : C_i \to C_j\}$ is a λ-directed diagram with C_is are $\otimes$-pure acyclic of N-complexes then for each $S \in \mathcal{G}$ we have

$$\mathrm{colim}(C_i) \otimes S = \mathrm{colim}(C_i \otimes S)$$

For each i, $C_i \otimes S$ is acyclic and by the fact that colimts in $\mathbb{C}_N(\mathcal{G})$ are computed pointwise so we can say that $\mathrm{colim}(C_i \otimes S)$ is acyclic. So it is enough to show that there is a regular cardinal κ such that for each $\mathbf{X} \subseteq \mathbf{Y} \neq 0$ in $\mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})$ with $\mathbf{X}$ is κ-presentable, there exists a κ-presentable object $\mathbf{X}' \neq 0$ with $\mathbf{X} \subseteq \mathbf{X}' \subseteq \mathbf{Y}$, and $\mathbf{X}' \in \mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})$, and also $X'^n \subseteq X^n$ is $\otimes$-pure for each $n \in \mathbb{Z}$. For this purpose we use the Theorem 2.13. So let $0 \neq \mathbf{Y} \in \mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})$. By Theorem 2.13 there exists a regular cardinal $\kappa > \lambda$ such that each subcomplex $\mathbf{X}$ of $\mathbf{Y}$ can be embedded in κ-presentable object $\mathbf{X}'$ which is a λ-pure embedding. All that remains is to show $\mathbf{X}'$ belongs to $\mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})$. According to the Lemma 3.4 we need to check that $\mathbf{X}'$ is an N-acyclic complex and $\mathrm{Z}_r^n(\mathbf{X}') \subseteq \mathrm{Z}_r^n(\mathbf{Y})$ is $\otimes$-pure for all $n \in \mathbb{Z}$. For any λ-presentable object $A \in \mathcal{G}$ we can say that $D_r^n(A)$ is an λ-presentable N-complex for any $n \in \mathbb{Z}$ and $1 \leq r \leq N-1$, since $|D_r^n(A)| < \lambda$. So if we consider the short exact sequence $0 \to \mathbf{X}' \to \mathbf{Y} \to \mathbf{Y}/\mathbf{X}' \to 0$ and apply $\mathrm{Hom}_{\mathbb{C}_N(\mathcal{G})}(D_r^n(A), -)$ on it then one can see that

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{C}_N(\mathcal{G})}(D_r^n(A), \mathbf{X}') \longrightarrow \mathrm{Hom}_{\mathbb{C}_N(\mathcal{G})}(D_r^n(A), \mathbf{Y}) \longrightarrow \mathrm{Hom}_{\mathbb{C}_N(\mathcal{G})}(D_r^n(A), \mathbf{Y}/\mathbf{X}') \longrightarrow 0$$

is an exact sequence. By using Lemma 2.8 we can say that the following sequence is exact.

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{G}}(A, \mathrm{Z}_r^n(\mathbf{X}')) \longrightarrow \mathrm{Hom}_{\mathcal{G}}(A, \mathrm{Z}_r^n(\mathbf{Y})) \longrightarrow \mathrm{Hom}_{\mathcal{G}}(A, \mathrm{Z}_r^n(\mathbf{Y}/\mathbf{X}')) \longrightarrow 0\,.$$

This shows that $0 \to \mathrm{Z}_r^n(\mathbf{X}') \to \mathrm{Z}_r^n(\mathbf{Y}) \to \mathrm{Z}_r^n(\mathbf{Y}/\mathbf{X}') \to 0$ is a λ-pure exact sequence, thus $\mathrm{Z}_r^n(\mathbf{X}') \subseteq \mathrm{Z}_r^n(\mathbf{Y})$ is λ-pure monomorphism and hence it is $\otimes$-pure monomorphism as well. Now we show that $\mathbf{X}'$ is an N-acyclic complex. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Z}_{N-r}^{n-N+r}(\mathbf{X}') & \longrightarrow & \mathrm{Z}_{N-r}^{n-N+r}(\mathbf{Y}) & \longrightarrow & \mathrm{Z}_{N-r}^{n-N+r}(\mathbf{Y}/\mathbf{X}') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X'^{n-N+r} & \longrightarrow & Y^{n-N+r} & \longrightarrow & (Y/X')^{n-N+r} & \longrightarrow & 0\,. \end{array}$$

The snake lemma tells us the sequence $0 \to \mathrm{B}_{N-r}^n(\mathbf{X}') \to \mathrm{B}_{N-r}^n(\mathbf{Y}) \to \mathrm{B}_{N-r}^n(\mathbf{Y}/\mathbf{X}') \to 0$ is exact. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{B}_{N-r}^n(\mathbf{X}') & \longrightarrow & \mathrm{B}_{N-r}^n(\mathbf{Y}) & \longrightarrow & \mathrm{B}_{N-r}^n(\mathbf{Y}/\mathbf{X}') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Z}_r^n(\mathbf{X}') & \longrightarrow & \mathrm{Z}_r^n(\mathbf{Y}) & \longrightarrow & \mathrm{Z}_r^n(\mathbf{Y}/\mathbf{X}') & \longrightarrow & 0\,. \end{array}$$

