Transport coefficients of a heated granular gas

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(August 1, 2013)

Abstract

The Navier-Stokes transport coefficients of a granular gas are obtained from the Chapman-Enskog solution to the Boltzmann equation. The granular gas is heated by the action of an external driving force (thermostat) which does work to compensate for the collisional loss of energy. Two types of thermostats are considered: (a) a deterministic force proportional to the particle velocity (Gaussian thermostat), and (b) a random external force (stochastic thermostat). As happens in the free cooling case, the transport coefficients are determined from linear integral equations which can be approximately solved by means of a Sonine polynomial expansion. In the leading order, we get those coefficients as explicit functions of the restitution coefficient α. The results are compared with those obtained in the free cooling case, indicating that the above thermostat forces do not play a neutral role in the transport. The kinetic theory results are also compared with those obtained from Monte Carlo simulations of the Boltzmann equation for the shear viscosity. The comparison shows an excellent agreement between theory and simulation over a wide range of values of the restitution coefficient. Finally, the expressions of the transport coefficients for a gas of inelastic hard spheres are extended to the revised Enskog theory for a description at higher densities.

Keywords: Granular gas; Thermostat forces; Kinetic theory; Direct simulation Monte Carlo method.

PACS number(s): 45.70.Mg, 05.20.Dd, 51.10.+y, 47.50.+d

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I. INTRODUCTION

The usefulness of fluid-like type equations to describe systems of granular particles in rapid, dilute flow has been recognized for many years. The essential difference from ordinary fluids is the absence of energy conservation, yielding subtle modifications of the conventional Navier-Stokes equations for states with small spatial gradients of the hydrodynamic fields. Although many efforts have been made in the past few years in the understanding of these systems, the analysis of the influence of dissipation on the transport coefficients still remains a topic of interest and controversy. For a low density gas, these coefficients may be determined from the Boltzmann equation modified to account for inelastic binary collisions. The idea is to extend the Chapman-Enskog method [1] to the inelastic case by expanding around the local version of the homogeneous cooling state (HCS), i.e., a reference state in which all the time dependence occurs through the granular temperature. In the first-order of the expansion, explicit expressions for the transport coefficients as functions of the restitution coefficient have been obtained in the case of hard spheres [2] as well as for a d-dimensional system [3]. This analysis has been also extended to higher densities in the context of the Enskog equation [4]. The results obtained in the latter theory describe very well the hydrodynamic profiles obtained in a recent experimental study of a three-dimensional system of mustard seeds fluidized by vertical vibrations of the container [5].

One of the main difficulties in obtaining the above transport coefficients lies in the fact that, in contrast to what happens for molecular fluids, the reference state (zeroth-order solution of the Chapman-Enskog expansion) depends on time due to the dissipation of energy through collisions. In addition, it is also well-known that this state is unstable to long enough wavelength perturbations so that the state becomes inhomogeneous for long times [6]. To overcome such difficulties, one possibility is to introduce external forces to accelerate the particles and hence compensate for collisional cooling. As a consequence, the corresponding reference state is stationary and linearly stable against spatial inhomogeneities. This mechanism of energy input (different from those in shear flows or flows through vertical pipes) has been used by many authors [7–12] in the past years to analyze different problems, such as non-Gaussian properties (cumulants, high energy tails) of the velocity distribution function [9,12], long-range correlations [10], and collisional statistics and short scale structure [11]. Since the latter requires the solution of the corresponding linearized hydrodynamic equations around the homogeneous state, the explicit expressions for the transport coefficients are needed. Given that the dependence of these coefficients on the restitution coefficient is not known, the expressions of these coefficients are usually assumed to be the same as those for the *elastic* gas. However, according to the results derived in the free cooling case [2–4], the above assumption could be only justified in the *small* inelasticity limit.

The goal of this paper is to determine the transport coefficients of a heated granular gas. This allows us to measure the new effects induced by the external force on transport by comparison with the results derived in the unforced case [2,3]. In addition, one could also assess to what extent the previous results on short and large structure [10,11] are indicative of what happens for finite degree of dissipation. There are different mechanisms to inject energy to the gas. Here, the fluidization is driven by the action of external forces (thermostats) acting locally on each particle. In this paper we will consider two types of thermostats: the Gaussian and the stochastic thermostats. In the case of the Gaussian thermostat, the gas is

heated by the action of an external force proportional to the peculiar velocity. This type of "anti-drag" force can be justified by Gauss's principle of least constraints [13] and has been widely used in nonequilibrium molecular dynamics simulations of molecular fluids. Another mechanism for thermostatting the system is to assume that the particles are subjected to a random external force, which gives frequent kicks to each particle between collisions. If this stochastic force has the properties of a white noise, it gives rise to a Fokker-Planck diffusion term in the Boltzmann equation [9].

It must be remarked that in most experiments, energy is added to the granular gas through a boundary, causing gradients in the energy perpendicular to that boundary. In this sense, we do not claim that the above forcing terms are the most suited to model any particular real system. However, they have the advantage of that it can be incorporated into the kinetic theory very easily. This allows for instance, to test the assumptions of the Chapman-Enskog method through a direct comparison with computer simulations, such as Monte Carlo or molecular dynamics simulations [14].

The plan of the paper is as follows. In Sec. II, we review the Boltzmann equation and associated macroscopic conservation laws in the presence of the external forces discussed above. The Chapman-Enskog method for solving this equation is presented in Sec. III and subsequently applied to the cases of Gaussian and stochastic thermostats. As happens in the free cooling case [2], the transport coefficients are determined from linear integral equations which can be approximately solved by means of a Sonine polynomial expansion. In the leading order, we get the transport coefficients as explicit functions of the restitution coefficient. Section III ends with a comparison between the results derived here in both driven cases with those previously obtained in the unforced case [3]. Such a comparison shows that, in general, the thermostats do not play a neutral role in the transport since they clearly affect the dependence of the Navier-Stokes transport coefficients on the dissipation. In order to check the degree of reliability of the Sonine approximation, a comparison with direct Monte Carlo simulation of the Boltzmann equation is carried out in Sec. IV. More specifically, the simulations are performed for a gas undergoing uniform shear flow, using the Gaussian and the stochastic thermostats to control inelastic cooling. In the long time limit, a (reduced) shear viscosity can be measured in both simulations. The comparison with the Chapman-Enskog solution shows an excellent agreement, indicating that the Sonine results have an accuracy comparable to that for elastic collisions. The paper is closed in Sec. V with a brief summary and discussion of the results presented. In addition, the corresponding expressions of the transport coefficients for a granular gas of hard spheres in the framework of Enskog kinetic theory are also displayed in Appendix B.

II. HEATED GRANULAR GASES

We consider a granular gas composed by smooth inelastic disks ($d = 2$) or spheres ($d = 3$) of mass m and diameter σ. The inelasticity of collisions among all pairs is characterized by a constant restitution coefficient $\alpha \leq 1$. In the low-density regime, the evolution of the one-particle velocity distribution function $f(\mathbf{r}, \mathbf{v}; t)$ is given by the Boltzmann kinetic equation [15,16]

$$\left(\partial_t + \mathbf{v}_1 \cdot \nabla + \mathcal{F}\right) f(\mathbf{r}, \mathbf{v}_1, t) = J\left[\mathbf{v}_1 | f(t), f(t)\right] \ , \tag{1}$$

where the Boltzmann collision operator $J[\mathbf{v}_1|f,f]$ is

$$J\left[\mathbf{v}_1|f,f\right] = \sigma^{d-1}\int d\mathbf{v}_2\int d\widehat{\boldsymbol{\sigma}}\,\Theta(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{g}) \times\left[\alpha^{-2}f(\mathbf{r},\mathbf{v}_1',t)f(\mathbf{r},\mathbf{v}_2',t)-f(\mathbf{r},\mathbf{v}_1,t)f(\mathbf{r},\mathbf{v}_2,t)\right]\ . \tag{2}$$

In Eq. (1), $\mathcal{F}$ is an operator representing the effect of an external forcing which injects energy into the granular gas allowing it to reach a steady state. Furthermore, d is the dimensionality of the system, $\widehat{\boldsymbol{\sigma}}$ is a unit vector along their line of centers, Θ is the Heaviside step function, and $\mathbf{g}=\mathbf{v}_1-\mathbf{v}_2$. The primes on the velocities denote the initial values $\{\mathbf{v}_1',\mathbf{v}_2'\}$ that lead to $\{\mathbf{v}_1,\mathbf{v}_2\}$ following a binary collision:

$$\mathbf{v}_1'=\mathbf{v}_1-\tfrac{1}{2}\left(1+\alpha^{-1}\right)(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{g})\widehat{\boldsymbol{\sigma}},\quad \mathbf{v}_2'=\mathbf{v}_2+\tfrac{1}{2}\left(1+\alpha^{-1}\right)(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{g})\widehat{\boldsymbol{\sigma}} \tag{3}$$

The macroscopic balance equations for density n, momentum $m\mathbf{u}$, and energy $\frac{d}{2}nT$ follow directly from Eq. (1) by multiplying with 1, $m\mathbf{v}_1$, and $\frac{m}{2}v_1^2$ and integrating over $\mathbf{v}_1$:

$$D_t n + n\nabla\cdot\mathbf{u}=0\ , \tag{4}$$

$$D_t u_i + (mn)^{-1}\nabla_j P_{ij}=0\ , \tag{5}$$

$$D_t T+\frac{2}{dn}\left(\nabla\cdot\mathbf{q}+P_{ij}\nabla_j u_i\right)=-(\zeta-\Lambda)T\ . \tag{6}$$

In the above equations, $D_t=\partial_t+\mathbf{u}\cdot\nabla$ is the material derivative,

$$\mathsf{P}=\int d\mathbf{v}\,m\mathbf{V}\mathbf{V}\,f(\mathbf{v}) \tag{7}$$

is the pressure tensor,

$$\mathbf{q}=\int d\mathbf{v}\,\tfrac{1}{2}mV^2\mathbf{V}\,f(\mathbf{v}) \tag{8}$$

is the total heat flux, and $\mathbf{V}=\mathbf{v}-\mathbf{u}$ is the peculiar velocity. In the right hand side of the temperature equation (6), the cooling rate ζ (measuring the rate of energy loss due to dissipation) and the source term Λ (measuring the rate of heating due to the external force) are given by

$$\zeta=-\frac{1}{dnT}\int d\mathbf{v}mv^2J[f,f], \tag{9}$$

$$\Lambda=-\frac{1}{dnT}\int d\mathbf{v}mv^2\mathcal{F}f(\mathbf{v}). \tag{10}$$

It is assumed that the external driving does not change the number of particles or the momentum, i.e.,

$$\int d\mathbf{v}\,\mathcal{F}f(\mathbf{v})=\int d\mathbf{v}\,\mathbf{v}\mathcal{F}f(\mathbf{v})=0. \tag{11}$$

In the case of elastic particles ($\alpha = 1$) and in the absence of external forcing ($\mathcal{F} = 0$), it is well known that the long-time *uniform* solution of Eq. (1) is the Maxwell-Boltzmann distribution function. However, if the particles collide inelastically ($\alpha < 1$) and $\mathcal{F} = 0$, a steady state is not possible in uniform situations since the temperature decreases monotonically in time. In this case, Goldshtein and Shapiro [15] showed that Eq. (1) admits an isotropic solution, describing the homogeneous cooling state (HCS), in which all the time dependence of f occurs only through the thermal velocity $v_0(t) = \sqrt{2T(t)/m}$: $f(\mathbf{v},t) \to n\pi^{-d/2}v_0^{-d}(t)\Phi(\mathbf{v}/v_0(t))$. So far, the exact form of Φ has not been found, although a good approximation for thermal velocities can be obtained from an expansion in Sonine polynomials. In the leading order, Φ is given by

$$\Phi(\mathbf{v}^*) \to \left\{1 + \frac{c}{4}\left[v^{*4} - (d+2)v^{*2} + \frac{d(d+2)}{4}\right]\right\} e^{-v^{*2}}, \quad \mathbf{v}^* = \mathbf{v}/v_0, \tag{12}$$

where the estimated value of c is [9]

$$c = \frac{32(1-\alpha)(1-2\alpha^2)}{9 + 24d - \alpha(41-8d) + 30(1-\alpha)\alpha^2}. \tag{13}$$

The estimate (13) presents a quite good agreement with Monte Carlo simulations of the Boltzmann equation [12,17].

However, by driving a granular gas by boundaries or external fields it can reach a steady state. The energy injected in the gas may exactly compensate for the energy dissipated by collisional cooling. The same effect can be obtained by means of external forces acting locally on each particle. These forces, which we will call *thermostats*, are represented by the operator $\mathcal{F}$ in Eq. (1) and depend on the state of the system. Several types of thermostats can be used. Here, we will consider two. One of them is a deterministic thermostat widely used in nonequilibrium molecular dynamics simulations of elastic particles [13], which is based on Gauss's principle of least constraints. In this case, $\mathcal{F}$ is given by [12]

$$\mathcal{F}f(\mathbf{v}) = \frac{1}{2}\zeta\frac{\partial}{\partial\mathbf{v}}\cdot\left[\mathbf{v}f(\mathbf{v})\right], \tag{14}$$

where, according to Eqs. (6) and (10), the thermostat has been adjusted to get a constant temperature in the long time limit. It must be pointed out that the corresponding Boltzmann equation (1) for this Gaussian thermostat force is formally identical with the Boltzmann equation in the HCS (i.e. with $\mathcal{F} = 0$) when both equations are written in terms of the reduced distribution $\Phi(v^*)$. As a consequence, the result (13) applies to this thermostatted case as well.

Another way of heating the gas is by means of a stochastic force assumed to have the form of a Gaussian white noise [7]. The corresponding operator $\mathcal{F}$ has a Fokker-Planck form [9]

$$\mathcal{F}f(\mathbf{v}) = -\frac{1}{2}\frac{T}{m}\zeta\left(\frac{\partial}{\partial\mathbf{v}}\right)^2 f(\mathbf{v}), \tag{15}$$

where again the strength of the correlation has been chosen to achieve a time independent temperature. By using this thermostat, van Noije and Ernst [9] have studied the stationary solution to the uniform Boltzmann equation (1) and found for the coefficient c the value

$$c=\frac{32(1-\alpha)(1-2\alpha^2)}{73+56d-3\alpha(35+8d)+30(1-\alpha)\alpha^2}. \tag{16}$$

As happened in the Gaussian case, recent Monte Carlo simulations of the Boltzmann equation [12] agree quite well with the Sonine estimate (16).

III. CHAPMAN-ENSKOG SOLUTION: NAVIER-STOKES TRANSPORT COEFFICIENTS

As said in the Introduction, our goal is to get the Navier-Stokes transport coefficients in the presence of the thermostats introduced in the previous Section. This allows one to assess the influence of these thermostats on transport by comparison with the results obtained in the free cooling case [2,3]. To do that, we consider now a spatially *inhomogeneous* state, created by initial preparation or by boundary conditions. We assume that the spatial variations of n, $\mathbf{u}$, and T are small on the scale of the mean free path. Under these conditions, the Chapman-Enskog method [1] provides a solution to the Boltzmann equation based on an expansion around the *local* version of the heated homogeneous state induced by the thermostat forces. This is obtained from the above states by replacing the temperature, density, and flow velocity by their nonequilibrium local values. As a consequence, the local version of the operators $\mathcal{F}$ consists of replacing $\mathbf{v}\to\mathbf{V}=\mathbf{v}-\mathbf{u}$ in Eqs. (14) and (15).

The Chapman-Enskog method assumes the existence of a *normal* solution in which all the space and time dependence of the distribution function appears through a functional dependence on the hydrodynamic fields

$$f(\mathbf{r},\mathbf{v},t)=f\left[\mathbf{v}|n(t),\mathbf{u}(t),T(t)\right]. \tag{17}$$

For small spatial variations, this functional dependence can be made local in space and time through an expansion in gradients of the fields. To generate the expansion, it is convenient to write f as a series expansion in a formal parameter ϵ measuring the nonuniformity of the system,

$$f=f^{(0)}+\epsilon\, f^{(1)}+\epsilon^2 f^{(2)}+\cdots\ , \tag{18}$$

where each factor of ϵ means an implicit gradient of a hydrodynamic field. The local reference state $f^{(0)}$ is chosen such that it has the same first moments as the exact distribution f, or equivalently, the remainder of the expansion must obey the orthogonality conditions

$$\int d\mathbf{v}\left[f(\mathbf{v})-f^{(0)}(\mathbf{v})\right]=0\ ,\quad \int d\mathbf{v}\mathbf{v}\left[f(\mathbf{v})-f^{(0)}(\mathbf{v})\right]=\mathbf{0}\ , \tag{19}$$

$$\int d\mathbf{v}v^2\left[f(\mathbf{v})-f^{(0)}(\mathbf{v})\right]=0\ . \tag{20}$$

The time derivatives of the fields are also expanded as $\partial_t=\partial_t^{(0)}+\epsilon\partial_t^{(1)}+\cdots$. The coefficients of the time derivative expansion are identified from the balance equations (4)–(6) with a representation of the fluxes, the cooling rate ζ and the heating term Λ in the macroscopic balance equations as a similar series through their definitions as functionals of the

distribution f. This is the usual Chapman-Enskog method for solving kinetic equations [1]. The main difference with respect to the unforced case [2] is that now (as happens in the elastic case) the sink term in the energy equation is zero, so that the terms coming from the time derivative $\partial_t^{(0)}$ vanish.