Since $\mathbf{Y}$ is an N-acyclic complex then $\mathrm{B}_{N-r}^n(\mathbf{Y}) = \mathrm{Z}_r^n(\mathbf{Y})$ for any $n \in \mathbb{Z}$ and $1 \le r \le N-1$. Now by applying snake lemma on the above commutative diagram we get that $\mathrm{B}_{N-r}^n(\mathbf{X}') = \mathrm{Z}_r^n(\mathbf{X}')$ for any $n \in \mathbb{Z}$ and $1 \le r \le N-1$. So $\mathbf{X}'$ is an N-acyclic complex. ∎

PROPOSITION 3.6. *The pair* $(\mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G}), \mathrm{dg}_N \otimes\text{-PInj})$ *is a complete cotorsion pair in the exact category* $(\mathbb{C}_N(\mathcal{G}), \mathcal{P}_{\otimes_{dw}})$.

Proof. According to the Proposition 3.5 $\mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})$ is deconstructible. Therefore by [29, Theorem 5.16] we can say that $(\mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G}), \mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})^{\perp})$ is a complete cotorsion pair. So it is enough to show that $\mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})^{\perp}$ is exactly $\mathrm{dg}_N \otimes$-PInj. By Definition 3.2 if $\mathbf{X} \in \mathrm{dg}_N \otimes$-PInj then $\mathrm{Hom}_{\mathbb{K}_N(\mathcal{G})}(\mathbf{Y}, \mathbf{X}) = 0$

for any $\mathbf{Y} \in \mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})$. But Remark 2.7 tells us $\Sigma^{-1}\mathbf{Y}$ is also $\otimes$-acyclic, so $\mathrm{Hom}_{\mathbb{K}_N(\mathcal{G})}(\Sigma^{-1}\mathbf{Y}, \mathbf{X}) = 0$. Therefore by using Lemma 2.4 we can say that $\mathrm{Ext}^1_{dw}(\mathbf{Y}, \mathbf{X}) = 0$. Note that since $\mathbf{X} \in \mathrm{dg}_N \otimes$-PInj, then $\mathrm{Ext}^1_{\mathcal{P}_{\otimes_{dw}}}(\mathbf{Y}, \mathbf{X}) = \mathrm{Ext}^1_{dw}(\mathbf{Y}, \mathbf{X}) = 0$. This follows that $\mathbf{X} \in \mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})^{\perp}$. Conversely, suppose $\mathbf{X} \in \mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})^{\perp}$. It is enough to show that each X^i is $\otimes$-pure injective. Let $0 \to X^i \xrightarrow{f} Y \to Z \to 0$ be $\otimes$-pure exact sequence in $\mathcal{G}$. We construct a short exact sequence $0 \to \mathbf{X} \to \mathbf{A} \to D_N^{i+N-1}(Z) \to 0$ in $\mathbb{C}_N(\mathcal{G})$ as follows:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X^{i-2} & \xrightarrow{id} & X^{i-2} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d^{i-2}} & & \downarrow{\scriptstyle d^{i-2}} & & \downarrow & & \\
0 & \longrightarrow & X^{i-1} & \xrightarrow{id} & X^{i-1} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d^{i-1}} & & \downarrow{\scriptstyle f\circ d^{i-1}} & & \downarrow & & \\
0 & \longrightarrow & X^{i} & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d^{i}} & PO & \downarrow{\scriptstyle \delta^{i}} & & \| & & \\
0 & \longrightarrow & X^{i+1} & \xrightarrow{g^{i+1}} & A^{i+1} & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d^{i+1}} & PO & \downarrow{\scriptstyle \delta^{i+1}} & & \| & & \\
0 & \longrightarrow & X^{i+2} & \xrightarrow{g^{i+2}} & A^{i+2} & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle d^{i+N-2}} & & \downarrow{\scriptstyle \delta^{i+N-2}} & & \| & & \\
0 & \longrightarrow & X^{i+N-1} & \xrightarrow{g^{i+N-1}} & A^{i+N-1} & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d^{i+N-1}} & & \downarrow{\scriptstyle \varphi} & & \downarrow & & \\
0 & \longrightarrow & X^{i+N} & \xrightarrow{id} & X^{i+N} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & .
\end{array}
$$

Note that for $i \leq j < i+N-1$, A^{j+1}s are defined based on the pushout of

$X^j \to X^{j+1}$ along with $X^j \to A^j$ and the morphism $\varphi : A^{i+N-1} \to X^{i+N}$ is defined according to the universal property of pushout. In fact, consider the following pushout diagram:

$$\begin{array}{ccc} X^i & \xrightarrow{f} & Y \\ {\scriptstyle d^i_{\{N-1\}}}\downarrow & & \downarrow{\scriptstyle \delta^i_{\{N-1\}}} \\ X^{i+N-1} & \xrightarrow{g^{i+N-1}} & A^{i+N-1}. \end{array}$$

Now consider two morphisms $0 : Y \to X^{i+N}$ and $d^{i+N-1} : X^{i+N-1} \to X^{i+N}$. By the universal property of pushout there is a $\varphi : A^{i+N-1} \to X^{i+N}$ such that $\varphi \circ \delta^i_{\{N-1\}} = 0$ and $\varphi \circ g^{i+N-1} = d^{i+N-1}$. Clearly $\mathbf{A}$ is an N-complex. Since pushout preserves ⊗-pure monomorphism, therefore the above sequence is an exact sequence in $(\mathbb{C}_N(\mathcal{G}), \mathcal{P}_{\otimes_{dw}})$.