Now, we derive the corresponding hydrodynamic equations in the presence of the Gaussian and the stochastic thermostats.

A. Gaussian thermostat

In the case of the Gaussian thermostat (14), the Boltzmann equation becomes

$$\partial_t f + \mathbf{v}\cdot\nabla f + \frac{1}{2}\zeta\frac{\partial}{\partial \mathbf{V}}\cdot(\mathbf{V}f) = J\left[f,f\right] \ . \tag{21}$$

Substitution of the Chapman-Enskog solution (18) into Eq. (21) leads to different kinetic equations for the distributions $f^{(k)}$. To zeroth order in ϵ, the Boltzmann equation reads

$$\frac{1}{2}\zeta^{(0)}\frac{\partial}{\partial \mathbf{V}}\cdot\left(\mathbf{V}f^{(0)}\right) = J\left[f^{(0)},f^{(0)}\right], \tag{22}$$

where use has been made of the macroscopic balance equations at this order

$$\partial_t^{(0)} n = 0, \quad \partial_t^{(0)}\mathbf{u} = 0, \quad \partial_t^{(0)} T = 0. \tag{23}$$

Here, the cooling rate $\zeta^{(0)}$ is determined by Eq. (9) to zeroth order

$$\zeta^{(0)} = -\frac{1}{dnT}\int d\mathbf{v} m v^2 J[f^{(0)},f^{(0)}]. \tag{24}$$

The solution to Eq. (22) $f^{(0)} = f^{(0)}(V)$ is isotropic so that the zeroth order approximations to the pressure tensor and heat flux are

$$P_{ij}^{(0)} = p\delta_{ij}, \quad \mathbf{q}^{(0)} = 0, \tag{25}$$

where $p = nT$ is the hydrostatic pressure. The distribution $f^{(0)}$ is essentially given by the Sonine approximation (12) with the cumulant c given by Eq. (13).

The analysis to first order in ϵ is similar to the one woked out in Ref. [2] for the free cooling case. We only display here the final expressions for the fluxes with some details being given in Appendix A. The final result to first order in the spatial gradients is

$$P_{ij}^{(1)} = -\eta\left(\nabla_i u_j + \nabla_j u_i - \frac{2}{d}\delta_{ij}\nabla\cdot\mathbf{u}\right), \tag{26}$$

$$\mathbf{q}^{(1)} = -\kappa\nabla T - \mu\nabla n, \tag{27}$$

where η is the shear viscosity, κ is the thermal conductivity, and μ is an additional transport coefficient not present in the elastic case. In dimensionless form, the transport coefficients are given by

$$\eta^* = \frac{\eta}{\eta_0} = \frac{1}{\nu_\eta^* - \zeta^*}, \tag{28}$$

$$\kappa^* = \frac{\kappa}{\kappa_0} = \frac{d-1}{d}\frac{1+c}{\nu_\kappa^* - \frac{3}{2}\zeta^*}, \tag{29}$$

$$\mu^* = \frac{n\mu}{T\kappa_0} = \frac{d-1}{2d}\frac{c}{\nu_\mu^* - \frac{3}{2}\zeta^*}. \tag{30}$$

Here, η_0 and κ_0 are the elastic values of the shear viscosity and thermal conductivity [1], respectively,

$$\eta_0 = \frac{d+2}{8}\pi^{-(d-1)/2}\Gamma\left(d/2\right)(mT)^{1/2}\sigma^{-(d-1)}, \quad \kappa_0 = \frac{d(d+2)}{2(d-1)}\frac{\eta_0}{m}, \tag{31}$$

$\zeta^* = \zeta^{(0)}/\nu_0$, $\nu_0 = p/\eta_0$ is a characteristic collision frequency, and $c(\alpha)$ is related to the fourth moment of $f^{(0)}$ by

$$c(\alpha) = \frac{8}{d(d+2)}\left[\left(\frac{m}{2T}\right)^2\frac{1}{n}\int d\mathbf{v}V^4 f^{(0)} - \frac{d(d+2)}{4}\right]. \tag{32}$$

The coefficient $c(\alpha)$ is a measure of the deviation of the reference state from that of a gas with elastic collisions. As said above, a good estimate of $c(\alpha)$ is given by the Sonine approximation (13). Furthermore, in Eqs. (28)–(30), we have introduced the reduced collision frequencies

$$\nu_\eta^* = \frac{\int d\mathbf{v}D_{ij}(\mathbf{V})\mathcal{L}\mathcal{C}_{ij}(\mathbf{V})}{\nu_0\int d\mathbf{v}D_{ij}(\mathbf{V})\mathcal{C}_{ij}(\mathbf{V})}, \tag{33}$$

$$\nu_\kappa^* = \frac{\int d\mathbf{v}\mathbf{S}(\mathbf{V})\cdot\mathcal{L}\mathcal{A}(\mathbf{V})}{\nu_0\int d\mathbf{v}\mathbf{S}(\mathbf{V})\cdot\mathcal{A}(\mathbf{V})}, \quad \nu_\mu^* = \frac{\int d\mathbf{v}\mathbf{S}(\mathbf{V})\cdot\mathcal{L}\mathcal{B}(\mathbf{V})}{\nu_0\int d\mathbf{v}\mathbf{S}(\mathbf{V})\cdot\mathcal{B}(\mathbf{V})}, \tag{34}$$

where

$$D_{ij}(\mathbf{V}) = m\left(V_iV_j - \frac{1}{d}\delta_{ij}V^2\right), \quad \mathbf{S}(\mathbf{V}) = \left(\frac{m}{2}V^2 - \frac{d+2}{2}T\right)\mathbf{V}. \tag{35}$$

So far, all the results are exact but not explicit because of ζ^*, ν_η^*, ν_κ^*, and ν_μ^*. To get more explicit expressions for the dependence of the transport coefficients on α it is convenient to use the leading Sonine polynomial approximations for $\mathcal{A}(\mathbf{V})$, $\mathcal{B}(\mathbf{V})$, $\mathcal{C}_{ij}(\mathbf{V})$, and $f^{(0)}$. In the case of $f^{(0)}$, we take the approximation (12) while the remaining quantities are given by

$$\begin{pmatrix}\mathcal{A}(\mathbf{V})\\ \mathcal{B}(\mathbf{V})\\ \mathcal{C}_{ij}(\mathbf{V})\end{pmatrix} \to f_M(V)\begin{pmatrix}c_T\mathbf{S}(\mathbf{V})\\ c_n\mathbf{S}(\mathbf{V})\\ c_uD_{ij}(\mathbf{V})\end{pmatrix}, \qquad f_M(V) = n\pi^{-d/2}v_0^{-d}e^{-(V/v_0)^2}. \tag{36}$$

The factor $f_M(V)$ occurs since these polynomials are defined relative to a Gaussian scalar product. The coefficients are the projections of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}_{ij}$ along $\mathbf{S}(\mathbf{V})$, and $D_{ij}(\mathbf{V})$,

$$\begin{pmatrix} c_T \\ c_n \end{pmatrix} = \frac{2}{d(d+2)} \frac{m}{nT^3} \int d\mathbf{v} \begin{pmatrix} \mathcal{A}(\mathbf{V}) \cdot \mathbf{S}(\mathbf{V}) \\ \mathcal{B}(\mathbf{V}) \cdot \mathbf{S}(\mathbf{V}) \end{pmatrix} = \begin{pmatrix} -\frac{2}{d+2} \frac{m}{nT^2} \kappa \\ -\frac{2}{d+2} \frac{m}{T^3} \mu \end{pmatrix} , \tag{37}$$

$$c_u = \frac{1}{(d+2)(d-1)} \frac{1}{nT^2} \int d\mathbf{V} \mathcal{C}_{ij}(\mathbf{V}) D_{ij}(\mathbf{V}) = -\frac{1}{nT^2} \eta \, . \tag{38}$$

where use has been made of the definitions (A11), (A12), and (A13). With these expressions, the cooling rate ζ^* and the collision frequencies ν_η^*, ν_κ^*, and ν_μ^* can be explicitly computed. These calculations have been made by Brey and Cubero [3], with the result

$$\zeta^* = \frac{d+2}{4d}(1-\alpha^2) \left(1 + \frac{3}{32} c\right), \tag{39}$$

$$\nu_\eta^* = \frac{3}{4d} \left(1 - \alpha + \tfrac{2}{3} d\right) (1+\alpha) \left(1 - \frac{c}{64}\right), \tag{40}$$

$$\nu_\kappa^* = \nu_\mu^* = \frac{1+\alpha}{d} \left[\frac{d-1}{2} + \frac{3}{16}(d+8)(1-\alpha) + \frac{4+5d-3(4-d)\alpha}{1024} c\right], \tag{41}$$

where c is given by Eq. (13).

Substitution of the above expressions into Eqs. (28)–(30) gives finally the explicit dependence of the transport coefficients on the restitution coefficient α. In order to gain some insight into the behavior of η^*, κ^*, and μ^* it is convenient to consider the weak dissipation limit. In this limit, the above coefficients can be expanded in powers of the inelasticity parameter $1-\alpha^2$. The leading contributions are

$$\eta^* \approx 1 + \frac{64d^2 - 97d + 32}{128 d(d-1)} (1-\alpha^2) + \cdots, \tag{42}$$

$$\kappa^* \approx 1 + \frac{56d - 191}{128(d-1)} (1-\alpha^2) + \cdots, \tag{43}$$

$$\mu^* \approx \frac{1}{4(1-d)} (1-\alpha^2) + \cdots. \tag{44}$$

B. Stochastic thermostat

In the case of the stochastic thermostat (15), the Boltzmann equation becomes

$$\partial_t f + \mathbf{v} \cdot \nabla f - \frac{1}{2} \frac{T}{m} \zeta \left(\frac{\partial}{\partial \mathbf{V}}\right)^2 f = J[f,f] \, . \tag{45}$$

As before, this equation can be solved by means of the Chapman-Enskog method. Since the procedures to arrive at the expressions of the transport coefficients are identical to the ones

employed in the case of the Gaussian force, here we only quote the final expressions of the transport coefficients. In dimensionless form, they are

$$\eta^* = \frac{1}{\nu_\eta^*}, \tag{46}$$

$$\kappa^* = \frac{d-1}{d}\frac{1+c}{\nu_\kappa^*}, \tag{47}$$

$$\mu^* = \frac{d-1}{2d}\frac{c}{\nu_\mu^*}. \tag{48}$$

Here, in the first Sonine approximation, the expressions of ζ^*, ν_η^*, and ν_κ^* are also given by Eqs. (39)–(41), respectively, with $c(\alpha)$ given by Eq. (16). In the quasielastic limit, one has

$$\eta^* \approx 1 + \frac{32d^2 - 129d + 96}{128d(d-1)}(1-\alpha^2) + \cdots, \tag{49}$$

$$\kappa^* \approx 1 + \frac{8d - 287}{128(d-1)}(1-\alpha^2) + \cdots, \tag{50}$$

$$\mu^* \approx \frac{1}{4(1-d)}(1-\alpha^2) + \cdots. \tag{51}$$

C. Comparison with the free cooling case

The transport coefficients for a dilute *unforced* granular gas in d dimensions has been recently obtained by Brey and Cubero [3]. These authors generalize a previous derivation made from the Boltzmann equation in the three dimensional case [2]. The results are

$$\eta^* = \frac{1}{\nu_\eta^* - \frac{1}{2}\zeta^*}, \tag{52}$$

$$\kappa^* = \frac{d-1}{d}\frac{1+c}{\nu_\kappa^* - 2\zeta^*}, \tag{53}$$

$$\mu^* = \frac{2\zeta^*}{2\nu_\mu^* - 3\zeta^*}\left[\kappa^* + \frac{(d-1)c}{2d\zeta^*}\right], \tag{54}$$

where $c(\alpha)$ is given by Eq. (13). In the quasielastic limit, these coefficients behave as

$$\eta^* \approx 1 + \frac{48d^2 - 113d + 64}{128d(d-1)}(1-\alpha^2) + \cdots, \tag{55}$$

$$\kappa^* \approx 1 + \frac{72d - 159}{128(d-1)}(1-\alpha^2) + \cdots, \tag{56}$$

$$\mu^* \approx \frac{d+1}{4(d-1)}(1-\alpha^2) + \cdots. \tag{57}$$

Comparison between Eqs. (42)–(44), (49)–(51), and (55)–(57) shows that the leading order corrections to the elastic values of the transport coefficients are clearly different in the free cooling and driven cases. This illustrates the fact that the transport properties are affected by the thermostat introduced so that the latter does not play a neutral role in the problem. In Figs. 1, 2, and 3 we plot the reduced coefficients η^*, κ^*, and μ^*, respectively, for hard spheres ($d = 3$) in the cases of the unforced gas, the Gaussian thermostat, and the stochastic thermostat. In general, we observe that the predictions obtained in the Gaussian case for the shear viscosity and the thermal conductivity are closer to those of the unforced case than to the ones obtained with the stochastic force. This is in part motivated by the fact that, in dimensionless form, the results of the unforced and Gaussian cases are identical in the uniform problem (reference state). In the case of the shear viscosity, all the theories give the same trends for η^* since this coefficient increases with the dissipation. However, at a qualitative level, the influence of dissipation on the viscosity in the stochastic case is much less significant than in the other two cases. Thus, for instance, for $\alpha = 0.8$ (moderate dissipation), the shear viscosity of the granular gas η has only changed about 1% with respect to its elastic value when the gas is heated by means of the stochastic force. This could justify the use of the elastic values of the transport coefficients in the analysis of the long-range correlations in a randomly driven granular fluid [10]. Discrepancies between unforced and the two driven systems are more important in the case of the thermal conductivity κ^*, as Fig. 2 shows. In addition, while in the free cooling and Gaussian cases κ^* increases with α, the opposite happens in the stochastic case. Finally, Fig. 3 shows the dependence of the coefficient μ^* on α. We observe that $\mu^* \simeq 0$ for both thermostats, while this coefficient is clearly different from zero in the free cooling problem. This is basically due to the fact that $\mu^* \propto c(\alpha)$ in the Gaussian and stochastic cases, Eqs. (30) and (48), and so it vanishes exactly if one takes the Maxwellian approximation (which is known to give a very accurate description) for the reference state. This means that, for practical purposes, one can neglect the contribution to the heat flux coming from the term proportional to the density gradient in the heated gas case: $\mathbf{q}^{(1)} \to -\kappa\nabla T$. As is apparent from Fig. 3, this cannot be assumed in the unforced case since μ^* is clearly different from zero even in the quasielastic limit.

IV. COMPARISON WITH MONTE CARLO SIMULATIONS

As has been discussed above, the practical evaluation of the transport coefficients requires the truncation of an expansion for the solutions of the integral equations in Sonine polynomials. In the case of elastic collisions, the leading order truncation is known to be a very good approximation [1]. A natural question is whether the above degree of accuracy is also maintained in the inelastic case. To answer this question, one has to resort to numerical solutions of the Boltzmann equation, such as those obtained from the Direct

Simulation Monte Carlo (DSMC) method [19]. Although this method was originally devised for molecular fluids, its extension to deal with inelastic collisions is straightforward [12,18].

Recently, the shear viscosity of a low-density granular gas of freely evolving hard spheres has been determined from the DSMC method [20]. The results show a very good agreement with the predictions based on the Boltzmann equation in the first Sonine approximation, Eq. (52). This experiment consists in preparing an initial inhomogeneous nonequilibrium state corresponding to a transverse shear wave, and then analyzing its subsequent evolution in time. The shear wave decays exponentially with a time scale inversely proportional to the viscosity. An alternative route to measuring the shear viscosity consists of preparing a state of uniform shear flow using Lees-Edwards boundary conditions [21]. Macroscopically, this state is characterized by constant density n, a uniform temperature T, and a linear velocity profile $a = \partial u_x/\partial y \equiv \text{const}$. In a molecular fluid, unless a termostatting force is introduced, the temperature increases in time due to viscous heating. The corresponding energy balance equation can be used to determine the shear viscosity for sufficiently long times [22]. In a granular fluid, the relationship between the temperature and the shear viscosity is not simple since there is a competition between viscous heating and collisional cooling [23,24]. However, if external forces of the form (14) and (15) that exactly compensate for the collisional energy loss are introduced, the viscous heating effect is still able to heat the system. Under these conditions, the Boltzmann equation to be solved is

$$\partial_t f - aV_y \frac{\partial}{\partial V_x} f + \frac{1}{2}\zeta \frac{\partial}{\partial \mathbf{V}} \cdot (\mathbf{V} f) = J[f,f] \tag{58}$$

in the case of the Gaussian thermostat, and

$$\partial_t f - aV_y \frac{\partial}{\partial V_x} f - \frac{1}{2}\frac{T}{m}\zeta \left(\frac{\partial}{\partial \mathbf{V}}\right)^2 f = J[f,f] \tag{59}$$

in the case of the stochastic thermostat.