On the other hand, $D_N^{i+N-1}(Z)$ is ⊗-acyclic so by assumption the sequence $0 \to \mathbf{X} \to \mathbf{A} \to D_N^{i+N-1}(Z) \to 0$ is split, in particular it is degree-wise split, hence X^i is ⊗-pure injective. ∎

Definition 3.7. Consider the exact category $(\mathbb{C}_N(\mathcal{G}), \mathcal{P}_{\otimes_{dw}})$. Let $\mathbb{K}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})$ be a full subcategory of $\mathbb{K}_N(\mathcal{G})$ consisting of ⊗-pure acyclic N-complexes. Notice that ⊗-pure acyclic N-complexes are closed under homotopy equivalences, so $\mathbb{K}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})$ is well defined. If $f : \mathbf{X} \to \mathbf{Y}$ is a morphism between ⊗-pure acyclic N-complexes, then $\mathrm{Con}(f)$ is again ⊗-pure acyclic. Thus $\mathbb{K}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})$ is a triangulated subcategory of $\mathbb{K}_N(\mathcal{G})$. Because ⊗-pure acyclic N-complexes are closed under direct summands, $\mathbb{K}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})$ is a thick subcategory of $\mathbb{K}_N(\mathcal{G})$. Then by the Verdier's correspondence, we get the ⊗-pure derived category

$$\mathbb{D}_{N\text{-}\otimes\text{-pur}}(\mathcal{G}) := \mathbb{K}_N(\mathcal{G})/\mathbb{K}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})$$

Now we are ready to introduce our main result.

Theorem 3.8. *Let $\mathcal{G}$ be as above. Then there is a model structure on the exact category $(\mathbb{C}_N(\mathcal{G}), \mathcal{P}_{\otimes_{dw}})$ where $\mathbb{C}_N(\mathcal{G})$ (resp. $\mathbb{C}_{N\text{-}\otimes\text{-ac}}(\mathcal{G})$) is the class of cofibrant (resp. trivially cofibrant) objects, $\mathrm{dg}_N \otimes$ -PInj (resp. $(\widetilde{\otimes\text{-PInj}})_N$) is the class of fibrant (resp. trivially fibrant) objects and trivial objects are ⊗-pure acyclic N-complexes. we call this model structure the ⊗-pure injective model structure on $(\mathbb{C}_N(\mathcal{G}), \mathcal{P}_{\otimes_{dw}})$ and its homotopy category is $\mathbb{D}_{N\text{-}\otimes\text{-pur}}(\mathcal{G})$.*

Proof. The proof is obtained using Proposition 3.3, Proposition 3.6 and Theorem 2.1. ∎

Example 3.9. Let $\mathbb{X}$ be a scheme with associated structure sheaf $\mathcal{O}_{\mathbb{X}}$.

(a) $(\text{Mod-}(\mathbb{X}), \otimes_{\mathbb{X}}, \mathcal{O}_{\mathbb{X}}, \mathcal{H}om_{\mathbb{X}})$ is a closed symmetric monoidal category, where Mod-($\mathbb{X}$) is the abelian category of all the sheaves (of $\mathcal{O}_{\mathbb{X}}$-modules) on $\mathbb{X}$ (see [19, Chapter II, Section 5]). It is well known that this is a Grothendieck category, see [18, Proposition 3.1.1]. We can define $\otimes_{\mathbb{X}}$-pure monomorphisms as in Definition 2.14. Then by Theorem 3.8 we can say that there is a model structure on the exact category $\mathbb{C}_N(\text{Mod-}(\mathbb{X}))$ in which its homotopy category is $\mathbb{D}_{N\text{-}\otimes_{\mathbb{X}}\text{-pur}}(\text{Mod-}(\mathbb{X}))$.

(b) The category $\mathcal{Q}coh(\mathbb{X})$ of quasi-coherent sheaves on $\mathbb{X}$ is an abelian subcategory of Mod-($\mathbb{X}$), (see [19, Chapter II, Proposition 5.7]). $\mathcal{Q}coh(\mathbb{X})$ is a closed symmetric monoidal Grothendieck category, with the closed structure coming from the coherator functor Q applied to the usual sheafhom, see [30, Tag 08D6] and [2, Lemma 1.3]. We can define $\otimes_{\mathbb{X}}$-pure monomorphisms as in Definition 2.14. Then by Theorem 3.8 we can say that there is a model structure on the exact category $\mathbb{C}_N(\mathcal{Q}coh(\mathbb{X}))$ in which its homotopy category is $\mathbb{D}_{N\text{-}\otimes_{\mathbb{X}}\text{-pur}}(\mathcal{Q}coh(\mathbb{X}))$.

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