Equations (58) and (59) have been numerically solved by means of the DSMC method for a three-dimensional system ($d = 3$). At given values of the shear rate a and the restitution coefficient α, we start from a local equilibrium state and monitor the time evolution of $a^* = a/\nu_0$ and $P_{xy}^* = P_{xy}/p$. Here, $\nu_0 = p/\eta_0 \propto T^{1/2}$ is an effective collision frequency, η_0 is the elastic shear viscosity. The simulations show that, after a transient regime, the ratio $-P_{xy}^*/a^*$ reaches a constant value (independent of the shear rate), which can be identified as the shear viscosity in the linear hydrodynamic regime. Details of the simulation for dense fluids will be published elsewhere [25], and here only compare the Monte Carlo simulations for the shear viscosity with the Sonine approximations given by Eqs. (28) and (46).

In our simulations we have typically taken 10^5 particles and have averaged over 5 replicas. Since the thermal velocity v_0 is not constant in the transient regime, we have taken a time-dependent time step given by $0.01\ell/v_0(t)$, where $\ell = (\sqrt{2}\pi n\sigma^2)^{-1}$ is the mean free path. The simulation results are shown in Fig. 1. In general, we observe a very good agreement between the predictions of the Chapman-Enskog theory in the first Sonine approximation and the simulation data. This agreement is similar to the one previously found in the free cooling case [20]. At a quantitative level, we see that the discrepancies between theory and simulation tend to increase as the dissipation increases, although these differences are quite small (less than 3%). As a final conclusion, it is important to remark that the agreement

extends over a wide range of values of the restitution coefficient ($\alpha \geq 0.6$), indicating the reliability of the Sonine approximation for describing granular flows beyond the quasielastic domain.

V. SUMMARY AND DISCUSSION

In this paper we have addressed the derivation of the hydrodynamic equations of a granular gas from the Boltzmann kinetic theory. The system is heated by the action of "thermostatting" external forces which exactly compensate for cooling effects associated with the inelasticity of collisions. Two different types of thermostats have been considered: (a) an "anti-drag" force proportional to the particle peculiar velocity (Gaussian force), and (b) a stochastic force, which gives frequent kicks to each particle between collisions. The introduction of these thermostats has the advantage of avoiding the intrinsic time dependence of the homogeneous cooling state (unforced gas), but at the price of introducing unknown new effects induced by the external forcing. While in the homogeneous problem the results obtained with and without a Gaussian thermostat are completely equivalent (when one scales the particle velocity with the thermal velocity), this equivalence does not hold in the stochastic case and the non-Maxwellian properties of the distribution function (cumulants) are different from those obtained in the unforced case. Here, our goal has been to assess the influence of thermostats on the transport properties of the gas.

The transport processes considered are those for a fluid with small spatial gradients of the hydrodynamic fields. For this reason, the Boltzmann equation has been solved perturbatively using an adaptation of the Chapman-Enskog method recently proposed for inelastic collisions [2,4]. By using similar procedures as those made in the free cooling case [2], we calculate the distribution function to first order in the gradients. Its use in the functionals for the pressure tensor and heat flux provides a representation of these as linear combinations of the gradients. The corresponding coefficients in these expressions are the shear viscosity η (defined in Eq. (26)), the thermal conductivity κ and the new coefficient μ (both coefficients defined in Eq. (27)). These transport coefficients are in general functions of the restitution coefficient α. Their expressions are given by Eqs. (28)–(30) in the case of the Gaussian thermostat and Eqs. (46)–(48) in the case of the stochastic thermostat. A practical evaluation of these coefficients is possible by means of a Sonine polynomial approximation and the derivation and approximate results are not limited to weak inelasticity.

The dependence of η, κ, and μ on α has been illustrated in the case of hard spheres ($d = 3$). As Figs. 1–3 show, the thermostats affect the transport properties since the discrepancies between the driven and free cooling results are quite significant. Although not widely recognized, the above conclusion illustrates the fact that generally the inclusion of an external force depending on the state of the system changes the apparent transport coefficients. This has been demonstrated as well for uniform shear flow with elastic collisions where thermostats are used to produce a steady state. In that case, the Navier-Stokes shear viscosity is unchanged, but nonlinear rheological properties are affected by the thermostat [26]. In the context of granular fluids, the effects already occur at the Navier-Stokes order. Notice that the above conclusion only affects to this type of external forcing mechanisms (termostats), since driving the system by shaking, vibration, and even the action of a weak external field (such as the gravity field) does not modify the transport coefficients of the

gas. Concerning the influence of dissipation on transport, we observe that in general the deviation from the functional form for elastic collisions is more significant in the unforced gas than in the driven cases. In particular, the coefficient μ (which is zero in the elastic limit) is clearly different from zero in the unforced gas (for instance, $\mu^* \simeq 0.27$ at $\alpha = 0.8$) while is negligible in the Gaussian and stochastic cases (for instance, at $\alpha = 0.5$, $\mu^* \simeq 0.064$ for the Gaussian force while $\mu^* \simeq 0.012$ for the stochastic force).

To check the accuracy of the Sonine approximation, we have numerically solved the Boltzmann equation by means of the Direct Simulation Monte Carlo (DSMC) method [19] for a granular gas under uniform shear flow. To control inelastic cooling, a thermostat force is introduced in the system. In these conditions, the (apparent) Navier-Stokes shear viscosity can be measured directly in the long time limit just as for the case of elastic collisions. This simulation method has been recently proved [25] to be an efficient way of measuring the shear viscosity of a moderately dense granular gas. The comparison carried out here in the low-density regime shows that the Chapman-Enskog results in the first Sonine approximation exhibit an excellent agreement with the simulation data, even for moderate dissipation (say, for instance $\alpha = 0.7$). These results indicate clearly the reliability of the quantitative predictions for transport coefficients from the Chapman-Enskog method with small spatial gradients but including strong dissipation.

As said before, a study of long-range correlations in a granular system fluidized by the random stochastic force (13) has been recently made [10]. In order to analyze the decay of fluctuations, it was assumed that the transport coefficients are equal to the corresponding quantities given by the elastic Enskog theory [1]. According to the results obtained in this paper for a heated granular gas in the low-density regime, although this assumption can be considered as a quite good approximation for the shear viscosity η and the coefficient μ, this is not true for the thermal conductivity since κ^* clearly differs from 1, even for small inelasticity. In this context, it would be interesting to reexamine the conclusions obtained in Ref. [10] when the true Enskog transport coefficients are considered. As said in the Introduction, the Enskog equation for a gas of inelastic hard spheres ($d = 3$) in the absence of thermostats has been recently solved [4] up to Navier-Stokes order. Taking into account these results [4], the extension of the calculations carried out in this paper to higher densities can be easily made. The final expressions of the corresponding Enskog transport coefficients in the three dimensional case are displayed in Appendix B. We plan to extend these expressions for a d dimensional granular fluid in the near future.

ACKNOWLEDGMENTS

V.G. acknowledges partial support from the Ministerio de Ciencia y Tecnología (Spain) through Grant No. BFM2001-0718.

APPENDIX A: CHAPMAN-ENSKOG EXPANSION

In the case of the Gaussian thermostat, the velocity distribution $f^{(1)}$ obeys the kinetic equation

$$\left(\mathcal{L}+\frac{1}{2}\zeta^{(0)}\frac{\partial}{\partial\mathbf{V}}\cdot\mathbf{V}\right)f^{(1)}+\frac{1}{2}\zeta^{(1)}\frac{\partial}{\partial\mathbf{V}}\cdot\left(\mathbf{V}f^{(0)}\right)=-\left(D_t^{(1)}+\mathbf{V}\cdot\nabla\right)f^{(0)}, \tag{A1}$$

with $D_t^{(1)}=\partial_t^{(1)}+\mathbf{u}\cdot\nabla$, $\mathcal{L}$ is the linear operator given by

$$\mathcal{L}f^{(1)}=-\left(J[f^{(0)},f^{(1)}]+J[f^{(1)},f^{(0)}]\right), \tag{A2}$$

and $\zeta^{(1)}$ is a linear functional of $f^{(1)}$ defined as

$$\zeta^{(1)}=-\frac{1}{dnT}\int d\mathbf{v}mV^2\left(J[f^{(0)},f^{(1)}]+J[f^{(1)},f^{(0)}]\right). \tag{A3}$$

The macroscopic balance equations to first order in the gradients are

$$D_t^{(1)}n=-n\nabla\cdot\mathbf{u},\quad D_t^{(1)}u_i=-(mn)^{-1}\nabla_i p,\quad D_t^{(1)}T=-\frac{2T}{d}\nabla\cdot\mathbf{u}. \tag{A4}$$

Use of these in Eq. (A1) yields

$$\left(\mathcal{L}+\frac{1}{2}\zeta^{(0)}\frac{\partial}{\partial\mathbf{V}}\cdot\mathbf{V}\right)f^{(1)}+\frac{1}{2}\zeta^{(1)}\frac{\partial}{\partial\mathbf{V}}\cdot\left(\mathbf{V}f^{(0)}\right)=\mathbf{A}\cdot\nabla\ln T+\mathbf{B}\cdot\nabla\ln n+C_{ij}\nabla_i u_j, \tag{A5}$$

where the expressions of $\mathbf{A}$, $\mathbf{B}$, and C_{ij} are the same as those obtained in the Appendix A of Ref. [2], i.e.,

$$\mathbf{A}(\mathbf{V})=\frac{1}{2}\mathbf{V}\frac{\partial}{\partial\mathbf{V}}\cdot\left(\mathbf{V}f^{(0)}\right)-\frac{T}{m}\frac{\partial}{\partial\mathbf{V}}f^{(0)}, \tag{A6}$$

$$\mathbf{B}(\mathbf{V})=-\mathbf{V}f^{(0)}-\frac{T}{m}\frac{\partial}{\partial\mathbf{V}}f^{(0)}, \tag{A7}$$

$$C_{ij}(\mathbf{V})=\frac{\partial}{\partial V_i}\left(V_j f^{(0)}\right)-\frac{1}{d}\delta_{ij}\frac{\partial}{\partial\mathbf{V}}\cdot\left(\mathbf{V}f^{(0)}\right). \tag{A8}$$

The fact that C_{ij} is traceless implies that the scalar $\zeta^{(1)}=0$ by symmetry. This is special to the low density Boltzmann equation, since at higher densities [4] there is a contribution to $f^{(1)}$ proportional to $\nabla\cdot\mathbf{u}$ leading to a nonzero value of $\zeta^{(1)}$. This can be seen in Appendix B. Comparison with the kinetic equation obeying $f_1^{(1)}$ in the unforced case (Eq. (A5) of Ref. [2]) shows that both kinetic equations only differ in the terms $\partial_t^{(0)}f_1^{(1)}$ (unforced description) and $(\zeta^{(0)}/2)(\partial/\partial\mathbf{V})\cdot\mathbf{V}f_1^{(1)}$ (driven case). These terms give different contributions to the transport coefficients. The solution to Eq. (A5) is of the form

$$f^{(1)}=\mathcal{A}\cdot\nabla\ln T+\mathcal{B}\cdot\nabla\ln n+\mathcal{C}_{ij}\nabla_i u_j. \tag{A9}$$

Substituting Eq. (A9) into Eq. (A1) and identifying coefficients of independent gradients, one gets the following three integral equations determining the unknowns $\mathcal{A}(\mathbf{V})$, $\mathcal{B}(\mathbf{V})$, and $\mathcal{C}_{ij}(\mathbf{V})$:

$$\left(\mathcal{L}+\frac{1}{2}\zeta^{(0)}\frac{\partial}{\partial\mathbf{V}}\cdot\mathbf{V}\right)\begin{pmatrix}\mathcal{A}\\ \mathcal{B}\\ \mathcal{C}_{ij}\end{pmatrix}=\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ C_{ij}\end{pmatrix}. \tag{A10}$$

The expressions (28), (29), and (30) for the transport coefficients follow directly from the integral equations (A10). The transport coefficients are defined as [3]

$$\eta=-\frac{1}{(d-1)(d+2)}\int d\mathbf{V}D_{ij}(\mathbf{V})\mathcal{C}_{ij}(\mathbf{V}), \tag{A11}$$

$$\kappa=-\frac{1}{dT}\int d\mathbf{v}\mathbf{S}(\mathbf{V})\cdot\mathcal{A}(\mathbf{V}), \tag{A12}$$

$$\mu=-\frac{1}{dn}\int d\mathbf{v}\mathbf{S}(\mathbf{V})\cdot\mathcal{B}(\mathbf{V}). \tag{A13}$$

The thermal conductivity κ and the coefficient μ can be easily obtained by multiplying the two first equations of (A10) by $\mathbf{S}(\mathbf{V})$ and integrating over the velocity. The result is

$$\left(\nu_\kappa-\tfrac{3}{2}\zeta^{(0)}\right)\kappa=\frac{d+2}{2}\frac{nT}{m}(1+c), \tag{A14}$$

$$\left(\nu_\mu-\tfrac{3}{2}\zeta^{(0)}\right)\mu=\frac{d+2}{4}\frac{T^2}{m}c. \tag{A15}$$

Upon deriving these expressions, use has been made of the relation

$$\frac{1}{2}\frac{\partial}{\partial\mathbf{V}}\cdot\left(\mathbf{V}f^{(0)}\right)=-T\partial_T f^{(0)} \tag{A16}$$

which follows from the temperature dependence of $f^{(0)}$. The shear viscosity η can be obtained in a similar way by multiplying the third equation of (A10) by $D_{ij}(\mathbf{V})$:

$$\left(\nu_\eta-\zeta^{(0)}\right)\eta=nT. \tag{A17}$$

Equations (A14), (A15), and (A17) lead directly to expressions (28)–(30) appearing in the text.

The analysis in the case of the stochastic thermostat (14) is similar to that made above for the Gaussian one. To first order in gradients, one has

$$\left[\mathcal{L}-\frac{1}{2}T\frac{\zeta^{(0)}}{m}\left(\frac{\partial}{\partial\mathbf{V}}\right)^2\right]f^{(1)}=\mathbf{A}\cdot\nabla\ln T+\mathbf{B}\cdot\nabla\ln n+C_{ij}\nabla_i u_j. \tag{A18}$$

Here, the velocity dependence on the right side of Eq. (A18) is given by Eqs. (A6)–(A8) and we have taken into account that $\zeta^{(1)}=0$. The solution to Eq. (A18) is of the form (A9), where now the corresponding integral equations are

$$\left[\mathcal{L}-\frac{1}{2}T\frac{\zeta^{(0)}}{m}\left(\frac{\partial}{\partial\mathbf{V}}\right)^2\right]\begin{pmatrix}\mathcal{A}\\ \mathcal{B}\\ \mathcal{C}_{ij}\end{pmatrix}=\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ C_{ij}\end{pmatrix}. \tag{A19}$$

Now, we multiply Eq. (A19) by $\mathbf{S}(\mathbf{V})$ and $D_{ij}(\mathbf{V})$ and integrate over the velocity to get the expressions of κ, μ, and η:

$$\nu_\kappa\kappa=\frac{d+2}{2}\frac{nT}{m}(1+c), \tag{A20}$$

$$\nu_\mu\mu=\frac{d+2}{4}\frac{T^2}{m}c, \tag{A21}$$

$$\nu_\eta\eta=nT. \tag{A22}$$

Here, we have made use of the results

$$\int d\mathbf{v}\,\mathbf{S}(\mathbf{V})\cdot\left(\frac{\partial}{\partial\mathbf{V}}\right)^2\mathcal{A}=0, \tag{A23}$$

$$\int d\mathbf{v}\,\mathbf{S}(\mathbf{V})\cdot\left(\frac{\partial}{\partial\mathbf{V}}\right)^2\mathcal{B}=0, \tag{A24}$$

$$\int d\mathbf{v}\,D_{ij}(\mathbf{V})\left(\frac{\partial}{\partial\mathbf{V}}\right)^2\mathcal{C}_{ij}=0, \tag{A25}$$

which follow directly from the solubility conditions (19) and (20).

APPENDIX B: RESULTS FOR A DENSE FLUID OF HARD SPHERES

In this Appendix, we display the results derived for inelastic hard spheres ($d=3$) in the framework of the Enskog equation when the gas is heated by the Gaussian and the stochastic thermostats. To first order in the gradients, the momentum and heat fluxes are

$$P_{ij}^{(1)}=-\eta\left(\nabla_i u_j+\nabla_j u_i-\frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{u}\right)-\gamma\delta_{ij}\nabla\cdot\mathbf{u}, \tag{B1}$$

$$\mathbf{q}^{(1)}=-\kappa\nabla T-\mu\nabla n, \tag{B2}$$

where γ is the bulk viscosity coefficient which vanishes in the low density limit. In a compact form, the transport coefficients can be written as

$$\eta=\eta^k\left[1+\frac{2\pi n^*\chi(1+\alpha)}{15}\right]+\frac{3}{5}\gamma, \tag{B3}$$

$$\gamma = \frac{2}{9}\sqrt{\pi m T} n^* n \chi \sigma (1+\alpha)\left(1-\frac{c}{32}\right), \tag{B4}$$

$$\kappa = \kappa^k \left[1+\frac{\pi n^* \chi (1+\alpha)}{5}\right] + \frac{1}{3}\sqrt{\frac{\pi T}{m}} n^* \chi n \sigma \left(1+\frac{7c}{32}\right), \tag{B5}$$

$$\mu = \mu^k \left[1+\frac{\pi n^* \chi (1+\alpha)}{5}\right], \tag{B6}$$

where χ is the pair correlation function at contact, $n^* = n\sigma^3$ is a reduced density, and the superscript k denotes the contributions from the kinetic parts of the fluxes [4]. These kinetic parts are given by

$$\eta^k = nT\left(\nu_\eta - b_\eta \zeta^{(0)}\right)^{-1}\left[1-\frac{1}{15}(1+\alpha)(1-3\alpha)\pi n^* \chi\right], \tag{B7}$$

$$\begin{aligned}\kappa^k = \frac{5}{2}\frac{nT}{m}\left(\nu_\kappa - b_\kappa \zeta^{(0)}\right)^{-1}&\left\{1+\left[1+c\left(1+\frac{1+\alpha}{6}\pi n^* \chi\right)\right]\right.\\ &\left.+\frac{1}{10}\pi n^* \chi (1+\alpha)^2\left[2\alpha - 1 + \left(\frac{1}{2}(1+\alpha)-\frac{5}{3(1+\alpha)}\right)c\right]\right\},\end{aligned} \tag{B8}$$

$$\begin{aligned}\mu^k = \frac{T}{n}\left(\nu_\mu - b_\mu \zeta^{(0)}\right)^{-1}&\left\{\frac{5}{4}\frac{nT}{m}\left(1+\frac{1+\alpha}{3}\pi n^* \chi\right)\left[1+n\partial_n \ln\left(1+\frac{1+\alpha}{3}\pi n^* \chi\right)\right]c\right.\\ &\left.-\frac{nT}{2m}\pi n^* \chi (1+\alpha)\left(1+\frac{1}{2}n\partial_n \ln \chi\right)\left[\alpha(1-\alpha)+\frac{1}{4}\left(\alpha(1-\alpha)+\frac{4}{3}\right)c\right]\right\}.\end{aligned} \tag{B9}$$

In these equations, $b_\eta = -1$, and $b_\kappa = b_\mu = -\frac{3}{2}$ in the case of the Gaussian thermostat while $b_\eta = b_\kappa = b_\mu = 0$ for the stochastic thermostat.

Up to first order in gradients, the cooling rate ζ is given by

$$\zeta \to \frac{5}{12}\nu_0 \chi (1-\alpha^2)\left(1+\frac{3c}{32}\right) + \left[-\frac{1}{3}\pi n^* \chi + \frac{5}{32}\nu_0\left(1+\frac{3c}{64}\right)\chi\beta\right](1-\alpha^2)\nabla\cdot\mathbf{u}, \tag{B10}$$

where the expression of the quantity β depends on the thermostat used. In the Gaussian case, one has

$$\beta = \pi n^* \chi \frac{(2/45)\lambda - (c/6)(1+\alpha)(5-3\alpha)}{\nu_\gamma - 2\zeta^{(0)} - (5c/64)(1-\alpha^2)\left(1+\frac{3c}{64}\right)\chi}, \tag{B11}$$

while in the stochastic case one gets

$$\beta = \pi n^* \chi \frac{(2/45)\lambda - (c/3)(1+\alpha)}{\nu_\gamma}. \tag{B12}$$

Here, we have introduced the quantities [4]

$$\lambda = \frac{3}{8}(1-\alpha^2)\left(5\alpha^2+4\alpha-1+\frac{c}{12}\frac{159\alpha+3\alpha^2-19\alpha-15\alpha^3}{1-\alpha}\right), \tag{B13}$$

$$\nu_\gamma = \frac{1+\alpha}{48}\nu_0\chi\left[128-96\alpha+15\alpha^2-15\alpha^3+\frac{c}{64}\left(15\alpha^3-15\alpha^2+498\alpha-434\right)\right]. \tag{B14}$$

To get the explicit dependence of the transport coefficients and the cooling rate on the reduced density n^*, one can take for instance the Carnahan-Starling approximation for χ given by

$$\chi = \frac{1-\frac{1}{12}\pi n^*}{1-\frac{1}{6}\pi n^*} \tag{B15}$$

It is easy to check that all results of Appendix B reduce to those presented in the text for the low-density limit ($n^* = 0$).

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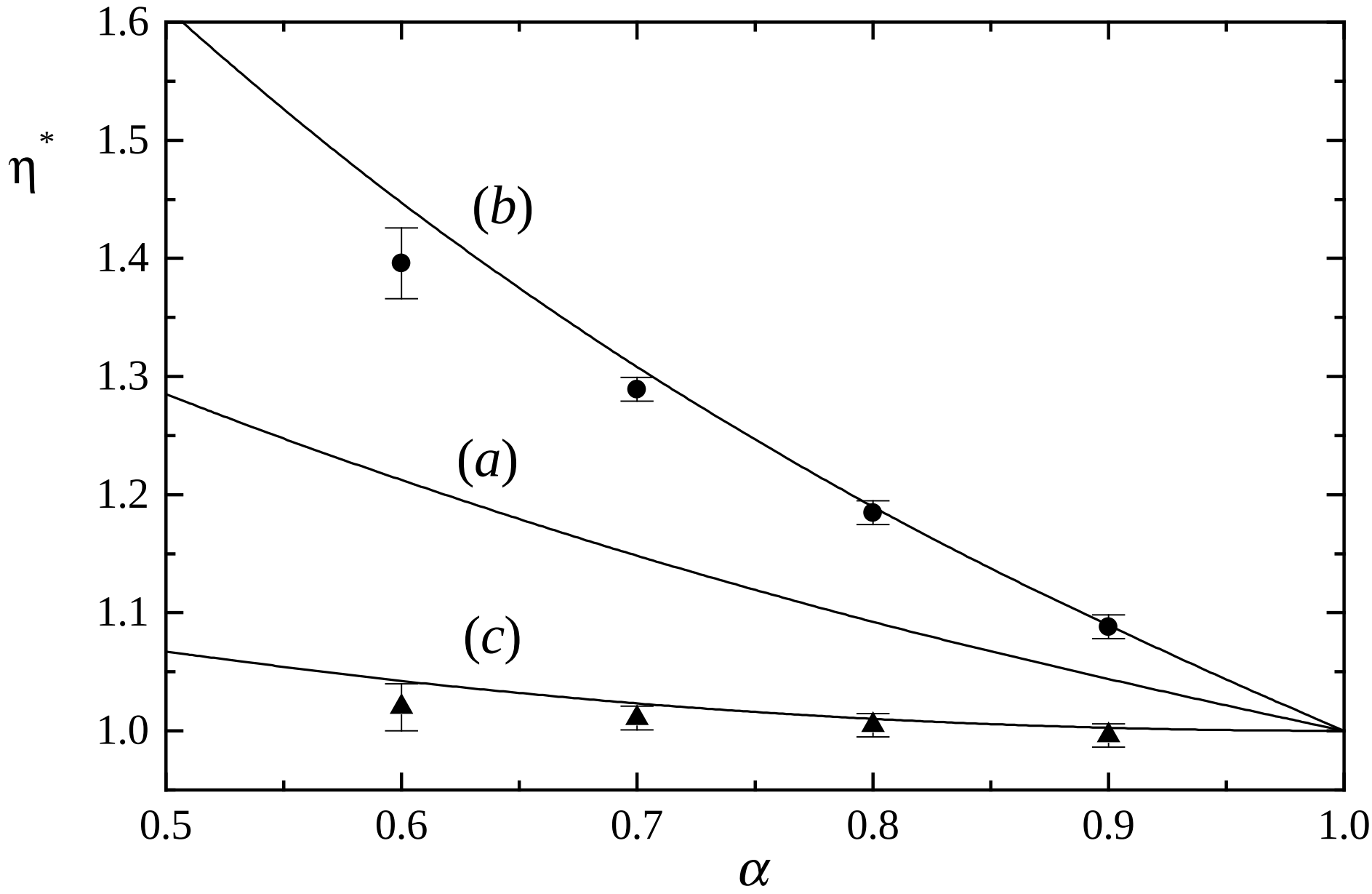
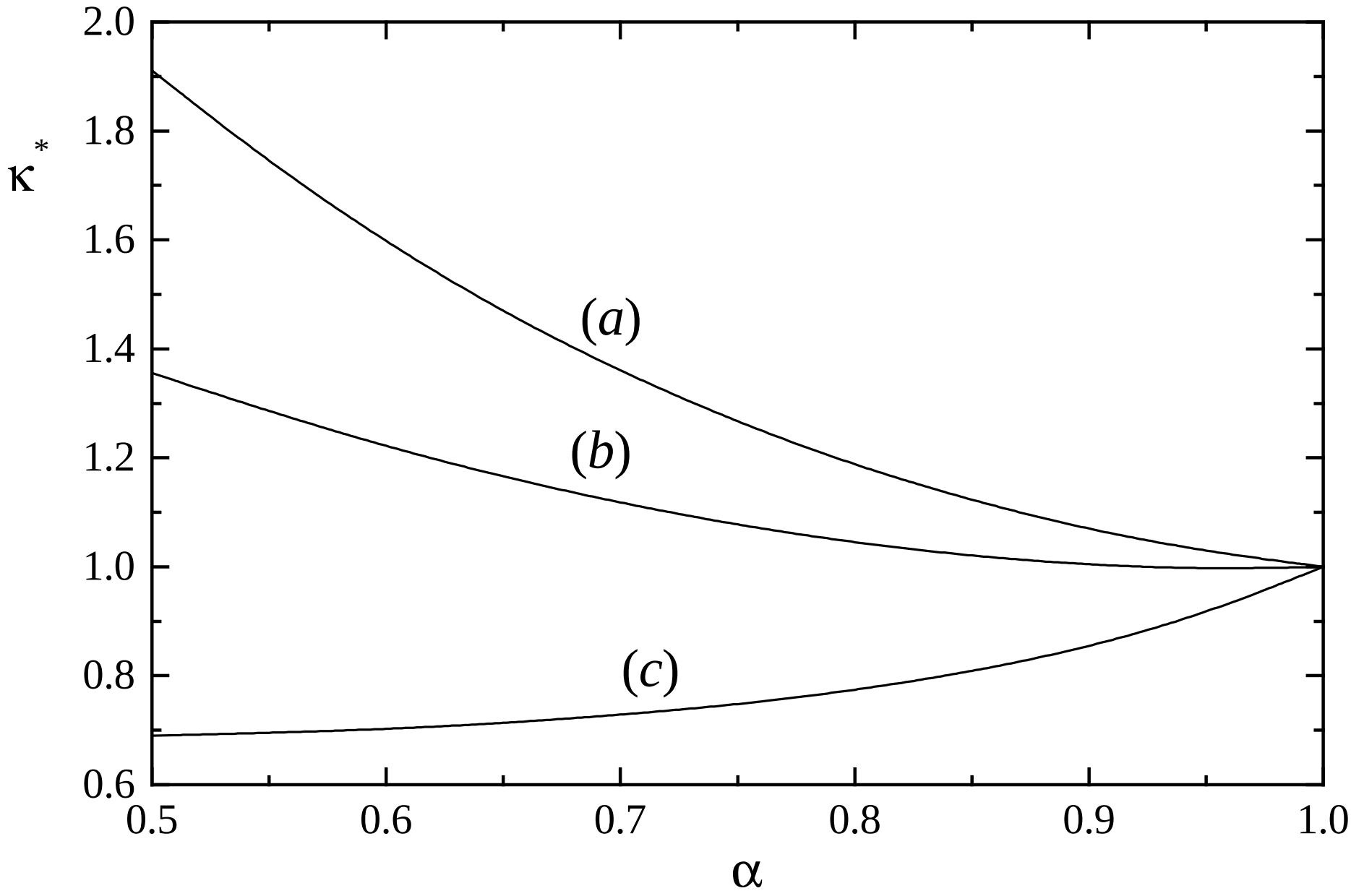
FIGURES

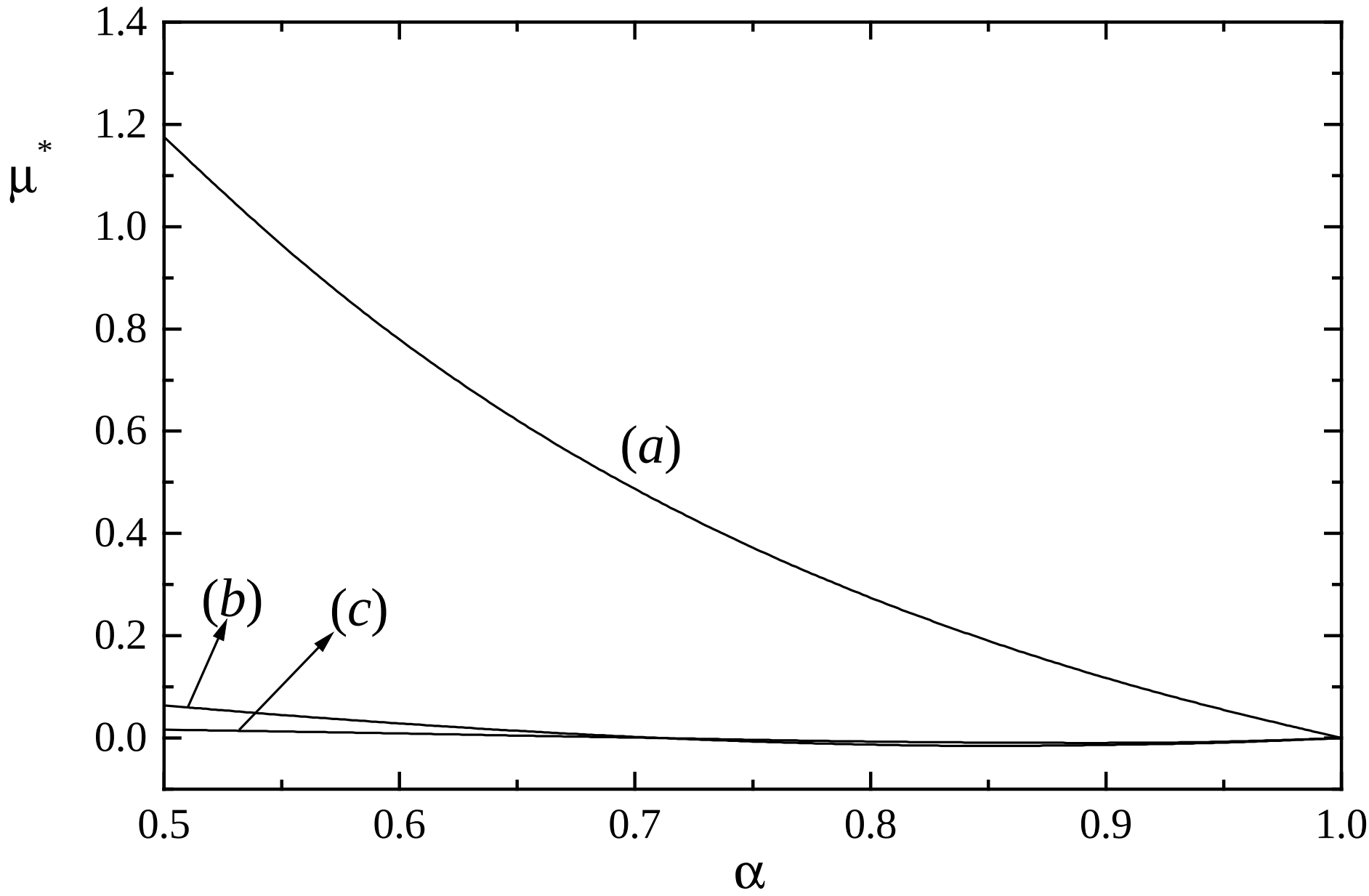
FIG. 1. Reduced shear viscosity $\eta^* = \eta/\eta_0$ as a function of the restitution coefficient α for a three-dimensional system. The solid lines refer to the theoretical expressions derived in the unforced case (a), in the Gaussian case (b), and in the stochastic case (c). The symbols are the results obtained from the Direct Simulation Monte Carlo method in the Gaussian (circles) and stochastic (triangles) cases.

FIG. 2. Reduced thermal conductivity $\kappa^* = \kappa/\kappa_0$ as a function of the restitution coefficient α for a three-dimensional system in the unforced case (a), in the Gaussian case (b), and in the stochastic case (c).

FIG. 3. Reduced coefficient $\mu^* = n\mu/T\kappa_0$ as a function of the restitution coefficient α for a three-dimensional system in the unforced case (a), in the Gaussian case (b), and in the stochastic case (c).


1.6
1.5
1.4
1.3
1.2
1.1
1.0
η*
(b)
(a)
(c)
0.5
0.6
0.7
0.8
0.9
1.0
α


2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
κ*
0.5
0.6
0.7
0.8
0.9
1.0
α
(a)
(b)
(c)


1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
μ*
(a)
(b)
(c)
0.5
0.6
0.7
0.8
0.9
1.0
α